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Finite rank commutators of Toeplitz operators on the bidisk

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Abstract. We study some algebraic properties of commutators of Toeplitz operators on the Hardy space of the bidisk. First, for two symbols where one is arbitrary and the other is (co-)analytic with respect to one fixed variable, we show that there is no nontrivial finite rank commutator. Also, for two symbols with separated variables, we prove that there is no nontrivial finite rank commutator or compact commutator in certain cases.

1. Introduction. Let $\mathbb{T}$ be the boundary of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. The bidisk $\mathbb{D}^2$ and the torus $\mathbb{T}^2$ are the cartesian products of two copies of $\mathbb{D}$ and $\mathbb{T}$ respectively. We let $L^2(\mathbb{T}^2) = L^2(\mathbb{T}^2, \sigma_2)$ be the usual Lebesgue space of $\mathbb{T}^2$ where σ_2 is the normalized Haar measure on $\mathbb{T}^2$. The Hardy space $H^2(\mathbb{D}^2)$ is the closure of the analytic polynomials in $L^2(\mathbb{T}^2)$. Let P denote the Hilbert space orthogonal projection from $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{D}^2)$. For $u \in L^\infty(\mathbb{T}^2)$, the *Toeplitz operator* T_u *with symbol* u is defined by

$$T_u f = P(uf)$$

for $f \in H^2(\mathbb{D}^2)$. Then clearly T_u is a bounded linear operator on $H^2(\mathbb{D}^2)$.

In this paper, we study the problems of when the commutator of two Toeplitz operators is zero, of finite rank or compact on $H^2(\mathbb{D}^2)$. On the Hardy space of the unit disk, these problems have been well studied. Brown and Halmos [1] obtained a complete description of commuting Toeplitz operators. Later, Gorkin and Zheng [6] characterized compactness of commutators of two Toeplitz operators. Also, Ding and Zheng [5] have recently obtained a characterization for the commutator of two Toeplitz operators to have finite rank.

In the setting of the Hardy space over the polydisk, the corresponding problems appear to be wide open in general. Gu and Zheng [8] studied the problems on the bidisk and observed that compactness and being zero property for the commutator of two Toeplitz operators are the same in

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certain cases. More explicitly, they proved that given $u, v \in L^\infty(\mathbb{T}^2)$ for which one of u, v is analytic or co-analytic, $T_uT_v - T_vT_u$ is compact on $H^2(\mathbb{D}^2)$ if and only if $T_uT_v = T_vT_u$ on $H^2(\mathbb{D}^2)$ (see Corollary 2 of [8]). But it turns out that the general situation is entirely different. In the same paper, Gu and Zheng considered symbols u, v of the form $u(z,w) = f(z)g(w)$ and $v(z,w) = h(z)k(w)$ where f, g, h, k are nonzero continuous on $\mathbb{T}$ and f, h have zero sets with an intersection of a positive Lebesgue measure. They proved that if $fh = 0$ and $gk = 0$ on $\mathbb{T}$, then $T_uT_v - T_vT_u$ is compact but $T_uT_v \neq T_vT_u$ on $H^2(\mathbb{D}^2)$.

Motivated by the above results of Gu and Zheng, we consider two classes of symbols. First, in Section 3, we consider symbols which are slightly more general than (co-)analytic symbols considered in the first result of Gu and Zheng. More explicitly, we will consider symbols $u, v \in L^\infty(\mathbb{T}^2)$ for which v is a general bounded symbol and u is a nonconstant (co-)analytic symbol with respect to *one fixed variable*. For such u and v, by using a completely different argument from that of Gu and Zheng, we show that $T_uT_v - T_vT_u$ has finite rank on $H^2(\mathbb{D}^2)$ if and only if v is (co-)analytic in the same fixed variable and $T_uT_v = T_vT_u$ on $H^2(\mathbb{D}^2)$. We also provide characterizations for symbols u, v for which $T_uT_v = T_vT_u$ on $H^2(\mathbb{D}^2)$ (see Theorem 4 and Corollary 5).

Next, in Section 4, we consider symbols u, v of the form $u(z,w) = f(z)g(w)$ and $v(z,w) = h(z)k(w)$ where $f, g, h, k \in L^\infty(\mathbb{T})$ are general bounded functions. In connection with the second result of Gu and Zheng mentioned above, we investigate how the assumptions that fh and gk are zero or not affect the behavior of the commutator $T_uT_v - T_vT_u$, concerning its finite rank or compactness on $H^2(\mathbb{D}^2)$. Using a different argument from that of Gu and Zheng, we show that if one of fh and gk is a nonzero function, then $T_uT_v - T_vT_u$ has finite rank if and only if $T_uT_v = T_vT_u$ on $H^2(\mathbb{D}^2)$. On the other hand, if both fh and gk are nonzero functions, then $T_uT_v - T_vT_u$ is compact if and only if $T_uT_v = T_vT_u$ on $H^2(\mathbb{D}^2)$. At the same time, we give characterizations for $T_uT_v = T_vT_u$ on $H^2(\mathbb{D}^2)$ (see Theorems 6 and 8). Also, some immediate consequences are obtained. But the cases of general symbols remain open.

2. Preliminaries. We let $L^2(\mathbb{T})$ denote the usual Lebesgue space on $\mathbb{T}$ and $H^2(\mathbb{D})$ be the well known Hardy space on $\mathbb{D}$. We write Q for the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. With the identification of a function in $H^2(\mathbb{D})$ with its holomorphic extension on $\mathbb{D}$, for each $z \in \mathbb{D}$, the reproducing kernel K_z for $H^2(\mathbb{D})$ is the well known Cauchy kernel given by

$$K_z(\zeta) = \frac{1}{1 - \bar{z}\zeta}, \qquad \zeta \in \mathbb{T}.$$

Thus the projection Q can be written as

$$Q\varphi(z) = \int_{\mathbb{T}} \varphi \overline{K_z}\, d\sigma$$

for $\varphi \in L^2(\mathbb{T})$ where σ is the normalized Lebesgue measure on $\mathbb{T}$.

We can also identify a function in $H^2(\mathbb{D}^2)$ with its holomorphic extension on $\mathbb{D}^2$. With this identification, given $x = (z, w) \in \mathbb{D}^2$, the reproducing kernel R_x for $H^2(\mathbb{D}^2)$ is given by

$$R_x(y) = \frac{1}{(1-\bar{z}\zeta)(1-\bar{w}\eta)}, \qquad y = (\zeta, \eta) \in \mathbb{T}^2,$$

and thus we can write the projection P as

$$P\varphi(x) = \int_{\mathbb{T}^2} \varphi \overline{R_x}\, d\sigma_2$$

for $\varphi \in L^2(\mathbb{T}^2)$. Since $R_{(z,w)}(\zeta, \eta) = K_z(\zeta)K_w(\eta)$, we have

$$P[f(\zeta)g(\eta)](z, w) = Qf(z)Qg(w) \tag{2.1}$$

for $f, g \in L^2(\mathbb{T})$. See Chapter 3 of [10] or Chapter 9 of [11] for details and related facts.

For $\varphi \in L^\infty(\mathbb{T})$, let $S_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ denote the *1-dimensional* Toeplitz operator with symbol φ defined by

$$S_\varphi f = Q(\varphi f)$$

for $f \in H^2(\mathbb{D})$. Clearly S_φ is a bounded linear operator on $H^2(\mathbb{D})$.

In the course of our proofs, we will often use several known results for Toeplitz operators on the Hardy space of the unit disk. Thus we need a couple of lemmas. First we recall the following characterization for zero sums of products of Toeplitz operators (Theorem 3 of [9]).

Lemma 1. *Let $f_j, g_j \in L^\infty(\mathbb{T})$ for $j = 1, \ldots, N$. Then $\sum_{j=1}^N S_{f_j} S_{g_j} = 0$ on $H^2(\mathbb{D})$ if and only if $\sum_{j=1}^N f_j g_j = 0$ on $\mathbb{T}$ and the function $\sum_{j=1}^N [Qf_j][\overline{Q\overline{g_j}}]$ is harmonic on $\mathbb{D}$.*

We also need the following lemma whose part (a) was proved in [1], while (b) is Theorem 2.1 of [4] or Theorem 1 of [7].

Lemma 2. *Let $f, g \in L^\infty(\mathbb{T})$. Then the following statements hold:*

- (a) *Let a_j, b_j be the Fourier coefficients of f and g respectively. Then the following statements are all equivalent:*
  - (a1) *$S_f S_g = S_g S_f$ on $H^2(\mathbb{D})$.*
  - (a2) *$a_j b_{-k} = b_j a_{-k}$ for all $j, k = 1, \ldots$.*
  - (a3) *f and g are analytic, or f and g are co-analytic, or a nontrivial linear combination of f and g is constant.*
- (b) *$S_f S_g$ has finite rank on $H^2(\mathbb{D})$ if and only if either $f = 0$ or $g = 0$.*

3. (Co-)analytic symbols in one variable. In this section, we consider pairs of symbols one of which is arbitrary and the other is nonconstant analytic or co-analytic with respect to one fixed variable. We only consider the z-variable without loss of generality. Before doing this, we recall some basic facts on finite rank operators.

Given a Hilbert space K with an inner product $\langle\ ,\ \rangle$ and $\alpha, \beta \in K$, we let $\alpha \otimes \beta$ denote the rank one operator on K defined by $\alpha \otimes \beta(\gamma) = \langle \gamma, \beta\rangle \alpha$ for $\gamma \in K$. Recall that a bounded linear operator T has finite rank on K if and only if there exist $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N \in K$ such that

$$T = \sum_{j=1}^{N} \alpha_j \otimes \beta_j.$$

We start with a necessary condition for a commutator of two Toeplitz operators with general symbols to have finite rank on $H^2(\mathbb{D}^2)$.

LEMMA 3. *Let $u, v \in L^\infty(\mathbb{T}^2)$ and suppose $T_uT_v - T_vT_u$ has finite rank on $H^2(\mathbb{D}^2)$. If*

$$u(z,w) = \sum_{j=-\infty}^{\infty} u_j(w) z^j, \qquad v(z,w) = \sum_{j=-\infty}^{\infty} v_j(w) z^j$$

are the Fourier series expansions of u, v with respect to z, then $S_{u_j}S_{v_{-k}} - S_{v_j}S_{u_{-k}}$ has finite rank on $H^2(\mathbb{D})$ for all $j, k = 1, \dots$. Also, if

$$u(z,w) = \sum_{j=-\infty}^{\infty} u_j(z) w^j, \qquad v(z,w) = \sum_{j=-\infty}^{\infty} v_j(z) w^j$$

are the Fourier series expansions with respect to w, then $S_{u_j}S_{v_{-k}} - S_{v_j}S_{u_{-k}}$ has finite rank on $H^2(\mathbb{D})$ for all $j, k = 1, 2, \dots$.

Proof. We only prove the first part because the second is similar. Fix an integer $k \geq 1$ and $h \in H^2(\mathbb{D})$. Let M_1 be the multiplication operator on $H^2(\mathbb{D}^2)$ given by $M_1 f(a,b) = af(a,b)$. By Lemma 3 of [7], we have

$$[M_1^* T_u T_v M_1 - T_u T_v] M_1^{k-1} h(a,b) = \sum_{j \geq 1} S_{u_j} S_{v_{-k}} h(b) a^{j-1}$$

and similarly

$$[M_1^* T_v T_u M_1 - T_v T_u] M_1^{k-1} h(a,b) = \sum_{j \geq 1} S_{v_j} S_{u_{-k}} h(b) a^{j-1}$$

for all $(a,b) \in \mathbb{D}^2$. Putting $T = T_uT_v - T_vT_u$, we then have

$$(3.1) \qquad [M_1^* T M_1 - T] M_1^{k-1} h(a,b) = \sum_{j \geq 1} [S_{u_j} S_{v_{-k}} - S_{v_j} S_{u_{-k}}] h(b) a^{j-1}$$

for all $(a,b) \in \mathbb{D}^2$. Note $[M_1^* T M_1 - T] M_1^{k-1}$ has finite rank on $H^2(\mathbb{D}^2)$ because T has finite rank on $H^2(\mathbb{D}^2)$ by assumption. Thus we can write

$$[M_1^* T M_1 - T] M_1^{k-1} = \sum_{\ell=1}^{N} x_\ell \otimes y_\ell$$

for some integer $N \geq 0$ and $x_\ell, y_\ell \in H^2(\mathbb{D}^2)$. Then

$$[M_1^* T M_1 - T] M_1^{k-1} h = \sum_{\ell=1}^{N} x_\ell \int_{\mathbb{T}} h(\eta) \overline{\psi_\ell(\eta)}\, d\sigma(\eta),$$

where $\psi_\ell(\eta) = y_\ell(0,\eta)$. Now, for fixed $j \geq 1$, letting

$$\varphi_\ell(b) = \int_{\mathbb{T}} x_\ell(\zeta, b) \overline{\zeta^{j-1}}\, d\sigma(\zeta)$$

for each ℓ, we see from (3.1) that

$$\begin{aligned}[S_{u_j} S_{v_{-k}} - S_{v_j} S_{u_{-k}}] h &= \int_{\mathbb{T}} [M_1^* T M_1 - T] M_1^{k-1} h \overline{\zeta^{j-1}}\, d\sigma(\zeta) \\ &= \sum_{\ell=1}^{N} \varphi_\ell \int_{\mathbb{T}} h \overline{\psi_\ell}\, d\sigma = \sum_{\ell=1}^{N} [\varphi_\ell \otimes \psi_\ell] h\end{aligned}$$

and thus

$$S_{u_j} S_{v_{-k}} - S_{v_j} S_{u_{-k}} = \sum_{\ell=1}^{N} \varphi_\ell \otimes \psi_\ell.$$

Note $\psi_\ell, \varphi_\ell \in H^2(\mathbb{D})$ for each ℓ. It follows that $S_{u_j} S_{v_{-k}} - S_{v_j} S_{u_{-k}}$ has finite rank on $H^2(\mathbb{D})$ for all $j, k = 1, 2, \ldots$. ∎

Now we state and prove the main result of this section. We consider two symbols u, v one of which is nonconstant analytic or co-analytic with respect to z. Our results show that there is no nontrivial finite rank commutator of Toeplitz operators under consideration. We first consider the case when one symbol is analytic in z.

THEOREM 4. *Let $u, v \in L^\infty(\mathbb{T}^2)$. Suppose u is nonconstant and analytic in z. Let*

$$u(z,w) = \sum_{k=-\infty}^{\infty} u_k(w) z^k, \qquad v(z,w) = \sum_{k=-\infty}^{\infty} v_k(w) z^k$$

be the Fourier series expansions of u, v with respect to z. Then the following statements are all equivalent:

(a) *$T_u T_v - T_v T_u$ has finite rank on $H^2(\mathbb{D}^2)$.*

(b) *v is analytic in z and*

$$\sum_{j,k\geq 0,\, j+k=i} [S_{u_j} S_{v_k} - S_{v_j} S_{u_k}] = 0$$

for every $i = 0, 1, \ldots$.

(c) *v is analytic in z and the function*

$$\sum_{j,k\geq 0,\, j+k=i} [Q(u_j)\overline{Q(\overline{v_k})} - Q(v_j)\overline{Q(\overline{u_k})}]$$

is harmonic on $\mathbb{D}$ for each $i = 0, 1, 2, \ldots$.

(d) *v is analytic in z and $T_u T_v = T_v T_u$ on $H^2(\mathbb{D}^2)$.*

Proof. Since u is nonconstant analytic in z, we have $u_k = 0$ for all $k \leq -1$ and $u_N \neq 0$ for some $N \geq 1$. Thus

$$u(z, w) = \sum_{k=0}^{\infty} u_k(w) z^k.$$

First suppose (a) holds. By Lemma 3, we see that $S_{u_N} S_{v_{-k}} - S_{v_N} S_{u_{-k}} = S_{u_N} S_{v_{-k}}$ has finite rank on $H^2(\mathbb{D})$ for all $k > 0$. By Lemma 2(b), we have $v_{-k} = 0$ for all $k > 0$ because $u_N \neq 0$. Thus v is also analytic in z and hence

$$v(z, w) = \sum_{k=0}^{\infty} v_k(w) z^k.$$

Let $\psi = \psi(w) \in H^2(\mathbb{D})$ be arbitrary. By a simple application of (2.1), one can check that

$$T_u T_v(\varphi\psi) = \varphi \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} z^{j+k} S_{u_j} S_{v_k} \psi$$

and

$$T_v T_u(\varphi\psi) = \varphi \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} z^{j+k} S_{v_j} S_{u_k} \psi$$

for every $\varphi = \varphi(z) \in H^2(\mathbb{D})$. Hence

$$\begin{aligned} (3.2)\qquad & (T_u T_v - T_v T_u)(\varphi\psi)(z, w) \\ &= \varphi(z) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} z^{j+k} [S_{u_j} S_{v_k} - S_{v_j} S_{u_k}](\psi)(w) \\ &= \varphi(z) \sum_{i=0}^{\infty} z^i \Big(\sum_{j,k\geq 0,\, j+k=i} [S_{u_j} S_{v_k} - S_{v_j} S_{u_k}](\psi)(w) \Big) \end{aligned}$$

for every $\varphi \in H^2(\mathbb{D})$ and $z, w \in \mathbb{D}$. Since $T_u T_v - T_v T_u$ has finite rank on

$H^2(\mathbb{D}^2)$ by assumption, it follows from (3.2) that

$$\sum_{i=0}^{\infty} z^i \Big(\sum_{j,k\geq 0,\, j+k=i} [S_{u_j}S_{v_k} - S_{v_j}S_{u_k}](\psi)(w)\Big) = 0$$

and hence

$$\sum_{j,k\geq 0,\, j+k=i} [S_{u_j}S_{v_k} - S_{v_j}S_{u_k}] = 0$$

for every $i = 0, 1, \ldots,$ thus (b) holds.

The equivalence of (b) and (c) follows from Lemma 1. Now assume (b). Then

$$v(z,w) = \sum_{k=0}^{\infty} v_k(w) z^k$$

and (3.2) holds. By (b), we have $(T_uT_v - T_vT_u)(\varphi\psi) = 0$ for all $\varphi, \psi \in H^2(\mathbb{D})$. Hence T_u commutes with T_v on $H^2(\mathbb{D}^2)$, so (d) holds. The implication (d)⇒(a) is clear. ∎

In the case when one symbol is co-analytic with respect to z, we have the following immediate consequence of Theorem 4.

Corollary 5. *Let $u, v \in L^\infty(\mathbb{T}^2)$. Suppose u is nonconstant and co-analytic in z. Let*

$$u(z,w) = \sum_{k=-\infty}^{\infty} u_k(w) z^k, \qquad v(z,w) = \sum_{k=-\infty}^{\infty} v_k(w) z^k$$

be the Fourier series expansions of u and v with respect to z. Then the following statements are all equivalent:

(a) *$T_uT_v - T_vT_u$ has finite rank on $H^2(\mathbb{D}^2)$.*

(b) *v is co-analytic in z and*

$$\sum_{j,k\leq 0,\, j+k=i} [S_{u_j}S_{v_k} - S_{v_j}S_{u_k}] = 0$$

for every $i = 0, -1, -2, \ldots.$

(c) *v is co-analytic in z and the function*

$$\sum_{j,k\leq 0,\, j+k=i} [Q(u_j)\overline{Q(\overline{v_k})} - Q(v_j)\overline{Q(\overline{u_k})}]$$

is harmonic on $\mathbb{D}$ for each $i = 0, -1, -2, \ldots.$

(d) *v is co-analytic in z and $T_uT_v = T_vT_u$ on $H^2(\mathbb{D}^2)$.*

Proof. Since u is nonconstant co-analytic in z, it follows that $\bar{u}$ is nonconstant analytic in z. Note that

$$\bar{u}(z,w) = \sum_{k=0}^{\infty} \overline{u_{-k}(w)} z^k, \qquad \bar{v}(z,w) = \sum_{k=-\infty}^{\infty} \overline{v_{-k}(w)} z^k.$$

Recall that $T_uT_v - T_vT_u$ has finite rank if and only if $[T_uT_v - T_vT_u]^*$ has finite rank. Also $[T_uT_v - T_vT_u]^* = T_{\bar{v}}T_{\bar{u}} - T_{\bar{u}}T_{\bar{v}}$ and $[S_{\bar{u}_j}S_{\bar{v}_k} - S_{\bar{v}_j}S_{\bar{u}_k}]^* = S_{v_k}S_{u_j} - S_{u_k}S_{v_j}$ for every j, k. Thus, using Theorem 4, we have the desired results. ∎

4. Symbols with separated variables. In this section, we consider two symbols with separated variables and study the problem of when the commutator of the corresponding Toeplitz operators is of finite rank or compact on $H^2(\mathbb{D}^2)$. Before doing this, we first recall the well known Berezin transform.

Given a bounded linear operator L on $H^2(\mathbb{D}^2)$, the *Berezin transform* $B[L]$ of L is the function on $\mathbb{D}^2$ defined by

$$B[L](x) = \int_{\mathbb{T}^2} (Lr_x)\overline{r_x}\, d\sigma_2, \qquad x = (a,b) \in \mathbb{D}^2,$$

where r_x is the normalized kernel on $H^2(\mathbb{D}^2)$ given by

$$r_x(y) = \frac{\sqrt{(1-|a|^2)(1-|b|^2)}}{(1-\bar{a}\zeta)(1-\bar{b}\eta)}, \qquad y = (\zeta,\eta) \in \mathbb{T}^2.$$

Also, we will use the same notation $B[S]$ to denote the *1-dimensional* Berezin transform for a bounded linear operator S on $H^2(\mathbb{D})$:

$$B[S](a) = \int_{\mathbb{T}} (Sk_a)\overline{k_a}\, d\sigma, \qquad a \in \mathbb{D},$$

where

$$k_a(\zeta) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}\zeta}, \qquad \zeta \in \mathbb{T}.$$

It is well known that the Berezin transform is one-to-one (see [3] for example). Also, if L is compact on $H^2(\mathbb{D}^2)$, then $B[L](x) \to 0$ as $x \to \partial\mathbb{D}^2$, the topological boundary of $\mathbb{D}^2$, because r_x converges weakly to 0 as $x \to \partial\mathbb{D}^2$. For symbols u, v of the form $u(z,w) = f(z)g(w)$ and $v(z,w) = h(z)k(w)$ where $f, g, h, k \in L^\infty(\mathbb{T})$, we see using (2.1) that

$$B[T_uT_v](a,b) = B[S_fS_h](a)B[S_gS_k](b) \tag{4.1}$$

for all $a, b \in \mathbb{D}$. Also, for a product of two Toeplitz operators, the boundary value of the Berezin transform can be obtained in terms of the product of symbols. More explicitly, given $f, g \in L^\infty(\mathbb{T})$, it is known that $B[S_fS_g]$ has an admissible limit and

$$B[S_f S_g] = fg \tag{4.2}$$

at almost all points of $\mathbb{T}$; see Theorem 3.2 of [3] for details and related facts.

The following is the first result of this section.

THEOREM 6. *Let $f, g, h, k \in L^\infty(\mathbb{T})$ be nonzero functions and put $u(z, w) = f(z)g(w)$ and $v(z, w) = h(z)k(w)$. Suppose either $fh \neq 0$ or $gk \neq 0$. Then the following statements are all equivalent:*

(a) *$T_u T_v - T_v T_u$ has finite rank on $H^2(\mathbb{D}^2)$.*
(b) *$S_f S_h = S_h S_f$ and $S_g S_k = S_k S_g$ on $H^2(\mathbb{D})$.*
(c) *The following two conditions hold:*
  (c1) *f and h are analytic, or f and h are co-analytic, or a nontrivial linear combination of f and h is constant.*
  (c2) *g and k are analytic, or g and k are co-analytic, or a nontrivial linear combination of g and k is constant.*
(d) *$T_u T_v = T_v T_u$ on $H^2(\mathbb{D}^2)$.*

Proof. We first note from (4.1) that

$$\begin{aligned} B[T_u T_v](a, b) &= B[S_f S_h](a) B[S_g S_k](b), \\ B[T_v T_u](a, b) &= B[S_h S_f](a) B[S_k S_g](b) \end{aligned} \tag{4.3}$$

for all $(a, b) \in \mathbb{D}^2$. Thus, if we assume (b), then (4.3) shows that $B[T_u T_v] = B[T_v T_u]$ and hence $T_u T_v = T_v T_u$ because B is one-to-one on $H^2(\mathbb{D}^2)$. Thus, implication (b)$\Rightarrow$(d) holds. Also, (b)$\Leftrightarrow$(c) is a consequence of Lemma 2(a). Since (d)$\Rightarrow$(a) is clear, it remains to prove (a)$\Rightarrow$(b). So suppose (a) holds. First assume gk is not a zero function. Let

$$f(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \qquad h(z) = \sum_{j=-\infty}^{\infty} b_j z^j$$

be the Fourier series expansions of f and h. Then

$$u(z, w) = \sum_{j=-\infty}^{\infty} a_j g(w) z^j, \qquad v(z, w) = \sum_{j=-\infty}^{\infty} b_j k(w) z^j.$$

Since T has finite rank on $H^2(\mathbb{D}^2)$, Lemma 3 shows that $S_{a_i g} S_{b_{-j} k} - S_{b_i k} S_{a_{-j} g}$ is a finite rank operator and hence compact on $H^2(\mathbb{D})$ for each $i, j \geq 1$. Thus $B[S_{a_i g} S_{b_{-j} k} - S_{b_i k} S_{a_{-j} g}](z) \to 0$ as $|z| \to 1$. By an application of (4.2), we obtain

$$(a_i b_{-j} - b_i a_{-j}) gk = 0, \qquad i, j = 1, 2, \ldots.$$

But, since $gk \neq 0$, we have $a_i b_{-j} = b_i a_{-j}$ for all $i, j \geq 1$, which is equivalent to $S_f S_h = S_h S_f$ on $H^2(\mathbb{D})$ by Lemma 2(a). Next, we show $S_g S_k = S_k S_g$ on

$H^2(\mathbb{D})$. Since $S_fS_h = S_hS_f$ on $H^2(\mathbb{D})$, it follows from (4.3) that

$$B[T_uT_v - T_vT_u](a,b) = B[S_fS_h](a)B[S_gS_k - S_kS_g](b)$$

for every $(a,b) \in \mathbb{D}^2$. On the other hand, since $T_uT_v - T_vT_u$ has finite rank on $H^2(\mathbb{D}^2)$ by assumption, we have

$$T_uT_v - T_vT_u = \sum_{\ell=1}^{N} x_\ell \otimes y_\ell \tag{4.4}$$

for some integer $N \geq 0$ and $x_\ell, y_\ell \in H^2(\mathbb{D}^2)$. Note that

$$B[F \otimes G](a,b) = (1-|a|^2)(1-|b|^2)F(a,b)\overline{G(a,b)}, \quad (a,b) \in \mathbb{D}^2,$$

for all $F, G \in H^2(\mathbb{D}^2)$. Thus, by taking the Berezin transform B of both sides of (4.4), we have

$$\begin{aligned} (1-|a|^2)(1-|b|^2)\sum_{\ell=1}^{N} x_\ell(a,b)\overline{y_\ell(a,b)} &= B\Big[\sum_{\ell=1}^{N} x_\ell \otimes y_\ell\Big](a,b) = B[T_uT_v - T_vT_u](a,b) \\ &= B[S_fS_h](a)B[S_gS_k - S_kS_g](b) \end{aligned} \tag{4.5}$$

for every $(a,b) \in \mathbb{D}^2$. Suppose that $S_gS_k \neq S_kS_g$ on $H^2(\mathbb{D})$. Then $B[S_gS_k - S_kS_g] \neq 0$ on $\mathbb{D}$ and hence

$$\beta := B[S_gS_k - S_kS_g](b_0) \neq 0$$

for some $b_0 \in \mathbb{D}$. For each ℓ, we put $X_\ell(a) = \beta^{-1}(1-|b_0|^2)x_\ell(a,b_0)$ and $Y_\ell(a) = y_\ell(a,b_0)$. Note $X_\ell, Y_\ell \in H^2(\mathbb{D})$ (see Lemma 3.3 of [2] for example). Also, for $\psi, \varphi \in H^2(\mathbb{D})$, we have

$$B[\psi \otimes \varphi](a) = (1-|a|^2)\psi(a)\overline{\varphi(a)}$$

for all $a \in \mathbb{D}$. It follows from (4.5) that

$$B[S_fS_h](a) = (1-|a|^2)\sum_{\ell=1}^{N} X_\ell(a)\overline{Y_\ell(a)} = B\Big[\sum_{\ell=1}^{N} X_\ell \otimes Y_\ell\Big](a)$$

for all $a \in \mathbb{D}$. Thus we obtain

$$S_fS_h = \sum_{\ell=1}^{N} X_\ell \otimes Y_\ell$$

because B is one-to-one on $H^2(\mathbb{D})$. Hence S_fS_h has finite rank on $H^2(\mathbb{D})$ and so $f = 0$ or $h = 0$ by Lemma 2(b), which is a contradiction. Therefore $S_gS_k = S_kS_g$ on $H^2(\mathbb{D})$ and (b) follows.

Now, assume fh is not a zero function and put

$$g(w)=\sum_{j=-\infty}^{\infty} c_j w^j, \qquad k(w)=\sum_{j=-\infty}^{\infty} d_j w^j.$$

Since $T_uT_v - T_vT_u$ has finite rank on $H^2(\mathbb{D}^2)$, we have $[c_i d_{-j} - d_i c_{-j}]fh = 0$ for all $i,j = 1,2,\ldots$ as before. Also, since $fh \neq 0$, we see S_g and S_k are commuting on $H^2(\mathbb{D})$. Now, using exactly the same argument as in the previous case, we conclude that S_f commutes with S_h on $H^2(\mathbb{D})$. ∎

Given analytic functions $g,k \in L^\infty(\mathbb{T})$, we see from Lemma 2(a) that $S_gS_{\bar{k}} = S_{\bar{k}}S_g$ if and only if one of g and k is constant. Also, note that $g\bar{k} = 0$ if and only if $g = 0$ or $k = 0$. Thus, the following is an immediate consequence of Theorem 6.

Corollary 7. *Let $f,g,h,k \in L^\infty(\mathbb{T})$ and suppose g,k are nonconstant analytic functions. Put $u(z,w) = f(z)g(w)$ and $v(z,w) = h(z)\bar{k}(w)$. Then $T_uT_v - T_vT_u$ has finite rank on $H^2(\mathbb{D}^2)$ if and only if either $u = 0$ or $v = 0$.*

We now consider the case when fh and gk are both nonzero. Our result below shows that there is no nontrivial compact commutator of two Toeplitz operators under consideration.

Theorem 8. *Let $f,g,h,k \in L^\infty(\mathbb{T})$ and suppose fh, gk are not zero functions. Put $u(z,w) = f(z)g(w)$ and $v(z,w) = h(z)k(w)$. Then the following statements are all equivalent:*

(a) *$T_uT_v - T_vT_u$ is compact on $H^2(\mathbb{D}^2)$.*
(b) *$S_fS_h = S_hS_f$ and $S_gS_k = S_kS_g$ on $H^2(\mathbb{D})$.*
(c) *The following two conditions hold;*
  (c1) *f and h are analytic, or f and h are co-analytic, or a nontrivial linear combination of f and h is constant.*
  (c2) *g and k are analytic, or g and k are co-analytic, or a nontrivial linear combination of g and k is constant.*
(d) *$T_uT_v = T_vT_u$ on $H^2(\mathbb{D}^2)$.*

Proof. Since the implications (b)⇒(c)⇒(d)⇒(a) have been shown in the proof of Theorem 6, we only need to prove (a)⇒(b). So assume (a) holds. Note $(a,b) \to \partial\mathbb{D}^2$ as $|a| \to 1$ for each $b \in \mathbb{D}$. Then the compactness of $T_uT_v - T_vT_u$ implies that for fixed $b \in \mathbb{D}$, $B[T_uT_v - T_vT_u](a,b) \to 0$ as $|a| \to 1$. Also, for each $a \in \mathbb{D}$, since $(a,b) \to \partial\mathbb{D}^2$ as $|b| \to 1$, we have $B[T_uT_v - T_vT_u](a,b) \to 0$ as $|b| \to 1$ for fixed $a \in \mathbb{D}$. It follows from (4.1) and (4.2) that

$$B[S_gS_k - S_kS_g](b)(fh)(\zeta) = 0,$$
$$B[S_fS_h - S_hS_f](a)(gk)(\eta) = 0,$$

for all $a, b \in \mathbb{D}$ and almost all $\zeta, \eta \in \mathbb{T}$. But, since both fg and gk are all nonzero by the assumption, we have

$$B[S_f S_h - S_h S_f] = B[S_g S_k - S_k S_g] = 0$$

on $\mathbb{D}$. Thus $S_g S_k = S_k S_g$ and $S_f S_h = S_h S_f$ on $H^2(\mathbb{D})$ because B is one-to-one, hence (b) holds. ∎

As an immediate consequence of Theorem 8 together with Lemma 2(a), we have the following corollary in the case when the two symbols depend on different variables.

Corollary 9. *Let $f, k \in L^\infty(\mathbb{T})$. Put $u(z, w) = f(z)$ and $v(z, w) = k(w)$. Then T_u and T_v are always commuting on $H^2(\mathbb{D}^2)$.*

On the other hand, if the two symbols depend on only one variable, the commutator of the corresponding Toeplitz operators is compact only in an obvious case as shown in the following.

Proposition 10. *Let $f, h \in L^\infty(\mathbb{T})$ be nonzero. Assume either $u(z, w) = f(z)$ and $v(z, w) = h(z)$, or $u(z, w) = f(w)$ and $v(z, w) = h(w)$. Then the following statements are all equivalent:*

(a) *$T_u T_v - T_v T_u$ is compact on $H^2(\mathbb{D}^2)$.*
(b) *f, h are analytic, or f, h are co-analytic, or a nontrivial combination of f and h is constant.*
(c) *$S_f S_h = S_h S_f$ on $H^2(\mathbb{D})$.*
(d) *$T_u T_v = T_v T_u$ on $H^2(\mathbb{D}^2)$.*

Proof. We only consider the case when $u(z, w) = f(z)$ and $v(z, w) = h(z)$ because the proof of the other case is similar. First by (4.1), we have

$$B[T_u T_v - T_v T_u](a, b) = B[S_f S_h - S_h S_f](a) \tag{4.6}$$

for all $(a, b) \in \mathbb{D}^2$, which implies the equivalence of (c) and (d). Also, since (b)⇔(c) holds by Lemma 2(a), and (d)⇒(a) is clear, we only prove (a)⇒(c). So assume (a) holds. Since $T_u T_v - T_v T_u$ is compact, $B[T_u T_v - T_v T_u](a, b) \to 0$ as $|b| \to 1$ for fixed $a \in \mathbb{D}$. It follows from (4.6) that $B[S_f S_h - S_h S_f] = 0$ and hence $S_f S_h = S_h S_f$, so we have (c). ∎

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