Disjoint hypercyclic operators

by

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Abstract. We introduce the concept of disjoint hypercyclic operators. These are operators performing the approximation of any given vectors with a common subsequence of iterates applied on a common vector. The notion is extended to sequences of operators, and applied to composition operators and differential operators on spaces of analytic functions.

1. Introduction. In the last 25 years there have been important advances in the research of hypercyclicity, from several points of view. Roughly speaking, hypercyclicity means existence of a dense orbit. In this paper, we are concerned with the existence of a common vector with dense orbit for several operators, such that approximation of any fixed vectors is also simultaneously performed by using a *common* subsequence. This will be formalized later. Although we have been inspired by an interesting work about universal Taylor series due to Costakis and Vlachou [20], the path of research opened in the current paper is, as far as we know, new. Below, we fix some related notation and terminology to be used in the present article. For a good account of concepts, results and history concerning hypercyclicity, the reader is referred to the surveys [26], [28] and [15].

By $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$, $B(a,r)$, $\overline{B}(a,r)$ ($a \in \mathbb{C}$, $r > 0$) we denote, respectively, the set of positive integers, the set $\mathbb{N} \cup \{0\}$, the real line, the complex plane, the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, the open disk with center a and radius r, and the corresponding closed disk. Assume that X and Y are two (Hausdorff) topological vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $L(X, Y)$ will stand for the space of all continuous linear mappings $T : X \to Y$. In

particular, if $X = Y$, then $L(X) := L(X, X) = \{\text{operators on } X\}$. If (T_n) is a sequence in $L(X, Y)$, then (T_n) is said to be *hypercyclic* whenever there exists a vector $x \in X$—called a *hypercyclic vector* for (T_n)—whose orbit $\{T_n x : n \in \mathbb{N}\}$ under (T_n) is dense in Y (so Y is necessarily separable). We denote by $\mathrm{HC}((T_n))$ the set of hypercyclic vectors for (T_n). An operator $T \in L(X)$ is said to be *hypercyclic* provided that the sequence (T^n) of iterates of T is hypercyclic. In this case the set $\mathrm{HC}(T) := \mathrm{HC}((T^n))$ is called the set of hypercyclic vectors for T. If $\mathrm{HC}((T_n))$ is dense in X, then (T_n) is said to be *densely hypercyclic*. Finally, recall that an *F-space* is a completely metrizable topological vector space.

Assume that X, Y, Z are separable F-spaces, and that $S_n : X \to Y$ and $T_n : X \to Z$ $(n \in \mathbb{N})$ are densely hypercyclic sequences. Since $\mathrm{HC}((S_n))$, $\mathrm{HC}((T_n))$ are dense G_δ subsets of X, it follows that $\mathrm{HC}((S_n)) \cap \mathrm{HC}((T_n))$ is also dense, so nonempty. Hence there is a common hypercyclic vector $x \in X$. So, for any given vectors $y \in Y$, $z \in Z$, there are sequences $\{n_1 < n_2 < \cdots\}$ and $\{m_1 < m_2 < \cdots\}$ in $\mathbb{N}$ such that $S_{n_j} x \to y$ and $T_{m_j} x \to z$ as $j \to \infty$. Now, we pose the following question (a similar conclusion as above can be obtained and a similar question can be posed for single hypercyclic operators on X, just by considering the sequences of their iterates):

> Under what conditions on (S_n) and (T_n) can one guarantee the existence of a vector $x \in X$ such that, for any given $y \in Y$ and $z \in Z$, there is *one* sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ such that $S_{n_j} x \to y$ and $T_{n_j} x \to z$ $(j \to \infty)$?

Motivated by the last question, the new concepts of jointly hypercyclic operators and of disjoint hypercyclic sequences of continuous linear mappings are introduced in Section 2. Also, two workable sufficient conditions for disjoint hypercyclicity will be provided in the setting of F-spaces (see Section 4). Before this, we introduce in Section 3 two new notions: supermixing operators and sequences of operators controlled by seminorms. Finally, in Section 5, the disjoint hyperciclity conditions will be applied on spaces of analytic functions to identify finite sets of sequences of either infinite order differential operators or composition operators that are disjoint hypercyclic.

2. Disjoint hypercyclic operators. Let us define the new concept that is the subject of this paper.

DEFINITION 2.1. Let $p \in \mathbb{N}$ and $X, Y_1, \ldots, Y_p$ be topological vector spaces. Assume that, for each $j \in \{1, \ldots, p\}$, $T_{j,n} : X \to Y_j$ $(n \in \mathbb{N})$ is a sequence of continuous linear mappings. We say that the sequences $(T_{1,n}), \ldots, (T_{p,n})$ are *disjoint hypercyclic* if the sequence $[T_{1,n}, \ldots, T_{p,n}]$:

$X \to Y_1 \times \cdots \times Y_p$ $(n \in \mathbb{N})$ defined as

$$[T_{1,n}, \ldots, T_{p,n}]x = (T_{1,n}x, \ldots, T_{p,n}x) \quad (x \in X)$$

is hypercyclic, where $Y_1 \times \cdots \times Y_p$ is assumed to be endowed with the product topology. We say that the sequences $(T_{1,n}), \ldots, (T_{p,n})$ are *densely disjoint hypercyclic* if the sequence $([T_{1,n}, \ldots, T_{p,n}])$ is densely hypercyclic. If $T_1, \ldots, T_p \in L(X)$ then $T_1, \ldots, T_p$ are said to be *disjoint hypercyclic* (*densely disjoint hypercyclic*, resp.) provided that the sequences $(T_1^n), \ldots, (T_p^n)$ are disjoint hypercyclic (densely disjoint hypercyclic, resp.).

REMARKS 2.1. 1. Clearly, the disjoint hypercyclicity of $(T_{1,n}), \ldots, (T_{p,n})$ remains unaltered under a change of order $(T_{\sigma(1),n}), \ldots, (T_{\sigma(p),n})$. Also, trivially, it implies the hypercyclicity of each sequence $(T_{j,n})$ $(j = 1, \ldots, p)$.

2. Let X, Y, Z be separable F-spaces. According to [26], being densely hypercyclic is equivalent to being topologically transitive, in the sense of Birkhoff. Thus, saying that two sequences $(S_n) \subset L(X, Y)$, $(T_n) \subset L(X, Z)$ are densely disjoint hypercyclic is the same as saying that if U, V, W are nonempty open subsets of X, Y, Z, respectively, there exists an n (equivalently, there exist infinitely many n) such that $U \cap S_n^{-1}(V) \cap T_n^{-1}(W) \neq \emptyset$.

3. The mere hypercyclicity of two sequences (S_n), (T_n) does not imply their disjoint hypercyclicity: take (T_n) hypercyclic and $(S_n) := (T_n)$.

4. The reader should be warned not to confuse the disjoint hypercyclicity of $(T_{1,n}), \ldots, (T_{p,n})$ with the hypercyclicity of the sequence

$$T_{1,n} \oplus \cdots \oplus T_{p,n} : (x_1, \ldots, x_p) \in X^p \mapsto (T_{1,n}x_1, \ldots, T_{p,n}x_p) \in Y_1 \times \cdots \times Y_p.$$

Disjoint hypercyclicity implies the hypercyclicity of the last sequence, but the converse is false.

5. It is well known that (T^n) is densely hypercyclic as soon as T is hypercyclic. But, in principle, nothing guarantees the dense hypercyclicity of the sequence $([S^n, T^n])$ if S, T are disjoint hypercyclic operators. Therefore we pose the problem of whether there exist disjoint hypercyclic operators S, T such that $([S^n, T^n])$ is not densely disjoint hypercyclic. For this reason, we have been forced in Definition 2.1 to define the dense disjoint hypercyclicity for several single operators, which would be superfluous for *one* operator.

6. In spite of the last remark, we have found a situation where the answer to the problem just posed is affirmative. It is inspired by [24, Section 4] and [28, Section 2]. Recall that if T is a hypercyclic operator on a topological vector space X, then HC(T) contains, except for zero, a dense linear submanifold (in the terminology of [2], HC(T) is *algebraically generic* in X); see [10], [17], [30] and [41]. In this order of ideas, we give the following statement, which can be easily proved just by taking into account (see [41]) that $P(T)$ has dense range if T is hypercyclic and P is a nonzero polynomial with

coefficients in $\mathbb{K}$: Suppose that $T_1, \ldots, T_p$ are disjoint hypercyclic operators on a topological vector space X. Assume also that there is $i \in \{1, \ldots, p\}$ such that every T_j commutes with T_i. Then the sequence $([T_1^n, \ldots, T_p^n])$ is densely hypercyclic. Even more, $\mathrm{HC}(([T_1^n, \ldots, T_p^n]))$ is algebraically generic in X.

3. Control by seminorms and supermixing operators. In order to isolate the properties that will be needed, we introduce two concepts concerning operators defined on Fréchet spaces. Recall that an F-space X is called a *Fréchet space* whenever X is locally convex. The family of continuous seminorms on such a space X will be denoted by $\mathrm{SN}(X)$. If $p \in \mathrm{SN}(X)$, then we define a *p-open set* of X as a union of open p-balls $\{x \in X : p(x - \alpha) < \varepsilon\}$ $(\alpha \in X, \varepsilon > 0)$.

DEFINITION 3.1. Let X, Y be Fréchet spaces, and (T_n) be a sequence in $L(X, Y)$. We say that (T_n) is *controlled by seminorms*, or that it is *SN-controlled*, if for every $q \in \mathrm{SN}(Y)$ there are $p \in \mathrm{SN}(X)$ and constants $C_n \in (0, \infty)$ $(n \in \mathbb{N})$ such that $q(T_n x) \le C_n p(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. Equivalently, for every $q \in \mathrm{SN}(X)$ there is $p \in \mathrm{SN}(X)$ such that $T_n^{-1}(V)$ is a p-open set in X for each q-open set V in Y and each $n \in \mathbb{N}$.

REMARK 3.1. The point in the above definition is that p depends on q, but *not* on n. Observe that if (T_n) is equicontinuous then (T_n) is SN-controlled (with a common constant $C_n = C$ for all n). Between Banach spaces X and Y, any sequence $(T_n) \subset L(X, Y)$ is SN-controlled. Hence the SN-control is only of interest for nonnormable Fréchet spaces.

Before giving some examples, let us fix some additional notation. By a *domain* we mean a nonempty connected open subset $G \subset \mathbb{C}$. Then $H(G)$ denotes the Fréchet space of holomorphic functions in G endowed with the topology of uniform convergence on compacta. It is known that $H(G)$ is separable and nonnormable. If $G = \mathbb{C}$, then we obtain in particular the space $\mathcal{E} = H(\mathbb{C})$ of entire functions.

Let $\Phi(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{E}$. Then Φ is said to be *of exponential type* (the set of these functions will be denoted by Exp) provided that there are positive constants A, B such that $|\Phi(z)| \le A \exp(B|z|)$ for all $z \in \mathbb{C}$. Consider its associated formal linear (in general, infinite order) differential operator $\Phi(D) = \sum_{n=0}^\infty a_n D^n$ defined as $\Phi(D)f = \sum_{n=0}^\infty a_n f^{(n)}$ $(f \in \mathcal{E})$. Then $\Phi(D)$ is a well-defined operator on $\mathcal{E}$. Moreover, an operator $T \in L(\mathcal{E})$ commutes with translations (that is, $T \circ \tau_a = \tau_a \circ T$ for all $a \in \mathbb{C}$, where $(\tau_a f)(z) := f(z + a)$ for $z \in \mathbb{C}$ and $f \in \mathcal{E}$) if and only if it commutes with the derivative operator D $(Df := f')$, and if and only if there exists $\Phi \in \mathrm{Exp}$ such that $T = \Phi(D)$ (see [23, Section 5]). Note that D and the translations τ_a are special cases (take $\Phi(z) \equiv z$ and $\Phi(z) \equiv e^{az}$, resp.).

Assume that Φ is an entire function *of subexponential type*, that is, for given $\varepsilon > 0$ there is a constant $A > 0$ such that $|\Phi(z)| \leq A \exp(\varepsilon|z|)$ for all $z \in \mathbb{C}$ (so, in particular, $\Phi \in \mathrm{Exp}$); equivalently, $\lim_{n \to \infty}(n!|a_n|)^{1/n} = 0$ (see [14]). Suppose that G is any domain in $\mathbb{C}$. Then $\Phi(D)$ is an operator on $H(G)$, not only on $\mathcal{E}$ (see [3]).

EXAMPLES 3.2.

1. Suppose that G is a domain in $\mathbb{C}$. If (Φ_n) is any sequence of entire functions of subexponential type, then the sequence $(T_n) := (\Phi_n(D))$ is SN-controlled in $H(G)$. To see this, take a sequence of compact sets (K_k) such that $G = \bigcup_{k=1}^{\infty} K_k$, $K_k \subset K_{k+1}^{\circ}$ ($k \in \mathbb{N}$) (A° denotes the interior of a subset $A \subset \mathbb{C}$) and the boundary ∂K_k of each K_k is the union of finitely many rectifiable Jordan curves, which can be assumed to be oriented counterclockwise. It is even possible to construct (K_k) so that, for every k, each bounded connected component of $\mathbb{C} \setminus K_k$ (i.e., each "hole" of K_k) contains some bounded connected component of $\mathbb{C} \setminus G$ (see [19] and [37]). The increasing sequence $\mathcal{P} := (p_k)$ of seminorms given by

$$p_k(f) := \sup\{|f(z)| : z \in K_k\} \quad (k \in \mathbb{N},\ f \in H(G))$$

defines the topology of $H(G)$. Fix a seminorm $q \in \mathrm{SN}(H(G))$. Then there are $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $q(f) \leq \varepsilon p_k(f)$ for all $f \in H(G)$. Let $\Phi_n(z) = \sum_{j=0}^{\infty} a_{j,n} z^j$ ($z \in \mathbb{C}$, $n \in \mathbb{N}$). Since $(j!|a_{j,n}|)^{1/j} \to 0$ ($j \to \infty$), there is a constant $A_n > 0$ such that $j!|a_{j,n}| \leq A_n \varepsilon_1^j$ ($j \in \mathbb{N}_0$), where $\varepsilon_1 := (1/2)\mathrm{dist}(K_k, \partial K_{k+1})$. Let $p := p_{k+1}$ and

$$C_n := \frac{\varepsilon A_n \, \mathrm{length}(\partial K_{k+1})}{\mathrm{dist}(K_k, \partial K_{k+1})} \quad (n \in \mathbb{N}).$$

Fix $f \in H(G)$. Then the Cauchy integral formula for derivatives shows that, for every $z \in K_k$,

$$|T_n f(z)| = \left| \sum_{j=0}^{\infty} a_{j,n} f^{(j)}(z) \right| = \left| \sum_{j=0}^{\infty} \frac{a_{j,n} \, j!}{2\pi i} \oint_{\partial K_{k+1}} \frac{f(t)}{(t-z)^{j+1}} \, dt \right|$$

$$\leq \sum_{j=0}^{\infty} \frac{A_n \varepsilon_1^j}{2\pi} \frac{\sup_{t \in \partial K_{k+1}} |f(t)| \, \mathrm{length}(\partial K_{k+1})}{(\mathrm{dist}(K_k, \partial K_{k+1}))^{j+1}}$$

$$\leq \frac{A_n \, \mathrm{length}(\partial K_{k+1}) \sup_{w \in K_{k+1}} |f(w)|}{2 \, \mathrm{dist}(K_k, \partial K_{k+1})} \sum_{j=0}^{\infty} \left(\frac{1}{2} \right)^j$$

$$= \frac{A_n \, \mathrm{length}(\partial K_{k+1})}{\mathrm{dist}(K_k, \partial K_{k+1})} p_{k+1}(f) = \frac{C_n}{\varepsilon} p(f).$$

Therefore, $q(T_n f) \leq \varepsilon p_k(T_n f) \leq \varepsilon \sup\{|T_n f(z)| : z \in K_k\} \leq C_n p(f)$, as desired.

2. However, if the Φ_n are merely of exponential type, then the sequence $(\Phi_n(D))$ $(\subset L(\mathcal{E}))$ is not always SN-controlled. For instance, the sequence (τ_n) of translations (case $\Phi_n(z) = e^{nz}$) is never SN-controlled. Indeed, let $q := p_1$ and fix $p \in \mathrm{SN}(\mathcal{E})$ and $C_n > 0$ $(n \in \mathbb{N})$, where $p_n(f) := \sup\{|f(z)| : |z| \le n\}$. We wish to find a function $F \in \mathcal{E}$ and a positive integer N such that $q(\tau_N F) > C_N p(F)$. Since p is continuous, there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ satisfying $\{f \in \mathcal{E} : p(f) < 1\} \supset \{f \in \mathcal{E} : p_n(f) < \varepsilon\}$. Since the balls $\overline{B}(0, n)$ and $\overline{B}(n+2, 1)$ are disjoint, we can apply Runge's approximation theorem (see [22]) to obtain a polynomial F (so $F \in \mathcal{E}$) such that $|F(z) - 0| < \varepsilon$ on $\overline{B}(0, n)$ and $|f(z) - (C_{n+2} + 1)| < 1$ on $\overline{B}(n+2, 1)$. Therefore $p(f) < 1$ and $q(\tau_{n+2} F) > C_{n+2}$. Hence, for $N := n + 2$, we get $q(\tau_N F) > C_N p(F)$, as required.

The second new concept that will be introduced has to do with the "improvement of quality" of the hypercyclicity. First, let us recall a number of related notions. Assume that X, Y are separable F-spaces and that $(T_n) \subset L(X, Y)$. Then (T_n) is said to be *weakly mixing* whenever the sequence

$$T_n \oplus T_n : (x, y) \in X^2 \mapsto (T_n x, T_n y) \in Y^2 \quad (n \in \mathbb{N})$$

is hypercyclic. The sequence (T_n) is called *mixing* if given a pair U, V of nonempty open subsets of X, there is an $N \in \mathbb{N}$ such that $T_n(U) \cap V \ne \emptyset$ for every $n \ge N$. If (T_n) is mixing, then it is weakly mixing, which in turn implies that (T_n) is densely hypercyclic. For sharper relations among these three concepts, see [23], [11], [26], [15], [7], [5], [35], [4] and [25].

Now, we present a property that is a kind of mixing property depending only on the "shape" of the neighborhood. The concept is suggested by the notion of "Runge-transitive operators" introduced by Bonilla and Grosse-Erdmann in [16]. Observe that if X, Y are two Fréchet spaces and $(T_n) \subset L(X, Y)$, then (T_n) is mixing if and only if, given $p \in \mathrm{SN}(X)$, $q \in \mathrm{SN}(Y)$, a p-open set U in X and a q-open set V in Y, there is $N = N(U, V) \in \mathbb{N}$ such that $T_n(U) \cap V \ne \emptyset$ for every $n \ge N$. Compare this with the definition below, which introduces a concept stronger than mixing.

DEFINITION 3.2. If X and Y are Fréchet spaces and $(T_n) \subset L(X, Y)$, then we say that (T_n) is *supermixing* if, given $p \in \mathrm{SN}(X)$ and $q \in \mathrm{SN}(Y)$, there exists $N = N(p, q) \in \mathbb{N}$ such that $T_n(U) \cap V \ne \emptyset$ for every p-open set U in X, every q-open set V in Y and every $n \ge N$. An operator $T \in L(X)$ is said to be *supermixing* whenever the sequence (T^n) of its iterates is supermixing.

EXAMPLES 3.3. 1. If X is a Banach space, then no sequence $(T_n) \subset L(X)$ can be supermixing. Indeed, supermixing in the Banach space case means that, for some N, the set $T_N(U) \cap V$ is nonempty for every pair U, V of nonempty open sets, which is clearly impossible.

2. Each translation operator τ_a ($a \in \mathbb{C} \setminus \{0\}$) is supermixing on $\mathcal{E}$. Indeed, let (p_k) be the sequence of seminorms $p_k(z) = \sup\{|f(z)| : |z| \leq k\}$ and fix $p, q \in \mathrm{SN}(\mathcal{E})$. Then there exist $k \in \mathbb{N}$ and $\delta > 0$ such that $\{f : p(f) < 1\} \cap \{f : q(f) < 1\} \supset \{f : p_k(f) < \delta\}$. Choose $N > 2k/|a|$, and fix an $n \geq N$, a p-open set U and a q-open set V. Then there are $\alpha, \beta \in \mathcal{E}$ and $\varepsilon > 0$ with $U \supset \{f : p_k(f - \alpha) < \varepsilon\}$ and $V \supset \{f : p_k(f - \beta) < \varepsilon\}$. By Runge's approximation theorem applied to α, β on the compact set $\overline{B}(0, k) \cup \overline{B}(na, k)$, there exists a polynomial P such that $|P(z) - \alpha(z)| < \varepsilon$ on $\overline{B}(0, k)$ and $|P(z) - \beta(z - na)| < \varepsilon$ on $\overline{B}(na, k)$. Then $p_k(P - \alpha) < \varepsilon$ and $|P(z + na) - \beta(z)| < \varepsilon$ on $\overline{B}(0, k)$, so $p_k(\tau_a^n P - \beta) < \varepsilon$. Hence $P \in U \cap (\tau_a^n)^{-1}(V)$, which finishes the proof. This example will be generalized later (see Section 5).

3. It is easy to check that the backward shift operator $B(x_k) = (x_{k+1})$ on the Fréchet space $\omega = \mathbb{K}^{\mathbb{N}}$ of all scalar sequences (endowed with the product topology) is supermixing. Just use the denseness of the sequences having only finitely many nonzero entries, and take into account that, for every $p \in \mathrm{SN}(\omega)$, there are $k \in \mathbb{N}$ and $\varepsilon > 0$ such that $\{x = (x_j) \in \omega : p(x) < 1\} \supset \{x \in \omega : \sup_{j \leq k} |x_j| < \varepsilon\}$.

4. Two criteria for disjoint hypercyclicity. We are now ready to state the first of our main results.

THEOREM 4.1. *Let X, Y and Z be three separable Fréchet spaces. Assume that $(S_n) \subset L(X, Y)$ and $(T_n) \subset L(X, Z)$. Suppose that the following conditions are satisfied*:

(a) (S_n) *is densely hypercyclic.*
(b) (S_n) *is SN-controlled.*
(c) (T_n) *is supermixing.*

Then (S_n) and (T_n) are densely disjoint hypercyclic.

Proof. We will use Remark 2.1.2. Fix nonempty open subsets U, V, W of X, Y, Z, respectively. Then there are seminorms $p \in \mathrm{SN}(X)$, $q \in \mathrm{SN}(Y)$, $r \in \mathrm{SN}(Z)$ as well as a p-open set U_0, a q-open set V_0 and an r-open set W_0 satisfying $U \supset U_0$, $V \supset V_0$, $W \supset W_0$. Since (S_n) is SN-controlled, there is $\widetilde{p} \in \mathrm{SN}(X)$ such that $S_n^{-1}(V_0)$ is a $\widetilde{p}$-open set for all $n \in \mathbb{N}$. If $p_0 := \max\{p, \widetilde{p}\}$, then $p_0 \in \mathrm{SN}(X)$ and each of the sets $U_0, S_n^{-1}(V_0)$ is p_0-open in X, hence so also is their intersection $U_0 \cap S_n^{-1}(V_0)$, for every $n \in \mathbb{N}$. Since (T_n) is supermixing, there is $N = N(p_0, r) \in \mathbb{N}$ such that $A \cap T_n^{-1}(W_0) \neq \emptyset$ for every $n \geq N$ and every nonempty p_0-open set A in X. Finally, by the dense hypercyclicity of (T_n), there is $n_0 \geq N$ such that $U_0 \cap S_{n_0}^{-1}(V_0) \neq \emptyset$. Hence we can take $A := U_0 \cap S_{n_0}^{-1}(V_0)$. It follows that $U_0 \cap S_{n_0}^{-1}(V_0) \cap T_{n_0}^{-1}(W_0) \neq \emptyset$. Consequently, $U \cap S_{n_0}^{-1}(V) \cap T_{n_0}^{-1}(W) \neq \emptyset$, as required. ∎

REMARK 4.2. As a consequence, if X is a Fréchet space and $S, T \in L(X)$ are such that S is hypercyclic, the sequence (S^n) is SN-controlled and T is supermixing, then S and T are densely disjoint hypercyclic.

The second criterion has to do with the Godefroy–Shapiro idea that an operator can be hypercyclic *if it possesses a good supply of eigenvectors.* This is the core of the proof of their theorem that an operator $\Phi(D)$ is hypercyclic (see [23, Section 5] and Section 5 below). Their idea was developed and formalized by the author in [6] to obtain hypercyclic sequences of differential operators. Now we extend Theorem 7 of [6] (see also [9]) to obtain a sufficient condition for disjoint hypercyclicity. Recall that, in a topological vector space, a subset is said to be *total* whenever its linear span is dense. If T is an operator and e is an eigenvector, then we denote by $\lambda(T, e)$ its corresponding eigenvalue.

THEOREM 4.3. *Let* $T_1, \dots, T_p$ *be operators on a separable F-space* X. *Assume that there are* $p+1$ *total subsets* $D_0, D_1, \dots, D_p$ *of* X *satisfying the following properties for all* $j \in \{1, \dots, p\}$:

(a) *Each vector* $e \in D_0 \cup D_1 \cup \dots \cup D_p$ *is an eigenvector of* T_j.
(b) $|\lambda(T_j, e)| < 1$ *for all* $e \in D_0$.
(c) $|\lambda(T_j, e)| > 1$ *for all* $e \in D_j$.
(d) $|\lambda(T_j, e)| < |\lambda(T_i, e)|$ *for all* $e \in D_i$ *and all* $i \in \{1, \dots, p\} \setminus \{j\}$.

Then the sequence $([T_1^n, \dots, T_p^n])$ *is mixing. In particular, the operators* $T_1, \dots, T_p$ *are densely disjoint hypercyclic.*

Proof. First, note that $\lambda(S^n, e) = (\lambda(S, e))^n$ $(n \in \mathbb{N})$ whenever e is an eigenvector of the operator S. If $A \subset X$ denote by $s(A)$ the linear span of A. In view of the Birkhoff transitivity theorem—applied to the sequence $T_n = [T_1^n, \dots, T_p^n] : X \to X^p$ $(n \in \mathbb{N})$—it should be proved that, given nonempty open subsets $U, V_1, \dots, V_p$ in X, there is $N \in \mathbb{N}$ such that $T_j^n(U) \cap V_j \neq \emptyset$ for all $j = 1, \dots, p$ and all $n \geq N$. Due to the denseness of $s(D_0), s(D_1), \dots, s(D_p)$ in X, it is enough to prove that, given $p+1$ vectors $u_j \in s(D_j)$ $(j = 0, \dots, p)$, there is a sequence $(x_n) \subset X$ such that, as $n \to \infty$,

(1) $$x_n \to u_0 \quad \text{and} \quad T_j^n x_n \to u_j \quad (j = 1, \dots, p).$$

For fixed vectors u_j $(j = 0, 1, \dots, p)$ as above, there are respective finite sets $F_j = \{e_{j,1}, \dots, e_{j,m(j)}\} \subset D_j$ and scalars $c_{j,1}, \dots, c_{j,m(j)}$ such that

$$u_j = \sum_{l=1}^{m(j)} c_{j,l} e_{j,l}.$$

By hypotheses (a)–(c), for every $j \in \{1, \ldots, p\}$,

(2) $\quad \lambda(T_j^n, e_{0,l}) = (\lambda(T_j, e_{0,l}))^n \to 0 \qquad$ as $n \to \infty$ $(l = 1, \ldots, m(0))$,

(3) $\quad \lambda(T_j^n, e_{j,l}) = (\lambda(T_j, e_{j,l}))^n \to \infty \qquad$ as $n \to \infty$ $(l = 1, \ldots, m(j))$.

From condition (d) we obtain, for $i, j \in \{1, \ldots, p\}$ with $i \neq j$,

(4) $\quad \dfrac{\lambda(T_j^n, e_{i,l})}{\lambda(T_i^n, e_{i,l})} = \left(\dfrac{\lambda(T_j, e_{i,l})}{\lambda(T_i, e_{i,l})}\right)^n \to 0 \qquad$ as $n \to \infty$ $(l = 1, \ldots, m(i))$.

Now, for each $n \in \mathbb{N}$, we define the vector

$$x_n = u_0 + \sum_{i=1}^{p} \sum_{l=1}^{m(i)} \frac{c_{i,l}}{\lambda(T_i^n, e_{i,l})} \, e_{i,l}.$$

By (3), $c_{i,l}/\lambda(T_i^n, e_{i,l}) \to 0$ $(n \to \infty)$ for every $i = 1, \ldots, p$ and every $l = 1, \ldots, m(i)$, so $x_n \to u_0$. The first part of (1) has been proved. Finally, for each $j = 1, \ldots, p$, we deduce from (2) and (4) that, as $n \to \infty$,

$$T_j^n x_n = T_j^n \left(u_0 + \sum_{i=1}^{p} \sum_{l=1}^{m(i)} \frac{c_{i,l}}{\lambda(T_i^n, e_{i,l})} \, e_{i,l} \right)$$

$$= \sum_{l=1}^{m(0)} c_{0,l} T_j^n e_{0,l} + \sum_{i=1}^{p} \sum_{l=1}^{m(i)} \frac{c_{i,l}}{\lambda(T_i^n, e_{i,l})} \, T_j^n e_{i,l}$$

$$= \sum_{l=1}^{m(0)} c_{0,l} \lambda(T_j^n, e_{0,l}) e_{0,l} + \sum_{i=1}^{p} \sum_{l=1}^{m(i)} \frac{c_{i,l}}{\lambda(T_i^n, e_{i,l})} \, \lambda(T_j^n, e_{i,l}) e_{i,l}$$

$$= \sum_{l=1}^{m(0)} c_{0,l} \lambda(T_j^n, e_{0,l}) e_{0,l} + \sum_{l=1}^{m(j)} c_{j,l} \, e_{j,l}$$

$$+ \sum_{i \in \{1,\ldots,p\} \setminus \{j\}} \sum_{l=1}^{m(i)} \frac{c_{i,l}}{\lambda(T_i^n, e_{i,l})} \, \lambda(T_j^n, e_{i,l}) e_{i,l}$$

$$\to 0 + \sum_{l=1}^{m(j)} c_{j,l} e_{j,l} + 0 = u_j.$$

This is the second part of (1), concluding the proof. ∎

As in [6] and [9], there is a generalization of the last theorem to the case of sequences of operators $(T_{1,n}), \ldots, (T_{p,n})$. This is left as an exercise for the interested reader.

5. Examples of disjoint hypercyclic operators. In this section, we furnish a number of examples of disjoint hypercyclic operators or sequences of operators. This will be done either directly or by applying Theorems 4.1

and 4.3. We will consider composition operators and differential operators on spaces of analytic functions.

We recall that by Birkhoff's theorem [13] and MacLane's theorem [33], respectively, each nontrivial translation operator τ_a and the derivative operator D are hypercyclic on $\mathcal{E}$. Moreover, as a link between D and τ_a, Godefroy and Shapiro [23] proved that $\Phi(D)$ is hypercyclic on $\mathcal{E}$ for any nonconstant entire function $\Phi \in \text{Exp}$.

Let us start with a simple result concerning composition operators. If $G \subset \mathbb{C}$ is a domain and $\varphi : G \to G$ is a holomorphic selfmap, then the composition operator $C_\varphi \in L(H(G))$ generated by φ is defined as $C_\varphi f = f \circ \varphi$. We denote by $\text{Aut}(G)$ the group of its automorphisms, that is, of its bijective holomorphic selfmaps. Recall that a domain G is called simply connected whenever $\mathbb{C}_\infty \setminus G$ is connected, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the extended plane.

THEOREM 5.1. *Assume that G is a simply connected domain, $m \in \mathbb{N}$, and $(\varphi_{j,n}) \subset \text{Aut}(G)$ $(j = 1, \ldots, m)$. Suppose that these sequences "leave compacta separately", that is, for every compact subset $K \subset G$, there is $N = N(K) \in \mathbb{N}$ such that, for each $n \geq N$, the sets $K, \varphi_{1,n}(K), \ldots, \varphi_{m,n}(K)$ are pairwise disjoint. Then the sequence $[C_{\varphi_{1,n}}, \ldots, C_{\varphi_{m,n}}] : H(G) \to (H(G))^m$ $(n \in \mathbb{N})$ is supermixing. In particular, the sequences $(C_{\varphi_{1,n}}), \ldots, (C_{\varphi_{m,n}})$ are densely disjoint hypercyclic.*

Proof. Let $p \in \text{SN}(H(G))$ and $q \in \text{SN}((H(G))^m)$. Since G is simply connected, one can find a compact set $K \subset G$ without holes and a $\delta > 0$ such that $\{f : p(f) < 1\} \supset \{f : \sup_K |f| < \delta\}$ and $\{F = (f_1, \ldots, f_m) : q(F) < 1\} \supset \{F : \sup_K |f_j| < \delta \ (j = 1, \ldots, m)\}$. Let $N = N(K)$ be the natural number furnished by the "leaving" hypothesis. Fix $n \geq N$. Fix also a nonempty p-open set $U \subset H(G)$ and a nonempty q-open set $V \subset (H(G))^m$. Then we derive the existence of functions $\alpha, \beta_1, \ldots, \beta_m \in H(G)$ and of a number $\varepsilon > 0$ such that $U \supset \{f \in H(G) : |f(z) - \alpha(z)| < \varepsilon$ for all $z \in K\}$ and $V \supset \{F \in (H(G))^m : |f_j(z) - \beta_j(z)| < \varepsilon$ for all $z \in K$ and all $j = 1, \ldots, m\}$. Since each $\varphi_{j,n}$ is an automorphism, the compact set $\varphi_{j,n}(K)$ has no holes. Hence the disjoint union $K \cup \varphi_{1,n}(K) \cup \cdots \cup \varphi_{m,n}(K)$ is a compact set without holes. Therefore, Runge's theorem yields a polynomial f (so $f \in H(G)$) such that $|f(z) - \alpha(z)| < \varepsilon$ on K and $|f(z) - \beta_j(\varphi_{j,n}^{-1}(z))| < \varepsilon$ on $\varphi_{j,n}(K)$ $(j = 1, \ldots, m)$. The last set of inequalities is equivalent to $|(C_{\varphi_{j,n}} f)(z) - \beta_j(z)| < \varepsilon$ on K $(j = 1, \ldots, m)$. Therefore $f \in U$ and $[C_{\varphi_{1,n}}, \ldots, C_{\varphi_{m,n}}]f \in V$. Hence $[C_{\varphi_{1,n}}, \ldots, C_{\varphi_{m,n}}](U) \cap V \neq \emptyset$, as required. ∎

REMARKS 5.2. 1. The "leaving" condition of Theorem 5.1 is fulfilled if the sequences $(\varphi_{j,n})$ tend uniformly on compacta in G—in the sense of the

chordal distance—to m distinct points of ∂G, the boundary of G in $\mathbb{C}_\infty$. This is not at all necessary (see Proposition 5.5).

2. In the case $m = 1$, the supermixing conclusion of Theorem 5.1 still holds just on assuming that G is a domain of $\mathbb{C}$ which is *not isomorphic to the punctured plane* $\mathbb{C} \setminus \{0\}$. That is, if $(\varphi_n) \subset \mathrm{Aut}(G)$ and $\varphi_n \to \partial G$ compactly, then the sequence (C_{φ_n}) is supermixing. Indeed, by adapting the proof of Lemma 2.12 in [8] from the so-called "run-away" sequences of automorphisms to our (φ_n) (the lemma dealt with domains of infinite connectivity; its conclusion is evident for simply connected domains, while for domains with finite connectivity ≥ 3 or isomorphic to an annulus or to $\mathbb{D} \setminus \{0\}$, its conclusion is trivial because there is no sequence of automorphisms tending to ∂G; in the case of finite connectivity ≥ 3, this is so by Heins' theorem [29]), we find that if K is a compact subset of G such that each hole of K contains some hole of G, then there is $N = N(K) \in \mathbb{N}$ such that, for all $n \geq N$, we have $K \cap \varphi_n(K) = \emptyset$ and $K \cup \varphi_n(K)$ is a compact set each of whose holes contains at least one hole of G. This yields, via Runge's theorem and similarly to the final part of the proof of Theorem 5.1, a rational function f with poles outside G (so $f \in H(G)$) such that $|f(z) - \alpha(z)| < \varepsilon$ on K and $|f(z) - \beta(\varphi_n^{-1}(z))| < \varepsilon$ on $\varphi_n(K)$, for any preassigned $\varepsilon > 0$ and $\alpha, \beta \in H(G)$. This completes the proof.

Next, we turn our attention to differential operators. In the following statement, which concerns operators $\Phi(D)$ on $H(G)$, we may suppose merely that $\Phi \in \mathrm{Exp}$ if G is the whole plane $\mathbb{C}$. In addition, the statement has a multi-dimensional version—with the same proof except for small changes of notation—if the expression "simply connected domain" is replaced by "Runge domain" (i.e., a domain in $\mathbb{C}^N$ such that the set of polynomials is dense in $H(G)$; see [31] or [32, Chap. 5] for properties and characterizations). In the definition of entire function of exponential or subexponential type, one should interpret the modulus $|z|$ of a point $z = (z_1, \ldots, z_N)$ as its euclidean norm $(|z_1|^2 + \cdots + |z_N|^2)^{1/2}$.

THEOREM 5.3. *Assume that G is a simply connected domain of $\mathbb{C}$ and that $\Phi_1, \ldots, \Phi_p$ are entire functions of subexponential type. Assume also that there are $p + 1$ points $z_0, z_1, \ldots, z_p \in \mathbb{C}$ such that*:

(a) $\max\{|\Phi_j(z_0)| : j = 1, \ldots, p\} < 1$.

(b) $\max\{1, |\Phi_j(z_i)| : j \in \{1, \ldots, p\} \setminus \{i\}\} < |\Phi_i(z_i)|$ *for all* $i = 1, \ldots, p$.

Then the operators $\Phi_1(D), \ldots, \Phi_p(D)$ are densely disjoint hypercyclic on $H(G)$.

Proof. For $a \in \mathbb{C}$ and $A \subset \mathbb{C}$, set $e_a(z) := \exp(az)$ and $M(A) := \{e_a : a \in A\}$. If A is a nonempty open set, then $M(A)$ is total in $H(G)$ (see for instance [23, Section 5] or [9, Lemma 2.3]). Observe that $\Phi(D)e_a =$

$\Phi(a)e_a$ for all $a \in \mathbb{C}$ and all $\Phi \in \operatorname{Exp}$, so e_a is an eigenvector of $\Phi(D)$ with eigenvalue $\Phi(a)$. By the continuity of the functions $|\Phi_j|$, there are nonempty open subsets $U_j \subset \mathbb{C}$ such that, for every $j \in \{1, \ldots, p\}$, one has $|\Phi_j(z)| < 1$ on U_0, $|\Phi_j(z)| > 1$ on U_j, and $|\Phi_i(z)| > |\Phi_j(z)|$ on U_i whenever $i \neq j$. Thus, it suffices to apply Theorem 4.3 to $T_j := \Phi_j(D)$ and $D_j := M(U_j)$. ∎

EXAMPLES 5.4. 1. Approximation by Birkhoff-universal functions and MacLane-universal functions can be simultaneously performed with the same functions and with the same approximating iterates. In other words, the operators τ_1 and D are disjoint hypercyclic on $\mathcal{E}$. Indeed, $\tau_1 = \Phi_1(D)$ and $D = \Phi_2(D)$, where $\Phi_1(z) := e^z$ and $\Phi_2(z) := z$. Then just apply Theorem 5.3 with $p = 2$, $z_0 = -1/2$, $z_1 = 1$ and $z_2 = -1$. This result can also be derived from Example 3.2.1, Theorem 4.1 and Remark 4.2, because τ_1 is supermixing, D is hypercyclic and (D^n) is SN-controlled.

2. Of course, one can mix Theorems 5.1 and 5.3 to obtain (with the help of Theorem 4.1) more examples of disjoint hypercyclic operators.

The final part of this section is devoted to several examples (in the cases $G = \mathbb{C}$ or $\mathbb{D}$) of disjoint hypercyclic sequences of composition operators, via an application of Theorem 5.1.

Recall that the members of $\operatorname{Aut}(\mathbb{C})$ are the similarities $z \mapsto az + b$ $(a, b \in \mathbb{C}, a \neq 0)$, while $\operatorname{Aut}(\mathbb{D})$ consists of the Möbius transformations of the form $z \mapsto k\frac{z-a}{1-\bar{a}z}$ ($|a| < 1 = |k|$). As a generalization of Birkhoff's theorem and of Seidel–Walsh's theorem (the latter is the $\mathbb{D}$-version of the Birkhoff theorem, see [39]), it is known that if $(\varphi_n(z) := a_n z + b_n) \in \operatorname{Aut}(\mathbb{C})$ $((\varphi_n(z) := k_n\frac{z-a_n}{1-\bar{a}_n z}) \in \operatorname{Aut}(\mathbb{D})$, resp.) then (C_{φ_n}) is hypercyclic on $\mathcal{E}$ (on $H(\mathbb{D})$, resp.) if and only if $\sup_{n \in \mathbb{N}} \min\{|b_n|, |b_n/a_n|\} = +\infty$ ($\sup_{n \in \mathbb{N}} |a_n| = 1$, resp.). Moreover, if $\varphi(z) := az + b \in \operatorname{Aut}(\mathbb{C})$, then C_φ is hypercyclic on $\mathcal{E}$ if and only if φ is a translation, that is, $a = 1$ and $b \neq 0$. And if $\varphi(z) := e^{i\theta}\frac{z-a}{1-\bar{a}z} \in \operatorname{Aut}(\mathbb{D})$, then C_φ is hypercyclic on $H(\mathbb{D})$ if and only if φ has no fixed point in $\mathbb{D}$, or equivalently, if and only if $|a| \geq |\sin(\theta/2)|$ and $a \neq 0$ (for this and extensions, see [8], [18] and [40]). In particular, every "non-Euclidean translation" $\varphi(z) := z * a = \frac{z+a}{1+\bar{a}z}$ ($a \in \mathbb{D} \setminus \{0\}$) generates a hypercyclic composition operator on $H(\mathbb{D})$.

PROPOSITION 5.5.

(A) *If $b_1, \ldots, b_p$ are pairwise different nonzero complex numbers, then the translation operators $\tau_{b_1}, \ldots, \tau_{b_p}$ are densely disjoint hypercyclic on the space $\mathcal{E}$.*

(B) *Assume that $\varphi_{j,n}(z) := a_{j,n}z + b_{j,n}$ $(j = 1, \ldots, p; n \in \mathbb{N})$ are automorphisms of $\mathbb{C}$ satisfying, for every $j \in \{1, \ldots, p\}$, the following properties:*

(a) $\lim\limits_{n\to\infty} |b_{j,n}| = \infty = \lim\limits_{n\to\infty} \left| \dfrac{b_{j,n}}{a_{j,n}} \right|.$

(b) $\lim\limits_{n\to\infty} \dfrac{|b_{i,n} - b_{j,n}|}{|a_{i,n}| + |a_{j,n}|} = \infty$ *for every* $i \in \{1, \ldots, p\} \setminus \{j\}.$

Then the sequence $[C_{\varphi_{1,n}}, \ldots, C_{\varphi_{p,n}}] : \mathcal{E} \to \mathcal{E}^p$ $(n \in \mathbb{N})$ *is supermixing. In particular, the sequences* $(C_{\varphi_{1,n}}), \ldots, (C_{\varphi_{p,n}})$ *are densely disjoint hypercyclic on* $\mathcal{E}.$

Proof. Observe that (A) follows from (B) applied to $a_{j,n} := 1$, $b_{j,n} := nb_j$. Therefore, it suffices to demonstrate (B). For this, fix a compact set $K \subset \mathbb{C}$. Choose $r > 0$ such that $K \subset \overline{B}(0, r) =: B$. By hypothesis, there is a positive integer N such that, for every $j \in \{1, \ldots, p\}$ and every $n \geq N$, we have $|b_{j,n}| > 2r$, $|b_{j,n}/a_{j,n}| > 2r$, and $|b_{i,n} - b_{j,n}| > |a_{i,n}|r + |a_{j,n}|r$ for all $i \in \{1, \ldots, p\} \setminus \{j\}$. Then, if $n \geq N$, the p balls $\overline{B}(b_{j,n}, |a_{j,n}|r) = \varphi_{j,n}(B)$ are pairwise disjoint. Furthermore, we have

$$|\varphi_{j,n}(z)| = |b_{j,n}| \left| 1 + \frac{a_{j,n}z}{b_{j,n}} \right| \geq |b_{j,n}| \left(1 - \left| \frac{a_{j,n}z}{b_{j,n}} \right| \right) > 2r \left(1 - \frac{r}{2r} \right) = r$$

for all $z \in B$, all $j \in \{1, \ldots, p\}$ and all $n \geq N$. Then $B \cap \varphi_{n,j}(B) = \emptyset$ for all j and all $n \geq N$. Since $K \subset B$, we infer that, for each $n \geq N$, the sets $K, \varphi_{1,n}(K), \ldots, \varphi_{p,n}(K)$ are pairwise disjoint. The conclusion follows from Theorem 5.1. ∎

If $a \in \mathbb{C}$, then we denote by a^* the point symmetric to a with respect to the unit circle $\partial\mathbb{D}$, that is, $a^* = 1/\overline{a}$.

PROPOSITION 5.6.

(A) *If* $a_1, \ldots, a_p$ *are pairwise different points in* $\mathbb{D}$ *and* $\varphi_j(z) := z * a_j$ *$(j = 1, \ldots, p)$ are their respective non-Euclidean translations, then the composition operators* $C_{\varphi_1}, \ldots, C_{\varphi_p}$ *are densely disjoint hypercyclic.*

(B) *Suppose that* $\{\varphi_{j,n} : j \in \{1, \ldots, p\}, n \in \mathbb{N}\} \subset \mathrm{Aut}(\mathbb{D})$. *Assume also that, for each* $j \in \{1, \ldots, p\}$, *the following properties are satisfied:*

(a) $\lim\limits_{n\to\infty} |\varphi_{j,n}(0)| = 1.$

(b) $\lim\limits_{n\to\infty} \left| \dfrac{\varphi_{i,n}(0) - \varphi_{j,n}(0)}{\varphi_{i,n}(0) - (\varphi_{j,n}(0))^*} \right| = 1$ *for all* $i \in \{1, \ldots, p\} \setminus \{j\}.$

Then the sequences $(C_{\varphi_{1,n}}), \ldots, (C_{\varphi_{p,n}})$ *are densely disjoint hypercyclic.*

Proof. Assuming that (B) is proved, let us demonstrate (A). Let φ be a non-Euclidean translation $\varphi(z) = z * a$ with $a = re^{i\theta}$, where $r \in (0, 1)$ and $\theta \in [0, 2\pi)$. Define α as the simplest Möbius transformation sending $1, -1$

to $0, \infty$, respectively, and β as the rotation of angle θ, that is,

$$\alpha(z) = \frac{z-1}{z+1} \quad \text{and} \quad \beta(z) = e^{i\theta} z.$$

Then $\alpha^{-1}(z) = (1+z)/(1-z)$ and $\beta^{-1}(z) = e^{-i\theta} z$. If ψ is defined as $\psi = \alpha^{-1} \circ \beta^{-1} \circ \varphi \circ \beta \circ \alpha$, then an easy calculation shows that ψ is the dilation $\psi = r_0 z$, where $r_0 = (1+r)/(1-r)$ (> 1). Let φ_n, ψ_n denote the nth iterate of φ, ψ, respectively. Since $\varphi = \beta \circ \alpha \circ \psi \circ \alpha^{-1} \circ \beta^{-1}$, one obtains $\varphi_n = \beta \circ \alpha \circ \psi_n \circ \alpha^{-1} \circ \beta^{-1}$. But $\psi_n(z) = r_0^n z$, whence

$$\varphi_n(0) = \beta(\alpha(\psi_n(1))) = e^{i\theta} \frac{r_0^n - 1}{r_0^n + 1}.$$

Note that, as $n \to \infty$, we get $\varphi_n(0) \to e^{i\theta}$, so $|\varphi_n(0)| \to 1$.

Consequently, according to (B), it is enough to prove that, if $\varphi(z) := z * a$, $\Phi(z) := z * b$ are two non-Euclidean translations (with $a = r e^{i\theta} \neq R e^{i\Theta} = b$, $0 < r \le R < 1$ and $\theta, \Theta \in [0, 2\pi)$) and φ_n, Φ_n denote their respective iterates, then

$$A_n := \left| \frac{\varphi_n(0) - \Phi_n(0)}{\varphi_n(0) - (\Phi_n(0))^*} \right| \to 1 \quad \text{as } n \to \infty.$$

At this point, we distinguish two cases. If $\theta \neq \Theta$, then

$$A_n \to \left| \frac{e^{i\theta} - e^{i\Theta}}{e^{i\theta} - (e^{i\Theta})^{-1}} \right| = 1.$$

Finally, if $\theta = \Theta$, then $r < R$. We set

$$r_0 := \frac{1+r}{1-r} \quad \text{and} \quad R_0 := \frac{1+R}{1-R},$$

so that $R_0 > r_0 > 1$. Then

$$A_n = \left| \frac{\frac{r_0^n - 1}{r_0^n + 1} - \frac{R_0^n - 1}{R_0^n + 1}}{\frac{r_0^n - 1}{r_0^n + 1} - \frac{R_0^n + 1}{R_0^n - 1}} \right| = \frac{R_0^n - 1}{R_0^n + 1} \frac{(R_0/r_0)^n - 1}{(R_0/r_0)^n + 1} \to 1 \cdot 1 = 1,$$

and we are done.

Next, we prove (B). For each pair (j, n), we can write

$$\varphi_{j,n}(z) = k_{j,n} \frac{z - a_{j,n}}{1 - \overline{a}_{j,n} z}$$

for certain numbers $a_{j,n}, k_{j,n}$ satisfying $|a_{j,n}| < 1 = |k_{j,n}|$. Then $\varphi_{j,n}(0) = -k_{j,n} a_{j,n}$ and $|\varphi_{j,n}(0)| = |a_{j,n}|$. Fix a compact set $K \subset \mathbb{D}$. Choose a ball $B = \overline{B}(0, r)$ with $r \in (0, 1)$ and $K \subset B$. As $|\varphi_{j,n}(0)| \to 1$ $(n \to \infty)$ for all j, we can find $N_1 \in \mathbb{N}$ such that $|a_{j,n}| > (2r - r^2)^{1/2}$ for all $j \in \{1, \ldots, p\}$ and

all $n \geq N_1$. Therefore, if $z \in B$ and j, n are as before, we get

$$|\varphi_{j,n}(z)| = \left| \frac{z - a_{j,n}}{1 - \overline{a}_{j,n}z} \right| = \frac{1}{|\overline{a}_{j,n}|} \left| \frac{|a_{j,n}|^2 - \overline{a}_{j,n}z}{1 - \overline{a}_{j,n}z} \right| = \frac{1}{|a_{j,n}|} \left| 1 + \frac{|a_{j,n}|^2 - 1}{1 - \overline{a}_{j,n}z} \right|$$

$$\geq 1 - \frac{1 - |a_{j,n}|^2}{|1 - \overline{a}_{j,n}z|} \geq 1 - \frac{1 - |a_{j,n}|^2}{1 - |z|} \geq 1 - \frac{1 + r^2 - 2r}{1 - r} = r.$$

Hence $B \cap \varphi_{j,n}(B) = \emptyset$, so $K \cap \varphi_{j,n}(K) = \emptyset$ $(n \geq N_1, j \in \{1, \ldots, p\})$. According to Theorem 5.1, and since $K \subset B$, it is enough to show the existence of a positive integer $N \geq N_1$ such that $\varphi_{i,n}(B) \cap \varphi_{j,n}(B) = \emptyset$ for all i, j with $i \neq j$ and all $n \geq N$.

For this, consider the hyperbolic distance ϱ on $\mathbb{D}$ (see [21]), given by

$$\varrho(z, w) = \tanh^{-1} \left| \frac{z - w}{1 - z\overline{w}} \right| \qquad (z, w \in \mathbb{D}).$$

Since ϱ is invariant under automorphisms, we deduce from the triangle inequality that, for $z, w \in B$, $n \in \mathbb{N}$ and $i, j \in \{1, \ldots, p\}$,

$\varrho(\varphi_{i,n}(z), \varphi_{j,n}(w))$

$\quad \geq \varrho(\varphi_{i,n}(0), \varphi_{j,n}(0)) - \varrho(\varphi_{i,n}(z), \varphi_{i,n}(0)) - \varrho(\varphi_{j,n}(w), \varphi_{j,n}(0))$

$\quad = \varrho(\varphi_{i,n}(0), \varphi_{j,n}(0)) - \varrho(z, 0) - \varrho(w, 0) \geq \varrho(\varphi_{i,n}(0), \varphi_{j,n}(0)) - 2\tanh^{-1} r.$

Since (a) and (b) hold and $\lim_{x \to 1^-} \tanh^{-1} x = \infty$, for every pair i, j with $i \neq j$ we obtain

$$\lim_{n \to \infty} \varrho(\varphi_{i,n}(0), \varphi_{j,n}(0)) = \lim_{n \to \infty} \tanh^{-1} \left| \frac{\varphi_{i,n}(0) - \varphi_{j,n}(0)}{1 - \varphi_{i,n}(0)\,\overline{\varphi_{j,n}(0)}} \right|$$

$$= \lim_{n \to \infty} \tanh^{-1} \left(\frac{1}{|\varphi_{j,n}(0)|} \left| \frac{\varphi_{i,n}(0) - \varphi_{j,n}(0)}{\varphi_{i,n}(0) - (\varphi_{j,n}(0))^*} \right| \right)$$

$$= \lim_{n \to \infty} \tanh^{-1} \left| \frac{\varphi_{i,n}(0) - \varphi_{j,n}(0)}{\varphi_{i,n}(0) - (\varphi_{j,n}(0))^*} \right| = \infty.$$

Hence $\lim_{n \to \infty} \varrho(\varphi_{i,n}(0), \varphi_{j,n}(0)) = \infty$, so there exist $N \in \mathbb{N}$ (we can choose $N \geq N_1$) such that $\varrho(\varphi_{i,n}(0), \varphi_{j,n}(0)) > 2\tanh^{-1} r$ for all $n \geq N$ and all i, j with $i \neq j$. Consequently, $\varrho(\varphi_{i,n}(z), \varphi_{j,n}(w)) > 0$ for such i, j, n and all $z, w \in B$. But this implies $\varphi_{i,n}(B) \cap \varphi_{j,n}(B) = \emptyset$, as required. ∎

We finish this paper by providing an example of disjoint hypercyclic operators in the setting of *Banach* spaces. In 1969, S. Rolewicz [36] gave the first example of a hypercyclic operator on a Banach space. Namely, he proved that if $c \in \mathbb{K}$ has modulus > 1 and B is the backward shift on any of the spaces l^p (= the space of scalar sequences $x = (x_j)$ with $\|x\| := (\sum_{j=1}^{\infty} |x_j|^p) < \infty$) $(1 \leq p < \infty)$ or c_0 (= the space of scalar sequences $x = (x_j)$ tending to 0, endowed with the norm $\|x\| := \sup_{j \geq 1} |x_j|$), then the operator cB is hypercyclic. In 1995, H. Salas [38] improved this

result by showing that, if $a = (a_j)$ is a bounded sequence of nonzero scalars, then the *weighted backward shift* operator T_a defined on l^p or c_0 as

$$T_a x = (a_{j+1} x_{j+1}) \quad \text{for } x = (x_j)$$

is hypercyclic if and only if the sequence $\{A_n := \prod_{j=1}^n a_j\}_{n \geq 1}$ of partial products of the weights is unbounded. Extensions of Salas' result can be found for example in [11], [27] and [34]. Since every power of a hypercyclic operator is also hypercyclic (even with the same hypercyclic vectors, see [1]), it follows that, if a is as above, then T_a^m is hypercyclic for every $m \in \mathbb{N}$. In fact, we have the following statement.

PROPOSITION 5.7. *Let X be c_0 or l^p $(1 \leq p < \infty)$, and let $m \in \mathbb{N}$. Assume that $a = (a_j)$ is a bounded sequence of nonzero scalars such that*

$$(5) \qquad \sup\{\min\{|A_n|, |A_{2n}|, \ldots, |A_{mn}|\}\} = \infty.$$

Then the operators $T_a, T_a^2, \ldots, T_a^m$ are densely disjoint hypercyclic on X.

Proof. The proof will be given when $m = 2$ and $X = l^p$, the general case being similar. If $T := T_a$, then it must be proved that the sequence $([T^n, T^{2n}])$ is densely disjoint hypercyclic. From the Birkhoff transitivity theorem it is enough to find dense subsets D_0, D_1, D_2 of X such that, for $x \in D_0$, $y \in D_1$ and $z \in D_2$, there are a sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ and a sequence $(x_k) \subset X$ satisfying:

(i) $T^{n_k} x \to 0$ and $T^{2n_k} x \to 0$ as $k \to \infty$.
(ii) $x_k \to 0$, $T^{n_k} x_k \to y$ and $T^{2n_k} x_k \to z$ as $k \to \infty$.

We choose $D_0 = D_1 = D_2 := D$, where D is the set of sequences with finitely many nonzero entries. Given $x \in D$, there is $n_0 \in \mathbb{N}$ such that $T^n x = 0$ for every $n \geq n_0$, so condition (i) is trivially satisfied for any (n_k). Now, fix $y = (y_1, \ldots, y_q, 0, 0, 0, \ldots)$ and $z = (z_1, \ldots, z_r, 0, 0, 0, \ldots)$ in D. Without loss of generality, we may suppose that $q = r$. By (5), there is a sequence $\{m_1 < m_2 < \cdots\} \subset \mathbb{N}$ such that $A_{m_k} \to \infty$ and $A_{2m_k} \to \infty$. We can assume that $m_1 > 2q$ (so $m_k - q > q$ for all k). For $k \in \mathbb{N}$, define $n_k \in \mathbb{N}$ and $x_k \in X$ by $n_k := m_k - q$ and

$$x_k := (0, \ldots, 0, A_1 A_{n_k+1}^{-1} y_1, A_2 A_{n_k+2}^{-1} y_2, \ldots, A_q A_{n_k+q}^{-1} y_q, 0, \ldots, 0,$$
$$A_1 A_{n_{2k}+1}^{-1} z_1, A_2 A_{n_{2k}+2}^{-1} z_2, \ldots, A_q A_{n_{2k}+q}^{-1} z_q, 0, 0, 0, \ldots)$$

where there are n_k zeros as starting entries and $n_k - q$ zeros immediately after $A_q A_{n_k+q}^{-1} y_q$. Observe that, by construction, $T^{2n_k} x_k = z \to z$ $(k \to \infty)$. So part of (ii) has been shown. Hence, it remains to prove that $x_k \to 0$ and $T^{n_k} x_k \to y$. Fix $j \in \{1, \ldots, q\}$. By hypothesis, there is $M \in (0, \infty)$ such that $|a_l| \leq M$ for all l. Therefore

$$|A_j A_{n_k+j}^{-1}| \leq M^j |A_{m_k-q+j}^{-1}| \leq M^j |A_{m_k}^{-1}| M^{q-j} = M^q |A_{m_k}^{-1}|$$

and, analogously, $|A_j A_{2n_k+j}^{-1}| \leq M^{2q}|A_{2m_k}^{-1}|$. Consequently,

$$\|x_k\|^p \leq (M^q|A_{m_k}^{-1}|)^p \sum_{j=1}^{q} |y_j|^p + (M^{2q}|A_{2m_k}^{-1}|)^p \sum_{j=1}^{q} |z_j|^p \to 0$$

and

$$\|T^{n_k}x_k - y\|^p = |(A_1 A_{n_k+1}^{-1}z_1, \ldots, A_q A_{n_k+q}^{-1}z_q, 0, 0, 0, \ldots)|^p$$
$$\leq (M^q|A_{m_k}^{-1}|)^p \sum_{j=1}^{q} |z_j|^p \to 0.$$

But this shows that $x_k \to 0$ and $T^{n_k}x_k \to y$, as required. ∎

Final questions. 1. If T is a hypercyclic operator, are T, T^2 always disjoint hypercyclic?

2. Is there any Banach space example where Theorem 4.3 can be applied?

3. Are finitely many composition operators generated by nonelliptic (= without fixed points in $\mathbb{D}$) automorphisms φ of $\mathbb{D}$ disjoint hypercyclic? Is this true for the Hardy space $H^p(\mathbb{D})$, or even for other Banach spaces $X \subset H(\mathbb{D})$ where C_φ is a well-defined operator?

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