STUDIA MATHEMATICA 182 (2) (2007)

Amenability and weak amenability of l^1-algebras of polynomial hypergroups

by

RUPERT LASSER (Neuherberg and Garching)

Abstract. We investigate amenability and weak amenability of the l^1-algebra of polynomial hypergroups. We derive conditions for (weak) amenability adapted to polynomial hypergroups and show that these conditions are often not satisfied. However, we prove amenability for the hypergroup induced by the Chebyshev polynomials of the first kind.

1. Introduction. The L^1-algebras of hypergroups show very distinctive properties compared with those of L^1-algebras of groups. We will investigate amenability and weak amenability of the l^1-algebra of polynomial hypergroups. We will show that the l^1-algebra is very seldom amenable or weakly amenable, whereas for every Abelian group the L^1-algebra is amenable. To have a reference we briefly recall the basic facts on polynomial hypergroups. For more details and the proofs we refer to [6, 7].

Let $(R_n)_{n\in\mathbb{N}_0}$ be a polynomial sequence defined by a recurrence relation

$$R_1(x)\,R_n(x) = a_nR_{n+1}(x) + b_nR_n(x) + cR_{n-1}(x) \tag{1}$$

for $n \in \mathbb{N}$ and $R_0(x) = 1$, $R_1(x) = (1/a_0)(x - b_0)$, where $a_n > 0$, $b_n \geq 0$ for all $n \in \mathbb{N}_0$, and $c_n > 0$ for $n \in \mathbb{N}$. We assume $a_n + b_n + c_n = 1$ for $n \in \mathbb{N}$ and $a_0 + b_0 = 1$. It follows from this assumption that $R_n(1) = 1$ for all $n \in \mathbb{N}_0$. By the theorem of Favard there is a (unique) probability measure π on $\mathbb{R}$ with bounded support such that $(R_n)_{n\in\mathbb{N}_0}$ is orthogonal with respect to π, i.e. $\int_{\mathbb{R}} R_n(x)R_m(x)\,d\pi(x) = (1/h(n))\delta_{n,m}$. The recurrence relation (1) is a special case of the linearization formula

$$R_m(x)R_n(x) = \sum_{k=|n-m|}^{n+m} g(m,n;k)R_k(x), \tag{2}$$

for $m, n \in \mathbb{N}_0$. We suppose throughout that the coefficients $g(m,n;k)$ are nonnegative. There are many orthogonal polynomial systems which have

2000 *Mathematics Subject Classification*: Primary 43A62, 46H20; Secondary 43A07, 46H25.

Key words and phrases: hypergroups, weak amenability, amenability.

this property (see [2, 6, 7]). We then define a convolution on $\mathbb{N}_0$ by

$$\omega(m,n) = \sum_{k=|n-m|}^{n+m} g(m,n;k)\delta_k, \tag{3}$$

where δ_k is the point measure at $k \in \mathbb{N}_0$. With this convolution, the involution $\tilde{n} = n$ and the discrete topology the set of natural numbers $\mathbb{N}_0$ is a commutative hypergroup, called the *polynomial hypergroup* induced by $(R_n)_{n\in\mathbb{N}_0}$ (see [6]). The basic notations and tools of commutative harmonic analysis are available. The Haar measure on the polynomial hypergroup $\mathbb{N}_0$ is the counting measure with weights $h(n) = g(n,n;0)^{-1}$ at $n \in \mathbb{N}_0$. They satisfy $h(0) = 1$, $h(n+1) = (a_n/c_{n+1})h(n)$, $n \in \mathbb{N}_0$. The translation of a sequence $\beta = (\beta(n))_{n\in\mathbb{N}_0}$ reads as

$$T_n\beta(m) = \sum_{k=|n-m|}^{n+m} g(m,n;k)\beta(k),$$

and the convolution of two sequences $f, g \in l^1(h)$ is given as

$$f * g = \sum_{k=0}^{\infty} T_n f(k) g(k) h(k)$$

($l^1(h) = \{f = (f(n))_{n\in\mathbb{N}_0} : \sum_{n=0}^{\infty} |f(n)|h(n) < \infty\}$). With this operation as multiplication, and $f^*(n) = \overline{f(n)}$ as involution, the Banach space $l^1(h)$ is a commutative Banach $*$-algebra with unit δ_0. The hermitian dual space $\widehat{\mathbb{N}}_0$ of $\mathbb{N}_0$ (i.e. the hermitian structure space of $l^1(h)$) can be identified with

$$\{x \in \mathbb{R} : |R_n(x)| \leq 1 \text{ for all } n \in \mathbb{N}_0\} \tag{4}$$

via the mapping $x \mapsto \alpha_x, \alpha_x(n) := R_n(x)$ (see [6]). Hence we consider $\widehat{\mathbb{N}}_0$ as a compact subset of $\mathbb{R}$ which contains $1 \in \mathbb{R}$ (since $R_n(1) = 1$). (We note that in general there exist homomorphisms on $l^1(h)$ which are not hermitian.) The support of the orthogonalization measure π is contained in $\widehat{\mathbb{N}}_0$. The Fourier transform of $f \in l^1(h)$ is defined by

$$\widehat{f}(x) = \sum_{k=0}^{\infty} f(k) R_k(x) h(k), \qquad x \in \widehat{\mathbb{N}}_0.$$

$\widehat{f}$ is a continuous bounded function on $\widehat{\mathbb{N}}_0$ and satisfies $\widehat{f * g} = \widehat{f}\,\widehat{g}$.

2. Weak amenability. Let $D : l^1(h) \to X$ be a continuous derivation, where X is a commutative Banach $l^1(h)$-bimodule (i.e. D is a continuous linear operator such that $D(a * b) = a \cdot D(b) + b \cdot D(a)$). We denote the module operation by $a \cdot x$ for $a \in l^1(h)$ and $x \in X$. Our main example will be $X = l^\infty$ (the space of all bounded sequences) with $a \cdot x = a * x$, the action of $l^1(h)$ on l^∞ by convolution.

Define $\varepsilon_n(m) = (1/h(n))\delta_{n,m}$. Obviously $\varepsilon_n \in l^1(h)$, $\|\varepsilon_n\|_1 = 1$ and $\varepsilon_1 * \varepsilon_n = a_n\varepsilon_{n+1} + b_n\varepsilon_n + c_n\varepsilon_{n-1}$. Define recursively the sequence $(\kappa_n)_{n\in\mathbb{N}_0}$ in $l^1(h)$ by $\kappa_0 = 0, \kappa_1 = \varepsilon_0$ and

$$\kappa_{n+1} = \frac{1}{a_n}(\varepsilon_n + \varepsilon_1 * \kappa_n - b_n\kappa_n - c_n\kappa_{n-1}). \tag{5}$$

PROPOSITION 1. *Let $D : l^1(h) \to X$ be a derivation as above. Then*

$$D(\varepsilon_n) = \kappa_n \cdot D(\varepsilon_1) \quad \textit{for } n \in \mathbb{N}_0. \tag{6}$$

Proof. For $n = 0$ and $n = 1$ the identity (6) is easily checked. Suppose (6) is valid for $k = n-1, n$. Then by the assumption

$$D(\varepsilon_1 * \varepsilon_n) = \varepsilon_1 \cdot D(\varepsilon_n) + \varepsilon_n \cdot D(\varepsilon_1) = (\varepsilon_1 * \kappa_n) \cdot D(\varepsilon_1) + \varepsilon_n \cdot D(\varepsilon_1)$$

and

$$D(\varepsilon_1 * \varepsilon_n) = c_n\kappa_{n-1} \cdot D(\varepsilon_1) + b_n\kappa_n \cdot D(\varepsilon_1) + a_nD(\varepsilon_{n+1}).$$

It follows that

$$D(\varepsilon_{n+1}) = \frac{1}{a_n}(\varepsilon_n + \varepsilon_1 * \kappa_n - b_n\kappa_n - c_n\kappa_{n-1}) \cdot D(\varepsilon_1) = \kappa_{n+1} \cdot D(\varepsilon_1). \blacksquare$$

We will derive conditions sufficient to decide whether $l^1(h)$ is weakly amenable or not. This means we have to determine when there exist no nonzero continuous derivations $D : l^1(h) \to l^\infty$ (see [1]).

We consider the Fourier transformation of κ_{n+1}. We have $\widehat{\varepsilon}_k(x) = R_k(x)$ and $\widehat{\kappa}_{n+1}(x) = a_0R'_{n+1}(x)$ for all $x \in \widehat{\mathbb{N}}_0$. The latter identity follows immediately by differentiating the three-term recurrence relation for $R_n(x)$ and comparing it with (5) (see also [8]). Since $R'_{n+1}(x)$ is a polynomial of degree n, we can write

$$a_0R'_{n+1}(x) = \sum_{k=0}^{n} d_{n,k}R_k(x). \tag{7}$$

Applying the uniqueness theorem for the Fourier transformation yields

$$\kappa_{n+1} = \sum_{k=0}^{n} d_{n,k}\varepsilon_k. \tag{8}$$

In particular, $\kappa_{n+1}(k) = d_{n,k}/h(k)$ for $k = 0, \ldots, n$.

THEOREM 1. *Assume $\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$ is bounded. Then $l^1(h)$ is not weakly amenable.*

Proof. Put $D(\varepsilon_1) = \varepsilon_0 \in l^\infty$. Then $\kappa_n * D(\varepsilon_1) = \kappa_n$, and by Proposition 1 we obtain $\|D(\varepsilon_n)\|_\infty = \|\kappa_n\|_\infty$. By the assumptions $\|D(\varepsilon_n)\|_\infty \le M$ for all $n \in \mathbb{N}_0$. The linear extension of equation (6) is a bounded map on the linear span of $\{\varepsilon_n : n \in \mathbb{N}_0\}$. This linear span is dense in $l^1(h)$, and hence D can be extended to a (nonzero) continuous derivation $D : l^1(h) \to l^\infty$. ∎

We apply Theorem 1 to the class of polynomial hypergroups induced by the ultraspherical polynomials $R_n^{(\alpha)}(x)$, $\alpha \geq -1/2$. From [9, (7.32.5)] we obtain

$$(R_n^{(\alpha)})'(x) = \frac{n(n+2\alpha+1)}{2+2\alpha}\, R_{n-1}^{(\alpha+1)}(x).$$

Hence we can calculate $\kappa_{n+1}(k)$ from the so-called connection coefficients which connect $R_n^{(\alpha+1)}(x)$ with $R_k^{(\alpha)}(x)$. We write the ultraspherical polynomials $R_n^{(\alpha)}(x)$ as Gegenbauer polynomials $C_n^{(\alpha+1/2)}(x)$. In fact, by (4.5.1) and (4.1.6) of [5] we have

$$R_n^{(\alpha)}(x) = \frac{n!}{(2\alpha+1)_n}\, C_n^{(\alpha+1/2)}(x).$$

Applying formula (9.1.2) of [5] we obtain, for $\alpha > -1/2$,

$$\begin{aligned} R_n^{(\alpha+1)}(x) &= \frac{n!}{(2\alpha+3)_n}\, C_n^{(\alpha+3/2)}(x) \\ &= \frac{n!}{(2\alpha+3)_n} \sum_{k=0}^{[n/2]} \frac{\alpha+1/2+n-2k}{\alpha+1/2}\, C_{n-2k}^{(\alpha+1/2)}(x) \\ &= \frac{n!}{(2\alpha+3)_n} \sum_{k=0}^{[n/2]} \frac{(\alpha+1/2+n-2k)(2\alpha+1)_{n-2k}}{(\alpha+1/2)(n-2k)!}\, R_{n-2k}^{(\alpha)}(x). \end{aligned}$$

Hence

$$(R_{n+1}^{(\alpha)})'(x) = \frac{(n+1)(n+2\alpha+2)n!}{(\alpha+1/2)(2\alpha+3)_n(2+2\alpha)} \cdot \sum_{k=0}^{[n/2]} \frac{(2\alpha+1)_{n-2k}(\alpha+1/2+n-2k)}{(n-2k)!}\, R_{n-2k}^{(\alpha)}(x) \tag{9}$$

The Haar weights are $h(n) = (2n+2\alpha+1)(2\alpha+1)_n/((2\alpha+1)n!)$.

It is now straightforward to determine $\kappa_{n+1}(k)$ for $k = 0, 1, \ldots, n$ from (7) and (9). For $n = 2m$ it follows that

$$\kappa_{2m+1}(k) = 0 \quad \text{if } k = 1, 3, \ldots, 2m-1$$

and

$$\kappa_{2m+1}(2j) = d_{2m,2j}\, \frac{1}{h(2j)} = \frac{(2m+1)(2m+2\alpha+2)(2m)!}{(2+2\alpha)(2\alpha+3)_{2m}} \tag{10}$$

for $j = 0, 1, 2, \ldots, m$.

For $n = 2m+1$ it follows that

$$\kappa_{2m+2}(k) = 0 \quad \text{if } k = 0, 2, 4, \ldots, 2m$$

and

$$\kappa_{2m+2}(2j+1) = \frac{(2m+2)(2m+2\alpha+3)(2m+1)!}{(2+2\alpha)(2\alpha+3)_{2m+1}} \tag{11}$$

for $j = 0, 1, \ldots, m$. It is easy to check that formulas (10) and (11) also hold for the limit $\alpha = -1/2$ in which case $\kappa_{2m+1}(2j) = 2m+1$ and $\kappa_{2m+2}(2j+1) = 2m+2$.

Corollary 1. *For the polynomial hypergroup induced by the ultraspherical polynomials $R_n^{(\alpha)}(x)$ the Banach algebra $l^1(h)$ is not weakly amenable whenever $\alpha \geq 0$.*

Proof. We know that every derivation $D : l^1(h) \to l^\infty$ satisfies $D(\varepsilon_n) = \kappa_n * D(\varepsilon_1)$. If we choose $D(\varepsilon_1) = \varepsilon_0$, we obtain $D(\varepsilon_n) = \kappa_n$. By (10) and (11) the asymptotic behaviour of the gamma function yields

$$\|\kappa_n\|_\infty = \frac{n(n+2\alpha+1)(n-1)!}{(2+2\alpha)(2\alpha+3)_{n-1}} = O(n^{-2\alpha}).$$

If $\alpha \geq 0$ then $\{\|\kappa_n\|_\infty : n \in \mathbb{N}\}$ is bounded, and hence $l^1(h)$ is not weakly amenable. ∎

Remark. Corollary 1 improves a result of [8], where it is shown that $l^1(h)$ is not weakly amenable if $\alpha \geq 1/2$, since in that case there exist nonzero bounded point derivations on $l^1(h)$.

Another consequence of Proposition 1 is the following result.

Theorem 2. *The following conditions are equivalent*:

(i) *$l^1(h)$ is weakly amenable.*
(ii) *For every $\varphi \in l^\infty$, $\varphi \neq 0$, the set $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is unbounded.*

Proof. By Proposition 1 we know that each derivation $D : l^1(h) \to l^\infty$ satisfies $D(\varepsilon_n) = \kappa_n * \varphi$, where $\varphi = D(\varepsilon_1)$. Moreover, if there is some $\varphi \in l^\infty, \varphi \neq 0$, such that $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}\}$ is bounded, then the continuous linear extension of $D(\varepsilon_n) = \kappa_n * \varphi$ defines a nonzero bounded derivation $D : l^1(h) \to l^\infty$. In particular, (i) and (ii) are equivalent. ∎

A sequence of elements $\sigma_n \in l^1(h)$ related to the κ_n is derived by the Christoffel–Darboux formula for $R_n(x)$. In fact, we have

$$\frac{1}{a_n h(n)} \sum_{k=0}^{n} R_k^2(x) h(k) = a_0 R'_{n+1}(x) R_n(x) - a_0 R'_n(x) R_{n+1}(x) \tag{12}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Define $\sigma_n \in l^1(h)$, $n \in \mathbb{N}$, by

$$\sigma_n = \kappa_{n+1} * \varepsilon_n - \kappa_n * \varepsilon_{n+1}. \tag{13}$$

By (12) it follows that

$$\sigma_n = \frac{1}{a_n h(n)} \sum_{k=0}^{n} \varepsilon_k * \varepsilon_k h(k). \tag{14}$$

Applying (14) we can easily show that $l^1(h)$ is weakly amenable if the polynomial hypergroup on $\mathbb{N}_0$ is induced by the Chebyshev polynomials $T_n(x)$ of the first kind. Note that $T_n(x) = \cos(nt)$ for $x = \cos t$, and moreover $T_n(x) = R_n^{(-1/2)}(x)$ belongs to the class of ultraspherical polynomials with $\alpha = -1/2$.

COROLLARY 2. *For the polynomial hypergroup induced by the Chebyshev polynomials $T_n(x)$, the Banach algebra $l^1(h)$ is weakly amenable.*

Proof. For the polynomial hypergroup induced by $T_n(x)$ we have

$$\varepsilon_m * \varepsilon_n = \frac{1}{2}\,\varepsilon_{|n-m|} + \frac{1}{2}\,\varepsilon_{n+m},$$

and $h(0) = 1$, $h(n) = 2$ for $n \in \mathbb{N}$. Hence

$$\sigma_n = \frac{1}{a_n h(n)} \sum_{k=0}^{n} \varepsilon_k * \varepsilon_k h(k) = n\varepsilon_0 + \sum_{k=0}^{n} \varepsilon_{2k}$$

and for $\varphi \in l^\infty$ it follows that

$$\sigma_n * \varphi = n\varphi + \sum_{k=0}^{n} T_{2k}\varphi.$$

Now suppose that $l^1(h)$ is not weakly amenable. By Theorem 2 there exists some $\varphi \in l^\infty$, $\varphi \neq 0$, such that $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}\}$ is bounded. From formula (13) it follows that $\{\|\sigma_n * \varphi\|_\infty : n \in \mathbb{N}\}$ is also bounded. Furthermore $\kappa_{2m+1} = (2m+1)\sum_{k=0}^{m} \varepsilon_{2k} h(2k)$ and hence

$$\kappa_{2m+1} * \varphi = (2m+1) \sum_{k=0}^{m} T_{2k}\varphi\, h(2k).$$

Since $\{|\kappa_{2m+1} * \varphi(0)| : m \in \mathbb{N}_0\}$ is bounded, it follows that $|\sum_{k=0}^{m} \varphi(2k)h(2k)| \to 0$ as $m \to \infty$.

Let $\sup\{\|\sigma_n * \varphi\|_\infty : n \in \mathbb{N}\} = M < \infty$. Then

$$\begin{aligned} M \geq |\sigma_n * \varphi(0)| &= \left| n\varphi(0) + \frac{1}{2}\,\varphi(0) + \frac{1}{2} \sum_{k=0}^{m} \varphi(2k)h(2k) \right| \\ &\geq \frac{2n+1}{2}\,|\varphi(0)| - \frac{1}{2}\left| \sum_{k=0}^{n} \varphi(2k)h(2k) \right|, \end{aligned}$$

which is only possible provided $\varphi(0) = 0$. Replacing φ above by $T_j\varphi$ it also follows that $\varphi(j) = T_j\varphi(0)$ has to be zero, which is a contradiction. ∎

3. Amenability. We follow the construction of [1] to prove that $l^1(h)$ is not amenable whenever $h(n) \to \infty$ as $n \to \infty$. We point out that the Banach algebra $l^1(h)$ is very different from the Beurling algebra $l^1(\omega)$ studied in [1]. In particular, the convolution in $l^1(h)$ is rather involved.

Given the polynomial hypergroup on $\mathbb{N}_0$ induced by $(R_n(x))_{n\in\mathbb{N}_0}$ consider the direct product hypergroup $\mathbb{N}_0 \times \mathbb{N}_0$ (see [2, 1.5.28]). The Haar weights on $\mathbb{N}_0 \times \mathbb{N}_0$ are given by $H(m,n) = h(m)h(n)$. The space of the hermitian characters of $l^1(\mathbb{N}_0\times\mathbb{N}_0, H)$ can be identified with the compact subset $\widehat{\mathbb{N}}_0\times\widehat{\mathbb{N}}_0$ of $\mathbb{R}^2$. The Fourier transform and the Fourier–Stieltjes transform of $F \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ and $\mu \in M(\mathbb{N}_0 \times \mathbb{N}_0)$ are given by

$$\widehat{F}(x,y) = \sum_{m,n=0}^{\infty} F(m,n)R_m(x)R_n(y)h(n)h(m)$$

and

$$\widehat{\mu}(x,y) = \sum_{m,n=0}^{\infty} R_m(x)R_n(y)\mu(m,n),$$

respectively, for $x,y \in \widehat{\mathbb{N}}_0$. Both $\widehat{F}$ and $\widehat{\mu}$ are continuous functions on $\widehat{\mathbb{N}}_0\times\widehat{\mathbb{N}}_0$. Obviously $(1,1) \in \widehat{\mathbb{N}}_0 \times \widehat{\mathbb{N}}_0$. We need the following auxiliary result.

LEMMA 1. *$x = 1$ is not isolated in $\widehat{\mathbb{N}}_0$.*

Proof. $\operatorname{supp}\pi$ is a subset of $\widehat{\mathbb{N}}_0$. Hence we have to consider two cases. If $1 \in \operatorname{supp}\pi$, then 1 is not an isolated point of $\operatorname{supp}\pi$ (see [7, Lemma (2.1)]), and hence 1 is not isolated in $\widehat{\mathbb{N}}_0$. If $1 \notin \operatorname{supp}\pi$ then 1 is not contained in the true interval I of orthogonality, which contains all the n simple zeroes of $R_n(x)$ (see [4]). Since $R_n(1) = 1$ the range of $R_n(x)$ for $x \in [\max\operatorname{supp}\pi, 1]$ is contained in $[0,1]$. Hence $[\max\operatorname{supp}\pi, 1] \subseteq \widehat{\mathbb{N}}_0$. ∎

For $f \in l^1(h)$ define elements Uf and Vf of $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ by

$$(15)\quad Uf(m,n) = \begin{cases} f(m) & \text{for } n = 0,\\ 0 & \text{for } n \neq 0,\end{cases} \qquad Vf(m,n) = \begin{cases} f(n) & \text{for } m = 0,\\ 0 & \text{for } m \neq 0.\end{cases}$$

The resulting mappings U and V are isometric isomorphisms from $l^1(h)$ into $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$. In fact, $T_{(m,n)}Uf(k,l) = 0$ if $l \neq n$ and $T_{(m,n)}Uf(k,l) = g(n,n;0)T_m f(k)$ if $l = n$. Hence

$$\begin{aligned} Uf * Ug(m,n) &= \sum_{k,l=0}^{\infty} Uf(k,l)T_{(m,n)}Ug(k,l)h(k)h(l) \\ &= \sum_{k=0}^{\infty} Uf(k,n)T_m g(k)h(k) = U(f*g)(m,n). \end{aligned}$$

Similarly one shows $Vf * Vg = V(f*g)$. For the Fourier transform we obtain $\widehat{Uf}(x,y) = \widehat{f}(x)$ and $\widehat{Vf}(x,y) = \widehat{f}(y)$ for all $x,y \in \widehat{\mathbb{N}}_0$.

THEOREM 3. *Suppose that $h(n) \to \infty$ as $n \to \infty$. Then $l^1(h)$ is not amenable.*

Proof. Let $Y := c_0(\mathbb{N}_0 \times \mathbb{N}_0) = \{y = y(m,n) : y(m,n) \to 0 \text{ as } (m,n) \to \infty \text{ in } \mathbb{N}_0 \times \mathbb{N}_0\}$. Then Y is a Banach space with respect to the sup-norm. For $f \in l^1(h)$ and $y \in Y$ set

$$\begin{aligned} f \cdot y = Uf * y &= \sum_{m,n=0}^{\infty} Uf(m,n) T_{(m,n)} y\, h(m)h(n) \\ &= \sum_{m=0}^{\infty} f(m) T_{(m,0)} y\, h(m) \end{aligned} \tag{16}$$

and

$$y \cdot f = Vf * y = \sum_{n=0}^{\infty} f(n) T_{(0,n)} y\, h(n). \tag{17}$$

The space Y is a Banach $l^1(h)$-bimodule with respect to these operations. The dual Y^* is the Banach space

$$Y^* = M(\mathbb{N}_0 \times \mathbb{N}_0) = \Big\{ \mu = \mu(m,n) : \|\mu\| = \sum_{m,n=0}^{\infty} |\mu(m,n)| < \infty \Big\}.$$

The space $M(\mathbb{N}_0 \times \mathbb{N}_0)$ can be identified with $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ via the mapping $\lambda \mapsto \lambda H$, $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H) \to M(\mathbb{N}_0 \times \mathbb{N}_0)$. The duality is given by

$$\langle y, \lambda \rangle = \sum_{m,n=0}^{\infty} y(m,n) \lambda(m,n) h(m) h(n).$$

The dual $l^1(h)$-bimodule operations on $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ are given by

$$f \cdot \lambda = Va * \lambda \quad \text{and} \quad \lambda \cdot f = Ua * \lambda \tag{18}$$

for $f \in l^1(h)$ and $\lambda \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$. Note that for $y \in Y$,

$$\langle y, \lambda \cdot f \rangle = \langle f \cdot y, \lambda \rangle = \langle Uf * y, \lambda \rangle = \langle y, Uf * \lambda \rangle$$

and

$$\langle y, f \cdot \lambda \rangle = \langle y \cdot f, \lambda \rangle = \langle Vf * y, \lambda \rangle = \langle y, Vf * \lambda \rangle.$$

Let $y_0(n,m) = 1/h(n)h(m)$. Then $y_0 \in Y$ since $h(n) \to \infty$ as $n \to \infty$.

Define

$$X = \Big\{ \lambda \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H) : \langle y_0, \lambda \rangle = \sum_{m,n=0}^{\infty} \lambda(m,n) = 0 \Big\}.$$

Then X is a weak $*$-closed subspace of $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$. Note that $\lambda \in X$ says that the Fourier–Stieltjes transform $\widehat{\lambda}(1,1)$ of λ (seen as an element of

$M(\mathbb{N}_0 \times \mathbb{N}_0))$ is zero at the point $(1,1)$. Moreover, X is an $l^1(h)$-submodule of $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$. In fact, if $\lambda \in X$ and $f \in l^1(h)$ then $f \cdot \lambda = Vf * \lambda$, $\lambda \cdot f = Uf * \lambda$ and

$$\widehat{f \cdot \lambda}(1,1) = \widehat{Vf}(1,1)\widehat{\lambda}(1,1) = 0, \qquad \widehat{\lambda \cdot f}(1,1) = \widehat{Uf}(1,1)\widehat{\lambda}(1,1) = 0.$$

By Proposition 1.3 of [1], X is a dual $l^1(h)$-module.

Define $D : l^1(h) \to X$ by $Df = Uf - Vf$. Obviously D maps into X, and D is linear and continuous. For $f, g \in l^1(h)$,

$$\begin{aligned} D(f * g) &= U(f * g) - V(f * g) = Uf * Ug - Vf * Vg \\ &= (Uf - Vf) * Ug + Vf * (Ug - Vg) = Ug * Df + Vf * Dg \\ &= f \cdot Dg + g \cdot Df, \end{aligned}$$

so that D is a derivation.

Now suppose that $l^1(h)$ is amenable. Then D is an inner derivation, so there is some $\lambda \in X$ such that $Df = f \cdot \lambda - \lambda \cdot f$ for every $f \in l^1(h)$. The Fourier transformation gives, for $(x, y) \in \widehat{\mathbb{N}}_0 \times \widehat{\mathbb{N}}_0$ and $f \in l^1(h)$,

$$\begin{aligned} \widehat{f}(x) - \widehat{f}(y) &= \widehat{Uf}(x,y) - \widehat{Vf}(x,y) = \widehat{Df}(x,y) \\ &= \widehat{Vf}(x,y)\widehat{\lambda}(x,y) - \widehat{Uf}(x,y)\widehat{\lambda}(x,y) = \widehat{\lambda}(x,y)(\widehat{f}(y) - \widehat{f}(x)). \end{aligned}$$

Given $x, y \in \widehat{\mathbb{N}}_0$, $x \neq y$, there exists $f \in l^1(h)$ such that $\widehat{f}(x) \neq \widehat{f}(y)$, and so $\widehat{\lambda}(x,y) = 1$. By Lemma 1, this is a contradiction to $\widehat{\lambda}(1,1) = 0$. Thus $l^1(h)$ is not amenable. ■

Our aim now is to construct an approximate diagonal in $l^1(h)$ (see [3]). So we want to construct a bounded sequence $(F_N(k,l))_{N \in \mathbb{N}}$ with $F_N \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ such that for $f \in l^1(h)$,

$$\lim_{N \to \infty} (f \cdot F_N - F_N \cdot f) = 0 \quad \text{and} \quad \lim_{N \to \infty} \pi(F_N) = \delta_0, \tag{19}$$

where

$$f \cdot F_N = Uf * F_N = \sum_{m=0}^{\infty} f(m) T_{(m,0)} F_N h(m),$$

$$F_N \cdot f = Vf * F_N = \sum_{n=0}^{\infty} f(n) T_{(0,n)} F_N h(n)$$

and

$$\pi(F_N)(k) = \sum_{n=0}^{\infty} T_{(k,0)} F_N(n,n) h(n).$$

If such a bounded sequence $(F_N)_{N \in \mathbb{N}}$ exists, then $l^1(h)$ is amenable (see [3]).

LEMMA 2. *Assume that for each $\varepsilon > 0$ there exists $G \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ such that*

$$\|T_{(1,0)}G - T_{(0,1)}G\|_1 < \varepsilon.$$

Then for each $n \in \mathbb{N}$ there exists a constant $\gamma = \gamma(n) > 0$ such that

$$\|T_{(k,0)}G - T_{(0,k)}G\|_1 < \gamma(n)\varepsilon \tag{20}$$

for $k = 0, \ldots, n$.

Proof. By the recursion formula for $R_n(x)$ we have, for $n \geq 1$,

$$T_{(n+1,0)} = \frac{1}{a_n} T_{(1,0)} \circ T_{(n,0)} - \frac{b_n}{a_n} T_{(n,0)} - \frac{c_n}{a_n} T_{(n-1,0)}$$

as well as

$$T_{(0,n+1)} = \frac{1}{a_n} T_{(0,1)} \circ T_{(0,n)} - \frac{b_n}{a_n} T_{(0,n)} - \frac{c_n}{a_n} T_{(0,n-1)}.$$

Now suppose that we have already found $\gamma(n)$ such that (20) holds true for $k = 0, \ldots, n$. (By assumption (20) is valid for $n = 1$.) Then

$$\begin{aligned}\|T_{(n+1,0)}G - T_{(0,n+1)}G\|_1 \leq{} & \frac{1}{a_n}\|T_{(1,0)}T_{(n,0)}G - T_{(0,1)}T_{(0,n)}G\|_1 \\ & + \frac{b_n}{a_n}\|T_{(n,0)}G - T_{(0,n)}G\|_1 + \frac{c_n}{a_n}\|T_{(n-1,0)}G - T_{(0,n-1)}G\|_1,\end{aligned}$$

and

$$\begin{aligned}\|T_{(1,0)}T_{(n,0)}G - T_{(0,1)}T_{(0,n)}G\|_1 &\leq \|T_{(1,0)}T_{(n,0)}G - T_{(1,0)}T_{(0,n)}G\|_1 \\ &\quad + \|T_{(0,n)}T_{(1,0)}G - T_{(0,n)}T_{(0,1)}G\|_1 \\ &\leq \|T_{(n,0)}G - T_{(0,n)}G\|_1 + \|T_{(1,0)}G - T_{(0,1)}G\|_1.\end{aligned}$$

Hence

$$\|T_{(n+1,0)}G - T_{(0,n+1)}G\|_1 \leq \frac{\gamma(n)}{a_n}\,(2\varepsilon + b_n\varepsilon + c_n\varepsilon),$$

and so we can choose $\gamma(n+1) = (\gamma(n)/a_n)(2 + b_n + c_n)$. ∎

LEMMA 3. *Assume there is some $M > 0$ such that for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there is some $G = G(\varepsilon, n) \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ with $\|G\|_1 \leq M$ such that*

$$\|T_{(k,0)}G - T_{(0,k)}G\|_1 < \varepsilon \quad \text{for } k = 0, \ldots, n.$$

Then for each $\varepsilon > 0$ and $f \in l^1(h)$ there is some $F = F(\varepsilon, f) \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ with $\|F\| \leq M$ such that

$$\|f \cdot F - F \cdot f\|_1 < \varepsilon.$$

Proof. Given $f \in l^1(h)$ choose $n \in \mathbb{N}$ such that $\sum_{m=n+1}^{\infty} |f(m)|h(m) < \varepsilon/4M$. Put $\tilde{f}(m) = f(m)$ for $m = 0, \ldots, n$ and $\tilde{f}(m) = 0$ for $m \geq n+1$. For each of the functions G we obtain

$$\|f \cdot G - \tilde{f} \cdot G\|_1 \leq \|f - \tilde{f}\|_1 \|G\|_1 \leq \frac{\varepsilon}{4}, \qquad \|G \cdot f - G \cdot \tilde{f}\|_1 \leq \frac{\varepsilon}{4}.$$

Let $C := \sup_{0 \leq m \leq n} |f(m)|h(m)$ and put $F = G(\varepsilon/2C, n)$. Then

$$\|f \cdot F - F \cdot f\|_1 \leq \|\tilde{f} \cdot F - F \cdot \tilde{f}\|_1 + \frac{\varepsilon}{2} \leq C\|T_{(k,0)}F - T_{(0,k)}F\|_1 + \frac{\varepsilon}{2} < \varepsilon. \ ∎$$

By Lemmas 2 and 3 we have the following sufficient condition for the amenability of $l^1(h)$.

Theorem 4. *If there is a bounded sequence $(F_N)_{N\in\mathbb{N}}$ in $l^1(\mathbb{N}_0\times\mathbb{N}_0, H)$ such that*

(i) *for each $\varepsilon > 0$ there exists $N\in\mathbb{N}$ such that*

$$\|T_{(1,0)}F_N - T_{(0,1)}F_N\|_1 < \varepsilon,$$

(ii) $\lim_{N\to\infty}\sum_{n=0}^{\infty} T_{(m,0)}F_N(n,n)h(n) = \delta_{0,m}$ *for each $m\in\mathbb{N}_0$,*

then $l^1(h)$ is amenable.

We apply Theorem 4 to show that $l^1(h)$ is amenable if the polynomial hypergroup $\mathbb{N}_0$ is induced by the Chebyshev polynomials $T_n(x)$ of the first kind.

Corollary 3. *For the polynomial hypergroup induced by the Chebyshev polynomials $T_n(x)$, the Banach algebra $l^1(h)$ is amenable.*

Proof. We have to construct a bounded sequence $(F_N)_{N\in\mathbb{N}}$ with the properties (i) and (ii) in Theorem 4. Define $F_N\in l^1(\mathbb{N}_0\times\mathbb{N}_0, H)$ by

$$F_N(k,l) = \begin{cases} \dfrac{1}{2N+1} & \text{if } k = l \le N,\\ \dfrac{-1}{(2N+1)(2N-2)} & \text{if } k\ne l;\ k+l \text{ even};\ k,l\le N,\\ 0 & \text{otherwise.}\end{cases}$$

We begin by calculating an upper bound for all F_N, $N\in\mathbb{N}$. Obviously summing along columns we get

$$\sum_{l=0}^{N} |F_N(k,l)| \le \frac{2}{2N+1} \qquad \text{for } 0\le k\le N.$$

Hence

$$\|F_N\|_1 \le \sum_{k,l=0}^{N} |F_N(k,l)|h(k)h(l) \le 4\sum_{k,l=0}^{N}|F_N(k,l)| \le \frac{8(N+1)}{2N+1} \le 6.$$

To check condition (i) we state that

$$T_{(1,0)}F_N(k,l) = T_{(0,1)}F_N(k,l) \qquad \text{if } 1\le k,l\le N-1.$$

In fact, comparing the direct neighbours of entry (k,l) in its column and row we see immediately that

$$\begin{aligned} T_{(1,0)}F_N(k,l) &= \frac{1}{2}\,F_N(k-1,l) + \frac{1}{2}\,F_N(k+1,l)\\ &= \frac{1}{2}\,F_N(k,l-1) + \frac{1}{2}\,F_N(k,l+1) = T_{(0,1)}F_N(k,l)\end{aligned}$$

whenever $1 \le k, l \le N-1$. For row $l = 0$ we obtain, for $0 \le k \le N-1$, $k \ne 1$,

$$T_{(1,0)}F_N(k,0) - T_{(0,1)}F_N(k,0) = 0,$$

and

$$\begin{aligned} T_{(1,0)}F_N(1,0) - T_{(0,1)}F_N(1,0) &= \frac{1}{2}\,F_N(0,0) + \frac{1}{2}\,F_N(2,0) - F_N(1,1) \\ &= \frac{-N}{(2N+1)(2N-2)}. \end{aligned}$$

For $N \ge 2$ we consider column $k = N$. Once again, comparing the neighbours along the column and row we get

$$T_{(1,0)}F_N(N,N) - T_{(0,1)}F_N(N,N) = \frac{1}{2}\,F_N(N-1,N) - \frac{1}{2}\,F_N(N,N-1) = 0$$

and

$$\begin{aligned} &T_{(1,0)}F_N(N,N-1) - T_{(0,1)}F_N(N,N-1) \\ &\qquad = \frac{1}{2}\,F_N(N-1,N-1) - \frac{1}{2}\,F_N(N,N-2) - \frac{1}{2}\,F_N(N,N) \\ &\qquad = \frac{1}{2}\,\frac{1}{(2N+1)(2N-2)}. \end{aligned}$$

For $l = 0, \dots, N-2$ we have

$$\begin{aligned} &T_{(1,0)}F_N(N,l) - T_{(0,1)}F_N(N,l) \\ &\quad = \frac{1}{2}\,F_N(N-1,l) - \frac{1}{2}\,F_N(N,l-1) - \frac{1}{2}\,F_N(N,l+1) \\ &\quad = \omega_{N,l}\,\frac{1}{2}\,\frac{1}{(2N+1)(2N-2)}, \quad \text{where} \quad \omega_{N,l} = \frac{1}{2}(1 + (-1)^{N+l+1}). \end{aligned}$$

Finally, for column $k = N+1$ it follows that

$$T_{(1,0)}F_N(N+1,N) - T_{(0,1)}F_N(N+1,N) = \frac{1}{2}\,F_N(N,N) = \frac{1}{2}\,\frac{1}{2N+1},$$

$$T_{(1,0)}F_N(N+1,N-1) - T_{(0,1)}F_N(N+1,N-1) = \frac{1}{2}\,F_N(N,N-1) = 0.$$

If $l = 0, \dots, N-2$, then

$$\begin{aligned} T_{(1,0)}F_N(N+1,l) - T_{(0,1)}F_N(N+1,l) &= \frac{1}{2}\,F_N(N+1,l) \\ &= \omega_{N,l}\,\frac{1}{2}\,\frac{-1}{(2N+1)(2N-2)}. \end{aligned}$$

Furthermore, we note the following symmetry. From $F_N(l,k) = F_N(k,l)$ it follows that

$$\begin{aligned} T_{(1,0)}F_N(k,l) - T_{(0,1)}F_N(k,l) &= T_{(0,1)}F_N(l,k) - T_{(1,0)}F_N(l,k) \\ &= -[T_{(1,0)}F_N(l,k) - T_{(0,1)}F_N(l,k)]. \end{aligned}$$

Summing up,

$$\begin{aligned}\|T_{(1,0)}F_N - T_{(0,1)}F_N\|_1 &\le 8 \sum_{0\le l\le k\le N+1} |T_{(1,0)}F_N(k,l) - T_{(0,1)}F_N(k,l)| \\ &\le 8\Big(\frac{N}{(2N+1)(2N-2)} + \frac{1}{2}\,\frac{N}{(2N+1)(2N-2)} + \frac{1}{2}\,\frac{N}{(2N+1)(2N-2)}\Big) \\ &= \frac{16N}{(2N+1)(2N-2)}.\end{aligned}$$

Selecting N large enough we see that

$$\|T_{(1,0)}F_N - T_{(0,1)}F_N\|_1 < \varepsilon.$$

It remains to verify (ii). We have

$$\pi(F_N)(0) = \sum_{n=0}^{N} F_N(n,n)h(n) = 1$$

and $\pi(F_N)(2j-1) = 0$ for all $j \in \mathbb{N}$. Furthermore,

$$\begin{aligned}\pi(F_N)(2) &= F_N(2,0) + (F_N(1,1) + F_N(3,1)) \\ &\quad + \sum_{k=2}^{N-2} (F_N(k-2,k) + F_N(k+2,k)) \\ &\quad + F_N(N-3,N-1) + F_N(N-2,N) \\ &= \frac{1}{2N+1} - \sum_{k=1}^{N-1} \frac{2}{(2N+1)(2N-2)} = 0.\end{aligned}$$

For $m = 2j$, we assume that $N > m$. A straightforward counting of the relevant entries in the $F_N(k,l)$ matrix shows that

$$\pi(F_N)(2j) = F_N(j,j) - 2(N-j)\,\frac{1}{(2N+1)(2N-2)} = \frac{1}{2N+1}\cdot\frac{j-1}{N-1}.$$

Hence $\lim_{N\to\infty} \pi(F_N)(2j) = 0$ for every $j \in \mathbb{N}$, and so $l^1(h)$ is amenable. ■

REMARK.

(i) That $l^1(h)$ is amenable for polynomial hypergroups induced by the Chebyshev polynomials is already contained in the thesis of S. Wolfenstetter (1984) at the Technische Universität München [10]. However, this result was never published. Our construction of the $(F_N)_{N\in\mathbb{N}_0}$ differs in some points from that of [10].

(ii) Obviously amenability implies weak amenability, and hence Corollary 3 implies Corollary 2. However, the proof of Corollary 2 does not use the tensor product of $l^1(h)$. Only $\kappa_n, \sigma_n \in l^1(h)$ are applied, which might be useful for other polynomial hypergroups.

References

[1] W. G. Bade, P. C. Curtis Jr. and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. 55 (1987), 359–377.

[2] W. R. Bloom and H. Heyer, *Harmonic Analysis and Probability Measures on Hypergroups*, de Gruyter, Berlin, 1995.

[3] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973.

[4] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[5] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge Univ. Press, Cambridge, 2005.

[6] R. Lasser, *Orthogonal polynomials and hypergroups*, Rend. Mat. 3 (1983), 185–209.

[7] —, *Orthogonal polynomials and hypergroups II—The symmetric case*, Trans. Amer. Math. Soc. 341 (1994), 749–770.

[8] —, *Point derivations on the l^1-algebra of polynomial hypergroups*, preprint.

[9] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., Providence, RI, 1959.

[10] S. Wolfenstetter, *Jacobi-Polynome und Bessel-Funktionen unter dem Gesichtspunkt der harmonischen Analyse*, PhD thesis, Technische Univ. München, 1984.

GSF-National Research Center
for Environment and Health
Institute of Biomathematics and Biometry
Ingolstädter Landstraße 1
85764 Neuherberg, Germany
E-mail: lasser@gsf.de

Centre of Mathematics
Munich University of Technology
85748 Garching, Germany

Received February 8, 2007
Revised version June 23, 2007 (6102)