STUDIA MATHEMATICA 209 (2) (2012)

Contractive homomorphisms of measure algebras and Fourier algebras

by

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Abstract. We show that the dual version of our factorization [J. Funct. Anal. 261 (2011)] of contractive homomorphisms $\varphi : L^1(F) \to M(G)$ between group/measure algebras fails to hold in the dual, Fourier/Fourier–Stieltjes algebra, setting. We characterize the contractive w^*-w^* continuous homomorphisms between measure algebras and (reduced) Fourier–Stieltjes algebras. We consider the problem of describing all contractive homomorphisms $\varphi : L^1(F) \to L^1(G)$.

1. Introduction. Let F and G be locally compact groups. The convolution "homomorphism problem" in abstract harmonic analysis asks for a description of all bounded homomorphisms between group and measure algebras, $L^1(F)$ and $M(G)$ (and related convolution algebras); the dual version of the homomorphism problem asks for a description of all homomorphisms between Fourier and Fourier–Stieltjes algebras, $A(F)$ and $B(G)$. For some details regarding the history of the problem, the reader is referred to [3], [16], [8], [10], [11], [17] and the references therein. In general, both versions of the problem remain open.

Let $\mathrm{LUC}(F)$ denote the C^*-algebra of left uniformly continuous functions on F and give the dual Banach space $\mathrm{LUC}(F)^*$ its left Arens product. Then $\mathrm{LUC}(F)^*$ contains $M(F)$, and therefore $L^1(F)$, as a Banach subalgebra through the embedding

$$\Theta_F : M(F) \hookrightarrow \mathrm{LUC}(F)^*, \qquad \langle \Theta_F(\mu), f\rangle = \int f\, d\mu.$$

For any map $\theta : Y \subset F \to G$ and a function f on G, $j_\theta(f)$ is defined on F by putting $j_\theta f = f \circ \theta$ on Y and $j_\theta f = 0$ off Y. When $\theta : F \to G$ is a continuous homomorphism, j_θ maps $C_0(G)$, the continuous functions on F that vanish at infinity, into $\mathrm{LUC}(F)$. The dual map $j_\theta^* : \mathrm{LUC}(F)^* \to M(G)$

2010 *Mathematics Subject Classification*: Primary 43A20, 43A10, 43A30, 43A22; Secondary 43A25.

Key words and phrases: measure algebra, group algebra, Fourier algebra, Fourier–Stieltjes algebra, homomorphism.

DOI: 10.4064/sm209-2-3

is a weak* continuous contractive Banach algebra homomorphism mapping the point mass δ_x to $\delta_{\theta(x)}$. The restriction of j_θ^* to both $M(F)$ and $L^1(F)$ is also denoted j_θ^*.

For $\alpha \in \widehat{F}^1$, i.e. for α a continuous character on F, let $M_\alpha : C_0(F) \to C_0(F) : f \mapsto f\alpha$ and let $A_\alpha = M_\alpha^*$; then A_α is an isometric algebra automorphism of $M(F)$ and, by restriction, of $L^1(F)$. Let $\mathbb{T}$ denote the circle group and let H be a locally compact group. Then θ_H denotes the projection homomorphism mapping $\mathbb{T} \times H$ onto H, and $\alpha_{\mathbb{T}}$ is the projection character mapping $\mathbb{T} \times H$ onto $\mathbb{T}$. If K is a compact normal subgroup of H with normalized Haar measure m_K—often viewed as an element of $M(H)$—define $S_K : C_0(H) \to C_0(H/K)$ by $S_K f(xK) = \int_K f(xk)\, dm_K(k)$. Then $S_K^* : M(H/K) \hookrightarrow M(H)$ is an isometric algebra isomorphism that embeds $L^1(H/K)$ into $L^1(H)$ and satisfies $S_K^*(\delta_{xK}) = \delta_x * m_K$. *We refer the reader to* [17] *for additional information regarding these "basic" homomorphisms and any additional notation that we have not defined here.*

In [17], the author showed that there are contractive homomorphisms $\varphi : L^1(F) \to M(G)$ which fail to have the Cohen factorization which holds in the abelian case, and which was shown by Pham [15] to extend in a way that characterizes the contractive homomorphisms $\varphi : A(F) \to B(G)$. The following alternative factorization was established:

THEOREM 1.1. *Let $\varphi : L^1(F) \to M(G)$ be a mapping. The following statements are equivalent:*

(i) *φ is a contractive homomorphism;*
(ii) *there is a closed subgroup H of G, a compact normal subgroup Ω_0 of $\mathbb{T} \times H$ and a continuous homomorphism $\theta : F \to \mathbb{T} \times H/\Omega_0$ such that $\varphi = j_{\iota_H}^* \circ j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_0}^* \circ j_\theta^*$. That is, φ factors as*

$$\begin{tikzcd}
L^1(F) \arrow[rrr, "\varphi"] \arrow[d, "j_\theta^*"'] & & & M(G) \\
M(\mathbb{T} \times H/\Omega_0) \arrow[r, hook, "S_{\Omega_0}^*"] & M(\mathbb{T} \times H) \arrow[r, "A_{\alpha_{\mathbb{T}}}"] & M(\mathbb{T} \times H) \arrow[r, "j_{\theta_H}^*"] & M(H) \arrow[u, hook, "j_{\iota_H}^*"']
\end{tikzcd} \tag{1.1}$$

REMARK 1.2. 1. Suppose that $\varphi : M(F) \to M(G)$ has the factorization (1.1) with $j_\theta^* : M(F) \to M(\mathbb{T} \times H/\Omega_0)$; this is the unique *so*-w^* continuous extension of φ on $L^1(F)$ to $M(F)$ (see [8] or [17]), where *so* denotes the strict topology on $M(F)$ taken with respect to the ideal $L^1(F)$. Then we can take H equal to the support subgroup of $\{\varphi(\delta_x) : x \in F\}$; $\Omega_0 = \Omega_\rho = \{(\rho(k), k) : k \in K\}$ where $\varphi(\delta_{e_F}) = \rho m_K$ with K a compact normal subgroup of H and $\rho \in \widehat{K}^1$ such that $\ker\rho$ is normal in H and $K/\ker\rho$ lies in the centre of $H/\ker\rho$; and $\theta(x) = \phi^{-1}(\varphi(\delta_x))$ where $\phi : \mathbb{T} \times H/\Omega_\rho \to \Gamma_{\mathbb{T}\times H} = \{\alpha\delta_x * \rho m_K : (\alpha, x) \in \mathbb{T} \times H\}$ is the natural topological isomorphism (see [17, Theorems 5.7

and 5.11 and Corollary 4.3]). In this case we will say that the factorization (1.1) is in *canonical form.*

2. If we define $\theta_{H,G} : \mathbb{T}\times H \to G$ by $\theta_{H,G}(\alpha, h) = h$, then the factorization (1.1) may be written as $\varphi = j^*_{\theta_{H,G}} \circ A_{\alpha_{\mathbb{T}}} \circ S^*_{\Omega_0} \circ j^*_\theta$ [17, Remark 5.12].

In Section 2, we will characterize precisely when the dual version of the factorization (1.1) can be used to describe a contractive homomorphism $\varphi : A(F) \to B(G)$. In (unfortunate) perfect duality with the Cohen factorization—which was shown in [17] to fail for group algebras—we will show that the dual version of the factorization (1.1) often cannot be used to describe all contractive homomorphisms in the Fourier algebra setting.

Even when F and G are abelian, there is no known characterization of all homomorphisms $\varphi : M(F) \to M(G)$. When F is amenable, descriptions of all *so*-w^* continuous (respectively w^*-w^* continuous) completely bounded homomorphisms $\varphi : B(F) \to B(G)$ were obtained in [12]: $\varphi = j_\alpha$ for some piecewise affine continuous (open) map $\alpha : Y \subseteq G \to F$, where Y lies in the open coset ring of G. For further definitions and details, the reader is referred to [10]–[12]. Recently, Hung Le Pham proved that every contractive homomorphism $\varphi : A(F) \to B(G)$ has a Cohen factorization $\varphi = l_r \circ s \circ j_{\theta_1} \circ l_u$:

$$A(F) \xrightarrow{l_u} A(F) \xrightarrow{j_{\theta_1}} B(G_1) \overset{s}{\hookrightarrow} B(G) \xrightarrow{l_r} B(G) \tag{1.2}$$

where $l_r u(s) = u(rs)$; $\theta : G_1 \to F$ is a continuous homomorphism, or antihomomorphism, defined on an open subgroup G_1 of G; and $s : B(G_1) \hookrightarrow B(G)$ is given by $s(u)(h) = u(h)$ if $h \in G_1$, and $s(u)(h) = 0$ otherwise [15, Theorem 5.1].

In Section 3, we will apply Pham's theorem and a modification of the quoted theorem from [12] to obtain a description of the *so*-w^* and w^*-w^* continuous contractive homomorphisms $\varphi : B_r(F) \to B(G)$. Here, $B_r(F)$ is the reduced Fourier–Stieltjes algebra of F, which we identify with the dual of $C^*_r(F)$, the C^*-subalgebra of the bounded operators on $L^2(F)$ generated by the convolution operators $\lambda(f)\xi = f * \xi$, $f \in L^1(F)$, $\xi \in L^2(F)$; see [6] and [1] for details. By replacing $L^1(F)$ by $M(F)$ in Theorem 1.1, one obtains a characterization of the *so*-w^* continuous contractive homomorphisms $\varphi : M(F) \to M(G)$ [17, Corollary 6.1]. In Section 3, we will also characterize the w^*-w^* continuous contractive homomorphisms $\varphi : M(F) \to M(G)$: such maps have the factorization (1.1) with θ a continuous *proper* homomorphism. In special cases, the contractive homomorphisms $\varphi : L^1(F) \to L^1(G)$ have been described [8], [13], [17] but in general the problem remains open. In Section 4, we discuss this problem.

To avoid trivial cases, all algebra homomorphisms are assumed to be nonzero.

2. The dual form of the factorization (1.1). Observe that when Ω_ρ is normal in $\mathbb{T} \times G$, one can replace H by G in the factorization (1.1) and drop the term $j^*_{\iota_H}$. Using the Fourier and Fourier–Stieltjes transforms, one can also check that when F and G are abelian, the precise dual form of the factorization

$$L^1(F) \xrightarrow{j^*_\theta} M(\mathbb{T} \times G/\Omega_0) \xrightarrow{S^*_{\Omega_\rho}} M(\mathbb{T} \times G) \xrightarrow{A_{\alpha_\mathbb{T}}} M(\mathbb{T} \times G) \xrightarrow{j^*_{\theta_G}} M(G)$$

is given by

$$A(F) \xrightarrow{j_\theta} B(G_0) \xrightarrow{s} B(\mathbb{Z} \times G) \xrightarrow{l_{(-1,e_G)}} B(\mathbb{Z} \times G) \xrightarrow{j_{\iota_G}} B(G) \tag{2.1}$$

where G_0 is an open subgroup of $\mathbb{Z} \times G$, $\theta : G_0 \to F$ is a continuous homomorphism, and $\iota_G : G \to \mathbb{Z} \times G : x \mapsto (0, x)$. It was natural to ask if every contractive homomorphism $\varphi : L^1(F) \to M(G)$ has a Cohen factorization [13], [17], and it is similarly natural to ask if every contractive homomorphism $\varphi : A(F) \to B(G)$ has the factorization (2.1) where θ is either a homomorphism or an anti-homomorphism. In this section we shall characterize exactly when this is the case.

LEMMA 2.1. *Let G_1 be a subset of G, $r \in G$, $u \in F$, $\theta_1 : G_1 \to F$.*

(i) *$G_0 = \{(n, r^n t) : n \in \mathbb{Z},\ t \in G_1\}$ is a subgroup of $\mathbb{Z} \times G$ if and only if G_1 is a subgroup of G and $rG_1r^{-1} = G_1$.*
(ii) *Suppose that G_0 is a subgroup of $\mathbb{Z} \times G$ and define*

$$\theta : G_0 \to F : (n, r^n t) \mapsto u^{-n}\theta_1(t) \qquad [(n, r^n t) \mapsto u\theta_1(t)u^{-n-1}].$$

Then θ is an [anti-]homomorphism if and only if θ_1 is an [anti-] homomorphism and for each $t \in G_1$,

$$\theta_1(r^{\pm 1} t r^{\mp 1}) = u^{\mp 1}\theta_1(t)u^{\pm 1} \qquad [\theta_1(r^{\pm 1} t r^{\mp 1}) = u^{\pm 1}\theta_1(t)u^{\mp 1}].$$

Proof. (i) Suppose that G_0 is a subgroup of $\mathbb{Z} \times G$. As $(0, t) \in G_0$ if and only if $t \in G_1$, we see that G_1 is a subgroup of G. For $t \in G_1$, $(0, rtr^{-1}) = (1, rt)(-1, r^{-1}) \in G_0$, so $rG_1r^{-1} \subseteq G_1$. Similarly, $r^{-1}G_1r \subseteq G_1$, so $rG_1r^{-1} = G_1$. Conversely, suppose that G_1 is a subgroup of G and $rG_1r^{-1} = G_1$. For $(m, r^m s)$ and $(n, r^n t)$ in G_0, we have $(m, r^m s)^{-1} = (-m, r^{-m}(r^m s^{-1} r^{-m})) \in G_0$ and $(m, r^m s)(n, r^n t) = (m+n, r^{m+n}(r^{-n} s r^n)t) \in G_0$. Hence, G_0 is a subgroup of $\mathbb{Z} \times G$.

(ii) Suppose that G_0 is a subgroup of $\mathbb{Z} \times G$ and that $\theta(n, r^n t) = u\theta_1(t)u^{-n-1}$ is an anti-homomorphism of G_0 into F. For $s, t \in G_1$,

$$\begin{aligned}\theta_1(st) &= u^{-1}\theta(0, st)u = u^{-1}\theta((0, s)(0, t))u \\ &= u^{-1}\theta(0, t)uu^{-1}\theta(0, s)u = \theta_1(t)\theta_1(s).\end{aligned}$$

Note in particular that $\theta_1(e_G) = e_F$. Hence,

$$\begin{aligned}\theta_1(rtr^{-1}) &= u^{-1}\theta(0, rtr^{-1})u = u^{-1}\theta((1, rt)(-1, r^{-1}))u \\ &= u^{-1}\theta(-1, r^{-1})\theta(1, rt)u = u^{-1}u\theta_1(e_G)u^0 u\theta_1(t)u^{-2}u = u\theta_1(t)u^{-1}.\end{aligned}$$

Similarly, $\theta_1(r^{-1}tr) = u^{-1}\theta((-1, r^{-1}t)(1, r))u = u^{-1}\theta_1(t)u$. Conversely, suppose that θ_1 is an anti-homomorphism and $\theta_1(r^{\pm 1}tr^{\mp 1}) = u^{\pm 1}\theta_1(t)u^{\mp 1}$ for each $t \in G_1$. Then for $(m, r^m s)$ and $(n, r^n t)$ in G_0,

$$\begin{aligned}\theta((m, r^m s)(n, r^n t)) &= \theta(m+n, r^{m+n}(r^{-n}sr^n)t)) = u\theta_1((r^{-n}sr^n)t)u^{-m-n-1} \\ &= u\theta_1(t)u^{-n}\theta_1(s)u^n u^{-m-n-1} = u\theta_1(t)u^{-n-1}u\theta_1(s)u^{-m-1} \\ &= \theta(n, r^n t)\theta(m, r^m s).\end{aligned}$$

Thus, θ is an anti-homomorphism. One similarly proves the homomorphism case. ■

THEOREM 2.2. *Let $\varphi : A(F) \to B(G)$ be a contractive homomorphism. The following statements are equivalent:*

(i) *there is an open subgroup G_0 of $\mathbb{Z} \times G$ and a continuous [anti-]homomorphism $\theta : G_0 \to F$ such that $\varphi = j_{\iota_G} \circ l_{(-1,e_G)} \circ s \circ j_\theta$—see* (2.1)*;*

(ii) *φ has a Cohen factorization $\varphi = l_r \circ s \circ j_{\theta_1} \circ l_u$—see* (1.2)*—for some open subgroup G_1 of G, a continuous [anti-]homomorphism $\theta_1 : G_1 \to F$, some $r \in G$ and $u \in F$ such that $rG_1r^{-1} = G_1$ and $\theta_1(r^{\pm 1}tr^{\mp 1}) = u^{\mp 1}\theta_1(t)u^{\pm 1}$ $[\theta_1(r^{\pm 1}tr^{\mp 1}) = u^{\pm 1}\theta_1(t)u^{\mp 1}]$.*

Moreover, when (i) *holds, every Cohen factorization satisfies the conditions of* (ii)*, and we necessarily have $G_0 = \{(n, r^n t) : n \in \mathbb{Z},\ t \in G_1\}$ and $\theta : G_0 \to F : (n, r^n t) \mapsto u^{-n}\theta_1(t)$ $[u\theta_1(t)u^{-n-1}]$.*

Proof. Suppose that φ has the factorization $\varphi = j_{\iota_G} \circ l_{(-1,e_G)} \circ s \circ j_\theta$ described in statement (i). By [15, Theorem 5.1], φ also has a Cohen factorization $\varphi = l_r \circ s \circ j_{\theta_1} \circ l_u$ for some open subgroup G_1 of G, a continuous [anti-]homomorphism $\theta_1 : G_1 \to F$ and some $r \in G$, $u \in F$. Statement (ii) will follow from Lemma 2.1 if we can show that $G_0 = \{(n, r^n t) : n \in \mathbb{Z},\ t \in G_1\}$ and $\theta(n, r^n t) = u^{-n}\theta_1(t)$ $[\theta(n, r^n t) = u\theta_1(t)u^{-n-1}]$. For $v \in A(F)$ and $x \in G$, the two factorizations of φ give

$$\varphi(v)(x) = \begin{cases} v(u\theta_1(rx)) & \text{if } x \in r^{-1}G_1 \\ 0 & \text{if } x \in G \setminus r^{-1}G_1 \end{cases} \tag{2.2}$$

$$= \begin{cases} v(\theta(-1, x)) & \text{if } (-1, x) \in G_0 \\ 0 & \text{if } (-1, x) \in \mathbb{Z} \times G \setminus G_0. \end{cases}$$

As $A(F)$ vanishes at no point of F, we obtain $r^{-1}G_1 = \{x \in G : (-1, x) \in G_0\}$. Hence, $(-1, r^{-1}) \in G_0$ so $(-1, r^{-1})^{-n} = (n, r^n) \in G_0$ $(n \in \mathbb{Z})$. For any $n \in \mathbb{Z}$, we obtain

$$\begin{aligned}\{n\} \times r^n G_1 &= (n+1, r^{n+1}) \cdot (\{-1\} \times r^{-1} G_1) \\ &= (n+1, r^{n+1}) \cdot (G_0 \cap (\{-1\} \times G)) \\ &= (n+1, r^{n+1}) G_0 \cap (n+1, r^{n+1})(\{-1\} \times G) = G_0 \cap (\{n\} \times G)\end{aligned}$$

so

$$G_0 = \bigcup_{n \in \mathbb{Z}} G_0 \cap (\{n\} \times G) = \{(n, r^n t) : n \in \mathbb{Z},\, t \in G_1\}$$

as needed.

As $A(F)$ separates points of F, (2.2) gives $\theta(-1, r^{-1}t) = u\theta_1(t)$ $(t \in G_1)$. Hence, $\theta(-1, r^{-1}) = u$ because θ_1 is an [anti-]homomorphism, and $\theta(n, r^n) = u^{-n}$ because θ is an [anti-]homomorphism. In the case that θ is an anti-homomorphism,

$$\begin{aligned}\theta(n, r^n t) &= \theta((n+1, r^{n+1})(-1, r^{-1}t)) = \theta(-1, r^{-1}t)\theta(n+1, r^{n+1}) \\ &= u\theta_1(t)u^{-n-1}\end{aligned}$$

as claimed. When θ is a homomorphism, one similarly obtains $\theta(n, r^n t) = u^{-n}\theta_1(t)$.

Conversely, suppose that φ has Cohen factorization $\varphi = l_r \circ s \circ j_{\theta_1} \circ l_u$ with r, u, and $\theta : G_1 \to F$ as in (ii). By Lemma 2.1, $G_0 = \{(n, r^n t) : n \in \mathbb{Z},\, t \in G_1\}$ is an open subgroup of $\mathbb{Z} \times G$ and $\theta : G_0 \to F : (n, r^n t) \mapsto u^{-n}\theta_1(t)$ $[u\theta_1(t)u^{-n-1}]$ is an—obviously continuous—[anti-]homomorphism. The calculations giving (2.2) show that $\varphi = j_{\iota_G} \circ l_{(-1, e_G)} \circ s \circ j_\theta$. ∎

Corollary 2.3. *If G contains a nonnormal open subgroup, then for any locally compact group F, there is a contractive homomorphism $\varphi : A(F) \to B(G)$ that fails to have a factorization of the form* (2.1).

Proof. Let G_1 be be a nonnormal open subgroup of G, and choose $r \in G$ such that $rG_1r^{-1} \neq G_1$. Let $\theta_1 : G_1 \to F$ be any homomorphism (for example, the trivial one) and let $\varphi = l_r \circ s \circ j_{\theta_1}$. By Theorem 2.2, φ does not have a factorization of the form (2.1). ∎

Proposition 2.4. *For $i = 1, 2$, let G_i be an open subgroup of G, $\theta_i : G_i \to F$ a homomorphism or an anti-homomorphism, $r_i \in G$, $u_i \in F$.*

(i) $\varphi = l_{r_i} \circ s_i \circ j_{\theta_i} \circ l_{u_i} : A(F) \to B(G)$ $(i = 1, 2)$ *if and only if θ_1 is an [anti-]homomorphism and*

$$G_1 = G_2, \quad r_2 r_1^{-1} \in G_1, \quad \theta_1(r_2 r_1^{-1}) = u_2^{-1}u_1, \quad \theta_1 = \theta_2 \quad [\theta_1(t) = u_1^{-1}u_2\theta_2(t)u_2^{-1}u_1]. \tag{2.3}$$

(ii) *If* (2.3) *holds and $l_{r_1} \circ s \circ j_{\theta_1} \circ l_{u_1}$ satisfies the conditions of Theorem* 2.2(ii), *then so does $l_{r_2} \circ s \circ j_{\theta_2} \circ l_{u_2}$.*

Proof. For $v \in A(F)$ and $x \in G$,

$$\varphi(v)(x) = \begin{cases} v(u_i\theta_i(r_i x)) & \text{if } x \in r_i^{-1}G_i \\ 0 & \text{if } x \in G \setminus r_i^{-1}G_i \end{cases} \qquad (i = 1, 2).$$

As in the proof of Theorem 2.2, the separation properties of $A(F)$ give $r_1^{-1}G_1 = r_2^{-1}G_2$. Hence, $r_2r_1^{-1} \in G_1$, $G_2 = r_2(r_2^{-1}G_2) = r_2(r_1^{-1}G_1) = G_1$, and $u_1\theta_1(r_1x) = u_2\theta_2(r_2x)$ $(x \in r_i^{-1}G_i)$. For $t \in G_1$, one obtains

$$\theta_1(t) = u_1^{-1}u_2\theta_2(r_2r_1^{-1}t) = u_1^{-1}u_2\theta_2(r_2r_1^{-1})\theta_2(t)$$

when θ_2 is a homomorphism, and $\theta_1(t) = u_1^{-1}u_2\theta_2(t)\theta_2(r_2r_1^{-1})$ when θ_2 is an anti-homomorphism. As $\theta_1(e_G) = \theta_2(e_G) = e_F$, the forward implication of (i) follows. Conversely, supposing that condition (2.3) is satisfied, it suffices to show that $u_1\theta_1(r_1x) = u_2\theta_2(r_2x)$ for $x \in r_1^{-1}G_1 = r_2^{-1}G_2$. Assuming that θ_1 is an anti-homomorphism—the homomorphism case is similar—for $x \in r_1^{-1}G_1$,

$$\begin{aligned} u_1^{-1}u_2\theta_2(r_2x) &= (u_1^{-1}u_2\theta_2(r_2x)u_2^{-1}u_1)u_1^{-1}u_2 = \theta_1(r_2x)(\theta_1(r_2r_1^{-1}))^{-1} \\ &= \theta_1((r_2r_1^{-1})^{-1}r_2x) = \theta_1(r_1x). \end{aligned}$$

This proves (i). Statement (ii) follows immediately from Theorem 2.2, although it can also be verified directly. ∎

For a subset $\mathcal{T}$ of $A(F)$, let $h(\mathcal{T}) = \{x \in F : u(x) = 0 \text{ for all } u \in \mathcal{T}\}$ be the hull of $\mathcal{T}$ and let $s(\mathcal{T}) = F \setminus h(\mathcal{T})$. Observe that if $\varphi = l_r \circ s \circ j_{\theta_1} \circ l_u : A(F) \to B(G)$, then $h(\ker\varphi) = \overline{u\theta_1(G_1)}$ and $s(\operatorname{Im}\varphi) = r^{-1}G_1$. Hence, if φ is any contractive homomorphism, then $F_\varphi := h(\ker\varphi)^{-1}h(\ker\varphi)$ is a closed subgroup of F and $G_\varphi := s(\operatorname{Im}\varphi)^{-1}s(\operatorname{Im}\varphi)$ is an open subgroup of G. Assuming that φ has the Cohen factorization described above, $\varphi_b = l_r \circ s \circ j_{\theta_1} \circ l_u : B(F) \to B(G)$ is called the canonical extension of φ in [12].

Corollary 2.5. *Let $\varphi : A(F) \to B(G)$ be a contractive homomorphism.*

(i) *If $s(\operatorname{Im}\varphi)$ has nonempty intersection with the commutant, G'_φ, of G_φ and $h(\ker\varphi)$ is contained in F'_φ, then φ has a factorization of the form* (2.1).

(ii) *If $s(\operatorname{Im}\varphi)$ is a subgroup of G (for example, when G is connected or when $\varphi_b(1_F)$ is positive definite) and F_φ is contained in the centre of F (for example when F is abelian), then φ has a factorization of the form* (2.1).

Proof. Assume that φ has Cohen factorization $\varphi = l_r \circ s \circ j_{\theta_1} \circ l_u$ where $\theta_1 : G_1 \to F$, so $s(\operatorname{Im}\varphi) = r^{-1}G_1$, $G_\varphi = G_1$ and $h(\ker\varphi) = \overline{u\theta_1(G_1)} \subset \theta_1(G_1)'$. Choose $r_2 \in rG_1 \cap G'_1$ and let $u_2 = u\theta_1(rr_2^{-1})$. Observe that because u and u_2 belong to $\theta_1(G_1)'$, we have $u_2^{-1}u\theta_1(t)u^{-1}u_2 = \theta_1(t)$, so regardless of whether θ_1 is a homomorphism or anti-homomorphism, $\varphi = l_{r_2} \circ s \circ j_{\theta_1} \circ l_{u_2}$ by Proposition 2.4. It is obvious that this factorization satisfies the conditions of

Theorem 2.2(ii). The validity of most of statement (ii) should be clear. Note that if $\varphi_b(1_F) = 1_{r^{-1}G_1}$ is positive definite, then $s(\operatorname{Im}\varphi) = r^{-1}G_1 = G_1$ by [11, Theorem 2.1]. ∎

3. Weak*-continuous homomorphisms. In this section we will characterize the w^*-w^* continuous contractive homomorphisms on (reduced) Fourier–Stieltjes algebras and on measure algebras. The *so*-topologies on $B_r(F)$ and $M(F)$ refer to the strict topologies on these algebras taken with respect to the ideals $A(F)$ and $L^1(F)$, respectively.

LEMMA 3.1. *Let $\varphi : A(F) \to B(G)$ be a homomorphism. Then there is an open subset Y of G and a continuous mapping $\alpha : Y \to F$ such that $\varphi = j_\alpha$. If Y is also closed, then $j_\alpha : B_r(F) \to B(G)$ is the unique extension of φ to $B_r(F)$ that is so-w^* continuous on bounded subsets of $B_r(F)$.*

REMARK 3.2. 1. When $A(F)$ has a bounded approximate identity, i.e. when F is amenable, this can be seen to follow from [12, Theorem 5.6]—or [5, Theorem 4.2]—and the proof of [12, Corollary 5.8]. In this case, the extension is *so*-w^* continuous on $B_r(F) = B(F)$. The fact that every contractive homomorphism $\varphi : A(F) \to B(G)$ extends to $B(F)$ is [15, Corollary 5.6].

2. We believe that Y is always closed, but were unfortunately not able to show this.

3. If $u \in B_r(G)$ is positive definite, then the proof of [1, Proposition 2.22] gives a bounded net (u_i) of positive definite functions in $A(F)$ that converges uniformly to u on compact subsets of F. Hence, $u_i \to u$ *so* by [14, Theorem 3.2]. By considering the Jordan decomposition of an arbitrary $u \in B_r(F)$, one obtains a bounded net (u_i) in $A(F)$ such that $u_i \to u$ *so*. The uniqueness part of the above lemma follows, and this also shows that $A(F)$ is *so*-dense in $B_r(F)$.

Proof of Lemma 3.1. As noted by Pham [15], because $B(G)$ is semisimple, φ is automatically continuous by [4, Theorem 2.3.3]. The existence of Y and α is a consequence of the fact that the Gelfand spectrum of $A(F)$ is F; see for example the first paragraph of the proof of [11, Theorem 3.7]. Assume now that Y is also closed. Let $u \in B_r(F)$ and let (u_i) be a bounded net in $A(F)$ such that $u_i \to u$ *so*. By passing to a subnet if necessary, we may assume that $(j_\alpha u_i)$ has a w^*-limit, v, in $B(G)$. Take $f \in C_{00}(G)$ with compact support K. Regularity of $A(F)$ and the inequality $\|\cdot\|_{B(F)} \ge \|\cdot\|_\infty$ give uniform convergence of (u_i) to u on the compact set $\alpha(Y \cap K)$. Hence,

$$|\langle j_\alpha u_i - j_\alpha u, f\rangle_{L^\infty\text{-}L^1}| \le \int_{Y\cap K} |j_\alpha u_i - j_\alpha u|(x)|f(x)|\,dx \to 0. \tag{3.1}$$

As $(j_\alpha u_i)$ is bounded in $L^\infty(G)$, $j_\alpha u_i \to j_\alpha u$ weak* in $L^\infty(G)$. Also, $j_\alpha u_i \to v$ weak* in $L^\infty(G)$, so by continuity of both $j_\alpha u$ and v, $j_\alpha u = v \in B(G)$. Thus,

j_α maps $B_r(F)$ into $B(G)$ and once again j_α is necessarily bounded. It follows from this, the density of $C_{00}(G)$ in $C^*(G)$ and the calculation (3.1) that j_α is *so*-w^* continuous on bounded subsets of $B_r(F)$. ∎

THEOREM 3.3. *Let $\varphi : B_r(F) \to B(G)$ be a mapping. The following statements are equivalent:*

(i) *φ is a contractive homomorphism that is so-w^* continuous on bounded subsets of $B_r(F)$ (w^*-w^* continuous on $B_r(F)$);*
(ii) *φ has a Cohen factorization*

$$B_r(F) \xrightarrow{l_u} B_r(F) \xrightarrow{j_\theta} B(G_1) \overset{s}{\hookrightarrow} B(G) \xrightarrow{l_r} B(G) \tag{3.2}$$

for some continuous (open) homomorphism or anti-homomorphism, θ, mapping an open subgroup G_1 of G into F.

Proof. Assuming statement (i), $\varphi|_{A(F)}$ is a contractive homomorphism, and therefore has a factorization (1.2) where $\theta = \theta_1 : G_1 \to F$ is a continuous homomorphism or anti-homomorphism; equivalently, $\varphi|_{A(F)} = j_\alpha$ where $Y = r^{-1}G_1$ and $\alpha : Y \to F$ is given by $\alpha(y) = u\theta(ry)$ [15, Theorem 5.1]. Note that $w^* \subseteq so$, so in both cases $\varphi = j_\alpha$ by Lemma 3.1. Suppose that $\varphi = j_\alpha$ is w^*-w^* continuous. Suppose first that θ is a homomorphism. One can check that the proofs of Proposition 4.3 and Lemma 2.3 of [12] remain valid when $B(F)$ is replaced by $B_r(F)$. (Indeed, in the proof of [12, Lemma 4.2] one can replace the universal representation ω with the left regular representation, λ. To modify the proof of [12, Proposition 4.3], note that in §5 of [2], the authors actually proved that if the restriction map $\Phi_r : B_r(G) \to B(G_1)$ is w^*-w^* continuous, then G_1 is an open subgroup of G—this is stronger than [2, Theorem 5.1]. To see this, note that just as Lemma 5.2 of [2] holds, if Φ_r is w^*-w^* continuous, then the map $\Psi_r : C^*(G_1) \to C^*_r(M(G))$, as defined on page 2292 of [2], maps $C^*(G_1)$ into $C^*_r(G)$. By [2, Theorem 5.4], this implies that G_1 is open.) Thus, θ is an open mapping in this case. Finally, suppose that θ is an anti-homomorphism. Then $\check\theta : G_1 \to F : x \mapsto \theta(x)^{-1}$ is a homomorphism, and φ factors as

$$B_r(F) \xrightarrow{l_u} B_r(F) \xrightarrow{\check{\cdot}} B_r(F) \xrightarrow{j_{\check\theta}} B(G_1) \overset{s}{\hookrightarrow} B(G) \xrightarrow{l_r} B(G)$$

where $\check u(x) = u(x^{-1})$, a w^*-w^* homeomorphism of $B_r(F)$. As φ is w^*-w^* continuous on $B_r(F)$, so is $\varphi \circ l_{u^{-1}} \circ \check{\cdot} = l_r \circ s \circ j_{\check\theta}$. By the previous case, $\check\theta$ is an open map, and therefore θ is also an open map.

Conversely, suppose that φ has the factorization (3.2) and consider the case when θ is an open anti-homomorphism. By [12, Proposition 3.4], $l_r \circ s \circ j_{\check\theta}$ is w^*-w^* continuous on $B(F)$ and therefore on $B_r(F)$. Hence, $\varphi = l_r \circ s \circ j_{\check\theta} \circ \check{\cdot} \circ l_u$ is w^*-w^* continuous on $B_r(F)$. ∎

Throughout the remainder of this section, we consider an arbitrary contractive homomorphism $\varphi : L^1(F) \to M(G)$ and its *so*-w^* continuous extension $\varphi = \varphi_m : M(F) \to M(G)$ (also denoted φ).

THEOREM 3.4. *Let $\varphi : M(F) \to M(G)$ be a mapping. Then φ is a w^*-w^* continuous contractive homomorphism if and only if there is a closed subgroup H of G, a compact normal subgroup Ω_0 of $\mathbb{T} \times H$ and a continuous proper homomorphism $\theta : F \to \mathbb{T} \times H/\Omega_0$ such that $\varphi = j^*_{\iota_H} \circ j^*_{\theta_H} \circ \Lambda_{\alpha_\mathbb{T}} \circ S^*_{\Omega_0} \circ j^*_\theta$:*

$$\begin{tikzcd}
M(F) \arrow[rrr, "\varphi"] \arrow[d, "j^*_\theta"'] & & & M(G) \\
M(\mathbb{T} \times H/\Omega_0) \arrow[r, hook, "S^*_{\Omega_0}"] & M(\mathbb{T} \times H) \arrow[r, "\Lambda_{\alpha_\mathbb{T}}"] & M(\mathbb{T} \times H) \arrow[r, "j^*_{\theta_H}"] & M(H) \arrow[u, hook, "j^*_{\iota_H}"']
\end{tikzcd}$$

Proof. As the projection map $\theta_H : \mathbb{T} \times H \to H$ is proper, the backwards implication follows from Propositions 5.1 and 5.3 in [17]. Conversely, suppose that $\varphi : M(F) \to M(G)$ is w^*-w^* continuous. Then φ is *so*-w^* continuous, so by [17, Corollary 6.1 and Theorem 5.11], φ has the factorization illustrated above with $\Omega_0 = \Omega_\rho = \{(\rho(k), k) : k \in K\}$ for some compact normal subgroup K of H, $\rho \in \widehat{K}^1$, and θ a continuous homomorphism. Let $\varphi_* : C_0(G) \to C_0(F)$ be the predual map of φ. Then the diagram below commutes:

$$\begin{tikzcd}
C_0(G) \arrow[rr, "\varphi_*"] \arrow[d, "j_\iota"'] & & C_0(F) \arrow[r, hook] & E(F) \\
C_0(H) \arrow[r, "j_{\theta_H}"] & C_0(\mathbb{T} \times H) \arrow[r, "M_{\alpha_\mathbb{T}}"] & C_0(\mathbb{T} \times H) \arrow[r, "S_{\Omega_\rho}"] & C_0(\mathbb{T} \times H/\Omega_\rho) \arrow[u, "j_\theta"']
\end{tikzcd}$$

To see this, it suffices to note that the dual maps of both the top line of the diagram and $j_\theta \circ S_{\Omega_\rho} \circ M_{\alpha_\mathbb{T}} \circ j_{\theta_H} \circ j_\iota$ are w^*-w^* continuous extensions of φ to $E(F)^*$, and are therefore equal. Supposing (towards a contradiction) that θ is not proper, we can choose a compact subset L of $\mathbb{T} \times H/\Omega_\rho$ such that $\theta^{-1}(L)$ is not a compact subset of F.

For $\gamma \in C_0(G)$, put $f_\gamma = (S_{\Omega_\rho} \circ M_{\alpha_\mathbb{T}} \circ j_{\theta_H} \circ j_\iota)(\gamma)$. Then for $(\alpha, x)\Omega_\rho \in \mathbb{T} \times H/\Omega_\rho$—writing $\Lambda_H = M_{\alpha_\mathbb{T}} \circ j_{\theta_H}$—we have

$$\begin{aligned}
f_\gamma((\alpha, x)\Omega_\rho) &= S_{\Omega_\rho} \circ \Lambda_H(\gamma|_H)((\alpha, x)\Omega_\rho) = \langle m_{\Omega_\rho}, l_{(\alpha,x)} \Lambda_H(\gamma|_H)\rangle_{M\text{-}C_0(\mathbb{T}\times H)} \\
&= \alpha \langle m_{\Omega_\rho}, \Lambda_H(l_x(\gamma|_H))\rangle_{M\text{-}C_0(\mathbb{T}\times H)} \\
&= \alpha \langle \Lambda^*_H(m_{\Omega_\rho}), l_x(\gamma|_H)\rangle_{M\text{-}C_0(H)} \\
&= \alpha \langle \rho m_K, l_x(\gamma|_H)\rangle_{M\text{-}C_0(H)} \\
&\overset{(*)}{=} \alpha \int\limits_K \gamma(xk)\rho(k)\, dm_K(k),
\end{aligned}$$

where in the penultimate line we have used [17, Proposition 5.9].

Let $\mathcal{P}$ denote the collection of all finite products of functions in $\{f_\gamma,\ \overline{f_\gamma} : \gamma \in C_0(G)\}$, and let $\mathcal{A}$ be the collection of all finite sums of functions in $\mathcal{P}$. Observe that $\mathcal{A}$ is a self-adjoint subalgebra of $C_0(\mathbb{T} \times H/\Omega_\rho)$, and it is easy to see that $\mathcal{A}$ vanishes at no point of $\mathbb{T} \times H/\Omega_\rho$. Suppose that $(\alpha, x)\Omega_\rho \neq (\beta, y)\Omega_\rho$, so $(\overline{\alpha}\beta, x^{-1}y) \notin \{(\rho(k), k) : k \in K\}$. If $x^{-1}y \notin K$, take $\gamma \in C_0(G)$ such that $\gamma(xk) = \overline{\rho(k)}$ $(k \in K)$ and $\gamma|_{yK} = 0$; then $(*)$ gives $f_\gamma((\alpha, x)\Omega_\rho) = \alpha \neq 0 = f_\gamma((\beta, y)\Omega_\rho)$. Otherwise, $y = xk_0$ for some $k_0 \in K$ such that $\beta\overline{\rho(k_0)} \neq \alpha$. Choosing $\gamma \in C_0(G)$ such that $\gamma(xk) = \overline{\rho(k)}$ $(k \in K)$, we get

$$\begin{aligned} f_\gamma((\beta, y)\Omega_\rho) &= \beta \int_K \gamma(xk_0k)\rho(k)\, dm_K(k) = \beta\overline{\rho(k_0)} \int_K \gamma(xk)\rho(k)\, dm_K(k) \\ &= \beta\overline{\rho(k_0)} \neq \alpha = f_\gamma((\alpha, x)\Omega_\rho). \end{aligned}$$

By the Stone–Weierstrass Theorem, $\mathcal{A}$ is uniformly dense in $C_0(\mathbb{T} \times H/\Omega_\rho)$, so we can find $g \in \mathcal{A}$ such that $|g(x)| > 1$ $(x \in L)$. Writing g as $g = g_1 + \cdots + g_n$ with each $g_j \in \mathcal{P}$, we obtain

$$1 < |g(x)| \le |g_1(x)| + \cdots + |g_n(x)| \quad (x \in L),$$

so

$$L = \bigcup_{j=1}^{n} L_j \quad \text{where} \quad L_j = |g_j|^{-1}[1/n, \infty) \cap L.$$

As $\theta^{-1}(L) = \bigcup_j \theta^{-1}(L_j)$ is not compact, $\theta^{-1}(L_{j_0})$ is not compact for some j_0; put $L_0 = L_{j_0}$, $g_0 = g_{j_0}$. As $\theta^{-1}(L_0)$ is closed but not compact, and $|j_\theta(g_0(x))| = |g_0(\theta(x))| \ge 1/n$ $(x \in \theta^{-1}(L_0))$, it follows that $j_\theta g_0 \notin C_0(F)$. Because $g_0 \in \mathcal{P}$, there are functions $\gamma_1, \ldots, \gamma_m$ in $C_0(G)$ such that $g_0 = f_{\gamma_1} \cdots f_{\gamma_k} \overline{f_{\gamma_{k+1}}} \cdots \overline{f_{\gamma_m}}$. Hence, $j_\theta g_0 = j_\theta f_{\gamma_1} \cdots j_\theta f_{\gamma_k} j_\theta \overline{f_{\gamma_{k+1}}} \cdots j_\theta \overline{f_{\gamma_m}}$ does not belong to the ideal $C_0(F)$ in $E(F)$. It follows—in particular—that $\varphi_*(\gamma_1) = (j_\theta \circ S_{\Omega_\rho} \circ M_{\alpha_{\mathbb{T}}} \circ j_{\theta_H} \circ j_\iota)(\gamma_1) = j_\theta(f_{\gamma_1}) \notin C_0(F)$. This contradiction completes the proof. ∎

In the above, were it true that $\{f_\gamma : \gamma \in C_0(G)\} = C_0(\mathbb{T} \times H/\Omega_\rho)$, the proof of Theorem 3.4 would, like the proof of the following proposition, be quite easy. This, however, is usually not the case. We recall from Example 5.4 of [17] that not every w^*-w^* continuous contractive homomorphism $\varphi : M(F) \to M(G)$ has a Cohen factorization.

Proposition 3.5. *Suppose that $\varphi : M(F) \to M(G)$ has a Cohen factorization $\varphi = j_\iota^* \circ A_\rho \circ S_K^* \circ j_\theta^* \circ A_\alpha$,*

$$M(F) \xrightarrow{A_\alpha} M(F) \xrightarrow{j_\theta^*} M(H/K) \overset{S_K^*}{\hookrightarrow} M(H) \xrightarrow{A_\rho} M(H) \overset{j_\iota^*}{\hookrightarrow} M(G). \tag{3.3}$$

Then φ is w^*-w^* continuous if and only if $\theta : F \to H/K$ is a proper mapping.

Proof. If φ is w^*-w^* continuous, then so is $\varphi\circ A_{\overline{\alpha}}$, so we may assume that $\alpha\equiv 1$. The predual map $\varphi_*: C_0(G)\to C_0(F)$ factors as $\varphi_* = j_\theta\circ S_K\circ M_\rho\circ j_\iota$. As $S_K\circ M_\rho\circ j_\iota$ maps $C_0(G)$ onto $C_0(H/K)$, j_θ maps $C_0(H/K)$ into $C_0(F)$, and hence θ is proper. The converse follows from [17, Propositions 5.1 and 5.3]. ∎

We now fix some notation that will be used throughout the remainder of this paper: Let $\varphi: M(F)\to M(G)$ have the factorization (1.1),

$$\varphi = j_{\theta_H}^*\circ A_{\alpha_{\mathbb{T}}}\circ S_{\Omega_\rho}^*\circ j_\theta^*,$$

in canonical form (see Remark 1.2). We wish to consider φ in relation to the associated homomorphism

$$\varphi^+ = j_{\theta_H}^*\circ S_{\Omega_\rho}^*\circ j_\theta^*.$$

As in the proof of [17, Corollary 5.8], we consider the continuous homomorphisms

$$p_K:\mathbb{T}\times H/\Omega_\rho\to H/K:(\alpha,h)\Omega_\rho\mapsto hK \quad\text{and}\quad \theta_K = p_K\circ\theta: F\to H/K.$$

Although φ and φ^+ are closely related, φ^+ has a very simple factorization.

PROPOSITION 3.6. *The homomorphism $\varphi^+: M(F)\to M(G)$ is positive with $\varphi^+(\delta_{e_F}) = m_K$ and Cohen factorization $\varphi^+ = S_K^*\circ j_{\theta_K}^*$,*

$$M(F)\xrightarrow{j_{\theta_K}^*} M(H/K)\overset{S_K^*}{\hookrightarrow} M(H)\hookrightarrow M(G).$$

Proof. For $x\in F$, let $\theta(x) = (\alpha_x,h_x)\Omega_\rho$, so $\theta_K(x) = h_xK$. Observe that $\gamma: K\to\Omega_\rho$, $\gamma(k) = (\rho(k),k)$, is a topological isomorphism, so $j_\gamma^*: M(K)\to M(\Omega_\rho)$ maps m_K to m_{Ω_ρ}. It follows that

$$\begin{aligned}\varphi^+(\delta_x) &= j_{\theta_H}^*\circ S_{\Omega_\rho}^*\circ j_\theta^*(\delta_x) = j_{\theta_H}^*(\delta_{(\alpha_x,h_x)} * m_{\Omega_\rho})\\ &= \delta_{h_x} * j_{\theta_H}^*(j_\gamma^* m_K) = \delta_{h_x} * (j_{\theta_H\circ\gamma})^* m_K = \delta_{h_x} * m_K = S_K^*\circ j_{\theta_K}^*(\delta_x).\end{aligned}$$

By *so*-w^* continuity of both maps, $\varphi^+ = S_K^*\circ j_{\theta_K}^*$. ∎

COROLLARY 3.7. *Let $\varphi: M(F)\to M(G)$ be a contractive homomorphism. The following statements are equivalent:*

(i) *φ is w^*-w^* continuous;*
(ii) *φ^+ is w^*-w^* continuous;*
(iii) *$\theta_K: F\to H/K$ is a proper mapping.*

Proof. The equivalence of (ii) and (iii) follows from Propositions 3.5 and 3.6. Moreover, it is not difficult to show that $p_K:\mathbb{T}\times H/\Omega_\rho\to H/K$ is a proper mapping and from this that one of θ and $\theta_K = p_K\circ\theta$ is proper exactly when the other is. Thus, (i) and (ii) are equivalent by Theorem 3.4. ∎

We remark that the implication (i)⇒(iii) in the above corollary is not, without the aid of Theorem 3.4, obvious to us.

4. Contractive homomorphisms of group algebras. An open problem is to characterize all (contractive) homomorphisms $\varphi : L^1(F) \to L^1(G)$. Some special cases are dealt with in [8], [13] and [17]. We begin our discussion of this problem with the following result which is very closely related to Theorem 2 and Corollary 3 of [13]. This short proof is independent of [13].

PROPOSITION 4.1. *Suppose that $\varphi : L^1(F) \to M(G)$ has a Cohen factorization $\varphi = j^*_{\iota_H} \circ A_\rho \circ S^*_K \circ j^*_\theta \circ A_\alpha$. Then φ maps $L^1(F)$ into $L^1(G)$ if and only if H is an open subgroup of G and $\theta : F \to H/K$ is an open mapping.*

Proof. Suppose that φ maps $L^1(F)$ into $L^1(G)$. As the support of any measure in $j^*_{\iota_H}(M(H))$ is contained in H and we are assuming—as always—that φ is nonzero, H must have positive Haar measure in G, whence H is open in G [7, Corollary III.12.5]. Hence, $m_H = m_G|_H$ and the map $\mu \mapsto \mu|_H : M(G) \to M(H)$—where $\mu|_H(E) = \mu(E)$, $E \subseteq H$—maps $L^1(G)$ into $L^1(H)$. As $(j^*_{\iota_H}\mu)|_H = \mu$ we can conclude that $j^*_{\iota_H}$ maps $M(H) \setminus L^1(H)$ into $M(G) \setminus L^1(G)$. Hence, $A_\rho \circ S^*_K \circ j^*_\theta \circ A_\alpha$ maps $L^1(F)$ into $L^1(H)$ and thus, clearly, $S^*_K \circ j^*_\theta$ maps $L^1(F)$ into $L^1(H)$. Let $q_K : H \to H/K$ be the quotient map. Then $S_K \circ j_{q_K} = \mathrm{id}_{C_0(H/K)}$ and q_K is an open mapping so $S^*_K\mu \in L^1(H)$ implies that $\mu = j^*_{q_K} S^*_K \mu \in L^1(H/K)$. Hence, j^*_θ maps $L^1(F)$ into $L^1(H/K)$, so by [17, Proposition 5.1], θ is an open mapping. The converse follows from [17, Proposition 5.1] and the well-known fact that S^*_K maps $L^1(H/K)$ into $L^1(H)$; indeed, one can readily show that $S^*_K f = f \circ q_K \in L^1(H)$ whenever $f \in L^1(H/K)$. ■

In light of Theorem 3.4 and Propositions 3.5 and 4.1, it seems natural to wonder if $\varphi : L^1(F) \to M(G)$ with a factorization (1.1) maps into $L^1(G)$ exactly when $\theta : F \to \mathbb{T} \times H/\Omega_\rho$ is an open mapping and H is an open subgroup of G. Indeed, it is clear from [17, Proposition 5.1] that the second condition is sufficient for φ to map $L^1(F)$ into $L^1(G)$. However, the converse implication is false.

EXAMPLE 4.2. (a) Let F be a discrete group, $\theta_0 \in \widehat{F}^1$, H a compact group. Take $K = H$ and $\rho \in \widehat{K}^1$ the trivial character. Then $\Omega_\rho = \{1\} \times H$ and $\theta : F \to \mathbb{T} \times H/\Omega_\rho : x \mapsto (\theta_0(x), e_H)\Omega_\rho$ is a continuous homomorphism. Let $\varphi = j^*_{\theta_H} \circ A_{\alpha_\mathbb{T}} \circ S^*_{\Omega_\rho} \circ j^*_\theta : L^1(F) \to M(H)$; see (1.1). For $x \in F$,

$$\varphi(\delta_x) = j^*_{\theta_H} \circ A_{\alpha_\mathbb{T}}(\delta_{(\theta_0(x), e_H)} * m_{\Omega_\rho}) = \theta_0(x) m_K = \theta_0(x) m_H \in L^1(H)$$

because of [17, Proposition 5.9]. It is easy to see from this that $\varphi = j^*_{\theta_H} \circ A_{\alpha_\mathbb{T}} \circ S^*_{\Omega_\rho} \circ j^*_\theta$ is precisely the canonical factorization of φ. As F is discrete it also follows that φ maps $L^1(F) = \ell^1(F)$ into $L^1(H)$. Note however that $\theta : F \to \mathbb{T} \times H/\Omega_\rho \cong \mathbb{T}$ is not open. On the other hand, $\theta_K : F \to H/K = H/H$ is—trivially—an open map. (In this case, by Corollary 5.8 of [17], φ has a

Cohen factorization involving j_{θ_K}, so θ_K had to be open by Proposition 4.1, or by Proposition 4.4 below.)

(b) Let $F = \mathrm{SU}_2(\mathbb{C})$, $K = \mathbb{Z}_2$, the centre of F, and $\rho : K \to \mathbb{T} : t \mapsto t$. Then $\varphi : M(F) \to M(F) : \mu \mapsto \mu * \rho m_K$ does not have a Cohen factorization [17, Example 5.4]. The support subgroup of $\{\varphi(\delta_x) : x \in F\}$ is F, $\Omega_\rho = \{(\rho(k), k) : k \in K\}$, and $\theta : F \to \mathbb{T} \times F/\Omega_\rho : x \mapsto (1, x)\Omega_\rho$ (see Remark 1.2). If U is an open subset of F, then $q_{\Omega_\rho}^{-1}(\theta(U)) = (\{1\} \times U) \cup (\{-1\} \times -U)$ which is not necessarily open in $\mathbb{T} \times F$; here $q_{\Omega_\rho} : \mathbb{T} \times F \to \mathbb{T} \times H/\Omega_\rho$. Hence, θ is not an open mapping, even though φ maps $L^1(F)$ into $L^1(F)$. Note however that $\theta_K : F \to F/K : x \mapsto xK$ is an open mapping.

It is not difficult to show that θ_K is an open mapping whenever θ is. As the above examples show, the converse implication is false.

Proposition 4.3. *Let $\varphi : L^1(F) \to M(G)$ be a contractive homomorphism with the factorization* (1.1) *in canonical form. Consider the following statements:*

(i) *φ maps $L^1(F)$ into $L^1(G)$;*
(ii) *φ^+ maps $L^1(F)$ into $L^1(G)$;*
(iii) *$\theta_K : F \to H/K$ is an open mapping and H is an open subgroup of G.*

Then (i)$\Leftarrow$(ii)$\Leftrightarrow$(iii). *The implication* (i)$\Rightarrow$(iii) *holds when either F is discrete or ρ extends to a continuous character on H.*

Proof. The equivalence of statements (ii) and (iii) is a consequence of Propositions 3.6 and 4.1. Suppose that (ii) holds and let $f \in L^1(F)$; assume without loss of generality that $f \geq 0$. Then $\mu = S^*_{\Omega_\rho} \circ j^*_\theta(f) \in M(\mathbb{T} \times H)$, $\mu \geq 0$ and $\varphi^+(f) = j^*_{\theta_{H,G}}\mu \ll m_G$ (see Remark 1.2). Note that

$$|j^*_{\theta_{H,G}}(\alpha_{\mathbb{T}} \cdot \mu)| \leq j^*_{\theta_{H,G}}|\alpha_{\mathbb{T}} \cdot \mu| = j^*_{\theta_{H,G}}|\mu| = j^*_{\theta_{H,G}}(\mu);$$

this can be verified by using—for example—[9, (14.4) and (14.5)]. Hence, $\varphi(f) = j^*_{\theta_{H,G}}(\alpha_T \cdot \mu) \ll m_G$ as well. That is, $\varphi(f) \in L^1(G)$, as needed. When $\rho \in \widehat{K}^1$ extends to $\rho_H \in \widehat{H}^1$, φ has a Cohen factorization of the form $\varphi = j^*_{\iota_H} \circ A_{\rho_H} \circ S^*_K \circ j^*_{\theta_K} \circ A_\alpha$ by the proof of [17, Corollary 5.8]. Hence, (i) implies (iii) in this case by Proposition 4.1. That (i) implies (iii) when F is discrete follows from Proposition 4.4 below. ∎

Proposition 4.4. *Let $\varphi : M(F) \to M(G)$ be an so-w^* continuous contractive homomorphism with the factorization* (1.1) *in canonical form. The following statements are equivalent:*

(i) *φ maps $\ell^1(F)$ into $L^1(G)$;*
(ii) *K is open in G;*

(iii) $\theta_K : F_d \to H/K$ *is an open mapping and* H *is an open subgroup of* G.

Proof. Suppose that $\varphi(\delta_{e_F}) = \rho m_K \in L^1(G)$ and $m_G(K) = 0$. If $\theta(x) = (\alpha_x, h_x)\Omega_\rho$, then $\varphi(\delta_x) = \alpha_x \delta_{h_x} * \rho m_K$ is zero in $L^1(F)$ for each $x \in F$. By [17, Lemma 1.1], φ is the zero homomorphism in contradiction to our blanket assumption that all homomorphisms are nonzero. Hence, if (i) holds, then $m_G(K) > 0$ and hence (ii) holds. The implication (ii)⇒(iii) is clear. Observe that the restriction of φ to $\ell^1(F) = M(F_d) = L^1(F_d)$ also has the canonical form factorization (1.1)—see Remark 1.2—so by Proposition 4.3, statement (iii) implies (i). ■

We conclude with a question: Does the implication (i)⇒(iii) of Proposition 4.3 hold in general?

Acknowledgements. The author is grateful to the anonymous referee whose comments have improved the readability of this paper. The author also gratefully acknowledges support from an NSERC grant.

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Received October 3, 2011
Revised version April 2, 2012 (7316)