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Distances to convex sets

by

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Abstract. If X is a Banach space and C a convex subset of X^*, we investigate whether the distance $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K),C) := \sup\{\inf\{\|k-c\| : c \in C\} : k \in \overline{\mathrm{co}}^{w^*}(K)\}$ from $\overline{\mathrm{co}}^{w^*}(K)$ to C is M-controlled by the distance $\widehat{d}(K,C)$ (that is, if $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K),C) \le M\widehat{d}(K,C)$ for some $1 \le M < \infty$), when K is any weak*-compact subset of X^*. We prove, for example, that: (i) C has 3-control if C contains no copy of the basis of $\ell_1(c)$; (ii) C has 1-control when $C \subset Y \subset X^*$ and Y is a subspace with weak*-angelic closed dual unit ball $B(Y^*)$; (iii) if C is a convex subset of X and X is considered canonically embedded into its bidual X^{**}, then C has 5-control inside X^{**}, in general, and 2-control when $K \cap C$ is weak*-dense in C.

1. Introduction. If X is a Banach space and C a convex subset of X^*, we investigate in this paper whether the distance $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K),C) := \sup\{\inf\{\|k-c\| : c \in C\} : k \in \overline{\mathrm{co}}^{w^*}(K)\}$ from $\overline{\mathrm{co}}^{w^*}(K)$ to C is controlled by the distance $\widehat{d}(K,C)$, that is, if $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K),C) \le M\widehat{d}(K,C)$ for some constant $1 \le M < \infty$ independent of K, where K is any weak*-compact subset of X^*.

When C is a subspace of X, the control of C inside the bidual X^{**} of X has been studied in [10]–[13]. Actually the results obtained in those papers extend the classical Krein–Shmul'yan theorem. This theorem, in terms of distances, states the following (see [8, p. 29]): if X is a Banach space and K a weak*-compact subset of X^{**} such that $\widehat{d}(K,X) = 0$ (that is, K is a weak-compact subset of X), then $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K),X) = 0$, that is, $\overline{\mathrm{co}}^{w^*}(K) \subset X$ and so $\overline{\mathrm{co}}^{w^*}(K)$ is a weak-compact subset of X and $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$. Thus, looking at the Krein–Shmul'yan theorem in terms of distances, it is natural to ask the following: if K is a weak*-compact subset of X^{**}, does the equality $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K),X) = \widehat{d}(K,X)$ always hold? The answer to this question is negative. Actually, in [11] and [12] are constructed two weak*-compact

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subsets K_1, K_2 of a bidual Banach space X^{**} such that: (i) $K_1 \cap X$ is weak*-dense in K_1, $\widehat{d}(K_1, X) = 1/2$ and $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K_1), X) = 1$; (ii) $\widehat{d}(K_2, X) = 1/3$ and $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K_2), X) = 1$.

Thus, in general, a Banach space X fails to have 1-control inside its bidual X^{**}. However, it could be true that every Banach space X has M-control inside X^{**}, M being a universal constant greater than 1. So, we can ask the following question: does there exist a universal constant $1 < M < \infty$ such that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), X) \leq M\widehat{d}(K, X)$ for every weak*-compact subset K of X^{**} and every Banach space X? The answer to this question is affirmative. In [11] the following result is proved, which extends the Krein–Shmul'yan theorem: if K is a weak*-compact subset of X^{**} and Z a subspace of X, then

$$\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), Z) \leq 5\widehat{d}(K, Z);$$

moreover, if $Z \cap K$ is weak*-dense in K, then

$$\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), Z) \leq 2\widehat{d}(K, Z).$$

When H is a normal countably compact space and we look at the Banach space $Z = C(H)$ of continuous real functions on H as a subspace of $\ell_\infty(H)$, then the distances $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), Z)$ and $\widehat{d}(K, Z)$ behave analogously, K being any weak*-compact subset of $\ell_\infty(H)$ (see [3], [14]). So, in view of these results we have: (i) the smallest value M_0 of the universal constant of the extension of the Krein–Shmul'yan theorem satisfies $3 \leq M_0 \leq 5$; (ii) for the category of weak*-compact subsets K of X^{**} such that $Z \cap K$ is weak*-dense in K, Z being a subspace of X, the value $M = 2$ is optimal.

The purpose of this paper is to go a step further and investigate the control of $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C)$ by $\widehat{d}(K, C)$ when C is a convex subset of a dual Banach space X^* and K is a weak*-compact subset of X^*. The behavior of $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C)$ with respect to $\widehat{d}(K, C)$ varies. If C is a weak*-closed convex subset of X^*, it is very easy to see that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) = \widehat{d}(K, C)$. However, if $C \subset X^*$ is not weak*-closed, all situations are possible. In any case, as we will see later, the control of C inside X^* and the existence in C of a copy of the basis of $\ell_1(\mathfrak{c})$ are closely connected.

The paper is organized as follows. In Section 2 we study the control of convex subsets C of a Banach space X inside X^{**}. The results and constants obtained are similar to the ones obtained when C is a subspace of X.

In Section 3 we deal with the relation between the existence in C of a copy of the basis of $\ell_1(\mathfrak{c})$ and the control of C inside a dual Banach space X^*. We prove that every convex subset C of X^* has 3-control inside X^* whenever C contains no copy of the basis of $\ell_1(\mathfrak{c})$. Moreover, $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$ for every weak*-compact subset K of X^* that contains no copy of that basis.

Section 4 is devoted to the study of the control of a convex subset C inside a dual Banach space X^* when C is contained in a subspace Y of X^* with weak*-angelic closed dual unit ball $B(Y^*)$. This case is particularly favorable because always $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) = \widehat{d}(K, C)$ for every weak*-compact subset K of X^*.

Our notation is standard. If A and I are sets, $a \in A^I$ and $i \in I$ then a_i (or $a(i)$) denotes the ith coordinate of a and $\pi_i : A^I \to A$ the ith projection mapping such that $\pi_i(a) = a_i$. $|I|$ is the cardinality of I and $\mathfrak{c} := |\mathbb{R}|$. βI denotes the Stone–Čech compactification of I (for I endowed with the discrete topology) and $I^* := \beta I \setminus I$. If $f : I \to \mathbb{R}$ is a bounded function, then $\check{f} \in C(\beta I)$ is the Stone–Čech continuous extension of f to βI.

We shall consider only Banach spaces over the real field. If X is a Banach space, let $B(a; r) := \{x \in X : \|x - a\| \leq r\}$ be the closed ball with center at $a \in X$ and radius $r \geq 0$. $B(X)$ and $S(X)$ will be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual. If A is a subset of X, then $[A]$ and $\overline{[A]}$ denote the linear hull and the closed linear hull of A, respectively. A subset A of the Banach space X is said to *contain a copy of the basis of* $\ell_1(\mathfrak{c})$ if A contains a family of vectors $\{a_i : i < \mathfrak{c}\}$ which is equivalent to the canonical basis of $\ell_1(\mathfrak{c})$. The weak*-topology of the dual Banach space X^* is denoted by w^* and the weak topology of X by w. If C is a convex subset of X^*, for $x^* \in X^*$ and $A \subset X^*$, let $d(x^*, C) = \inf\{\|x^* - c\| : c \in C\}$ be the distance from x^* to C, and $\widehat{d}(A, C) = \sup\{d(a, C) : a \in A\}$ the distance from A to C. $\mathrm{co}(A)$ denotes the convex closure of the set A, $\overline{\mathrm{co}}(A)$ is the $\|\cdot\|$-closure of $\mathrm{co}(A)$ and $\overline{\mathrm{co}}^{w^*}(A)$ the w^*-closure of $\mathrm{co}(A)$. Given $1 \leq M < \infty$, a convex subset C of X^* is said to have *M-control inside* X^* if $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) \leq M\widehat{d}(K, C)$ for every w^*-compact subset K of X^*; and C is said to have control inside X^* if C has M-control inside X^* for some constant $1 \leq M < \infty$.

If K is a w^*-compact subset of a dual Banach space X^* and μ a Radon Borel probability on K, then $r(\mu)$ will denote the barycenter of μ (see [6, p. 115]). Recall that: (i) $r(\mu) \in \overline{\mathrm{co}}^{w^*}(K)$; (ii) $x^* \in \overline{\mathrm{co}}^{w^*}(K)$ if and only if there exists a Radon Borel probability μ on K such that $r(\mu) = x^*$; (iii) $r(\mu)(x) = \int_K x^*(x)\, d\mu(x^*)$ for all $x \in X$.

2. The control of convex subsets of X inside X^{**}. Convex subsets of a bidual Banach space X^{**}, in general, fail to have control inside X^{**}. For example, if X is a Banach space such that X^* contains a copy of ℓ_1, then there exists a w^*-compact subset H of X^{**} such that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(H), \overline{\mathrm{co}}(H)) > 0$ (see [15]). However, when we restrict ourselves to the convex subsets C of the Banach space X, we will see in this section that there exists control inside X^{**}. We begin with the calculation of the distance $d(x, C)$ when C is a convex subset of a Banach space X and $x \in X$.

Lemma 2.1. *Let X be a Banach space, C a convex subset of X and $x \in X$. Then*

$$d(x,C) = \sup_{\varphi \in S(X^*)} \inf\{|\varphi(x-c)| : c \in C\}.$$

Moreover, if $x \notin \overline{C}$, then even $d(x,C) = \sup_{\varphi\in S(X^*)} \inf \varphi(x - C)$.

Proof. If we assume that $x \notin \overline{C}$, the proof is a simple application of the Banach separation theorem. If $x \in \overline{C}$, then for every $\varphi \in S(X^*)$ we have $\inf\{|\varphi(x-c)| : c \in C\} = 0$, whence

$$d(x,C) = 0 = \sup_{\varphi \in S(X^*)} \inf\{|\varphi(x-c)| : c \in C\}. \ \blacksquare$$

The following lemmas are basic for the proofs of next propositions.

Lemma 2.2. *Let X be a Banach space and D a convex subset of X. Then for every $z \in \overline{D}^{w^*} \subset X^{**}$ we have*

$$d(z,D) \le 2d(z,X).$$

Proof. Suppose that $d(z,D) > 2d(z,X)$. Then

(i) for some $a > 0$ we have $d(z,D) > 2a > 2d(z,X)$,
(ii) there exists a vector $w \in X$ such that $\|w - z\| < a$ (because $d(z,X) < a$) and so $d(w,D) > a$ (otherwise, if $d(w,D) \le a$, we would get $d(z,D) \le \|w-z\| + d(w,D) < 2a$, a contradiction).

Since $d(w,D) > a$, by Lemma 2.1 there exists $x^* \in S(X^*)$ such that $\inf\{x^*(w-d) : d \in D\} > a$. Let $\{d_i\}_{i\in I} \subset D$ be a net such that $d_i \xrightarrow{w^*} z$. Then $w - d_i \xrightarrow{w^*} w - z$ and so $x^*(w-d_i) \to x^*(w-z)$. Hence $x^*(w-z) > a$ and so $\|w-z\| > a$, a contradiction. Thus, we get $d(z,D) \le 2d(z,X)$. $\blacksquare$

Lemma 2.3. *Let X be a Banach space, C a convex subset of X^*, K a w^*-compact subset of X^* and assume there exist $a, b > 0$ such that*

$$\widehat{d}(K,C) < a < b < \widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C).$$

Then there exist $z_0 \in \overline{\mathrm{co}}^{w^*}(K)$ and $\psi \in S(X^{**})$ with $\inf \psi(z_0 - C) > b$ such that, if μ is a Radon probability on K with barycenter $r(\mu) = z_0$ and $H = \mathrm{supp}(\mu)$, for every w^*-open subset V of X^* with $V \cap H \ne \emptyset$ there exists $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ such that $\inf \psi(\xi - C) > b$.

Proof. Without loss of generality, we suppose that $K \subset B(X^*)$. Choose $z \in \overline{\mathrm{co}}^{w^*}(K)$ such that $d(z,C) > b$. By Lemma 2.1 there exists $\psi \in S(X^{**})$ such that $\inf \psi(z - C) > b + \varepsilon$ for some $\varepsilon > 0$, that is, $\psi(z) > b + \varepsilon + \sup \psi(C)$. By the Bishop–Phelps theorem, there exists a vector $\phi \in S(X^{**})$ with $\|\psi - \phi\| \le \varepsilon/4$ such that ϕ attains its maximum on $\overline{\mathrm{co}}^{w^*}(K)$ at some point $z_0 \in \overline{\mathrm{co}}^{w^*}(K)$. So

$$\begin{aligned}(2.1)\qquad \phi(z_0) &\geq \phi(z) = \psi(z) + (\phi - \psi)(z)\\ &> \sup \psi(C) + b + \varepsilon - \tfrac{1}{4}\varepsilon = \sup \psi(C) + b + \tfrac{3}{4}\varepsilon,\end{aligned}$$

whence we get

$$\psi(z_0) = \phi(z_0) + (\psi - \phi)(z_0) > \sup \psi(C) + b + \tfrac{3}{4}\varepsilon - \tfrac{1}{4}\varepsilon = \sup \psi(C) + b + \tfrac{1}{2}\varepsilon,$$

that is,

$$(2.2)\qquad \inf \psi(z_0 - C) > b + \tfrac{1}{2}\varepsilon.$$

Thus $d(z_0, C) > b + \frac{1}{2}\varepsilon$ and so $z_0 \notin \overline{C}$ and $z_0 \notin K$ (because $\widehat{d}(K, C) < a < b$). Let μ be a Radon probability on K with $r(\mu) = z_0$ and let $H := \operatorname{supp}(\mu)$. Assume that there exists a w^*-open subset V of X^* with $V \cap H \neq \emptyset$ such that $\inf \psi(\xi - C) \leq b$ (that is, $\psi(\xi) \leq b + \sup \psi(C)$) for every $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$. Let $\mu_1 = \mu \restriction V \cap H$ denote the restriction of μ to $V \cap H$, that is, $\mu_1(B) = \mu(B \cap V \cap H)$ for every Borel subset $B \subset K$. Let $\mu_2 := \mu - \mu_1$. Observe that μ_1 and μ_2 are positive measures such that

(i) $\mu_1 \neq 0$, because $\emptyset \neq V \cap H = V \cap \operatorname{supp}(\mu)$,
(ii) $\mu_2 \neq 0$ because, if we assume $\mu_2 = 0$ (that is, $\mu = \mu_1 = \mu \restriction V \cap H$), then $z_0 = r(\mu) \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ and so $\inf \psi(z_0 - C) \leq b$, a contradiction to (2.2).

Thus, we have the decomposition $\mu = \mu_1 + \mu_2$ such that $1 = \|\mu\| = \|\mu_1\| + \|\mu_2\|$ with $\|\mu_1\| \neq 0 \neq \|\mu_2\|$. So, we can write

$$z_0 = r(\mu) = \|\mu_1\| \cdot r\left(\frac{\mu_1}{\|\mu_1\|}\right) + \|\mu_2\| \cdot r\left(\frac{\mu_2}{\|\mu_2\|}\right).$$

Since $r(\mu_1/\|\mu_1\|) \in \overline{\mathrm{co}}^{w^*}(V \cap H)$, we have $\psi(r(\mu_1/\|\mu_1\|)) \leq b + \sup \psi(C)$ by hypothesis. Hence $\phi(r(\mu_1/\|\mu_1\|)) \leq b + \frac{1}{4}\varepsilon + \sup \psi(C)$ (because $\|\psi - \phi\| \leq \varepsilon/4$). Thus, taking into account that $r(\mu_2/\|\mu_2\|) \in \overline{\mathrm{co}}^{w^*}(K)$, $\phi(r(\mu_2/\|\mu_2\|)) \leq \phi(z_0)$ and (2.1), we get

$$\begin{aligned}\phi(z_0) &= \|\mu_1\| \phi\left(r\left(\frac{\mu_1}{\|\mu_1\|}\right)\right) + \|\mu_2\| \phi\left(r\left(\frac{\mu_2}{\|\mu_2\|}\right)\right)\\ &\leq \|\mu_1\|(b + \tfrac{1}{4}\varepsilon + \sup \psi(C)) + \|\mu_2\| \phi(z_0)\\ &< \|\mu_1\| \phi(z_0) + \|\mu_2\| \phi(z_0) = \phi(z_0),\end{aligned}$$

a contradiction which completes the proof. ∎

Proposition 2.4. *Let X be a Banach space, C a convex subset of X and K a w^*-compact subset of X^{**}. Then*

$$\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) \leq 5\widehat{d}(K, C).$$

Proof. Without loss of generality, we assume that $0 \in C$. Suppose that the statement is not true, so there exists a w^*-compact subset K of X^{**} and $a, b > 0$ such that

$$\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) > b > 5a > 5\widehat{d}(K, C).$$

From Lemma 2.3 we have the following:

FACT. *There exists a functional* $\psi \in S(X^{***})$ *and a* w^**-compact subset* $\emptyset \neq H \subset K$ *such that for every* w^**-open subset* V *with* $V \cap H \neq \emptyset$ *there exists* $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ *with* $\inf \psi(\xi - C) > b$.

Now we carry out the following construction step by step:

STEP 1. Let $D_0 = \{0\}$. Applying the Fact to the w^*-open subset $V_0 := X^{**}$, we choose a vector $\xi_1 \in \overline{\mathrm{co}}^{w^*}(H)$ such that $\inf \psi(\xi_1 - C) > b$. So, $\psi(\xi_1) > b + \sup \psi(D_0) = b$. As $B(X^*)$ is w^*-dense in $B(X^{***})$, there exists $x_1^* \in S(X^*)$ such that $x_1^*(\xi_1) > b + \max x_1^*(D_0) = b$. Let $W_1 := \{u \in X^{**} : \langle u, x_1^* \rangle > b + \max x_1^*(D_0) = b\}$. Clearly, W_1 is a w^*-open halfspace of X^{**} such that $\xi_1 \in W_1 \cap \overline{\mathrm{co}}^{w^*}(H)$. Thus, $W_1 \cap H \neq \emptyset$ and so we can find a vector $\eta_1 \in W_1 \cap H$. Since $d(\eta_1, C) < a$, we have a decomposition $\eta_1 = \eta_1^1 + \eta_1^2$ such that $\eta_1^1 \in C$ and $\eta_1^2 \in aB(X^{**})$.

STEP 2. Let $D_1 = \{\eta_1^1\} \cup D_0 \subset C$ and $V_1 := W_1 \cap V_0 = W_1$. As V_1 is a w^*-open subset with $V_1 \cap H \neq \emptyset$, by the Fact there exists a vector $\xi_2 \in \overline{\mathrm{co}}^{w^*}(V_1 \cap H)$ such that $\inf \psi(\xi_2 - C) > b$, and also $\inf \psi(\xi_2 - D_1) \geq \inf \psi(\xi_2 - C) > b$ because $D_1 \subset C$. Since D_1 is finite and $\min \psi(\xi_2 - D_1) > b$, there exists a vector $x_2^* \in S(X^*)$ such that $\min x_2^*(\xi_2 - D_1) > b$, that is, $x_2^*(\xi_2) > b + \max x_2^*(D_1)$. Let $W_2 := \{u \in X^{**} : \langle u, x_2^* \rangle > b + \max x_2^*(D_1)\}$. Clearly, W_2 is a w^*-open halfspace of X^{**} such that $\xi_2 \in W_2 \cap \overline{\mathrm{co}}^{w^*}(V_1 \cap H)$. Thus $W_2 \cap V_1 \cap H \neq \emptyset$ and we can find $\eta_2 \in W_2 \cap V_1 \cap H$. So, $x_2^*(\eta_2) > b + \max x_2^*(D_1)$, that is, $\min x_2^*(\eta_2 - D_1) > b$. Moreover, $\min x_1^*(\eta_2 - D_0) > b$ because $\eta_2 \in V_1$. Since $d(\eta_2, C) < a$, we have a decomposition $\eta_2 = \eta_2^1 + \eta_2^2$ such that $\eta_2^1 \in C$ and $\eta_2^2 \in aB(X^{**})$.

By iteration, we get sequences $\{x_n^*\}_{n\geq 1} \subset S(X^*)$, $\{\eta_k\}_{k\geq 1} \subset H$, $D_k = \{\eta_k^1\} \cup D_{k-1}$ with $\eta_k = \eta_k^1 + \eta_k^2$, $\eta_k^1 \in C$ and $\eta_k^2 \in aB(X^{**})$, $k \geq 1$, such that $\min x_i^*(\eta_k - D_{i-1}) > b$ for every $k \geq i$.

Let $D = \overline{\mathrm{co}}(\bigcup_{k\geq 1} D_k) \subset \overline{C}$ and

$$K_1 = \overline{\{\eta_i^1 : i \geq 1\}}^{w^*} \subset (K + aB(X^{**})) \cap \overline{D}^{w^*}.$$

Let η_0 be a w^*-cluster point of $\{\eta_k\}_{k\geq 1}$.

CLAIM 1. $d(\eta_0, D) < 5a$.

Indeed, clearly $\eta_0 \in H \cap (K_1 + aB(X^{**}))$. Observe that:

(i) Since $K_1 \subset K + aB(X^{**})$, we get $\widehat{d}(K_1, C) \leq \widehat{d}(K, C) + a < 2a$.

(ii) Since $K_1 \subset \overline{D}^{w^*}$, by Lemma 2.2 we get $\widehat{d}(K_1, D) \leq 2\widehat{d}(K_1, X) \leq 2\widehat{d}(K_1, C) < 4a$.

Thus, as $\eta_0 \in K_1 + aB(X^{**})$, we finally get $d(\eta_0, D) < 5a$.

CLAIM 2. $d(\eta_0, D) \geq b$.

Indeed, let $\phi \in B(X^{***})$ be a w^*-cluster point of $\{x_n^*\}_{n\geq 1}$. Since we have $\min x_n^*(\eta_k - D_{n-1}) > b$ for every $k \geq n$, it follows that $\min x_n^*(\eta_0 - D_{n-1}) \geq b$ for all $n \geq 1$. Hence $\inf \phi(\eta_0 - D) \geq b$ and so $d(\eta_0, D) \geq b$ by Lemma 2.1.

Since $b > 5a$, we get a contradiction that completes the proof. ∎

PROPOSITION 2.5. *Let X be a Banach space, $C \subset X$ a convex subset of X, and K a w^*-compact subset of X^{**} such that $K \cap C$ is w^*-dense in K. Then $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) \leq 2\widehat{d}(K, C)$.*

Proof. Suppose that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) > b > 2a > 2\widehat{d}(K, C)$ for some $a, b > 0$. We follow the proof of Proposition 2.4 with the following changes. As $C \cap K$ is w^*-dense in K and $V_k \cap H \neq \emptyset$, $k \geq 0$, it follows that $V_k \cap C \cap K \neq \emptyset$ for all $k \geq 0$. Thus, we choose $\eta_k \in V_k \cap K \cap C$, $k \geq 1$, and put $\eta_k^1 = \eta_k$ and $\eta_k^2 = 0$. Hence, now $K_1 = \overline{\{\eta_k^1 : k \geq 1\}}^{w^*} = \overline{\{\eta_k : k \geq 1\}}^{w^*}$ satisfies $K_1 \subset K$ and so $\widehat{d}(K_1, C) \leq \widehat{d}(K, C) < a$, whence $\widehat{d}(K_1, D) < 2a$. Finally, every w^*-cluster point η_0 of $\{\eta_k : k \geq 1\}$ satisfies $\eta_0 \in K_1$, $d(\eta_0, D) < 2a$ and $d(\eta_0, D) \geq b$, a contradiction. ∎

REMARK 2.6. In Proposition 2.4 we have proved that there exists a constant M such that $1 \leq M \leq 5$ and $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) \leq M\widehat{d}(K, C)$ for every Banach space X, every convex subset C of X and every w^*-compact subset K of X^{**}. It is an open problem what is the best value M_0 of this constant, but $3 \leq M_0 \leq 5$ by the results of [12]. Concerning the constant of Proposition 2.5, the value $M = 2$ is optimal by [12].

3. Distances to convex subsets of dual Banach spaces. Let X be a Banach space, C a convex subset of X^*, and W a w^*-compact subset of X^*. In this section we study whether the distance $\widehat{d}(\overline{\mathrm{co}}^{w^*}(W), C)$ is controlled by the distance $\widehat{d}(W, C)$. The following proposition is an elementary result.

PROPOSITION 3.1. *Let C be a w^*-closed convex subset of the dual Banach space X^*. Then C has 1-control inside X^*, that is, for every w^*-compact subset W of X^* we have $\widehat{d}(\overline{\mathrm{co}}^{w^*}(W), C) = \widehat{d}(W, C)$.*

Proof. Let W be a w^*-compact subset of X^* and let $\widehat{d}(W, C) =: a$. Fix a point $w_0 \in \overline{\mathrm{co}}^{w^*}(W)$ and a number $\varepsilon > 0$; we prove that $d(w_0, C) \leq a + \varepsilon$. Let $\{w_\alpha : \alpha \in \mathcal{A}\} \subset \mathrm{co}(W)$ be a net such that $w_\alpha \xrightarrow{w^*} w_0$ for $\alpha \in \mathcal{A}$. Since $\widehat{d}(\mathrm{co}(W), C) = \widehat{d}(W, C)$, for each $\alpha \in \mathcal{A}$ we can choose $z_\alpha \in C$ such that $\|w_\alpha - z_\alpha\| < a + \varepsilon$. So, the net $\{w_\alpha - z_\alpha : \alpha \in \mathcal{A}\}$ is inside the ball $(a + \varepsilon)B(X^*)$, which is a w^*-compact subset. Thus, by passing to a subnet if necessary, we can suppose that $w_\alpha - z_\alpha \xrightarrow{w^*} u_0$ for some $u_0 \in (a+\varepsilon)B(X^*)$. Hence, we get $z_\alpha = w_\alpha - (w_\alpha - z_\alpha) \xrightarrow{w^*} w_0 - u_0$ and so $w_0 - u_0 =: z_0 \in C$, because C is w^*-closed. Therefore, we can write $w_0 = z_0 + u_0$ with $z_0 \in C$

and $u_0 \in (a+\varepsilon)B(X^*)$, that is, $d(w_0, C) \leq a+\varepsilon$. As $\varepsilon > 0$ is arbitrarily small, we conclude that $d(w_0, C) \leq a = \widehat{d}(W, C)$. So, $\widehat{d}(\overline{\mathrm{co}}^{w^*}(W), C) = \widehat{d}(W, C)$. ■

The following result is a consequence of Lemma 2.3.

LEMMA 3.2. *Let X be a Banach space and K a w^*-compact subset of X^* such that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)) > d > 0$. Then there exist $r_0 \in \mathbb{R}$, $z_0 \in \overline{\mathrm{co}}^{w^*}(K)$ and $\psi \in S(X^{**})$ with $\psi(z_0) > r_0 + d$ and $\psi(k) < r_0$ for all $k \in K$, and such that, if μ is a Radon probability on K with barycenter $r(\mu) = z_0$ and $H = \mathrm{supp}(\mu)$, then*:

(i) *for every w^*-open subset $V \subset X^*$ with $V \cap H \neq \emptyset$, there exist $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ such that $\psi(\xi) > r_0 + d$,*
(ii) *there exist a sequence $\{x_n : n \geq 1\} \subset B(X)$ and, for every pair of disjoint subsets M, N of $\mathbb{N}$, a point $\eta_{M,N} \in H$ such that*

$$\eta_{M,N}(x_m) \geq r_0 + d, \quad \forall m \in M, \quad \textit{and} \quad \eta_{M,N}(x_n) \leq r_0, \ \forall n \in N.$$

Proof. Find $\varepsilon > 0$ such that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)) > d+\varepsilon > 0 = \widehat{d}(K, \overline{\mathrm{co}}(K))$. By Lemma 2.3 there exist $z_0 \in \overline{\mathrm{co}}^{w^*}(K)$ and $\psi \in S(X^{**})$ such that $\inf \psi(z_0 - \overline{\mathrm{co}}(K)) > d + \varepsilon$, that is,

$$\psi(z_0) > \sup \psi(\overline{\mathrm{co}}(K)) + d + \varepsilon \geq \sup \psi(K) + \varepsilon + d.$$

So, if $r_0 := \sup \psi(K) + \varepsilon$, then $\psi(z_0) > r_0 + d$ and $\psi(k) < r_0$ for all $k \in K$. Let μ be a Radon Borel probability on K with $r(\mu) = z_0$ and let $H := \mathrm{supp}(\mu)$.

CLAIM. *For every w^*-open subset V of X^* with $V \cap H \neq \emptyset$ there exist $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ and $\eta \in \overline{\mathrm{co}}(V \cap H) \subset \overline{\mathrm{co}}^{w^*}(V \cap H)$ such that $\psi(\xi) > r_0 + d$ and $\psi(\eta) < r_0$.*

Indeed, by Lemma 2.3 there exists $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ with $\inf \psi(\xi - \overline{\mathrm{co}}(K)) > d + \varepsilon$, that is, $\psi(\xi) > r_0 + d$. On the other hand, as $\psi(k) < r_0$ for all $k \in K$, we have $\psi(\eta) < r_0$ for every $\eta \in \mathrm{co}(V \cap H)$.

Thus, by the Claim and the proof of [15, Lemma 2] we can find a sequence $\{x_n : n \geq 1\} \subset S(X)$ such that, if we define

$$A_n = \{\xi \in H : \xi(x_n) > r_0 + d\}, \qquad B_n = \{\eta \in H : \eta(x_n) < r_0\}, \qquad \forall n \geq 1,$$

then, for every pair of disjoint finite subsets M, N of $\mathbb{N}$, the w^*-open subset $V(M, N) := (\bigcap_{m \in M} A_m) \cap (\bigcap_{n \in N} B_n)$ of H is nonempty. In particular,

$$\emptyset \neq V(M, N) \subset \Big(\bigcap_{m \in M} \overline{A}_m^{w^*} \Big) \cap \Big(\bigcap_{n \in N} \overline{B}_n^{w^*} \Big) \subset H.$$

Since H is a w^*-compact subset, we conclude that for every pair of disjoint (finite or infinite) subsets M, N of $\mathbb{N}$,

$$\emptyset \neq \Big(\bigcap_{m \in M} \overline{A}_m^{w^*} \Big) \cap \Big(\bigcap_{n \in N} \overline{B}_n^{w^*} \Big) \subset H.$$

Since $\overline{A}_m^{w^*} \subset \{\xi \in H : \xi(x_m) \geq r_0 + d\}$ and $\overline{B}_n^{w^*} \subset \{\eta \in H : \eta(x_n) \leq r_0\}$, we finally deduce that for every pair of disjoint (finite or infinite) subsets M, N of $\mathbb{N}$ there exists $\eta_{M,N} \in H$ such that

$$\eta_{M,N}(x_m) \geq r_0 + d, \quad \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leq r_0, \quad \forall n \in N. \ \blacksquare$$

Definition 3.3. If X is a Banach space, a subset $\mathcal{F}$ of X^* is said to be a w^*-$\mathbb{N}$-*family* of width $d > 0$ if $\mathcal{F}$ is bounded and has the form

$$\mathcal{F} = \{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\},$$

so that there exist two sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(X)$ such that for every pair of disjoint subsets M, N of $\mathbb{N}$ we have

$$\eta_{M,N}(x_m) \geq r_m + d, \quad \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leq r_n, \quad \forall n \in N.$$

Moreover, if $r_m = r_0$ for all $m \geq 1$, we say that $\mathcal{F}$ is a *uniform* w^*-$\mathbb{N}$-*family* in X^*. We say that $A \subset X^*$ *has a* w^*-$\mathbb{N}$-*family* if there exists a w^*-$\mathbb{N}$-family $\mathcal{F} \subset A$.

Remark 3.4. (0) If Z is a set, a family $(A_i, B_i)_{i \in I}$ of pairs of nonempty subsets of Z is said to be an *independent family* if $A_i \cap B_i = \emptyset$ for all $i \in I$, and for every finite nonempty subset $F \subset I$ we have $\bigcap_{i \in F} \varepsilon_i A_i \neq \emptyset$, where $\varepsilon_i = \pm 1$, $(+1)A_i = A_i$ and $(-1)A_i = B_i$. In $\mathbb{N}$ there exists an independent family $(M_i, N_i)_{i < \mathfrak{c}}$ of cardinality $\mathfrak{c}$. Indeed, since $\beta\mathbb{N}$ is an extremally disconnected compact Hausdorff space with weight $\mathrm{w}(\beta\mathbb{N}) = \mathfrak{c}$ (see [21, p. 76]), by the Balcar–Franěk theorem (see [2], [7, p. 120]) there exists a continuous onto mapping $f : \beta\mathbb{N} \to \{0,1\}^{\mathfrak{c}}$. Let $\pi_i : \{0,1\}^{\mathfrak{c}} \to \{0,1\}$, $i < \mathfrak{c}$, be the projection onto the i-factor $\{0,1\}$ and put $M_i := (\pi_i \circ f)^{-1}(1) \cap \mathbb{N}$ and $N_i := (\pi_i \circ f)^{-1}(0) \cap \mathbb{N}$. Clearly, $\{(M_i, N_i) : i < \mathfrak{c}\}$ is an independent family in $\mathbb{N}$.

(1) If $(M_i, N_i)_{i < \mathfrak{c}}$ is an independent family in $\mathbb{N}$ of cardinality $\mathfrak{c}$ and $\mathcal{F} = \{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\}$ is a w^*-$\mathbb{N}$-family in the dual Banach space X^*, then a standard argument (see [8, p. 206]) proves that the family $\{\eta_{M_i,N_i} : i < \mathfrak{c}\}$ is equivalent to the basis of $\ell_1(\mathfrak{c})$. Moreover, the same argument shows that the sequence $\{x_n : n \geq 1\} \subset B(X)$ associated to $\mathcal{F}$ is equivalent to the basis of ℓ_1.

(2) So, if a dual Banach space X^* has a w^*-$\mathbb{N}$-family, then X has an isomorphic copy of ℓ_1. And vice versa, if X has a copy of ℓ_1, then X^* contains a w^*-$\mathbb{N}$-family. Indeed, let $i : \ell_1 \to X$ be an isomorphism between ℓ_1 and $i(\ell_1)$, and $i^* : X^* \to \ell_\infty$ its adjoint operator, which is a quotient mapping such that $B(\ell_\infty) \subset i^*(\|i^{-1}\| B(X^*))$. For each pair M, N of disjoint subsets of $\mathbb{N}$ choose $\eta_{M,N} \in \|i^{-1}\| B(X^*)$ such that $i^*(\eta_{M,N}) = \mathbf{1}_M - \mathbf{1}_N$. Then $\{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\}$ is a w^*-$\mathbb{N}$-family in X^*.

(3) Let $\mathcal{F} = \{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\}$ be a w^*-$\mathbb{N}$-family of width $\delta > 0$ in a dual Banach space X^*, associated to the sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(X)$. Then for every $0 <$

$\gamma < \delta$ there exists an infinite subset $\mathbb{N}_\gamma \subset \mathbb{N}$ such that $\mathcal{F}_\gamma := \{\eta_{M,N} : M, N$ disjoint subsets of $\mathbb{N}_\gamma\}$ is a uniform w^*-$\mathbb{N}$-family of width $\gamma > 0$ associated to the sequence $\{x_m : m \in \mathbb{N}_\gamma\} \subset B(X)$ and some number $r_0 \in \mathbb{R}$. Indeed, since the sequence $\{r_m : m \geq 1\} \subset \mathbb{R}$ is bounded, there exists some $r_0 \in \mathbb{R}$ such that $\mathbb{N}_\gamma := \{m \in \mathbb{N} : r_0 + \eta - \delta \leq r_m \leq r_0\}$ is infinite. Now, it is easy to see that $\mathcal{F}_\gamma := \{\eta_{M,N} : M, N$ disjoint subsets of $\mathbb{N}_\gamma\}$ is a uniform w^*-$\mathbb{N}$-family of width $\gamma > 0$ associated to r_0 and the sequence $\{x_m : m \in \mathbb{N}_\gamma\} \subset B(X)$.

(4) It is worth mentioning (and easy to see) that, if A is a subset of X^*, then A has a w^*-$\mathbb{N}$-family if and only if $\overline{A}$ does.

Proposition 3.5. *Let X be a Banach space.*

(1) *If K is a w^*-compact subset of X^* such that K fails to have a w^*-$\mathbb{N}$-family* (*in particular*, *if K contains no copy of the basis of $\ell_1(\mathfrak{c})$*), *then $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$.*

(2) *If C is a convex subset of X^* that fails to have a w^*-$\mathbb{N}$-family* (*in particular*, *if C contains no copy of the basis of $\ell_1(\mathfrak{c})$*), *then C has 3-control inside X^*, that is, for every w^*-compact subset K of X^* we have $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) \leq 3\widehat{d}(K, C)$.*

Proof. (1) Otherwise, there exists $d > 0$ such that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)) > d > 0$. By Lemma 3.2 there exist $\{x_n : n \geq 1\} \subset B(X)$, $r_0 \in \mathbb{R}$ and, for every pair of disjoint subsets M, N of $\mathbb{N}$, a vector $\eta_{M,N} \in K$ such that

$$\eta_{M,N}(x_m) \geq r_0 + d, \quad \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leq r_0, \quad \forall n \in \mathbb{N}.$$

Thus there exists a w^*-$\mathbb{N}$-family in K, a contradiction.

(2) Suppose that C fails to have 3-control inside X^*. Then there exist a w^*-compact subset K of X^* and $a, b > 0$ such that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) > b > 3a > 3\widehat{d}(K, C)$. So, as $\widehat{d}(\overline{\mathrm{co}}(K), C) = \widehat{d}(K, C) < a$, we have $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)) > b - a > 0$. By Lemma 3.2 there exist $r_0 \in \mathbb{R}$, $\{x_n : n \geq 1\} \subset B(X)$ and, for every pair of disjoint subsets M, N of $\mathbb{N}$, a vector $\eta_{M,N} \in K$ such that

$$\eta_{M,N}(x_m) \geq r_0 + b - a, \quad \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leq r_0, \quad \forall n \in N.$$

As $\widehat{d}(K, C) < a$, for each pair of disjoint subsets M, N of $\mathbb{N}$ there is $z_{M,N} \in C$ so that $\|z_{M,N} - \eta_{M,N}\| < a$. Thus, the family $\{z_{M,N} : M, N$ disjoint subsets of $\mathbb{N}\}$ is bounded and satisfies

$$z_{M,N}(x_m) \geq r_0 + b - 2a, \quad \forall m \in M, \quad \text{and} \quad z_{M,N}(x_n) \leq r_0 + a, \quad \forall n \in \mathbb{N}.$$

Since $r_0 + b - 2a = r_0 + a + (b - 3a) > r_0 + a$, the set $\{z_{M,N} : M, N$ disjoint subsets of $\mathbb{N}\}$ is a w^*-$\mathbb{N}$-family in C, a contradiction. ∎

Remark 3.6. For a convex subset C of a dual Banach space X^*, the statements "C has 3-control inside X^*" and "C contains no w^*-$\mathbb{N}$-family" are not equivalent, in general. For example, if $C := B(\ell_\infty)$, then C has a w^*-$\mathbb{N}$-family (this is trivial), and also C has 1-control (and so 3-control) inside

ℓ_∞ because C is w^*-closed (see Proposition 3.1). Concerning the statement "C contains no copy of the basis of $\ell_1(\mathfrak{c})$", it can be characterized as follows.

PROPOSITION 3.7. *Let X be a Banach space and C a convex subset of X^*. The following statements are equivalent*:

(i) *C contains no copy of the basis of $\ell_1(\mathfrak{c})$.*
(ii) *C has universal 3-control, that is, if $\overline{[C]}$ is (isomorphic to) a subspace of some dual Banach space V^*, then C has 3-control inside V^*.*
(iii) *C has universal control, that is, if $\overline{[C]}$ is (isomorphic to) a subspace of some dual Banach space V^*, then C has control inside V^*.*

Proof. (i)⇒(ii) follows from Proposition 3.5, and (ii)⇒(iii) is obvious.

(iii)⇒(i). Suppose that C contains a copy K of the basis of $\ell_1(\mathfrak{c})$ and let $Z := \overline{[C]}$. By the proof of [13, Prop. 3] there exists a closed subspace V of Z^* norming on Z such that K is $\sigma(Z,V)$-compact but $\overline{\mathrm{co}}^{\sigma(Z,V)}(K)$ is not $\sigma(Z,V)$-compact. Let $i : Z \to V^*$ be the canonical embedding such that $i(z)(v) = \langle v, z\rangle$ for all $z \in Z$ and $v \in V$. Clearly, i is a norm-isomorphism between Z and $i(Z)$. Moreover, $i : (Z, \sigma(Z,V)) \to (i(Z), w^*)$ is also an isomorphism. Then $i(K)$ is a w^*-compact subset of V^* such that $i(K) \subset i(C)$. Since $\overline{\mathrm{co}}^{\sigma(Z,V)}(K)$ is not $\sigma(Z,V)$-compact in $(Z, \sigma(Z,V))$, necessarily $\overline{\mathrm{co}}^{w^*}(i(K)) \setminus i(Z) \neq \emptyset$ and so $\widehat{d}(\overline{\mathrm{co}}^{w^*}(i(K)), i(C)) > 0$. Thus $i(C)$ does not have control inside V^*, a contradiction to (iii). ∎

A result of Talagrand [20] allows us to prove the following corollary:

COROLLARY 3.8. *Let X be a Banach space and A a subset of X^* that contains no copy of the basis of $\ell_1(\mathfrak{c})$. Then*:

(1) *For every w^*-compact subset $K \subset \overline{[A]}$ we have $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$.*
(2) *Every convex subset $C \subset \overline{[A]}$ has 3-control inside X^*.*

Proof. First, observe that $\overline{[A]}$ contains no copy of the basis of $\ell_1(\mathfrak{c})$, because, if τ is a cardinal with cofinality $\mathrm{cf}(\tau) > \aleph_0$, then Talagrand proved in [20, Theorem 4] that A contains a copy of the basis of $\ell_1(\tau)$ if and only if $\overline{[A]}$ has a copy of $\ell_1(\tau)$. Now it is enough to apply Proposition 3.5 and the fact that $\mathrm{cf}(\mathfrak{c}) > \aleph_0$ (see [16, p. 78]). ∎

COROLLARY 3.9. *Let X be a Banach space and let W be a subset of X^* which is either weakly Lindelöf or is closed, convex and has the property (C) of Corson. Then*

(i) *Every convex subset C of $\overline{[W]}$ has 3-control inside X^*.*
(ii) *For every w^*-compact subset K of $\overline{[W]}$ we have $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$.*

Proof. In both cases W cannot contain a copy of the basis of $\ell_1(\mathfrak{c})$ and so (i) and (ii) follow from Corollary 3.8. Indeed, if W is weakly Lindelöf, then

W fails to contain a copy of the basis of $\ell_1(\mathfrak{c})$ because such a copy would be a w-closed but non-w-Lindelöf subset.

Suppose now that W is closed, convex and has the property (C) of Corson. Recall that a closed convex subset F of a Banach space has the *property* (C) *of Corson* if $\bigcap_{i\in I} C_i \neq \emptyset$ whenever $\{C_i : i \in I\}$ is a family of closed convex subsets of F with the countable intersection property, that is, $\bigcap_{i\in J} C_i \neq \emptyset$ for every countable subset $J \subset I$. If a closed convex subset F of a Banach space has the property (C) of Corson, then F cannot contain a copy of the basis of $\ell_1(\mathfrak{c})$. Indeed, suppose $\mathcal{F} := \{u_\sigma : \sigma < \mathfrak{c}\} \subset F$ is equivalent to the basis of $\ell_1(\mathfrak{c})$ and $C_\sigma := \overline{\mathrm{co}}(\mathcal{F} \setminus \{u_\sigma\})$. Clearly, the family $\{C_\sigma : \sigma < \mathfrak{c}\}$ has the countable intersection property but $\bigcap_{\sigma<\mathfrak{c}} C_\sigma = \emptyset$. ∎

REMARK 3.10. In [19, Problem 4.5] Talagrand asks, among other things, if $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$ whenever K is a w^*-compact w-Lindelöf subset of a dual Banach space X^*. Cascales, Namioka and Vera proved in [5, Corollary E] (see also [4, Theorem 4.5]) that every w^*-compact w-Lindelöf subset of a dual Banach space X^* is fragmented by the dual norm. So, applying [18, Theorem 2.3], they gave an affirmative answer to the question posed by Talagrand. Clearly, this result is a particular case of Proposition 3.5 because a w-Lindelöf subset cannot contain a copy of the basis of $\ell_1(\mathfrak{c})$, and so it does not have a w^*-$\mathbb{N}$-family.

4. Convex subsets of Banach spaces with w^*-angelic closed dual unit ball . If Y is a Banach space, the closed dual unit ball $B(Y^*)$ is said to be w^*-*angelic* if given a subset A of $B(Y^*)$ and $a \in \overline{A}^{w^*}$, there exists a sequence $\{a_n : n \geq 1\} \subset A$ such that $a_n \xrightarrow{w^*} a$. In this section we consider a particularly favorable case of the problem of the control of the distance $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C)$ by the distance $\widehat{d}(K, C)$, C being a convex subset of X^* and K a w^*-compact subset of X^*. This case appears when C is a convex subset of some subspace Y of X^* such that the closed dual unit ball $(B(Y^*), w^*)$ is angelic. We prove that in this case there is 1-control.

LEMMA 4.1. *Let K be a compact Hausdorff space with* $\mathrm{card}(K) \geq 2$, *μ a Radon measure on K and $f \in C(K)$ a continuous real function on K. Let $\mu = \mu^+ - \mu^-$ be the decomposition of μ into its positive and negative parts. Then there exist distinct points $p_1, p_2 \in K$ such that*

$$\|\mu^+\| f(p_1) - \|\mu^-\| f(p_2) \geq \mu(f).$$

Proof. Let p_1, p_2 be two distinct points of K such that

$$f(p_1) = \max\{f(p) : p \in K\} \quad \text{and} \quad f(p_2) = \min\{f(p) : p \in K\}.$$

With this choice the statement holds because

$$\mu^+(f) = \int_K f(k)\, d\mu^+(k) \leq \int_K f(p_1)\, d\mu^+(k) = \|\mu^+\| f(p_1),$$

$$\mu^-(f) = \int\limits_K f(k)\,d\mu^-(k) \ge \int\limits_K f(p_2)\,d\mu^-(k) = \|\mu^-\| f(p_2),$$

whence $\|\mu^+\| f(p_1) - \|\mu^-\| f(p_2) \ge \mu^+(f) - \mu^-(f) = \mu(f)$. ∎

If I is an infinite set, let $\mathbf{c}(I)$ denote the subspace of $\ell_\infty(I) = C(\beta I)$ consisting of those elements which are constant on $I^* = \beta I \setminus I$.

PROPOSITION 4.2. *Let I be an infinite set and C a convex subset of $\mathbf{c}(I)$. Then for every w^*-compact subset K of $\ell_\infty(I)$ $(= \ell_1(I)^*)$ we have*

$$\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) = \widehat{d}(K, C).$$

Proof. Let K be a w^*-compact subset of $\ell_\infty(I)$. Without loss of generality (after a homothety if necessary), we suppose that $K \subset B(\ell_\infty(I))$.

The trivial case. Assume that $K \subset \mathbf{c}(I)$. Observe that $\mathbf{c}(I)$ is Asplund (see [9, p. 6]) because it is isomorphic to $\mathbf{c}_0(I)$. So, $\mathbf{c}(I)$ fails to contain a copy of $\ell_1(\mathfrak{c})$. Thus, from Proposition 3.5 we get $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$ and so $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) = \widehat{d}(\overline{\mathrm{co}}(K), C) = \widehat{d}(K, C)$.

The nontrivial case. Suppose that $K \setminus \mathbf{c}(I) \neq \emptyset$. This implies that $\widehat{d}(K, C) > 0$. Assume that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) > \widehat{d}(K, C)$. Thus, for some $a, b > 0$ we have

$$\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) > b > a > \widehat{d}(K, C).$$

Therefore, there exist vectors $w_0 \in \overline{\mathrm{co}}^{w^*}(K) \setminus \overline{C}$ and $\varphi \in S(\ell_\infty^*(I))$ (see Lemma 2.1) such that $\inf \varphi(w_0 - C) > b$. Let $\varepsilon > 0$ be such that $a + \varepsilon < b$. By the Riesz representation theorem (see [17, p. 46]) the dual $\ell_\infty^*(I) = C(\beta I)^*$ can be identified with the space of Radon Borel measures $M_{\mathrm{R}}(\beta I)$ on βI. On the other hand, if $\mu \in M_{\mathrm{R}}(\beta I)$, we have the decomposition $\mu = \mu_1 + \mu_2$, where:

(i) $\mu_1 = (\mu_{1i})_{i \in I} \in \ell_1(I)$ with $\mu_{1i} = \mu(\{i\}),\ i \in I$, and $\mu_2 = \mu\restriction I^*$, that is, μ_2 is the restriction of μ to the compact space I^*.

(ii) $\|\mu\| = \|\mu_1\| + \|\mu_2\|$.

So, $\ell_\infty^*(I)$ can be identified with the ℓ_1-direct sum $\ell_1(I) \oplus_1 M_{\mathrm{R}}(I^*)$, where $M_{\mathrm{R}}(I^*)$ is the space of Radon Borel measures on I^*. Thus, we have the decomposition $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in \ell_1(I),\ \varphi_2 \in M_{\mathrm{R}}(I^*)$ and $1 = \|\varphi_1\| + \|\varphi_2\|$. Let $\varphi_2 = \varphi_2^+ - \varphi_2^-$ be the decomposition of φ_2 into its positive and negative parts, and put $\lambda_1 := \|\varphi_2^+\|$ and $\lambda_2 := \|\varphi_2^-\|$. Now we apply Lemma 4.1 to the compact space I^*, the Radon Borel measure φ_2 on I^* and the continuous function $\check{w}_0$, where $\check{w}_0$ is the Stone–Čech continuous extension of w_0 to βI. So, there exist distinct points $p_1, p_2 \in I^*$ such that

$$\lambda_1 \check{w}_0(p_1) - \lambda_2 \check{w}_0(p_2) \ge \varphi_2(\check{w}_0).$$

Since $\check{w}_0$ is continuous on βI, there exist two infinite disjoint subsets V_1, V_2 of I such that

(i) $p_i \in \overline{V_i}^{\beta I}$, $i = 1, 2$.
(ii) For every $v_i \in \overline{V_i}^{\beta I}$, $i = 1, 2$,
$$\lambda_1 \check{w}_0(v_1) - \lambda_2 \check{w}_0(v_2) > \varphi_2(\check{w}_0) - \varepsilon/2.$$

Since V_1, V_2 are infinite disjoint subsets of I, we can choose two sequences of pairwise distinct points $\{d_n : n \geq 1\} \subset V_1$ and $\{e_n : n \geq 1\} \subset V_2$. Obviously,
$$\lambda_1 \check{w}_0(d_n) - \lambda_2 \check{w}_0(e_m) > \varphi_2(\check{w}_0) - \varepsilon/2, \quad \forall m, n \in \mathbb{N}. \tag{4.1}$$
Let μ be a Radon Borel probability on K with $r(\mu) = w_0$. Define the linear mapping $T_n : \ell_\infty(I) \to \mathbb{R}$ by $T_n(f) = \varphi_1(f) + \lambda_1 f(d_n) - \lambda_2 f(e_n)$ for every $n \in \mathbb{N}$ and every $f \in \ell_\infty(I)$. Clearly, T_n is $\|\cdot\|$-continuous and w^*-continuous with $\|T_n\| \leq 1$. By (4.1) for every $n \geq 1$ we have
$$\begin{aligned}\varphi(w_0) - \varepsilon/2 &= \varphi_1(w_0) + \varphi_2(\check{w}_0) - \varepsilon/2 \\ &< \varphi_1(w_0) + \lambda_1 \check{w}_0(d_n) - \lambda_2 \check{w}_0(e_n) = T_n(w_0),\end{aligned}$$
whence
$$\varphi(w_0) - \varepsilon/2 < T_n(w_0) = T_n(r(\mu)) = \int_K T_n(f)\, d\mu(f).$$
Let $A_n := \{f \in K : T_n(f) > \varphi(w_0) - \varepsilon\}$ for all $n \geq 1$. Observe that A_n is a relatively w^*-open subset of K for all $n \geq 1$.

Claim 1. *$\mu(A_n) \geq \varepsilon/2$ for all $n \geq 1$.*

Indeed, for every $n \geq 1$ we have
$$\begin{aligned}\varphi(w_0) - \varepsilon/2 < T_n(w_0) = \int_K T_n(f)\, d\mu(f) &= \Big(\int_{A_n} + \int_{K \setminus A_n}\Big) T_n(f)\, d\mu(f) \\ &\leq \mu(A_n) + \varphi(w_0) - \varepsilon.\end{aligned}$$
Thus $\mu(A_n) \geq \varepsilon/2$ for all $n \geq 1$.

Let $B_n := \bigcup_{m \geq n} A_m$ for every $n \geq 1$. The sequence $\{B_n\}_{n \geq 1}$ is decreasing and satisfies $\mu(B_n) \geq \varepsilon/2$ for every $n \geq 1$. Hence $\mu(\bigcap_{n \geq 1} B_n) \geq \varepsilon/2$ and so $\bigcap_{n \geq 1} B_n \neq \emptyset$. Choose $g \in \bigcap_{n \geq 1} B_n$ and, inductively, the sequence $\{A_{n_i}\}_{i \geq 1}$, $n_i < n_{i+1}$, such that $g \in A_{n_i}$ for every $i \geq 1$. Then
$$\varphi_1(g) + \lambda_1 g(d_{n_i}) - \lambda_2 g(e_{n_i}) = T_{n_i}(g) > \varphi(w_0) - \varepsilon, \quad \forall i \geq 1.$$
By a compactness argument, we can choose two distinct points $q_1 \in \overline{\{d_{n_i} : i \geq 1\}}^{\beta I} \setminus I \subset \overline{V_1}^{\beta I}$ and $q_2 \in \overline{\{e_{n_i} : i \geq 1\}}^{\beta I} \setminus I \subset \overline{V_2}^{\beta I}$ such that
$$\varphi_1(g) + \lambda_1 \check{g}(q_1) - \lambda_2 \check{g}(q_2) \geq \varphi(w_0) - \varepsilon. \tag{4.2}$$

Let $\psi := \varphi_1 + (\lambda_1 \delta_{q_1} - \lambda_2 \delta_{q_2})$. Observe that ψ belongs to $S(\ell_\infty^*(I))$.

Claim 2. $\inf \psi(g - C) \geq b - \varepsilon$.

Indeed, if $c \in C$ then $c \in \mathbf{c}(I)$ and so $\check{c}$ is constant on I^*. Thus
$$\begin{aligned}\psi(c) &= \varphi_1(c) + (\lambda_1 \delta_{q_1} - \lambda_2 \delta_{q_2})(\check{c}) = \varphi_1(c) + (\lambda_1 - \lambda_2)\check{c}(q_1) \\ &= \varphi_1(c) + \varphi_2(\check{c}) = \varphi(c).\end{aligned}$$

So, taking into account (4.2) and the fact that $\inf \varphi(w_0 - C) > b$, for every $c \in C$ we have

$$\begin{aligned}\langle \psi, g - c\rangle &= \varphi_1(g) + (\lambda_1 \check{g}(q_1) - \lambda_2 \check{g}(q_2)) - \varphi(c)\\ &\geq \varphi(w_0) - \varepsilon - \varphi(c) = \langle \varphi, w_0 - c\rangle - \varepsilon > b - \varepsilon.\end{aligned}$$

Therefore, we get $d(g, C) \geq b - \varepsilon$. On the other hand, as $g \in K$, we have $d(g, C) < a$ by hypothesis. So $b - \varepsilon < a$, which contradicts the choice of ε and completes the proof. ■

PROPOSITION 4.3. *Let X be a Banach space and Y a closed subspace of X^* with w^*-angelic closed dual unit ball $(B(Y^*), w^*)$. If C is a convex subset of Y, then $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) = \widehat{d}(K, C)$ for every w^*-compact subset K of X^*. Moreover, $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$ for every w^*-compact subset K of Y.*

Proof. Let C be a convex subset of Y and suppose that there exist a w^*-compact subset $K \subset B(X^*)$ and $0 < a, b < 1$ such that

$$\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) > b > a > \widehat{d}(K, C).$$

Let $w_0 \in \overline{\mathrm{co}}^{w^*}(K)$ be such that $d(w_0, C) > b$. By Lemma 2.1 there exists $\varphi_0 \in S(X^{**})$ such that $\inf \varphi_0(w_0 - C) > b$. Let $\varepsilon > 0$ be such that $b + \varepsilon < \inf \varphi_0(w_0 - C)$ and define

$$\begin{aligned}U &:= \{\varphi \in B(X^{**}) : \langle \varphi, w_0\rangle \geq \langle \varphi_0, w_0\rangle - \varepsilon\},\\ V &:= \{x \in B(X) : \langle w_0, x\rangle \geq \langle \varphi_0, w_0\rangle - \varepsilon\}.\end{aligned}$$

Observe that $\varphi_0 \in U$ and, as $\langle \varphi_0, w_0\rangle - \varepsilon < 1$, also $U = \overline{V}^{w^*}$. If $i : Y \to X^*$ is the canonical inclusion, then $i^* : X^{**} \to Y^*$ satisfies $i^*(\varphi_0) \in i^*(U) = \overline{i^*(V)}^{w^*} \subset B(Y^*)$. Since $(B(Y^*), w^*)$ is angelic, there exists a sequence $\{x_n : n \geq 1\} \subset V$ such that $i^*(x_n) \xrightarrow{w^*} i^*(\varphi_0)$ in the w^*-topology $\sigma(Y^*, Y)$. Let $T : X^* \to \ell_\infty$ be the continuous linear mapping such that $T(u) = (u(x_n))_{n\geq 1}$ for all $u \in X^*$. Then:

(1) $\|T\| \leq 1$ and, moreover, T is w^*-w^*-continuous on bounded subsets of X^*.
(2) As $i^*(x_n) \xrightarrow{w^*} i^*(\varphi_0)$, for every $y \in Y$ we have $y(x_n) = i^*(x_n)(y) \to i^*(\varphi_0)(y)$. Hence $T(Y) \subset \mathbf{c}(\mathbb{N}) = \{f \in \ell_\infty : \check{f}\restriction\mathbb{N}^* \text{ is constant}\}$.

Let $\tilde{C} := T(C)$, $T(K) =: H \subset B(\ell_\infty)$ and $v_0 := T(w_0)$. Clearly, H is a w^*-compact subset of $B(\ell_\infty)$ such that $\widehat{d}(H, \tilde{C}) \leq \widehat{d}(K, C) < a$ because $\|T\| \leq 1$, and $v_0 \in \overline{\mathrm{co}}^{w^*}(H)$. Let $e_n : \ell_\infty \to \mathbb{R}$, $n \geq 1$, be the nth canonical projection. Then $\{e_n : n \geq 1\} \subset B(\ell_\infty^*)$ and $T^*(e_n) = x_n$, $n \geq 1$. Let η_0 be a w^*-cluster point of $\{e_n : n \geq 1\}$ in $B(\ell_\infty^*)$. Clearly, $\eta_0 \in S(\ell_\infty^*)$.

CLAIM. $\inf \eta_0(v_0 - \tilde{C}) \geq b$.

Indeed, first $T^*(\eta_0)$ is a w^*-cluster point of $\{x_n : n \geq 1\}$ in $B(X^{**})$. Thus

(i) $T^*(\eta_0) \in U$, whence $\langle T^*(\eta_0), w_0\rangle \geq \langle \varphi_0, w_0\rangle - \varepsilon$.

(ii) If $c \in C$, for every $n \geq 1$ we have $\langle T^*(e_n), c\rangle = \langle x_n, c\rangle \to \langle \varphi_0, c\rangle$, whence $\langle T^*(\eta_0), c\rangle = \langle \varphi_0, c\rangle$. Therefore, for every $c \in C$ we have

$$\begin{aligned}\langle \eta_0, v_0 - Tc\rangle &= \langle \eta_0, Tw_0 - Tc\rangle = \langle T^*(\eta_0), w_0 - c\rangle \\ &\geq \langle \varphi_0, w_0\rangle - \varepsilon - \langle \varphi_0, c\rangle \\ &= \langle \varphi_0, w_0 - c\rangle - \varepsilon > b + \varepsilon - \varepsilon = b,\end{aligned}$$

and this proves the Claim.

Therefore we get $d(v_0, \tilde{C}) \geq b$. On the other hand, from Proposition 4.2 we deduce that $d(v_0, \tilde{C}) < a$. So, we get a contradiction and this proves that $\widehat{d}(\overline{\mathrm{co}}^{w^*}(K), C) = \widehat{d}(K, C)$. Finally, if K is a w^*-compact subset of Y, taking $C = \overline{\mathrm{co}}(K)$ in the above argument, we deduce that $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$. ∎

Corollary 4.4. *Let X be a Banach space and W a w^*-closed w-Lindelöf subset of X^*. Then C has 1-control inside X^* whenever C is a convex subset of $\overline{[W]}$.*

Proof. Let $W_n := \{z \in W : \|z\| \leq n\}$ for all $n \geq 1$. Then W_n is a w^*-compact w-Lindelöf subset of X^* and $\overline{[W_n]}$ is WLD (weakly Lindelöf determined) by [4, Corollary 6.4]. Therefore $\overline{[W]}$ is WLD because $\overline{[W]} = \overline{\bigcup_{n\geq 1} \overline{[W_n]}}$ (for instance, apply [1, Theorem 1.6]) and so the closed unit ball of $\overline{[W]}^*$ is w^*-angelic. Now apply Proposition 4.3. ∎

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