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Spectral analysis of unbounded Jacobi operators with oscillating entries

by

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Abstract. We describe the spectra of Jacobi operators J with some irregular entries. We divide $\mathbb{R}$ into three "spectral regions" for J and using the subordinacy method and asymptotic methods based on some particular discrete versions of Levinson's theorem we prove the absolute continuity in the first region and the pure pointness in the second. In the third region no information is given by the above methods, and we call it the "uncertainty region". As an illustration, we introduce and analyse the **O&P** family of Jacobi operators with weight and diagonal sequences $\{w_n\}$, $\{q_n\}$, where $w_n = n^\alpha + r_n$, $0 < \alpha < 1$ and $\{r_n\}$, $\{q_n\}$ are given by "essentially oscillating" weighted Stolz D^2 sequences, mixed with some periodic sequences. In particular, the limit point set of $\{r_n\}$ is typically infinite then. For this family we also get extra information that some subsets of the uncertainty region are contained in the essential spectrum, and that some subsets of the pure point region are contained in the discrete spectrum.

0. Introduction. In this work we are concerned with spectral properties of a new family of unbounded self-adjoint Jacobi operators acting in $\ell^2(\mathbb{N})$. Most papers dealing with unbounded Jacobi operators are restricted to "regular" sequences of weights and diagonals. However, recently, several works concerning the "irregular" case have appeared: see e.g. [10], [7], [11], [5], [4], [18]. The irregular sequences in those works were mostly given as periodic perturbations (modulations) of regular ones, but also more irregular behaviour of the entries has been studied lately in [2], [3], [12] and [15].

The main goal of studying various kinds of deformations of regular unbounded entries is to illustrate and understand the somewhat delicate influence of such deformations on spectral properties of the operator. This general idea of deformations allowed several examples with new spectral properties to be constructed. In particular, in [2] concrete classes of weights defining Jacobi operators having a few gaps in the essential spectrum were found.

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One can also mention here a dramatic difference between the cases of periodic perturbations of even and odd periods [7]. Recently, in [12], for a class of Jacobi operators many important spectral details on some intervals of $\mathbb{R}$ were observed, e.g., the appearance of dense point spectrum.

The family of Jacobi operators we study in the present work is defined in terms of the weighted Stolz class $D^2(\mu)$, a generalisation of the so-called Stolz class of slowly oscillating sequences (see [19]). The class $D^2(\mu)$ was introduced in [8] to formulate special new versions of discrete Levinson type theorems [8, Ths. 5.1 and 5.3] on asymptotics of solutions of difference equations. These theorems are the tools in the proof of our main result, Theorem 2.2. As we shall see, the family studied here exhibits spectral pictures which do not seem to have been observed before.

After introducing in Section 1 the necessary notation and the main abstract conditions for Theorem 2.2, in Section 2 we divide the real line into three "spectral regions" relative to a Jacobi operator J. This partition of $\mathbb{R}$ is determined mainly by the assumptions of the Levinson theorems we use. Theorem 2.3 states that the first region Σ^- is a subset of the a.c. spectrum of J, and that J is pure point in the second region Σ_+ (under the extra assumption that the diagonal sequence of J has only a finite or countable number of limit points). The third is the "uncertainty region" Σ_{un}, where the theorem gives no information.

Section 3 is devoted to the proof of Theorem 2.2, conducted in several steps, and based on subordinacy methods [13] and asymptotic methods (the Levinson type theorems mentioned above). We also use the H-class method for the transfer matrix sequence (see e.g. [14, 15]).

In Section 4 we study concrete families of Jacobi operators satisfying the general assumptions of Theorem 2.2. We compute the spectral regions for them, and we find additional spectral information concerning the essential and discrete spectrum. In particular, we obtain the discreteness of J in certain parts of pure point regions, and we prove that some subsets of the uncertainty region are included in the essential spectrum. The proofs are based on the Weyl sequences method of [15] and, following [4], on simple but tricky estimates of the quadratic form induced by J. Studying the wide class of J given by deformations which mix oscillations and periodicity (**O&P** family) turned out to be a fruitful idea (see notation in Section 1 and the definition of **O&P** in Section 4). The spectral results for this family are collected in Theorem 4.8. Some more general results are formulated in Theorem 4.4 and Proposition 4.5.

These more or less general studies are illustrated by several concrete examples. In particular, we study special cases with the main oscillatory term having the form

$$\sin(n^\gamma + \theta).$$

The same term was related to various interesting spectral phenomena in several papers (e.g. [12, 19]). The typical spectral information which can be obtained by the general methods described in our paper is as follows (see Example 4.10(2b)).

EXAMPLE 0.1. Consider the Jacobi operator J with weights w_n and diagonals q_n given by

$$w_n = n^\alpha + b_n + c_n \sin(n^\gamma + \theta), \qquad n \in \mathbb{N},$$

where

$$0 < \alpha < 1, \qquad 0 < \gamma < \frac{1-\alpha}{2},$$

and with $\{q_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}$ being real 2-periodic sequences defined by: $q_1 = 1/2$, $q_2 = -1/2$, $b_1 = 2$, $b_2 = 0$, $c_1 = 1$, $c_2 = 0$. Then J is absolutely continuous in $\Sigma^- = (-\infty; -\sqrt{37}/2) \cup (\sqrt{37}/2; +\infty) \subset \sigma_{\mathrm{ac}}(J)$ and pure point in $\Sigma_+ = (-\sqrt{5}/2; \sqrt{5}/2)$. Thus $\Sigma_{\mathrm{un}} = [-\sqrt{37}/2; -\sqrt{5}/2] \cup [\sqrt{5}/2; \sqrt{37}/2]$, however $\mathbb{R} \setminus (-\sqrt{17}/2; \sqrt{17}/2) \subset \sigma_{\mathrm{ess}}(J)$. Moreover J is discrete in $(-1/2; 1/2)$.

As one can see, even in such a particular case, much work remains to be done to get the full spectral picture. Hence, we finish Section 4 by some open problems and a conjecture. Several technical proofs and lemmas are collected in the Appendix.

1. Notation and some abstract conditions. Let us consider the Jacobi matrix

$$\begin{pmatrix} q_1 & w_1 & & & \\ w_1 & q_2 & w_2 & & \\ & w_2 & q_3 & w_3 & \\ & & w_3 & q_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

determined by some given real sequences $\{w_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$. The object of our studies is the *Jacobi operator* J, the maximal operator defined by the above matrix in the Hilbert space $\ell^2(\mathbb{N})$ of square-summable complex sequences on $\mathbb{N}$. So, J is the restriction of the formal Jacobi operator $\mathcal{J}$ to

$$D(J) := \{u \in \ell^2(\mathbb{N}) : \mathcal{J}u \in \ell^2(\mathbb{N})\},$$

where $\mathcal{J}$ acts in the vector space $\ell(\mathbb{N})$ of all complex sequences on $\mathbb{N}$ by

$$(\mathcal{J}u)(n) := w_{n-1}u(n-1) + q_n u(n) + w_n u(n+1), \qquad n \in \mathbb{N}, \tag{1.1}$$

for any complex sequence $u = \{u(n)\}_{n\geq 1}$, with the convention that $w_k = 0 = u(k)$ if $k < 1$. The J's studied here will, in fact, always be self-adjoint by appropriate assumptions (i.e., equivalently, the minimal Jacobi operator will be essentially self-adjoint). Note that we use both kinds of sequence notation: with subscript—like w_n—mainly to denote some "coefficients", and functional—like $u(n)$—mainly for u being "vectors". In this paper we shall usually assume that

$$w_n \neq 0, \quad n \in \mathbb{N}; \quad w_n \to +\infty. \tag{1.2}$$

We shall use the weighted D^k classes introduced in [8] (a generalisation of Stolz's D^k classes, see [19]). Let us recall the relevant notions for the convenience of the reader. Let $\mu := \{\mu_n\}_{n\geq 1}$ be a sequence of "weights", consisting of positive numbers, and let $n_0 \geq 1$. For $1 \leq p < \infty$ and a sequence $x := \{x(n)\}_{n\geq n_0}$ of elements of a normed space we write $x \in \ell^p(\mu)$ iff $\sum_{n=n_0}^{+\infty} \|x(n)\|^p \mu_n < +\infty$. In the case of μ constant equal to 1 we also write ℓ^p instead of $\ell^p(\mu)$, and as usual, ℓ^∞ is the set of bounded sequences. The same notation $\ell^p(\mu)$ is valid for any normed space and any starting index n_0, but recall that for $p = 2$ the similar symbol $\ell^2(\mathbb{N})$ denotes our basic Hilbert space (and $n_0 = 1$ then). The discrete right derivative of x is denoted by Δx, i.e. $(\Delta x)(n) = x(n+1) - x(n)$, and Δ^k is the kth power of Δ for $k = 1, 2, \ldots$. We denote by $D^k(\mu)$ the *weighted D^k class* with weight μ:

$$x \in D^k(\mu) \quad \text{iff} \quad x \in \ell^\infty \text{ and } \Delta^j x \in \ell^{k/j}(\mu),\ j = 1, \ldots k.$$

By $D^2(\boldsymbol{n^\alpha})$ we denote the class $D^2(\mu)$ with $\mu = \{n^\alpha\}_{n\geq 1}$.

The set of all limit points of a real sequence $x := \{x_n\}_{n\geq n_0}$ will be denoted by $\mathrm{LIM}(x)$, i.e., $\mathrm{LIM}(x)$ is the set of all $g \in \mathbb{R} \cup \{+\infty, -\infty\}$ for which there exists a sequence $\{k_n\}_{n\geq 1}$ of integers such that $k_n \to +\infty$ and $x_{k_n} \to g$.

We shall also use the class 0_α, introduced in [16], which consists of all real sequences x such that $(\Delta x)_n = o(n^{-\alpha})$ as $n \to \infty$, and $0 \in \mathrm{LIM}(x)$.

For a sequence $x = \{x_n\}_{n\geq 1}$ and $j = 0, 1$ denote by $x^{(j)}$ the sequence given by

$$x_n^{(j)} := x_{2n+j}, \quad n \in \mathbb{N}.$$

As usual, for a self-adjoint operator A in a Hilbert space, we denote by $\sigma_{\mathrm{ac}}(A)$, $\sigma_{\mathrm{pp}}(A)$, $\sigma_{\mathrm{ess}}(A)$, $\sigma_{\mathrm{d}}(A)$ its absolutely continuous, pure point, essential and discrete spectrum, respectively. To avoid confusion, let us explain the notions of absolute continuity, pure pointness and discreteness, used in this paper, as several other names are also used in similar situations in the literature. Denote by $\mathcal{H}_{\mathrm{ac}}(A)$, $\mathcal{H}_{\mathrm{pp}}(A)$ the space of absolute continuity of A, and the pure point space of A (i.e. the closure of the space spanned by all the eigenvectors of A), respectively. For any Borel subset G of $\mathbb{R}$ denote by $\mathcal{H}_G(A)$ the range of the spectral projection $E_G(A)$ of A corresponding to G. Recall that A is *absolutely continuous* (respectively, *pure point*) *in* G iff $\mathcal{H}_G(A) \subset \mathcal{H}_{\mathrm{ac}}(A)$ (respectively, $\mathcal{H}_G(A) \subset \mathcal{H}_{\mathrm{pp}}(A)$). Note that in the literature, when A is absolutely continuous in G, it is sometimes said that "A has purely absolutely continuous spectrum in G". When G is open and $\sigma_{\mathrm{ess}}(A) \cap G = \emptyset$, then we say that A is *discrete in* G. Note that (under the above definitions) if $\sigma(A) \cap G = \emptyset$, then A is both absolutely continuous and pure point in G, and if moreover G is open, then A is also discrete in G. So, our terminology differs from some of the others.

Our main results will be formulated for operators J which satisfy several abstract conditions. To write them in compact form, we define

$$l_n := w_{2n-1} - w_{2n-2}, \qquad r_n := w_{2n} - w_{2n-1}, \qquad n \in \mathbb{N} \tag{1.3}$$

(**l**eft and **r**ight difference of w at $2n-1$). We also choose the weight μ:

$$\mu_n := |w_{2n-1}|, \qquad n \in \mathbb{N}. \tag{1.4}$$

Surely, this gives $\mu_n = w_{2n-1}$ for n large enough, by (1.2).

The above choice of weight for the weighted D^k class obeys for the most part of our abstract considerations. In this paper we use $D^k(\mu)$ classes for $k = 2$ only.

We combine some of our assumptions into two groups:
conditions **(W)**:

$$\sum_{n=1}^{+\infty} \frac{((\Delta\mu)_n)^2}{\mu_n} < +\infty, \tag{1.5}$$

$$\sum_{n=1}^{+\infty} \frac{1}{\mu_n} = +\infty, \tag{1.6}$$

and *conditions* **(D^2)**:

$$\{l_n\}_{n\geq 1}, \{r_n\}_{n\geq 1} \in D^2(\mu), \tag{1.7}$$

$$\left\{\frac{1}{w_{2n}}\right\} \in D^2(\mu), \tag{1.8}$$

$$q^{(0)}, q^{(1)} \in D^2(\mu). \tag{1.9}$$

Observe that if (1.4) and (1.6) hold then J is self-adjoint, because the Carleman condition $\sum_{n=1}^{+\infty} 1/|w_n| = +\infty$ is satisfied.

Note that conditions **(W)** already appeared in our paper [8]. The reason for assuming conditions **(D^2)** will become more clear later; now one can just remark that they are related to grouping the transfer matrices in pairs (see (3.3), (3.4)), which can be convenient when the sequences defining the operator J contain some 2-periodic terms. The analogous assumptions for the more general, T-periodic case would be much more complicated.

2. Spectral regions and abstract results. Here we define and analyse some "spectral regions" of $\mathbb{R}$ for J, and we formulate our main results on spectral properties of J in these regions.

Define a sequence $\{\gamma_n\}_{n\geq 1}$ of quadratic polynomials on $\mathbb{R}$ by

$$\gamma_n(\lambda) := (r_n - l_n)^2 - 4(\lambda - q_{2n-1})(\lambda - q_{2n}), \qquad n \in \mathbb{N},\ \lambda \in \mathbb{R}. \tag{2.1}$$

The key role of γ_n for our further investigations is explained in Proposition 3.1(ii). The functions $\gamma^\uparrow, \gamma_\downarrow$ on $\mathbb{R}$ are given by

$$\gamma^\uparrow(\lambda) := \limsup_{n\to+\infty} \gamma_n(\lambda), \qquad \gamma_\downarrow(\lambda) := \liminf_{n\to+\infty} \gamma_n(\lambda), \qquad \lambda \in \mathbb{R}.$$

Note that $\gamma^\uparrow, \gamma_\downarrow : \mathbb{R} \to \mathbb{R}$ provided that

$$\{q_n\}_{n\geq 1} \quad \text{and} \quad \{l_n - r_n\}_{n\geq 1} \text{ are bounded.} \tag{2.2}$$

In particular, the above condition holds if we assume $(\mathbf{D}^2)$.

Let us define the following *spectral regions*:

$$\begin{aligned}
\Sigma^- &:= \{\lambda \in \mathbb{R} : \gamma^\uparrow(\lambda) < 0\},\\
\Sigma_+ &:= \{\lambda \in \mathbb{R} : \gamma_\downarrow(\lambda) > 0\},\\
\tilde{\Sigma}_+ &:= \{\lambda \in \Sigma_+ : \lambda \text{ is not a limit point of } \{q_n\}_{n\geq 1}\}.
\end{aligned}$$

Below we list some of their properties. Here, the notion of *interval* includes also the empty set and unbounded intervals.

PROPOSITION 2.1. *If* (2.2) *holds, then the functions $\gamma_\downarrow$, $\gamma^\uparrow$ are continuous, $\gamma_\downarrow$ is concave, Σ_+ is a bounded open interval, $\tilde{\Sigma}_+$ and Σ^- are open sets, and $\mathbb{R} \setminus (-R; R) \subset \Sigma^-$ for some $R > 0$.*

Proof. Each γ_n is concave, since it is a quadratic polynomial with a negative leading coefficient. Concavity is preserved under taking the infimum of a set of functions, and also under taking a pointwise limit, provided that a finite infimum or finite limit exists at each point. Hence, by (2.2), in our case the lower limit also preserves concavity, since $\liminf_{n\to+\infty} a_n = \lim_{n\to+\infty}(\inf_{k\geq n} a_k)$ for any $\{a_n\}$. Thus $\gamma_\downarrow$ is concave. This shows that Σ_+ is an interval, and it must be an open set, since $\gamma_\downarrow$ is also continuous, as a concave function defined on $\mathbb{R}$. The set $\tilde{\Sigma}_+$ is open, because the limit point set of any sequence is closed.

To see the continuity of $\gamma^\uparrow$ observe first that

$$\gamma^\uparrow(\lambda) = -4\lambda^2 + \varphi(\lambda), \qquad \varphi(\lambda) := \limsup_{n\to+\infty}(\alpha_n\lambda + \beta_n)$$

for some bounded sequences $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$, independent of λ. But any affine function is convex, so the argument above yields the continuity of φ, and thus of $\gamma^\uparrow$, and Σ^- is open.

By (2.2), there exists $R > 0$ such that $\gamma_n(\lambda) \leq -1$ for any $n \in \mathbb{N}$ and $|\lambda| \geq R$. Hence if $|\lambda| \geq R$, then $\lambda \in \Sigma^-$ and $\lambda \notin \Sigma_+$. ∎

Let

$$L_n = \sum_{k=1}^{n} \frac{l_k + r_k}{\mu_k}, \qquad M_n = \sum_{k=1}^{n} \frac{1}{\mu_k}, \qquad n \in \mathbb{N}. \tag{2.3}$$

Denote

$$\Sigma_P := \Big\{ \lambda \in \tilde{\Sigma}_+ : \exists_{0<t<\sqrt{\gamma_\downarrow(\lambda)}} \Big\{ \exp\Big(-\frac{1}{2}(L_n + tM_n) \Big) \Big\} \in \ell^2 \Big\}. \tag{2.4}$$

We are now ready to formulate our main result.

Theorem 2.2. *Assume that* (1.2), (1.4), **(W)** *and* **(D^2)** *hold. Then* $\overline{\Sigma^-} \subset \sigma_{\mathrm{ac}}(J)$, *$J$ is absolutely continuous in Σ^-, and J is pure point in Σ_P.*

The proof is given in the next section.

The definition of Σ_P is rather complicated, so it would be convenient to formulate conditions guaranteeing that Σ_P is the whole $\tilde{\Sigma}_+$. Denote by **(P)** the combination of the two additional conditions

$$\{L_n\}_{n\geq 1} \text{ is bounded from below} \quad \textbf{or} \quad \liminf_{n\to+\infty}(l_n + r_n) \geq 0, \tag{2.5}$$

$$\text{for any } \epsilon > 0,\ \{\exp(-\epsilon M_n)\}_{n\geq 1} \in \ell^2. \tag{2.6}$$

Assume **(P)** and let $\lambda \in \tilde{\Sigma}_+$. We have $\gamma_\downarrow(\lambda) > 0$, so choose an arbitrary t satisfying $0 < t < \sqrt{\gamma_\downarrow(\lambda)}$. If $\{L_n\}_{n\geq 1}$ is bounded from below, then by (2.6), $\big\{\exp\big(-\frac{1}{2}(L_n+tM_n)\big)\big\} \in \ell^2$. If $\liminf_{n\to+\infty}(l_n+r_n) \geq 0$, then $l_n+r_n \geq -t/2$ for t as above and n large enough. Thus for some $C \in \mathbb{R}$,

$$L_n \geq \frac{-t}{2} M_n + C, \qquad n \geq 1.$$

Hence $\exp\big(-\frac{1}{2}(L_n + tM_n)\big) \leq \exp\big(-\frac{t}{4}M_n\big) \cdot \exp(-C/2)$ for all n. Thus in both cases of (2.5) we have $\Sigma_P = \tilde{\Sigma}_+$, which allows us to formulate the following consequence of Theorem 2.2.

Theorem 2.3. *Assume that* (1.2), (1.4), **(W)**, **(D^2)** *and* **(P)** *hold. Then* $\overline{\Sigma^-} \subset \sigma_{\mathrm{ac}}(J)$, *$J$ is absolutely continuous in Σ^-, and J is pure point in $\tilde{\Sigma}_+$.*

We also immediately obtain the following (see e.g. [15, Prop. 5.15(ii)]):

Corollary 2.4. *Under the assumptions of Theorem* 2.3, *if the set* $\mathrm{LIM}(\{q_n\}_{n\geq 1})$ *is at most countable, then* $\overline{\Sigma^-} \subset \sigma_{\mathrm{ac}}(J)$, *$J$ is absolutely continuous in Σ^-, and J is pure point in Σ_+.*

Observe, however, that in our case the countability assumption is equivalent to the condition "$q^{(0)}$, $q^{(1)}$ are both convergent". This is a direct consequence of (1.9), [8, Lemma 5.2] and Lemma 5.1(2). Without this condition the limit point set is the union of two closed intervals (the limit point sets of $q^{(0)}$ and $q^{(1)}$), at least one of them non-trivial. So our "pure point" information can be essentially weaker.

Note that Theorem 2.3 says nothing about spectral properties of J in the set $\Sigma_{\mathrm{un}} := \mathbb{R} \setminus (\Sigma^- \cup \Sigma_+)$. This is why we call it the *uncertainty region.*

3. Transfer matrices and generalised eigenvectors in the spectral regions. For a fixed $\lambda \in \mathbb{C}$ we consider *generalised eigenvectors* of J for λ, i.e., $u = \{u(n)\}_{n\geq 1} \in \ell(\mathbb{N})$ such that

$$((\mathcal{J} - \lambda)u)(n) = 0, \qquad n \geq \mathbf{2}. \tag{3.1}$$

If $w_n \neq 0$ for all $n \geq 1$, then the above condition can be equivalently written as

$$\begin{pmatrix} u(n) \\ u(n+1) \end{pmatrix} = B_n(\lambda) \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix}, \qquad n \geq 2, \tag{3.2}$$

where $B_n(\lambda)$ is the transfer matrix for J and λ, given for $n \geq 2$ by

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -\dfrac{w_{n-1}}{w_n} & \dfrac{\lambda - q_n}{w_n} \end{pmatrix}. \tag{3.3}$$

The assumptions of Theorem 2.2 (see, e.g., (1.3)) are closely related to some regularity of the products

$$A_n(\lambda) = B_{2n}(\lambda) B_{2n-1}(\lambda), \qquad n \geq 2. \tag{3.4}$$

One can easily compute that

$$A_n(\lambda) = -\Big(I + \frac{1}{\mu_n} V_n(\lambda)\Big), \tag{3.5}$$

where μ_n is given by (1.4) and for $n \geq 2$,

$$V_n(\lambda) = -\begin{pmatrix} l_n & \lambda - q_{2n-1} \\ \dfrac{(q_{2n} - \lambda) w_{2n-2}}{w_{2n}} & \dfrac{r_n w_{2n-1}}{w_{2n}} + \dfrac{(\lambda - q_{2n})(\lambda - q_{2n-1})}{w_{2n}} \end{pmatrix}. \tag{3.6}$$

We denote by $\operatorname{discr} C$ the discriminant of the characteristic polynomial of the 2×2 matrix C, i.e.,

$$\operatorname{discr} C = (\operatorname{tr} C)^2 - 4 \det C = (C_{11} - C_{22})^2 + 4 C_{12} C_{21}. \tag{3.7}$$

In particular, the formula for $\operatorname{discr} V_n(\lambda)$ is somewhat related to the formula (2.1) for $\gamma_n(\lambda)$:

$$\begin{aligned} \operatorname{discr} V_n(\lambda) = {} & \Big[(r_n - l_n) + \Big(r_n\Big(1 - \frac{w_{2n-1}}{w_{2n}}\Big) - \frac{(\lambda - q_{2n})(\lambda - q_{2n-1})}{w_{2n}}\Big)\Big]^2 \\ & - 4(\lambda - q_{2n-1})(\lambda - q_{2n}) \frac{w_{2n-2}}{w_{2n}}. \end{aligned} \tag{3.8}$$

The following technical result will help us to study some properties of transfer matrices and generalised eigenvectors for λ in the spectral regions, and will be used in the proof of Theorem 2.2.

PROPOSITION 3.1. *Let* $\lambda \in \mathbb{R}$. *Suppose that* (1.2), (1.4), **(W)** *and* **(D**^{2}**)** *hold. Then*

(i) $\{V_n(\lambda)\}_{n\ge 2} \in D^2(\mu)$;

(ii) $\epsilon_n(\lambda) := \operatorname{discr} V_n(\lambda) - \gamma_n(\lambda) \to 0$, $\tilde{\epsilon}_n(\lambda) := \operatorname{tr} V_n(\lambda) + (l_n + r_n) \to 0$;

(iii) $\lambda \in \Sigma^-$ $(\in \Sigma_+)$ *iff* $\limsup_{n\to+\infty} \operatorname{discr} V_n(\lambda) < 0$ (> 0);

(iv) *if* $\lambda \in \tilde{\Sigma}_+$, *then*
$$\liminf_{n\to+\infty} |\nu_{n,\pm}(\lambda) - (V_n(\lambda))_{11}| > 0,$$
where for n *with* $\operatorname{discr} V_n(\lambda) > 0$ *we denote*
$$\nu_{n,\pm}(\lambda) := \frac{\operatorname{tr} V_n(\lambda) \pm \sqrt{\operatorname{discr} V_n(\lambda)}}{2}. \tag{3.9}$$

Proof. To prove (i) we should check that the sequences of matrix coefficients of $\{V_n(\lambda)\}_{n\ge 2}$ are in $D^2(\mu)$. We have
$$\frac{w_{2n-1}}{w_{2n}} = 1 - \frac{r_n}{w_{2n}}, \qquad \frac{w_{2n-2}}{w_{2n}} = 1 - \frac{l_n + r_n}{w_{2n}}. \tag{3.10}$$
Hence, it suffices to use **(D**^{2}**)** and the fact that the set of scalar $D^2(\mu)$ sequences is an algebra for the weight μ satisfying **(W)**—see [8, Section 2.1] (in particular the "shiftability" of μ follows from (1.5) and from the fact that $\lim_{n\to+\infty} \mu_n = +\infty$, because $\mu_{n+1}/\mu_n \to 1$ in this case).

Observe that the boundedness of $D^2(\mu)$ sequences, (1.2) and (3.10) give
$$\frac{w_{2n-1}}{w_{2n}} \to 1, \qquad \frac{w_{2n-2}}{w_{2n}} \to 1, \tag{3.11}$$
and by (2.1), (3.6), (3.8) this also gives (ii). From (ii) we immediately get (iii).

Now, let $\lambda \in \tilde{\Sigma}_+$. By (iii) we have $\operatorname{discr} V_n(\lambda) > 0$ for $n \ge n_0$ with n_0 large enough and then, using the boundedness of $\{V_n(\lambda)\}_{n\ge 2}$ (e.g., by (i) and (3.7)), we get
$$\begin{aligned}
|\nu_{n,\pm}(\lambda) - (V_n(\lambda))_{11}| &= \frac{1}{2}\big|\pm\sqrt{\operatorname{discr} V_n(\lambda)} - [(V_n(\lambda))_{11} - (V_n(\lambda))_{22}]\big| \\
&\ge \delta|\operatorname{discr} V_n(\lambda) - [(V_n(\lambda))_{11} - (V_n(\lambda))_{22}]^2| = 4\delta|(V_n(\lambda))_{12}(V_n(\lambda))_{21}| \\
&= 4\delta|q_{2n-1} - \lambda|\,|q_{2n} - \lambda|\left|\frac{w_{2n-2}}{w_{2n}}\right|
\end{aligned}$$
for some n-independent $\delta > 0$. Using now (3.11) and the fact that λ is not a limit point of $\{q_n\}_{n\ge 1}$, we get the assertion of (iv). ∎

We shall use the notion of the H class for sequences of complex 2×2 matrices (see, e.g., [14, 15]). Recall that $\{C_n\}_{n\ge n_0} \in H$ iff there exists $M > 0$ such that
$$\|C_n \cdots C_{n_0}\|^2 \le M \prod_{k=n_0}^{n} |\det C_k|, \qquad n \ge n_0.$$
This class is a convenient tool in studying the absolutely continuous part of some Jacobi operators, because of nonexistence of subordinate solutions

for a fixed spectral parameter λ, following from $\{B_n(\lambda)\}_{n\geq 2} \in H$. For this reason, several sufficient conditions for a matrix sequence to be in H have been proved in [14, 15]. We formulate here one more result of this kind. Its proof, presented in the Appendix, is based on a discrete version of the Levinson theorem, namely Theorem 5.1 of [8] (see e.g. [1, 7] for other discrete versions of the Levinson theorem). Note that below we **do not** assume (1.4), but we consider a more general case.

CRITERION 3.2. *Suppose that C_n are invertible complex 2×2 matrices for $n \geq n_0$, and that*

$$C_n = I + \frac{1}{\mu_n} V_n + R_n, \qquad n \geq n_0,$$

where the positive scalar sequence $\{\mu_n\}_{n\geq n_0}$ satisfies

$$\mu_n \to +\infty, \qquad \sum_{n=n_0}^{+\infty} \frac{((\Delta\mu)_n)^2}{\mu_n} < +\infty, \qquad \sum_{n=n_0}^{+\infty} \frac{1}{\mu_n} = +\infty,$$

*the **real** matrix sequence $\{V_n\}_{n\geq n_0}$ is in $D^2(\mu)$,*

$$\limsup_{n\to+\infty} \operatorname{discr} V_n < 0,$$

and the complex matrix sequence $\{R_n\}_{n\geq n_0}$ is in ℓ^1. Then $\{C_n\}_{n\geq n_0} \in H$.

The following result gives our main argument for the proof of the absolutely continuous part of Theorem 2.2.

PROPOSITION 3.3. *Suppose* (1.2), (1.4), **(W)** *and* **(D^2)** *hold. If $\lambda \in \Sigma^-$, then $\{B_n(\lambda)\}_{n\geq 2} \in H$.*

Proof. By (3.5) and Proposition 3.1(i), (iii), Criterion 3.2 implies that $\{-A_n(\lambda)\}_{n\geq 2} \in H$, thus also $\{A_n(\lambda)\}_{n\geq 2} \in H$. Observe $\{(\Delta w)_n\}_{n\geq 1} \in \ell^\infty$, by (1.7). Hence, by (1.2) and (1.9), we have

$$\frac{w_{n-1}}{w_n} \to 1 \quad \text{and} \quad \left\{\frac{\lambda - q_n}{w_n}\right\}_{n\geq 2} \in \ell^\infty.$$

Now, by (3.3), $\{B_n(\lambda)\}_{n\geq 2}$, $\{B_n(\lambda)^{-1}\}_{n\geq 2} \in \ell^\infty$, which gives the assertion by [15, Proposition 5.7(ii)]. ∎

The next proposition will be the base for the proof of the pure point part of Theorem 2.2.

From now on, the jth coordinate of a vector v is denoted by $[v]_j$, and $\cdot^\top$ is used for matrix or vector transposition.

PROPOSITION 3.4. *Suppose* (1.2), (1.4), **(W)** *and* **(D^2)** *hold. If $\lambda \in \Sigma_P$, then there exists a nonzero generalised eigenvector of J for λ, which belongs to $\ell^2(\mathbb{N})$.*

Proof. Fix $\lambda \in \Sigma_P$. We shall first prove that the recurrent vector equation

$$(x_{n+1})^\top = A_n(\lambda)(x_n)^\top, \qquad n \geq 2,$$

has a nonzero solution $\{x_n\}_{n\geq 2}$ belonging to ℓ^2. Once we prove it, the assertion follows, because defining

$$\begin{aligned} u(2n) &:= [x_{n+1}]_1, \quad u(2n+1) := [x_{n+1}]_2, \qquad n \geq 1, \\ u(1) &:= [(B_2(\lambda))^{-1}(x_2)^\top]_1, \end{aligned}$$

we check at once by (3.4) that (3.2) holds, so $\{u(n)\}_{n\geq 1}$ is the nonzero generalised eigenvector.

By (1.2) the matrices $A_n(\lambda)$ are all invertible and multiplication by the scalar sequence $\{(-1)^n\}_{n\geq 2}$ does not change the ℓ^2-norm, so it is sufficient to find a nonzero ℓ^2 solution of the equation

$$(x_{n+1})^\top = -A_n(\lambda)(x_n)^\top, \qquad n \geq n_0, \tag{3.12}$$

for some $n_0 \geq 2$. To do this we also use one of discrete versions of the Levinson theorems [8, Th. 5.3]. The assumption of that theorem holds for $\{-A_n(\lambda)\}_{n\geq 2}$ by (3.5) and by Proposition 3.1(i), (iii), (iv). We find, in particular, that there exists $n_0 \geq 2$ and a nonzero solution $\{x_n\}_{n\geq n_0}$ of (3.12) satisfying

$$x_n = \Big(\prod_{k=n_0}^{n-1}\Big(1 + \frac{\rho_k}{\mu_k}\Big)\Big) y_n, \qquad n \geq n_0 + 1, \tag{3.13}$$

where $\{y_n\}_{n\geq n_0}$ is a bounded sequence of $\mathbb{C}^2$ vectors, $\rho_n \in \mathbb{R}$ and

$$\rho_n - \nu_{n,-}(\lambda) \to 0 \tag{3.14}$$

(with $\nu_{n,-}(\lambda)$ given by (3.9)). The proof is completed by showing that $\{b_n\}_{n\geq n_0} \in \ell^2$ with $b_n := \prod_{k=n_0}^{n}(1 + \rho_k/\mu_k)$. By (3.14), (3.9) and by Proposition 3.1(i), (ii) we have $\rho_n = -\frac{1}{2}(l_n + r_n) - \frac{1}{2}\sqrt{\gamma_n(\lambda)} + \delta_n$ with $\delta_n \to 0$. Choose now t for λ according to the definition (2.4) of Σ_P. For some $N \geq n_0$,

$$-\mu_n < \rho_n \leq -\frac{1}{2}(l_n + r_n + t), \qquad n \geq N,$$

the left inequality following from $\mu_n \to +\infty$ (see (1.2)) and the boundedness of $\{\rho_n\}$ (see $(\mathbf{D}^2)$). Hence there exist constants $C, C' > 0$ such that for $n \geq N$,

$$\begin{aligned} |b_n| &\leq C \exp\Big(\sum_{k=N}^{n} \ln\Big(1 + \frac{\rho_k}{\mu_k}\Big)\Big) \leq C \exp\Big(\sum_{k=N}^{n} \frac{\rho_k}{\mu_k}\Big) \\ &\leq C \exp\Big(-\frac{1}{2}\sum_{k=N}^{n} \frac{l_k + r_k + t}{\mu_k}\Big) \leq C' \exp\Big(-\frac{1}{2}(L_n + tM_n)\Big), \end{aligned}$$

so $\{b_n\}_{n\geq n_0} \in \ell^2$ by the choice of t. ∎

Now we have all the tools necessary to prove the main theorem with the use of standard subordination theory techniques for Jacobi operators (see the basic paper [13] and some conclusions formulated in [6], [9], [14], [15]).

Proof of Theorem 2.2. As already mentioned, under our assumptions J is self-adjoint. Using [15, Theorem 5.6], by Propositions 2.1 and 3.3 we see that $\overline{\Sigma^-} \subset \sigma_{\mathrm{ac}}(J)$ and J is absolutely continuous in Σ^-. By [15, Lemma 5.13] and Proposition 3.4 we get the pure pointness of J in Σ_P. ∎

4. Essential oscillations and the O&P family. We study here some more concrete Jacobi operators satisfying the abstract assumptions of Theorem 2.2. We start with a result (Theorem 4.4) which will serve for all our examples presented here. It concerns J with w_n being a perturbation of n^α, with $0 < \alpha < 1$, by a bounded sequence, and with a bounded sequence $\{q_n\}$, where both sequences are given by formulae combining some 2-periodic sequences and weighted D^2 sequences. The weights μ_n for this D^2 class are chosen as w_{2n-1} for large n. Note that

$$D^2(\mu) = D^2(\boldsymbol{n^\alpha}) \tag{4.1}$$

in that case (however, usually we cannot replace μ_n by n^α, checking the assumptions **(W)**). The following lemma gives a convenient $D^2(\boldsymbol{n^\alpha})$ criterion for sequences defined by some C^2 functions.

LEMMA 4.1. *If $0 < \alpha < 1$, $c \geq 0$, $f : [c; +\infty) \to \mathbb{C}$ is a bounded C^2 function, and*

$$\int_c^{+\infty} |f^{(j)}(s)|^{2/j} s^\alpha \, ds < +\infty \quad \textit{for } j = 1, 2,$$

then the sequence x given for $n > c$ by $x_n = f(n)$ is a $D^2(\boldsymbol{n^\alpha})$ sequence.

The proof can be easily obtained from the integral estimate for $\Delta^k x$ in [19, p. 246].

EXAMPLE 4.2. By Lemma 4.1 the scalar sequences given for large n by the following formulae are in $D^2(\boldsymbol{n^\alpha})$ $(0 < \alpha < 1)$:

(1) $g(n^\gamma)$, where $0 < \gamma < (1-\alpha)/2$ and $g : [1; +\infty) \to \mathbb{C}$ is a bounded C^2 function with g' and g'' bounded; in particular, $\sin(n^\gamma + \theta)$ with any phase θ;
(2) $(n - r_1)^\alpha - (n - r_2)^\alpha$ for any fixed $r_1, r_2 \in \mathbb{R}$.

A general example of a 0_α sequence (see notation in Section 1) of the type of (1) above is worth mentioning. The sequence given for large n by

(3) $g(n^\gamma)$, where $0 < \gamma < 1 - \alpha$ and $g : [1; +\infty) \to \mathbb{C}$ is a periodic C^1 function with $\inf g([1; +\infty)) \leq 0 \leq \sup g([1; +\infty))$

is in the class 0_α (see [15]).

REMARK 4.3. If $0 < \alpha < 1$ and $x \in D^2(\boldsymbol{n^\alpha})$, then by [8, Lemma 5.2], $(\Delta x)_n = o(n^{-\alpha})$ as $n \to +\infty$. Thus to prove $x \in 0_\alpha$ for such an x, we only need to check the zero limit subsequence condition.

The result below, with several versions of assumptions, will be used to construct the examples presented at the end of the section.

THEOREM 4.4. *Let $0<\alpha<1$, and consider the Jacobi operator J given by*

$$w_n = n^\alpha + b_n + c_n h_n, \qquad q_n = a_n + y_n, \qquad n \in \mathbb{N}, \tag{4.2}$$

where $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}, \{h_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ are real sequences satisfying:

(i) $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}$ *are 2-periodic;*
(ii) $h^{(0)}, h^{(1)} \in D^2(\boldsymbol{n^\alpha})$*;*
(iii) $y_n \to 0$ *and* $y^{(0)}, y^{(1)} \in D^2(\boldsymbol{n^\alpha})$*;*
(iv) $n^\alpha + b_n + c_n h_n \neq 0$ *for all* $n \in \mathbb{N}$.

Let

$$d_{\mathrm{pp}} := \liminf_{n\to+\infty} |(b_1 - b_2) + \tilde{h}_n|, \qquad d_{\mathrm{ac}} := \limsup_{n\to+\infty} |(b_1 - b_2) + \tilde{h}_n|, \tag{4.3}$$

with $\tilde{h}_n = c_1 h_n^{(1)} - c_2 h_n^{(0)}$, and define

$$a_\pm := \lambda_\pm(d_{\mathrm{ac}}), \qquad p_\pm := \lambda_\pm(d_{\mathrm{pp}}), \qquad e_\pm := \lambda_\pm(b_1 - b_2), \tag{4.4}$$

where for any $t \in \mathbb{R}$, $\lambda_-(t) \leq \lambda_+(t)$ are the solutions of the equation

$$(\lambda - a_1)(\lambda - a_2) = t^2.$$

Then:

(A) *J is absolutely continuous in $\mathbb{R}\setminus[a_-; a_+]$, $\mathbb{R}\setminus(a_-; a_+) \subset \sigma_{\mathrm{ac}}(J)$ and J is pure point in $(p_-; p_+)$;*

(B) *we have*

$$[d_{\mathrm{pp}}; d_{\mathrm{ac}}] := \{|(b_1 - b_2) + s| : \liminf_{n\to+\infty} \tilde{h}_n \leq s \leq \limsup_{n\to+\infty} \tilde{h}_n\}; \tag{4.5}$$

(C) *if moreover $h \in 0_\alpha$ and $n^\alpha + b_n \neq 0$ for all $n \in \mathbb{N}$, then $\mathbb{R} \setminus (e_-; e_+) \subset \sigma_{\mathrm{ess}}(J)$;*

(D) *if we assume $h \in D^2(\boldsymbol{n^\alpha})$ instead of* (ii)*, then* (ii) *holds and*

$$[d_{\mathrm{pp}}; d_{\mathrm{ac}}] := \{|(b_1 - b_2) + t(c_1 - c_2)| : \liminf_{n\to+\infty} h_n \leq t \leq \limsup_{n\to+\infty} h_n\}; \tag{4.6}$$

(C + D) *if we assume $h \in D^2(\boldsymbol{n^\alpha})$ and $0 \in \mathrm{LIM}(h)$, then $|b_1 - b_2| \in [d_{\mathrm{pp}}; d_{\mathrm{ac}}]$ and $(e_-; e_+) \subset (a_-; a_+)$; if also $n^\alpha + b_n \neq 0$ for all $n \in \mathbb{N}$, then $\mathbb{R} \setminus (e_-; e_+) \subset \sigma_{\mathrm{ess}}(J)$.*

Proof. Let us check the assumptions of Theorem 2.3. For large n we have

$$\mu_n = (2n - 1)^\alpha + b_1 + c_1 h^{(1)}_{(n-1)}. \tag{4.7}$$

Obviously (1.2), (1.6), (1.9) hold by (4.1), (4.7), and by assumptions (ii)–(iv).

To get (1.5) it is enough to prove

$$\left\{\frac{(\Delta\mu)_n}{n^{\alpha/2}}\right\}_{n\ge 1}\in \ell^2. \tag{4.8}$$

Indeed, $(\Delta\mu)_n = (2n+1)^\alpha - (2n-1)^\alpha + c_1(\Delta h^{(1)})_{(n-1)}$, thus (4.8) holds from (ii) and from the estimate

$$\frac{(2n+1)^\alpha-(2n-1)^\alpha}{n^{\alpha/2}} \le \text{const}\,\frac{1}{n^{1-\alpha/2}}.$$

To obtain (1.7) observe that

$$l_n = (2n-1)^\alpha - (2n-2)^\alpha + b_1 - b_2 + c_1 h^{(1)}_{(n-1)} - c_2 h^{(0)}_{(n-1)}, \tag{4.9}$$

$$r_n = (2n)^\alpha - (2n-1)^\alpha + b_2 - b_1 + c_2 h^{(0)}_{(n)} - c_1 h^{(1)}_{(n-1)}, \tag{4.10}$$

hence (1.7) follows from (ii) and Example 4.2(2).

Now we prove (1.8). We shall use the following formulae for the discrete derivatives of the sequence $\frac{1}{x}$:

$$\left(\Delta\frac{1}{x}\right)_n = \frac{-(\Delta x)_n}{x_{n+1}x_n},\qquad \left(\Delta^2\frac{1}{x}\right)_n = \frac{(\Delta x)_n^2}{x_{n+1}^2 x_n} + \frac{(\Delta x)_{n+1}(\Delta x)_n}{x_{n+2}x_{n+1}^2} - \frac{(\Delta^2 x)_n}{x_{n+2}x_{n+1}}.$$

So, to obtain (1.8), it is enough to check

$$\text{(a)}\ \left\{\frac{(\Delta w^{(0)})_n}{n^{3\alpha/2}}\right\}_{n\ge1}\in\ell^2,\ \text{(b)}\ \left\{\frac{(\Delta w^{(0)})_n}{n^{\alpha}}\right\}_{n\ge1}\in\ell^2,\ \text{(c)}\ \left\{\frac{(\Delta^2 w^{(0)})_n}{n^{\alpha}}\right\}_{n\ge1}\in\ell^1.$$

We have $w_n^{(0)} = 2^\alpha n^\alpha + b_2 + c_2 h_n^{(0)}$, thus (b) follows from (ii), (a) follows from (b), and to get (c) we can use (ii) and the fact that for $\eta := \{n^\alpha\}_{n\ge1}$ we have $\{(\Delta^2\eta)_n/n^\alpha\}_{n\ge1}\in\ell^1$. So, we have checked **(W)** and **(D^2)**. To check **(P)** observe that by (ii),

$$l_n + r_n = (2n)^\alpha - (2n-2)^\alpha + c_2(\Delta h^{(0)})_{n-1} \to 0,$$

which gives (2.5). To get (2.6) we can estimate first

$$\mu_k = |w_{2k-1}| \le 2k^\alpha,\qquad k\ge k_0,$$

for some k_0 sufficiently large. Thus for $n \ge k_0$,

$$M_n \ge C_1 + \beta n^{1-\alpha}$$

with some constants $C_1\in\mathbb{R}$ and $0<\beta<+\infty$, and hence (2.6) follows. Now observe that by (ii) and by (2.1), (4.9), (4.10) there exists a sequence z convergent to 0 such that

$$\gamma_n(\lambda) = z_n + 4[(b_1-b_2)+\tilde h_n]^2 - 4(\lambda-a_1)(\lambda-a_2).$$

By Lemma 5.1(1), (2) we get

$$\gamma^{\uparrow}(\lambda) = 4[d_{\mathrm{ac}}^2 - (\lambda-a_1)(\lambda-a_2)] \quad\text{and}\quad \gamma_{\downarrow}(\lambda) = 4[d_{\mathrm{pp}}^2 - (\lambda-a_1)(\lambda-a_2)],$$

which gives $\Sigma^- = \mathbb{R} \setminus [a_-; a_+]$ and $\Sigma_+ = (p_-; p_+)$, and thus by Corollary 2.4 we obtain assertion (A).

To get (B) we use Lemma 5.1(1) for the function F given by $F(s) = |(b_1 - b_2) + s|$, and Lemma 5.1(2) for the sequences $\tilde{h}$ and $F \circ \tilde{h}$.

(C) follows immediately from [16, Corollary 4.2].

To prove (D) observe that

$$\begin{aligned}(\Delta h^{(j)})_n &= (\Delta h)_{2n+1+j} + (\Delta h)_{2n+j},\\ (\Delta^2 h^{(j)})_n &= (\Delta^2 h)_{2n+2+j} + 2(\Delta^2 h)_{2n+1+j} + (\Delta^2 h)_{2n+j}.\end{aligned}$$

Assuming $h \in D^2(\boldsymbol{n}^{\boldsymbol{\alpha}})$, from these formulae we obtain (ii). Moreover, $\tilde{h}_n = (c_1 - c_2)h_n^{(1)} + c_2(\Delta h)_{2n}$ and $(\Delta h)_n \to 0$, thus $\mathrm{LIM}(\tilde{h}) = \mathrm{LIM}((c_1 - c_2)h^{(1)})$. Using also Lemma 5.1(3) we get $\mathrm{LIM}(\tilde{h}) = \mathrm{LIM}((c_1 - c_2)h)$, and by Lemma 5.1(2),

- if $c_1 \geq c_2$:

$$\liminf_{n\to+\infty} \tilde{h}_n = (c_1 - c_2) \liminf_{n\to+\infty} h, \qquad \limsup_{n\to+\infty} \tilde{h}_n = (c_1 - c_2) \limsup_{n\to+\infty} h,$$

- if $c_1 < c_2$:

$$\liminf_{n\to+\infty} \tilde{h}_n = (c_1 - c_2) \limsup_{n\to+\infty} h, \qquad \limsup_{n\to+\infty} \tilde{h}_n = (c_1 - c_2) \liminf_{n\to+\infty} h.$$

Hence, from (4.5) we obtain (4.6).

To obtain (C + D) we apply Lemma 5.1(2) to the sequence h, and by (4.6) we get $|b_1 - b_2| \in [d_{\mathrm{pp}}; d_{\mathrm{ac}}]$. This gives $(e_-; e_+) \subset (a_-; a_+)$, by (4.4). The last part follows from (C) and Remark 4.3. ∎

The information from Theorem 4.4 on the pure pointness of J in $(p_-; p_+)$ is not very strong—in particular it does not say anything on discreteness or on the existence of regions with dense point spectrum. However, for some coefficients, the discreteness in a nonempty region can be obtained by the following result.

PROPOSITION 4.5. *Let $0 < \alpha < 1$, and consider the Jacobi operator J given by* (4.2)*, where $\{a_n\}_{n\geq1}, \{b_n\}_{n\geq1}, \{c_n\}_{n\geq1}, \{h_n\}_{n\geq1}, \{y_n\}_{n\geq1}$ are real sequences satisfying:*

(a) *$\{a_n\}_{n\geq1}, \{b_n\}_{n\geq1}, \{c_n\}_{n\geq1}$ are 2-periodic;*
(b) *h is bounded and $(\Delta h^{(j)})_n = o(n^{-\alpha})$ as $n \to +\infty$, for $j = 0, 1$;*
(c) *$y_n \to 0$.*

Let d_{pp} be as in (4.3) *and p_-, p_+ as in* (4.4)*. Then J is discrete in*

$$D := \bigcup_{\lambda\in\Lambda} (\lambda - \sqrt{r(\lambda)}; \lambda + \sqrt{r(\lambda)}), \tag{4.11}$$

where

$$\varLambda := \{\lambda \in \mathbb{R} : |a_1 - \lambda| < 1,\ |a_2 - \lambda| < 1,\ r(\lambda) > 0\},$$
$$r(\lambda) := \min\{r_{12}(\lambda); r_{21}(\lambda)\}$$

and for $i, j \in \{1, 2\}$, $i \neq j$,

$$r_{ji}(\lambda) := d_{\mathrm{pp}}^2(1 - |a_i - \lambda|) - |a_j - \lambda|(1 - |a_j - \lambda|).$$

In particular, if $a_1 = a_2$, *then* $D = (p_-; p_+) = (a_1 - d_{\mathrm{pp}}; a_1 + d_{\mathrm{pp}})$.

Proof. For any $\lambda \in \varLambda$ we use Lemma 5.2 for the Jacobi operator $J - \lambda$, and we get the discreteness of J in the set $(\lambda - \sqrt{r(\lambda)}; \lambda + \sqrt{r(\lambda)})$. But this is an open set, hence, summing, we obtain the discreteness in D. For the special case $a_1 = a_2$, to get $D \supset (a_1 - d_{\mathrm{pp}}; a_1 + d_{\mathrm{pp}})$ it suffices to consider $\lambda = a_1$, and the opposite inclusion is easy to obtain by the symmetry $r_{21} = r_{12}$. ∎

The explicit formula for the discreteness set D seems to be rather sophisticated in the general case. Using the above statement only for $\lambda = (a_1 + a_2)/2$, we can immediately formulate its simplified (but weaker—see Example 4.10(3)) version.

Corollary 4.6. *Under the assumptions of Proposition* 4.5, *if moreover*

$$|a_1 - a_2| < \min\{2; 2d_{\mathrm{pp}}^2\}, \tag{4.12}$$

then J *is discrete in the subinterval*

$$\left(\frac{a_1 + a_2}{2} - s;\ \frac{a_1 + a_2}{2} + s\right) \tag{4.13}$$

of $(p_-; p_+)$, *where* $s := \frac{1}{2}\sqrt{(2 - |a_1 - a_2|)(2d_{\mathrm{pp}}^2 - |a_1 - a_2|)}$.

Remarks 4.7. 1. Under assumptions (i)–(iv) of Theorem 4.4, assumptions (a)–(c) of Proposition 4.5 also hold (see Remark 4.3). Thus, if $\varLambda \neq \emptyset$, we obtain the discreteness of J in D (which is also nonempty in this case)—see e.g. Examples 4.9 and 4.10(2a), (2b), (3).

2. We obviously have $\varLambda \subset D$, and also $\varLambda = \emptyset$ iff $D = \emptyset$. However, in general, $\varLambda \neq D$ (even for $a_1 = a_2$), which may seem somewhat strange. This proves that the estimates of the quadratic form for J, the main tool in the proof of Proposition 4.5 (see Lemma 5.2), have been far from optimal.

3. Part (A) of Theorem 4.4 guarantees that the essential spectrum is at least $\mathbb{R}\setminus(a_-; a_+)$. However, under all the extra assumptions of (C + D) we can get stronger information: that the essential spectrum is at least $\mathbb{R} \setminus (e_-; e_+)$. In other words, we then get some important information on the spectrum in a subset $(a_-; a_+) \setminus (e_-; e_+)$ of the uncertainty region $\varSigma_{\mathrm{un}}$. And quite often this subset is non-empty—see e.g. Examples 4.9 and 4.10.

From the point of view of calculations, the most useful part of Theorem 4.4 is the case with the extra assumptions of (C + D). The family of all Jacobi operators satisfying the assumptions of this case is called the **O&P** family in this paper ("**O**scillations & **P**eriodicity"). More precisely, we say that J is in the **O&P** *family* iff (4.2) holds, $0 < \alpha < 1$, the sequences $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$, $\{h_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ are real, $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ are 2-periodic and

- $0 \in \mathrm{LIM}(h)$, $h \in D^2(\boldsymbol{n^\alpha})$;
- $y_n \to 0$, $y^{(0)}, y^{(1)} \in D^2(\boldsymbol{n^\alpha})$;
- $n^\alpha + b_n + c_n h_n \neq 0 \neq n^\alpha + b_n$ for all $n \in \mathbb{N}$.

In that case we also say that $\alpha, \{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}, \{h_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ *describe the entries of J in the* **O&P** *family.* A special case of **O&P** already appeared in [12].

A useful (and direct) consequence of Theorem 4.4 and Proposition 4.5 is the following result.

Theorem 4.8. *Suppose that the Jacobi operator J is in the* **O&P** *family and that α, $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$, $\{h_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ describe the entries of J. Let $d_{\mathrm{pp}} := \inf K$, $d_{\mathrm{ac}} := \sup K$, where*

$$K := \{|(b_1 - b_2) + t(c_1 - c_2)| : \liminf_{n\to+\infty} h_n \leq t \leq \limsup_{n\to+\infty} h_n\},$$

and let $a_\pm := \lambda_\pm(d_{\mathrm{ac}})$, $p_\pm := \lambda_\pm(d_{\mathrm{pp}})$, and $e_\pm := \lambda_\pm(b_1 - b_2)$, where for any $t \in \mathbb{R}$, $\lambda_-(t) \leq \lambda_+(t)$ are the solutions of the equation

$$(\lambda - a_1)(\lambda - a_2) = t^2.$$

Then $(p_-; p_+) \subset (e_-; e_+) \subset (a_-; a_+)$, J is absolutely continuous in $\mathbb{R} \setminus [a_-; a_+]$, $\mathbb{R} \setminus (a_-; a_+) \subset \sigma_{\mathrm{ac}}(J)$, $\mathbb{R} \setminus (e_-; e_+) \subset \sigma_{\mathrm{ess}}(J)$ and J is pure point in $(p_-; p_+)$. Moreover, if $a_1 = a_2$, then J is discrete in $(p_-; p_+)$.

Let us now consider several examples. We start with a direct application of Theorem 4.8, Proposition 4.5 and Example 4.2(1), (3).

Example 4.9. We consider Jacobi operators J with weights w_n and diagonals q_n given by

$$w_n = n^\alpha + b_n + c_n g(n^\gamma), \qquad q_n = a_n,$$

where

$$0 < \alpha < 1, \qquad 0 < \gamma < \frac{1-\alpha}{2}, \tag{4.14}$$

$\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}$ are real 2-periodic sequences and $g : [1; +\infty) \to \mathbb{R}$ is a periodic C^2 function with

$$g_{\min} := \inf g([1; +\infty)) \leq 0 \leq \sup g([1; +\infty)) =: g_{\max}, \tag{4.15}$$

and assume

$$n^\alpha + b_n \neq 0 \neq w_n \quad \text{for any } n \in \mathbb{N}. \tag{4.16}$$

So, the assumptions of Theorem 4.8 and also the assumptions of Proposition 4.5 are satisfied (with $h_n := g(n^\gamma)$ and $y_n := 0$). We have (see, e.g., [16, Lemma 4.4])

$$\liminf_{n\to+\infty} h_n = g_{\min}, \qquad \limsup_{n\to+\infty} h_n = g_{\max},$$

and hence we can write down explicit formulae for $d_{\mathrm{ac}}, d_{\mathrm{pp}}$.

Case 1: $c_1 = c_2$. Then $d_{\mathrm{pp}} = d_{\mathrm{ac}} = |b_1 - b_2|$.

Case 2: $c_1 \neq c_2$. Then defining

$$g_0 := -\frac{b_1 - b_2}{c_1 - c_2}$$

we get:

Case 2(a): $g_{\min} \le g_0 \le g_{\max}$. Then

$$d_{\mathrm{pp}} = 0, \qquad d_{\mathrm{ac}} = |c_1 - c_2| \max\{|g_{\min} - g_0|, |g_{\max} - g_0|\};$$

Case 2(b): $g_{\max} < g_0$. Then

$$d_{\mathrm{pp}} = |c_1 - c_2|\,|g_{\max} - g_0|, \qquad d_{\mathrm{ac}} = |c_1 - c_2|\,|g_{\min} - g_0|;$$

Case 2(c): $g_0 < g_{\min}$. Then

$$d_{\mathrm{pp}} = |c_1 - c_2|\,|g_{\min} - g_0|, \qquad d_{\mathrm{ac}} = |c_1 - c_2|\,|g_{\max} - g_0|.$$

Hence J is absolutely continuous in $\Sigma^- = \mathbb{R} \setminus [a_-; a_+]$, $\overline{\Sigma^-} = \mathbb{R} \setminus (a_-; a_+) \subset \sigma_{\mathrm{ac}}(J)$, J is pure point in $\Sigma_+ = (p_-; p_+)$ and $\mathbb{R} \setminus (e_-; e_+) \subset \sigma_{\mathrm{ess}}(J)$, where $a_\pm := \lambda_\pm(d_{\mathrm{ac}})$, $p_\pm := \lambda_\pm(d_{\mathrm{pp}})$, $e_\pm := \lambda_\pm(b_1 - b_2)$ and for any $t \in \mathbb{R}$,

$$\lambda_\pm(t) := \frac{1}{2}\Big(\pm\sqrt{(a_1 - a_2)^2 + 4t^2} + a_1 + a_2\Big). \tag{4.17}$$

Recall also that by Theorem 4.4(C + D) we always have

$$|b_1 - b_2| \in [d_{\mathrm{pp}}; d_{\mathrm{ac}}].$$

Observe that the situations where $|b_1 - b_2|$ is the right or left end of this interval have special meanings. If $|b_1 - b_2| = d_{\mathrm{pp}}$, then $e_\pm = p_\pm$, i.e., the whole uncertainty region is contained in $\sigma_{\mathrm{ess}}(J)$. This happens always in Case 1; in Case 2(a) iff $b_1 = b_2$; in Case 2(b) iff $g_{\max} = 0$; and in Case 2(c) iff $g_{\min} = 0$. On the other hand, $|b_1 - b_2| = d_{\mathrm{ac}}$ means that $e_\pm = a_\pm$, so we have **no** extra information on $\sigma_{\mathrm{ess}}(J)$ from Theorem 4.4(C + D). This second situation happens always in Case 1; in Case 2(a) iff one of $g_{\max}, g_{\min}$ equals 0 (the one farther away from g_0); in Case 2(b) iff $g_{\min} = 0$; and in Case 2(c) iff $g_{\max} = 0$.

Moreover, J is discrete in D given by (4.11), and if (4.12) holds, then in particular, by Corollary 4.6, J is discrete in the nonempty subinterval (4.13) of $(p_-; p_+)$.

Example 4.10. The following families of Jacobi operators are particular cases of the above general example (we use here the notation from Example 4.9). The common choice is here

$$g(x) = \sin(x+\theta), \qquad c_1 = 1,\, c_2 = 0,$$

where θ is an arbitrary real phase and for each family we only vary the parameters a_1, a_2, b_1, b_2. The parameters α, γ are as in (4.14) and for all the families below we assume that the free parameters α, γ, θ are such that the assumption (4.16) holds.

(1) $b_1 = b_2$. Then $g_0 = 0$, $d_{\mathrm{pp}} = 0$, $d_{\mathrm{ac}} = 1$.

  (a) $a_1 = 0 = a_2$. We obtain $a_\pm = \pm 1$, $p_\pm = 0 = e_\pm$, and $D = \emptyset$. Hence $\Sigma^- = (-\infty; -1) \cup (1; +\infty)$, $\Sigma_+ = \emptyset$, the whole uncertainty region $\Sigma_{\mathrm{un}} = [-1; 1]$ is in the essential spectrum, and the discrete spectrum is empty.

  (b) $a_1 = 1/2$, $a_2 = -1/2$. We obtain $a_\pm = \pm\sqrt{5}/2$, $p_\pm = \pm 1/2 = e_\pm$, and $D = \emptyset$. Hence $\Sigma^- = (-\infty; -\sqrt{5}/2) \cup (\sqrt{5}/2; +\infty)$, $\Sigma_+ = (-1/2; 1/2)$, and the whole $\Sigma_{\mathrm{un}} = [-\sqrt{5}/2; -1/2] \cup [1/2; \sqrt{5}/2]$ is in the essential spectrum. Compared with the previous picture, we now know that in the nonempty interval Σ_+ the operator J is pure point (but we do not know anything about essentiality or discreteness there).

(2) $b_1 = 2$, $b_2 = 0$. Then $g_0 = -2$, $d_{\mathrm{pp}} = 1$, $d_{\mathrm{ac}} = 3$.

  (a) $a_1 = 0 = a_2$. We obtain $a_\pm = \pm 3$, $p_\pm = \pm 1$, $e_\pm = \pm 2$, and $D = (-1; 1)$. Hence $\Sigma^- = (-\infty; -3) \cup (3; +\infty)$, $\Sigma_+ = (-1; 1)$, $\Sigma_{\mathrm{un}} = [-3; -1] \cup [1; 3]$. Now we have information on essentiality only of the part $[-3; -2] \cup [2; 3]$ of the uncertainty region—the character of the remaining $(-2; -1] \cup [1; 2)$ is unknown. But we have discreteness in the whole Σ_+.

  (b) $a_1 = 1/2$, $a_2 = -1/2$. We obtain $a_\pm = \pm\sqrt{37}/2$, $p_\pm = \pm\sqrt{5}/2$, $e_\pm = \pm\sqrt{17}/2$, and $D = (-1/2; 1/2)$. Thus $\Sigma^- = (-\infty; -\sqrt{37}/2) \cup (\sqrt{37}/2; +\infty)$, $\Sigma_+ = (-\sqrt{5}/2; \sqrt{5}/2)$, $\Sigma_{\mathrm{un}} = [-\sqrt{37}/2; -\sqrt{5}/2] \cup [\sqrt{5}/2; \sqrt{37}/2]$. The picture is similar to the previous one, excluding the Σ_+ region—we have information on discreteness only on its part $(-1/2; 1/2)$.

(3) $b_1 = 0$, $b_2 = 3/2$, $a_1 = -u$, $a_2 = u$, where $u = 1/4 + \epsilon$ with a small $\epsilon > 0$. In this case $g_0 = 3/2$, $d_{\mathrm{pp}} = 1/2$, $d_{\mathrm{ac}} = 5/2$, so we obtain: $a_\pm = \pm\sqrt{u^2 + 25/4} \simeq \pm\sqrt{26}/2$, $p_\pm = \pm\sqrt{u^2 + 1/4} \simeq \pm 1/2$, $e_\pm = \pm\sqrt{u^2 + 9/4} \simeq \pm\sqrt{10}/2$. But unlike the previous cases, when the set D given by (4.11) was $\emptyset$ or the interval (4.13), now, as one can

easily check, $D \neq \emptyset$ for ϵ small enough, while the condition (4.12), necessary to define (4.13), does not even hold.

The last example of the paper was presented in [8, Example 6.2] and it is also a specific instance of the general case of Theorem 4.4, but the extra assumptions of parts (C), (D), (C + D), $h \in 0_\alpha$ and $h \in D^2(\boldsymbol{n^\alpha})$, **do not** hold here. However, the assumptions of Proposition 4.5 do.

EXAMPLE 4.11. Consider Jacobi operators J with weights

$$w_n = n^\alpha + c_n h_n$$

and with zero diagonals, where

$$h_{2n} = 1, \qquad h_{2n+1} = \sin(n^\gamma),$$

under condition (4.14), and with $\{c_n\}_{n\geq 1}$ being a real 2-periodic sequence such that $w_n \neq 0$ for any $n \in \mathbb{N}$.

Using (4.5) and an argument similar to that from Example 4.9, we compute

$$d_{\mathrm{pp}} = \begin{cases} |c_2| - |c_1| & \text{for } |c_2| > |c_1|, \\ 0 & \text{for } |c_2| \leq |c_1|, \end{cases} \qquad d_{\mathrm{ac}} = |c_1| + |c_2|.$$

We also have $a_\pm = \pm d_{\mathrm{pp}}$, $p_\pm = \pm d_{\mathrm{ac}}$ and $D = (-d_{\mathrm{pp}}; d_{\mathrm{pp}})$. Hence $\Sigma^- = (-\infty; -d_{\mathrm{ac}}) \cup (d_{\mathrm{ac}}; +\infty)$, $\Sigma_+ = (-d_{\mathrm{pp}}; d_{\mathrm{pp}})$, $\Sigma_{\mathrm{un}} = [-d_{\mathrm{ac}}; -d_{\mathrm{pp}}] \cup [d_{\mathrm{pp}}; d_{\mathrm{ac}}]$, and by Theorem 4.4, J is absolutely continuous in $\mathbb{R} \setminus [-d_{\mathrm{ac}}; d_{\mathrm{ac}}]$ and in $\mathbb{R} \setminus (-d_{\mathrm{ac}}; d_{\mathrm{ac}}) \subset \sigma_{\mathrm{ac}}(J)$. Moreover, by Proposition 4.5, J is discrete in $(-d_{\mathrm{pp}}; d_{\mathrm{pp}})$ (which is stronger information than that on pure pointness from [8, Example 6.2]).

Let us finish with some questions and suppositions related to the results presented here.

Open problems and conjectures

- The main questions concern spectral problems only partially solved, at least for the various cases of Examples 4.9–4.11:

    1. What is the detailed spectral character of the region Σ_{un}? How to check there the existence and how to localise the discrete, dense point, singular continuous and absolutely continuous spectrum?
    2. What is the detailed spectral character of the region Σ_+? How to check there the existence, and how to localise the dense point spectrum?

- We have the following conjecture concerning the last question:

$$J \textit{ is discrete in } \Sigma_+.$$

Note that one of the motivations for the above questions is [12, Theorem 6.1], which gives a partial answer for some special cases of **O&P**.

5. Appendix. Here we have collected several more technical proofs and lemmas.

Proof of Criterion 3.2. We use [8, Th. 5.1] to study the recurrent $\mathbb{C}^2$ vector equation

$$(x_{n+1})^\top = C_n(x_n)^\top$$

for large n. Thus we can choose $n_0' \geq n_0$, $\delta > 0$ such that

$$\operatorname{discr} V_{n-1} < -\delta, \qquad n \geq n_0', \tag{5.1}$$

and the above equation considered for $n \geq n_0'$ has two linearly independent solutions $\{x_n^1\}_{n \geq n_0'}$, $\{x_n^2\}_{n \geq n_0'}$ of the form $x_n^m = \varphi_n^m y_n^m$, with $\varphi_n^m \in \mathbb{C}$, $y_n^m \in \mathbb{C}^2$, such that the scalar terms satisfy

$$\varphi_n^2 = \overline{\varphi_n^1} \neq 0, \tag{5.2}$$

and the vector terms satisfy

$$(y_n^m)^\top = S_n(\beta_n^m)^\top, \qquad m = 1, 2, \tag{5.3}$$

where S_n is the diagonalising matrix for V_{n-1} described in [8, Section 2.3.1], and $\beta_n^m \to e_m$, with $e_1 = (1, 0)$, $e_2 = (0, 1)$. By (3.7) we have

$$(V_{n-1})_{12}(V_{n-1})_{21} \leq \frac{1}{4} \operatorname{discr} V_{n-1},$$

hence, using the boundedness of $\{V_n\}_{n \geq n_0}$ and (5.1), we see that there exists $\delta' > 0$ such that $|(V_{n-1})_{12}| > \delta'$ for $n \geq n_0'$. Consequently, again by the boundedness of $\{V_n\}_{n \geq n_0}$ and by [8, Section 2.3.1], we get

$$\sup_{n \geq n_0'} \|S_n\| < +\infty, \tag{5.4}$$

$$\inf_{n \geq n_0'} |\det S_n| > 0. \tag{5.5}$$

By (5.2) and (5.3) we have $|\varphi_n^1/\varphi_n^2| = 1$, and

$$((y_n^1)^\top, (y_n^2)^\top) = S_n E_n, \qquad E_n := ((\beta_n^1)^\top, (\beta_n^2)^\top) \to I.$$

Thus, employing (5.4) and (5.5), we can use [15, Lemma 5.9] to get $\{C_n\}_{n \geq n_0'} \in H$. This finishes the proof by [15, Proposition 5.7(i)]. ■

We formulate here the following lemma which is used, e.g., in the proof of Theorem 4.4.

Lemma 5.1. *Let $x := \{x_n\}_{n \geq n_0}$ be a bounded real sequence.*

(1) *If K is a compact subset of $\mathbb{R}$ containing all the terms of x and $f : K \to \mathbb{R}$ is continuous, then* $\operatorname{LIM}(\{f(x_n)\}_{n \geq n_0}) = f(\operatorname{LIM}(x))$.

(2) *If* $(\Delta x)_n \to 0$, *then*

$$\mathrm{LIM}(x) = [\liminf_{n\to+\infty} x_n; \limsup_{n\to+\infty} x_n].$$

(3) *If* $(\Delta x)_n \to 0$ *and* $l = \{l_n\}_{n\geq 1}$ *is a sequence of integers* $(\geq n_0)$ *such that* $l_n \to +\infty$ *and* Δl *is bounded from above, then* $\mathrm{LIM}(x) = \mathrm{LIM}(\{x_{l_n}\}_{n\geq 1})$.

Proof. Parts (1) and (2) are rather well-known, and their proofs are standard, so we only prove (3). The inclusion "$\supset$" is obvious. Let A be the set of all terms of $\{l_n\}_{n\geq 1}$ and let $d := \max\{C, (\min A) - n_0\}$, where C is a fixed upper bound of $\Delta(l)$. From $l_n \to +\infty$, we can easily get

$$A \cap [k; k+d] \neq \emptyset \quad \text{for any integer } k \geq n_0. \tag{5.6}$$

Assume $g \in \mathrm{LIM}(x)$, and choose a sequence $\{k_n\}_{n\geq 1}$ of integers $\geq n_0$ such that $k_n \to +\infty$ and $x_{k_n} \to g$. For any $n \in \mathbb{N}$ define

$$a_n := \min\{a \in A : a \geq k_n\}, \qquad m_n = \min\{m \in \mathbb{N} : l_m = a_n\}.$$

In particular $l_{m_n} = a_n \geq k_n$ for any $n \in \mathbb{N}$. We have $[k_n; a_n - 1] \cap A = \emptyset$, hence by (5.6), $a_n - 1 - k_n < d$, which gives

$$k_n \leq l_{m_n} \leq k_n + d. \tag{5.7}$$

We also have $m_n \to +\infty$, since otherwise $m_n = p$ for some integer p and for infinitely many n, which by the definition of m_n leads to $l_p \geq k_n$ for infinitely many n, contrary to $k_n \to +\infty$. Moreover, by (5.7) we get

$$|x_{l_{m_n}} - x_{k_n}| \leq \max_{j=0,\dots,d} |x_{k_n+j} - x_{k_n}|,$$

which gives $x_{l_{m_n}} - x_{k_n} \to 0$, because for any $j \in \mathbb{N}$,

$$|x_{k_n+j} - x_{k_n}| \leq \sum_{s=0}^{j-1} |x_{k_n+s+1} - x_{k_n+s}| = \sum_{s=0}^{j-1} |(\Delta x)_{k_n+s}| \to 0.$$

Thus, finally, $x_{l_{m_n}} \to g$, and $g \in \mathrm{LIM}(\{x_{l_n}\}_{n\geq 1})$. ∎

Below we prove a lemma which is a basic tool in the proof of Proposition 4.5. The way of estimating the quadratic form in the proof of the lemma is inspired by the paper of Dombrowski [4].

LEMMA 5.2. *Let* $0 < \alpha < 1$, *and consider the Jacobi operator* J *given by* (4.2), *where* $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}, \{h_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ *are real sequences satisfying:*

(a) $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}$ *are* 2-*periodic;*
(b) h *is bounded and* $(\Delta h^{(j)})_n = o(n^{-\alpha})$ *as* $n \to +\infty$, *for* $j = 0, 1$;
(c) $y_n \to 0$.

Let $d_{\rm pp}$ *be as in* (4.3) *and* p_-, p_+ *as in* (4.4). *If* $|a_1|, |a_2| < 1$ *and*

$$r := \min\{d_{\rm pp}^2(1-|a_1|)-|a_2|(1-|a_2|); d_{\rm pp}^2(1-|a_2|)-|a_1|(1-|a_1|)\} > 0, \tag{5.8}$$

then J *is discrete in* $(-\sqrt{r}; \sqrt{r})$. *In particular, if* $a_1 = a_2 = 0$, *then* J *is discrete in the whole* $(p_-; p_+) = (-d_{\rm pp}; d_{\rm pp})$.

Proof. By the Weyl theorem on the invariance of the essential spectrum under compact perturbations, we can assume that $\{y_n\}_{n\geq 1}$ is a zero sequence. By the min-max principle (see e.g. [17, Vol. IV]) for the operator J^2, it suffices to check that for any $r' < r$ there exists $N \in \mathbb{N}$ such that for any $f \in D(J^2)$ with

$$f_1 = \cdots = f_{2N-1} = 0 \tag{5.9}$$

we have

$$(J^2 f, f) \geq r' \|f\|^2. \tag{5.10}$$

By Lemma 5.3 below it is enough to consider $f \in \ell_{\rm fin}(\mathbb{N})$, the set of sequences with only finitely many nonzero terms, instead of all the vectors from $D(J^2)$, since $\ell_{\rm fin}(\mathbb{N})$ is a domain of essential self-adjointness for J (because $\alpha < 1$), it is contained in the domain of J^2 and it contains all the standard basis vectors e_n.

Fix $r' < r$ and let $f \in \ell_{\rm fin}(\mathbb{N})$ satisfy (5.9) for some N.

Denote by J_0 and Z the operators given by the off-diagonal and the diagonal part of the Jacobi matrix for J, respectively. Denote also by $\ell^2_e(\mathbb{N})$, $\ell^2_o(\mathbb{N})$ the subspaces of $\ell^2(\mathbb{N})$ which are the closures of the linear spans of all the e_n's with n even and odd, respectively, and for any $g \in \ell^2(\mathbb{N})$ let g_e, g_o be the orthogonal projections of g onto these subspaces. We have $J_0 f_o \in \ell^2_e(\mathbb{N})$, $J_0 f_e \in \ell^2_o(\mathbb{N})$, $Z f_o \in \ell^2_o(\mathbb{N})$, $Z f_e \in \ell^2_e(\mathbb{N})$, hence

$$\begin{aligned}(J^2 f, f) &= (J_0^2 f, f) + 2\,\mathrm{Re}(J_0 f, Z f) + (Z^2 f, f) \\ &= \|J_0 f_e\|^2 + \|J_0 f_o\|^2 + 2a_1\,\mathrm{Re}(J_0 f_e, f_o) + 2a_2\,\mathrm{Re}(J_0 f_o, f_e) \\ &\quad + a_1^2\|f_o\|^2 + a_2^2\|f_e\|^2.\end{aligned} \tag{5.11}$$

We have

$$\begin{aligned}&2a_1\,\mathrm{Re}(J_0 f_e, f_o) + 2a_2\,\mathrm{Re}(J_0 f_o, f_e) \\ &\qquad \geq -2(|a_1|\,\|J_0 f_e\|\,\|f_o\| + |a_2|\,\|J_0 f_o\|\,\|f_e\|) \\ &\qquad \geq -|a_1|(\|J_0 f_e\|^2 + \|f_o\|^2) - |a_2|(\|J_0 f_o\|^2 + \|f_e\|^2).\end{aligned} \tag{5.12}$$

By (5.11) and (5.12) we get

$$\begin{aligned}(J^2 f, f) \geq{} & (1-|a_1|)\|J_0 f_e\|^2 + (1-|a_2|)\|J_0 f_o\|^2 \\ & + (a_1^2 - |a_1|)\|f_o\|^2 + (a_2^2 - |a_2|)\|f_e\|^2.\end{aligned} \tag{5.13}$$

Denote

$$\begin{aligned}A_k &:= w_{2k-1} = (2k-1)^\alpha + b_1 + c_1 h_{2k-1},\\ B_k &:= (2k-2)^\alpha + b_1 + c_1 h_{2k-1},\\ C_k &:= w_{2k-2} - B_k = -((b_1-b_2) + \tilde{h}_{k-1})\end{aligned}$$

(see (4.3) for the definition of $\tilde{h}$). Let $0 < \epsilon < d_{\mathrm{pp}}$. Using (5.9), (4.3) and the inequality $2\,\mathrm{Re}\,x\overline{y} \ge -(|x|^2 + |y|^2)$ we see that choosing N large enough we also have

$$\begin{aligned}\|J_0 f_e\|^2 &= \sum_{k=N}^{+\infty} |w_{2k-2} f(2k-2) + w_{2k-1} f(2k)|^2 \\ &= \sum_{k=N}^{+\infty} |(A_k f(2k) + B_k f(2k-2)) + C_k f(2k-2)|^2 \\ &= \sum_{k=N}^{+\infty} |A_k f(2k) + B_k f(2k-2)|^2 + \sum_{k=N}^{+\infty} C_k^2 |f(2k-2)|^2 \\ &\quad + \sum_{k=N}^{+\infty} 2\,\mathrm{Re}\big[(A_k f(2k) + B_k f(2k-2)) C_k \overline{f(2k-2)}\big] \\ &\ge (d_{\mathrm{pp}} - \epsilon/2)^2 \|f_e\|^2 \\ &\quad + \sum_{k=N}^{+\infty} [(A_k C_k) 2\,\mathrm{Re}\, f(2k)\overline{f(2k-2)} + (2B_k C_k)|f(2k-2)|^2)].\end{aligned} \tag{5.14}$$

The last sum is

$$\begin{aligned}&\ge \sum_{k=N}^{+\infty} [(2B_k - A_k) C_k |f(2k-2)|^2 - A_k C_k |f(2k)|^2] \\ &= \sum_{k=N}^{+\infty} (2B_k - A_k) C_k |f(2k-2)|^2 - \sum_{k=N}^{+\infty} A_{k-1} C_{k-1} |f(2k-2)|^2 \\ &= \sum_{k=N}^{+\infty} F_k |f(2k-2)|^2,\end{aligned}$$

where

$$\begin{aligned}F_k &= (2B_k - A_k - A_{k-1}) C_k + A_{k-1}(C_k - C_{k-1}) \\ &= [(2(2k-2)^\alpha - (2k-1)^\alpha - (2k-3)^\alpha) + c_1(h_{2k-1} - h_{2k-3})] C_k \\ &\quad + ((2k-3)^\alpha + b_1 + c_1 h_{2k-3})[\tilde{h}_{k-2} - \tilde{h}_{k-1}] \\ &= [(2(2k-2)^\alpha - (2k-1)^\alpha - (2k-3)^\alpha) + c_1(\Delta h^{(1)})_{k-2}] C_k \\ &\quad + ((2k-3)^\alpha + b_1 + c_1 h_{2k-3})[c_2(\Delta h^{(0)})_{k-2} - c_1(\Delta h^{(1)})_{k-2}].\end{aligned}$$

Hence, by (5.14),

$$\begin{aligned}\|J_0 f_e\|^2 &\ge (d_{\rm pp} - \epsilon/2)\|f\|^2 + \sum_{k=N}^{+\infty} F_k |f(2k-2)|^2 \\ &\ge \Big[\liminf_{n\to+\infty} F_n + (d_{\rm pp} - \epsilon)^2\Big] \|f_e\|^2.\end{aligned}$$

By (a) and (b) we have $\liminf_{n\to+\infty} F_n = 0$, so

$$\|J_0 f_e\|^2 \ge (d_{\rm pp} - \epsilon)^2 \|f_e\|^2.$$

An analogous argument yields

$$\|J_0 f_o\|^2 \ge (d_{\rm pp} - \epsilon)^2 \|f_o\|^2,$$

and by (5.13), choosing ϵ small enough, we finally get (5.10). ■

LEMMA 5.3. *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Suppose that X is a closed linear subspace and $\widetilde{D}$ a linear subspace of $\mathcal{H}$ such that $X \subset \widetilde{D} \subset D(A^2)$ and $\widetilde{D}$ is a core space for A. Denote*

$$M := \{\varphi \in D(A^2) : \varphi \perp X,\ \|\varphi\| = 1\}, \qquad \widetilde{M} := M \cap \widetilde{D}.$$

Then

$$\inf_{\varphi\in M} (A^2\varphi, \varphi) = \inf_{\varphi\in\widetilde{M}} (A^2\varphi, \varphi). \tag{5.15}$$

Proof. We have LHS $\le$ RHS in (5.15); if $M = \emptyset$, then both sides are $+\infty$. To get the assertion it suffices to prove that for any $\varphi \in M$ there exists a sequence $\{\varphi_n\}$ in $\widetilde{M}$ with

$$(A^2\varphi_n, \varphi_n) \to (A^2\varphi, \varphi). \tag{5.16}$$

Since $\varphi \in D(A)$ and $\widetilde{D}$ is a core space for A, we can first choose $\{\psi_n\}$ in $\widetilde{D}$ satisfying $\psi_n \to \varphi$ and $A\psi_n \to A\varphi$. Let P be the orthogonal projection onto $X^\perp$, Q the orthogonal projection onto X, and let $\eta_n := P\psi_n$. Using $\varphi \in X^\perp$ and $X \subset \widetilde{D}$ we obtain

$$\eta_n \to P\varphi = \varphi, \qquad \eta_n = \psi_n - Q\psi_n \in \widetilde{D}, \qquad \eta_n \perp X.$$

Moreover AQ is bounded on $\mathcal{H}$, since A is closed, Q is bounded on $\mathcal{H}$, and $\operatorname{Ran} Q = X \subset D(A)$. Thus $AQ\eta_n \to AQ\varphi$, and hence, using again $\varphi \in X^\perp$, we get

$$A\eta_n = AP\psi_n = A\psi_n - AQ\psi_n \to A\varphi - AQ\varphi = A\varphi.$$

We also have $\|\eta_n\| \to \|\varphi\| = 1$, so $\|\eta_n\| \ne 0$ for n large enough, and we can define $\varphi_n := \|\eta_n\|^{-1}\eta_n$. Then

$$\varphi_n \in \widetilde{M}, \qquad \varphi_n \to \varphi, \qquad A\varphi_n \to A\varphi.$$

Therefore, using $\widetilde{D} \subset D(A^2)$, we obtain

$$(A^2\varphi_n, \varphi_n) = \|A\varphi_n\|^2 \to \|A\varphi\|^2 = (A^2\varphi, \varphi). \ ■$$

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