STUDIA MATHEMATICA 203 (3) (2011)

On the structure of non-dentable subsets of $C(\omega^{\omega^k})$

by

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Dedicated to the memory of J. J. Uhl, Jr.

Abstract. It is shown that there is no closed convex bounded non-dentable subset K of $C(\omega^{\omega^k})$ such that on subsets of K the PCP and the RNP are equivalent properties. Then applying the Schachermayer–Rosenthal theorem, we conclude that every non-dentable K contains a non-dentable subset L so that on L the weak topology coincides with the norm topology. It follows from known results that the RNP and the KMP are equivalent on subsets of $C(\omega^{\omega^k})$.

Introduction. The study of subsets of Banach spaces with the RNP flourished in the 70's and 80's. For the history of the subject and results until 1977 one can see [14]. For more recent results (until 1983), in the form of a book, see [13]. Important results can be found in [9]. See also [5] for a concise exposition of RNP.

Numerous mathematicians worked on the Radon–Nikodym property, including: R. Phelps, R. C. James, J. Diestel, J. J. Uhl, Jr., M. Talagrand, C. Stegall, J. Bourgain, H. Rosenthal, W. Schachermayer, N. Ghoussoub, B. Maurey, G. Godefroy, S. Argyros.

Important papers in the field are: [7], [9], [10], [24], [15], [22], [19].

The cornerstone of our considerations in this paper is the paper [2] which can be considered as a localization of the results in [7] and a unification of Bourgain's and Schachermayer's theorems ([7], [22]).

Also [4] played an important role in the constructions of bushes in some of our theorems.

According to [19] the study of the structure of non-dentable sets of a Banach space is central in the geometry of Banach spaces.

2010 *Mathematics Subject Classification*: 46B20, 46B22.

Key words and phrases: Radon–Nikodym and Krein–Milman properties, operators on L^1, bush, $C(\alpha)$ space with α a countable ordinal.

This is part of the first author's Ph.D. thesis, which is in preparation at the Technical University of Crete under the supervision of the second author.

DOI: 10.4064/sm203-3-1

The Diestel conjecture has remained open since 1973 both globally and locally: Is the KMP equivalent to the RNP?

The most significant results related to this problem are the following:

THEOREM (Schachermayer, [22, Th. 2.1]). *If a convex, bounded, closed subset $D \subset X$ is strongly regular and fails to be an RN-set, then there is a closed, bounded, convex and separable subset C of D which does not have an extreme point.*

THEOREM (Rosenthal, [19, Th. 2]). *Let K be a closed bounded non-empty convex subset of X such that K is non-dentable and has SCS. Then there exists a closed convex non-empty subset W of K such that*

$(*)$ *W is non-dentable and the weak and norm topologies on W coincide.*

Moreover there exists a subspace Y of X such that Y has an FDD and a closed bounded convex non-empty subset W satisfying $(*)$.

It is known that for certain classes of spaces (or sets) the RNP is equivalent to the KMP: dual spaces (Huff–Morris [16], based on the work of Stegall [24]), subsets of the positive cone of L^1 (Argyros–Deliyanni [2]), spaces which can be embedded in a space with an unconditional FDD (James [17]), spaces with $X \equiv X \oplus X$ (Schachermayer [23]), Banach lattices (Bourgain–Talagrand [12]).

It is shown in [2] that in many of the above cases any convex, closed, bounded non-dentable set contains a subset with the $\mathcal{P}al$ representation.

We believe that a positive answer to the problem of equivalence of the RNP and KMP on closed convex bounded (c.c.b.) subsets of $C(a)$, where a is a countable ordinal, and a similar positive answer on c.c.b. subsets of L^1, are a strong indication that the RNP and KMP are equivalent on c.c.b. subsets of a general Banach space X.

In this paper we show that the RNP and KMP are equivalent on closed convex bounded subsets of $C(a)$ for ordinals $a < \omega^{\omega^\omega}$.

The main results in our paper are:

THEOREM 3.2. *Let X be a separable Banach space that contains no copy of $l_1(\mathbb{N})$, and $Q_n : X \to C(\omega^{\omega^k})$, $n \in \mathbb{N}$, be bounded linear operators. Suppose K is a closed, convex, bounded, non-PCP subset of X such that the PCP and RNP are equivalent on subsets of K. Then there exists a closed, convex, bounded, non-dentable subset L of K such that on $Q_n(L)$ the norm and weak topologies coincide for all $n \in \mathbb{N}$.*

THEOREM 3.3. *Let K be a closed, convex, bounded, non-dentable subset of $C(\omega^{\omega^k})$. Then there exists a convex closed subset L of K such that L has the PCP and fails the RNP. Therefore the KMP and RNP are equivalent on subsets of $C(\omega^{\omega^k})$.*

The set L mentioned in Theorem 3.2 is constructed as the closed convex hull of a δ-approximate bush which has a Convex Finite-Dimensional Schauder Decomposition (CFDSD) ([7], [2], [21]), and is given by the closed convex hull of the average back bush of a δ-approximate bush.

The result of Theorem 3.3 is best possible for spaces $C(\omega^{\omega^a})$, for a ordinal, since E. Odell [18] has proved, in unpublished work, that $C(\omega^{\omega^\omega})$ contains a convex, closed, bounded non-dentable subset L on which the PCP is equivalent to the RNP.

Preliminaries

RNP and related properties. Let K a closed, convex, bounded subset of a Banach space X. The set K has the *Radon–Nikodym property* (RNP) if for every probability space $(\Omega, \mathcal{B}, \mu)$ and every X-valued measure m on $\mathcal{B}$ which is absolutely continuous with respect to μ and whose average range is contained in K, there exists an $f \in L^1_X(\Omega, \mathcal{B}, \mu)$ such that $m(A) = \int_A f\,d\mu$ (Bochner integral) for each $A \in \mathcal{B}$. The set K has the *Krein–Milman Property* (KMP) if each closed, convex, bounded subset of K is the closed convex hull of its extreme points.

The *slice* $S(f,a,K)$ of K determined by $f \in X^*$ and $a > 0$ is the set $S(f,a,K) = \{x \in K : f(x) \geq \sup f(K) - a\}$.

The set K is said to be *strongly regular* if for every non-empty subset L of K and any $\varepsilon > 0$, there exist positive scalars $a_1, \ldots, a_n$ with $\sum_{i=1}^n a_i = 1$ and slices $S_1, \ldots, S_n$ of L such that the diameter of $\sum_{i=1}^n a_i S_i$ is less than ε. The set K has the *Point of Continuity Property* (PCP) if for every weakly closed non-empty subset L of K the identity map $i : (L, w) \to (L, \|\cdot\|)$ has a point of continuity. The set K has the *Convex PCP* (CPCP) if for every closed convex non-empty subset L of K the identity map $i : (L, w) \to (L, \|\cdot\|)$ has a point of continuity ([15]).

If K is non-PCP then there are $L \subseteq K$ and $\delta > 0$ such that L is δ-non-PCP (i.e. for every weak open subset W of L we have $\operatorname{diam} W > \delta$ [7]). Of course if K is δ-non-PCP, then K is non-PCP.

It is well known that if K has the PCP then K is strongly regular ([9]).

Operators on L^1 and RNP. Let

$$\mathcal{P}(\mu) = \left\{ f \in L^1(\mu) : f \geq 0 \text{ and } \int f\,d\mu = 1 \right\}$$

be the probability densities in $L^1(\mu)$.

It is well known that K has the RNP if and only if every bounded linear operator $T : L^1(\mu) \to X$ such that $Tf \in K$ for every $f \in \mathcal{P}(\mu)$ is representable. An operator $T : L^1(\mu) \to X$ is said to be *representable* if there is a function $g \in L^\infty_X(\mu)$ such that $Tf = \int fg\,d\mu$ (Bochner integral) for every $f \in L^1(\mu)$ ([14]).

The set K is strongly regular if and only if every bounded linear operator $T : L^1(\mu) \to X$ with $T(\mathcal{P}) \subseteq K$ is *strongly regular* (which means that if a net $(f_i)_{i\in I} \subseteq \mathcal{P}$ converges weakly to $f \in \mathcal{P}$ then $Tf_i \xrightarrow{\|\cdot\|} Tf$ [15]).

A bounded linear operator T from L^1 to a Banach space X is said to be a *Dunford–Pettis operator* if T maps every weakly compact subset of L^1 into a norm compact subset of X ([14]).

Indices, trees and bushes. In the notation we follow [2]. The set of all finite sequences of natural numbers of the form $a = (0, a_1, \ldots, a_n)$ is denoted by $\mathbb{N}^{(\mathbb{N})}$. Using the notion of length ($|0| = 0$, $|(0, a_1, \ldots, a_n)| = n$) and the notion of restriction ($a/n = |(0, a_1, \ldots, a_n)|$ if $|a| \geq n$) we can define a partial order in $\mathbb{N}^{(\mathbb{N})}$ by $a \leq \beta$ if and only if $|a| \leq |\beta|$ and $\beta/|a| = a$, when $a, \beta \in \mathbb{N}^{(\mathbb{N})}$. We also make use of the lexicographic total order of $\mathbb{N}^{(\mathbb{N})}$ and denote it by $<_{\mathrm{lex}}$. A subset $\mathcal{A}$ of $\mathbb{N}^{(\mathbb{N})}$ is called a *finitely branching tree* if the set $\{a \in \mathcal{A} : |a| = n\}$ is finite for every $n \in \mathbb{N}$. The set of immediate successors of $a \in \mathcal{A}$ is denoted by $S_a = \{\beta : a < \beta,\ |\beta| = |a| + 1\}$ and is finite when $\mathcal{A}$ is a finitely branching tree.

Let $(\varepsilon_n)_n \subset (0, 1)$ be such that $\sum_{n=0}^{\infty} \varepsilon_n < \delta/4$. A bounded subset $(x_a)_{a\in\mathcal{A}}$ of a Banach space X is called a *δ-approximate bush* with $\delta > 0$ if $\mathcal{A}$ is a finitely branching tree, for every $a, \beta \in \mathcal{A}$ with $\beta \in S_a$ we have $\|x_a - x_\beta\| > \delta$, and there exists $\{\lambda_\beta : \beta \in S_a\}$ with $\lambda_\beta \geq 0$, $\sum_{\beta\in S_a} \lambda_\beta = 1$ and $\|x_a - \sum_{\beta\in S_a} \lambda_\beta x_\beta\| < \varepsilon_{|a|}$. The vectors $y_\beta = x_\beta - x_a$, where $\beta \in S_a$, are called the *nodes* of the approximate bush.

We have the identity $\sum_{|\beta|=m} \lambda_\beta x_\beta = \sum_{n=0}^{m} \sum_{|a|=n} \mu_a y_a$ where $\mu_a = \lambda_a$ for $|a| = m$, and $\mu_a = \sum_{\beta\in S_a} \mu_\beta$ if $m > |a|$.

We can then define the notion of the *average back bush* $(\tilde{x}_a)_{a\in\mathcal{A}}$ corresponding to the approximate bush. Set $x_a^m = \sum_{|\beta|=m} \lambda_\beta^{(a)} x_\beta$ for $a \in \mathcal{A}$ and $m > |a|$, a convex combination, where the numbers $\lambda_\beta^{(a)}$ are defined inductively. If $m = |a|+1$ then $x_a^m = \sum_{\beta\in S_a} \lambda_\beta x_\beta$ with the numbers λ_β those in the definition of the δ-approximate bush, and if the numbers $\lambda_\beta^{(a)}$ with $|\beta| = n$ are defined for some n, then set $\lambda_\gamma^{(a)} = \lambda_\beta^{(a)} \lambda_\gamma$ for $|\gamma| = n + 1$ with the numbers λ_γ those in the definition of the δ-approximate bush. It can be shown that the sequence $\{x_a^m\}_{m>|a|}$ is norm Cauchy. Define $\tilde{x}_a = \lim_{m\to\infty} x_a^m$; then for $a \in \mathcal{A}$, $\beta \in S_a$ we have $\|\tilde{x}_a - \tilde{x}_\beta\| > \delta/2$, $\tilde{x}_a = \sum_{\beta\in S_a} \lambda_\beta \tilde{x}_\beta$ and every $\tilde{x}_a$ belongs to $\overline{\mathrm{co}}\,(x_a)_{a\in\mathcal{A}}$.

Let $(y_a)_{a\in\mathcal{A}}$ and $(\tilde{y}_a)_{a\in\mathcal{A}}$ be the nodes of the δ-approximate bush $(x_a)_{a\in\mathcal{A}}$ and of the corresponding regular bush $(\tilde{x}_a)_{a\in\mathcal{A}}$ respectively, when the family $(\mu_a)_{a\in\mathcal{A}}$ of real numbers is a normalized conditionally determined family (which means that $\mu_0 = 1$, $\mu_a \geq 0$, and $\mu_a = \sum_{\beta\in S_a} \mu_\beta$, [21]). Then $\sum_{n=0}^{\infty} \sum_{|a|=n} \mu_a y_a = \sum_{n=0}^{\infty} \sum_{|a|=n} \mu_a \tilde{y}_a$ whenever either series converges.

The spaces $C(\omega^{\omega^k})$. Let ω be the first infinite ordinal number corresponding to $\mathbb{N}$, and let $k \in \mathbb{N}^*$.

Then $\omega^{\omega^k} = \sum_{n=0}^{\infty} \omega^{\omega^{k-1}\cdot n}$ and if $C(K)$ is the space of continuous real functions defined on the set K, we have

$$C(\omega^{\omega^k}) = \Big(\sum_{n=0}^{\infty} \oplus C(\omega^{\omega^{k-1}\cdot n})\Big)_0.$$

This can be proved by using the result due to Bessaga and Pełczyński [6].

THEOREM (Bessaga–Pełczyński). *If* $a < \beta$ *are countable ordinals, then* $C(a)$ *and* $C(\beta)$ *are isomorphic Banach spaces if and only if* $\beta < a^{\omega}$.

Moreover, $C(\omega)$ is isomorphic to c_0, and $C(\omega^{\omega^k})$ is isomorphic to $C(\omega^{\omega^k\cdot n})$ for $n \in \mathbb{N}$.

Finally it is known that $C(\omega^{\omega})$, and hence $C(a)$ with $a > \omega^{\omega}$, cannot be embedded in a Banach space with an unconditional basis (in fact $C(\omega^{\omega})$ cannot be embedded in a Banach space with an unconditional FDD) (Pełczyński's thesis). See also [1, Theorem 4.5.2]. Of course, that means that no $C(\omega^{\omega^k})$ can be embedded in a Banach space with an unconditional basis since $C(\omega^{\omega})$ is a subspace of $C(\omega^{\omega^k})$ for every $k \in \mathbb{N}^*$.

The fundamental example. In [4] one can find two examples of closed bounded convex subsets of c_0. The first example has the CPCP but fails the PCP. The second is strongly regular but fails the CPCP.

These examples are the prototype for the following simplified example which is fundamental for our work.

We denote by $\mathscr{D}$ the dyadic tree (i.e. the family of all finite sequences consisting of 0's and 1's), ordered by the initial segment partial order, and we endow $c_{00}(\mathscr{D})$ with the supremum norm. Clearly its completion is $c_0(\mathscr{D})$. For $a \in \mathscr{D}$, we define by $x_a = \sum_{\gamma \le a} e_{\gamma}$, where $(e_a)_{a\in\mathscr{D}}$ is the natural basis of $c_{00}(\mathscr{D})$. We also set

$$\tilde{x}_a = x_a + \sum_{k=1}^{\infty} \sum_{\substack{|b|=|a|+k \\ b>a}} \frac{1}{2^k} e_b.$$

Then on the set $K = \overline{\mathrm{co}}\,(\tilde{x}_a)_{a\in\mathscr{D}}$ the weak and norm topologies coincide.

1. "Large" operators on L^1 with "small" projections

PROPOSITION 1.1. *Let* X, X_n, $n \in \mathbb{N}$, *be Banach spaces. Suppose that* $X = \sum_{n=1}^{\infty} \oplus X_n$ *and there exists a non-strongly regular operator* $T : L^1(0,1) \to X$ *such that the operators* $P_nT : L^1(0,1) \to X_n$ *are strongly regular for every* $n \in \mathbb{N}$ *where* P_n *denotes the projection* $P_n : X \to X_n$. *Then there exists an operator* $D : L^1(0,1) \to L^1(0,1)$ *such that* $TD : L^1(0,1) \to X$ *is* *non-representable and the operators $P_nTD : L^1(0,1) \to X_n$ are representable for every $n \in \mathbb{N}$.*

Proof. Since T is non-strongly regular, there exists a Borel set $U \subset (0,1)$ and $\delta > 0$ such that for every weak open subset W of $\mathcal{P}_U$ we have

$$\text{(1)} \qquad \operatorname{diam}(T(W)) > 2\delta \qquad ([15, \text{Theorem IV.10}])$$

(where $\mathcal{P}_U = \{f \in L^1(0,1) : f \geq 0, \int f = 1, \operatorname{supp} f \subset U\}$ are the densities supported in U). Since P_nT are strongly regular operators, for every $n \in \mathbb{N}$, we get

$$\text{(2)} \qquad \text{the maps } P_nT : \mathcal{P}_U \to X_n \text{ are weak-to-norm continuous.}$$

Inductively we define $(f_a)_{a\in\mathcal{A}}$ in $\mathcal{P}_U$ with the following properties:

(i) For every $a \in \mathcal{A}$ and $\beta \in S_a$, $\|Tf_a - Tf_\beta\| > \delta$.
(ii) For all $a \in \mathcal{A}$ there exists $(\lambda_\beta)_{\beta\in S_a}$ with $\lambda_\beta \geq 0$, $\sum_{\beta\in S_a}\lambda_\beta = 1$ and $\|f_a - \sum_{\beta\in S_a}\lambda_\beta x_\beta\| < 1/2^n$.
(iii) For all $n \in \mathbb{N}$ and $a \in \mathcal{A}$ with $|a| \geq n$, and all $\beta \in S_a$, we have $\|P_nTf_a - P_nTf_\beta\| < 1/2^n$.

The construction goes as follows. Assume that $(f_a)_{|a|\leq n}$ have been chosen satisfying the inductive assumptions. Then setting $\mathcal{A}_n = \{a : |a| = n\}$, for every $a \in \mathcal{A}_n$ we choose a net $(f_{a,i})_{i\in I_a} \subset \mathcal{P}_U$ such that $f_{a,i} \xrightarrow{\text{w}} f_a$ and $\|Tf_{a,i} - Tf_a\| > \delta$.

Since for $k = 1,\ldots,n+1$, $P_kTf_{a,i} \xrightarrow{\|\cdot\|} P_kTf_a$, we may assume that the net $(f_{a,i})_{i\in I_a}$ satisfies $\|P_kTf_{a,i} - P_kTf_a\| < 1/2^{n+1}$.

By Mazur's theorem there exists a finite subset F_a of I_a and $(\lambda_i)_{i\in F_a}$ with $\lambda_i \geq 0$ and $\sum_{i\in F_a}\lambda_i = 1$ such that $\|f_a - \sum_{i\in F_a}\lambda_i f_{a,i}\| < 1/2^{n+1}$.

We can write the finite set of immediate successors of a as $S_a = \{\beta : \beta = (a,i),\ i \in F_a\}$ and the family $(f_\beta)_{\beta\in S_a}$, $|a| = n$, is as desired.

Let us point out that if we do not require the f_β, $\beta \in S_a$, to be different, we may assume $\lim_{n\to\infty}\max\{\lambda_a : |a| = n\} = 0$. Let $(\xi_n)_{n\in\mathbb{N}}$ be the quasi-martingale which is determined by this bush. Then $\sigma(\bigcup_{n\in\mathbb{N}}\sigma(\xi_n)) = \mathcal{B}(0,1)$ (the Borel measurable sets).

For a $\bigcup_{n\in\mathbb{N}}\sigma(\xi_n)$-simple function φ the limit $D\varphi = \lim_{n\to\infty}\int \xi_n(t)\varphi(t)\,dt$ exists. By density we extend the operator D on $L^1(0,1)$. Then the operator $TD : L^1(0,1) \to X$ is non-representable, since for all a and $\beta \in S_a$ we have $\|Tf_\beta - Tf_a\| > \delta$, while the operators $P_nTD : L^1(0,1) \to X_n$ are representable for every $n \in \mathbb{N}$, since for $n \in \mathbb{N}$ and $|\gamma| = m + k > m = |a|$, $\gamma > a$, we have

$$\|P_nTf_\gamma - P_nTf_a\| \leq \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{m+k-1}} < \frac{1}{2^{m-1}}$$

and so

$$\int \|P_n T f_\gamma - P_n T f_a\| \, dt < \frac{1}{2^{m-1}}.$$

Taking m large enough implies that the bush $(P_n T f_a)_{a\in\mathcal{A}}$ is Cauchy in the Bochner norm in X_n, for every $n \in \mathbb{N}$, and therefore the operators $P_n TD$ are representable [14]. ∎

Of related interest is the following:

Proposition 1.2. *Let X, X_n, $n \in \mathbb{N}$, be Banach spaces. Suppose that $X = \sum_{n=1}^{\infty} \oplus X_n$ and let $T : L^1(0,1) \to X$ be a non-Dunford–Pettis operator such that the operators $P_n T : L^1(0,1) \to X_n$ are Dunford–Pettis. Then the conclusion of Proposition* 1.1 *is true.*

Proof. It is shown in [8] that if $T : L^1(0,1) \to X$ is non-Dunford–Pettis then there exists a dyadic tree $\{\psi_{n,k} : n = 0, 1, \ldots, 1 \le k \le 2^n\}$ in $L^1(0,1)$ such that $(T\psi_{n,k})$ is a δ-tree in X. The nodes $d_{n,k} = \psi_{n+1,2k-1} - \psi_{n+1,2k}$ of the tree $(\psi_{n,k})$ can be taken to be of the form $2\psi_{n,k} r_{n,k}$, where $r_{n,k}$ are elements of a weakly null sequence $(r_n)_{n\in\mathbb{N}}$ in $L^1(0,1)$ such that $\inf_n \|Tr_n\| > \delta'$ for some $\delta' > 0$. Since $P_i T$ are Dunford–Pettis for every $i \in \mathbb{N}$, we may choose the $\{r_{n,k} : n = 0, 1, \ldots, 1 \le k \le 2^n\}$ so that for every $i \in \mathbb{N}$ there exists an $n_i \in \mathbb{N}$ such that $\sum_{k=1}^{2^n} \|P_i T d_{n,k}\| < 1/2^n$ for $n > n_i$.

Let $D : L^1(0,1) \to L^1(0,1)$ be the operator defined by the tree $(\psi_{n,k})$. It follows that the operators $P_i TD : L^1(0,1) \to X$ are representable for every $i \in \mathbb{N}$ (in fact can be taken to be compact). ∎

2. Convex sets on which the norm and weak topologies coincide. In this section we show that under certain conditions there exist closed bounded convex sets on which the norm topology coincides with the weak topology.

Definition 2.1. Let $(x_a)_{a\in\mathcal{A}}$ be a δ-approximate bush with $(y_a)_{a\in\mathcal{A}}$ the corresponding nodes. Let also $(\tilde{x}_a)_{a\in\mathcal{A}}$ be the regular averaging back bush resulting from $(x_a)_{a\in\mathcal{A}}$. We say that the closed convex set $K = \overline{\mathrm{co}}\,(\tilde{x}_a)_{a\in\mathcal{A}}$ has the *non-atomic martingale coordinatization property* if every $x \in K$ can be represented as $x = \sum_{k=0}^{\infty} \sum_{|a|=k} \lambda_a^{(x)} y_a$ with $\lambda_\emptyset^{(x)} = 1$, $\lambda_a^{(x)} \ge 0$, $\lambda_a^{(x)} = \sum_{b\in S_a} \lambda_b^{(x)}$ for all $a \in \mathcal{A}$ and if we set $\lambda_k^{(x)} = \max\{\lambda_a^{(x)} : |a| = k\}$, then $\lim_{k\to\infty} \lambda_k^{(x)} = 0$.

Notation. In what follows, for a Banach space X admitting a (not necessarily finite) Schauder decomposition $(X_n)_{n\in\mathbb{N}}$ (i.e. $X = \sum_{k=1}^{\infty} \oplus X_n$) and $x \in X$, we say that $I \subset \mathbb{N}$ is the *support* of x, written $I = \operatorname{supp} x$, if I is minimal such that $x \in \sum_{n\in I} \oplus X_n$. Also for $X = \sum_{k=1}^{\infty} \oplus X_n$, a family

$(y_a)_{a\in\mathcal{A}}$ is said to be *block* if its elements have pairwise disjoint supports with respect to $(X_n)_{n\in\mathbb{N}}$.

DEFINITION 2.2. Let X be a Banach space with a Schauder decomposition $(X_n)_{n\in\mathbb{N}}$. A δ-approximate bush $(x_a)_{a\in\mathcal{A}}$ is said to be a *block δ-approximate bush* if there exists a family $(I_a)_{a\in\mathcal{A}}$ of disjoint intervals of $\mathbb{N}$ such that if $a <_{\mathrm{lex}} b$, then $I_a < I_b$, and for every $a \in \mathcal{A}$, $\operatorname{supp} y_a \subset I_a$.

LEMMA 2.3. *Let X be a Banach space with a Schauder decomposition $(X_n)_{n\in\mathbb{N}}$ and $(x_a)_{a\in\mathcal{A}}$ be a block δ-approximate bush in X. Then for every $x \in \overline{\mathrm{co}}\,(\tilde{x}_a)_{a\in\mathcal{A}}$ there exists a unique non-atomic martingale coordinatization.*

Proof. By definition, $\tilde{x}_a$ has a martingale coordinatization for all $a \in \mathcal{A}$, hence so does every $x \in \mathrm{co}\,(\tilde{x}_a)_{a\in\mathcal{A}}$.

Let $x \in \overline{\mathrm{co}}\,(\tilde{x}_a)_{a\in\mathcal{A}}$, and let $(x_n)_{n\in\mathbb{N}} \subset \mathrm{co}\,(\tilde{x}_a)_{a\in\mathcal{A}}$ be such that $x_n \xrightarrow{\|\cdot\|} x$. Assume that $x_n = \sum_{k=0}^{\infty}\sum_{|a|=k}\lambda_a^n y_a$ for each n and $(y_a^*)_{a\in\mathcal{A}}$ are the biorthogonal functionals of $(y_a)_{a\in\mathcal{A}}$, defined on $\overline{\langle (y_a)_{a\in\mathcal{A}}\rangle}$. Then $y_a^*(x_n) \to y_a^*(x)$ for all $a \in \mathcal{A}$. Therefore for each $a \in \mathcal{A}$, there exists $\lambda_a^{(x)}$ such that $\lambda_a^n \to \lambda_a^{(x)}$. Then $x = \sum_{k=0}^{\infty}\sum_{|a|=k}\lambda_a^{(x)} y_a$ and this coordinatization is unique.

Also, it is non-atomic, since if $x = \sum_{k=0}^{\infty}\sum_{|a|=k}\lambda_a^{(x)} y_a$ and $\varepsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\|\sum_{k=n}^{\infty}\sum_{|a|=k}\lambda_a^{(x)} y_a\| < \varepsilon$ for all $n \ge n_0$, thus if $|a| = n \ge n_0$, then

$$\lambda_a^{(x)}\|y_a\| \le C\Big\|\sum_{k=n}^{\infty}\sum_{|a|=k}\lambda_a^{(x)} y_a\Big\| \le C\varepsilon.$$

This yields $\lambda_a^{(x)} \le C\varepsilon/\delta$ and hence $\lambda_k^{(x)} = \max\{\lambda_a^{(x)} : |a| = k\} \to 0$. ∎

LEMMA 2.4. *Let X be a Banach space with a Schauder decomposition $(X_n)_{n\in\mathbb{N}}$ and $(x_a)_{a\in\mathcal{A}}$ be a δ-approximate bush in X. Assume moreover that there exists a block δ'-approximate bush $(x_a')_{a\in\mathcal{A}}$ such that if we denote by $(y_a)_{a\in\mathcal{A}}$, $(y_a')_{a\in\mathcal{A}}$ the corresponding families of nodes, we have $\|y_a - y_a'\| < \delta_a$ and $\sum_{a\in\mathcal{A}}\delta_a < \infty$. Then the set $K = \overline{\mathrm{co}}\,(\tilde{x}_a)_{a\in\mathcal{A}}$ has the non-atomic martingale coordinatization property.*

Proof. Let $x \in K$, and let $(x_n)_{n\in\mathbb{N}} \subset \mathrm{co}\,(\tilde{x}_a)_{a\in\mathcal{A}}$ with $x_n \xrightarrow{\|\cdot\|} x$. If

$$x_n = \sum_{k=0}^{\infty}\sum_{|a|=k}\lambda_a^n y_a,$$

then since the set $\mathcal{A}$ is countable and $(\lambda_a^n)_{n\in\mathbb{N}}$ is bounded for all $a \in \mathcal{A}$, by passing to a subsequence we may assume that $\lambda_a^n \to \lambda_a^{(x)}$ for all $a \in \mathcal{A}$.

Define $x_n' = \sum_{k=0}^{\infty}\sum_{|a|=k}\lambda_a^n y_a'$. As $\sum_{a\in\mathcal{A}}\|y_a - y_a'\| < \infty$, x_n' is well defined and $x_n' \in \overline{\mathrm{co}}\,(\tilde{x}_a')_{a\in\mathcal{A}}$. It will be shown that $(x_n')_{n\in\mathbb{N}}$ is a Cauchy sequence.

Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon/3$ for all $n, m \geq n_0$. There also exists $\ell_0 \in \mathbb{N}$ such that $\sum_{|a|>\ell_0} \|y_a - y'_a\| < \varepsilon/6$. Moreover, there exists $n_1 \geq n_0$ such that for all $n, m \geq n_1$, and all $a \in \mathcal{A}$ with $|a| \leq \ell_0$, we have

$$|\lambda_a^n - \lambda_a^m| < \frac{\varepsilon}{3M}, \quad \text{where} \quad M = \sum_{a \in \mathcal{A}} \|y_a - y'_a\|.$$

Then, for $n, m \geq n_1$,

$$\begin{aligned}
\|x'_n - x'_m\| &= \Big\|\sum_{k=0}^{\infty} \sum_{|a|=k} (\lambda_a^n - \lambda_a^m) y'_a\Big\| \\
&= \Big\|\sum_{k=0}^{\infty} \sum_{|a|=k} (\lambda_a^n - \lambda_a^m)(y'_a - y_a) + x_n - x_m\Big\| \\
&\leq \Big\|\sum_{k=0}^{\ell_0} \sum_{|a|=k} (\lambda_a^n - \lambda_a^m)(y'_a - y_a)\Big\| \\
&\quad + \Big\|\sum_{k=\ell_0+1}^{\infty} \sum_{|a|=k} (\lambda_a^n - \lambda_a^m)(y'_a - y_a)\Big\| + \|x_n - x_m\| \\
&\leq \max\{|\lambda_a^n - \lambda_a^m| : |a| \leq \ell_0\} \sum_{k=0}^{\ell_0} \sum_{|a|=k} \|y'_a - y_a\| \\
&\quad + \sup\{|\lambda_a^n - \lambda_a^m| : |a| > \ell_0\} \sum_{k=\ell_0+1}^{\infty} \sum_{|a|=k} \|y'_a - y_a\| + \|x_n - x_m\| \\
&\leq \frac{\varepsilon}{3M} M + 2\frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence $(x'_n)_{n\in\mathbb{N}}$ is converging to some $x' \in \overline{\mathrm{co}}\,(\tilde{x}'_a)_{a\in\mathcal{A}}$. As in the previous proof, if we let $(y'^*_a)_{a\in\mathcal{A}}$ be the biorthogonal functionals to $(y'_a)_{a\in\mathcal{A}}$ defined on the space $\overline{\langle (y'_a)_{a\in\mathcal{A}}\rangle}$, then $y'^*_a(x'_n) \to y'^*_a(x')$ for all $a \in \mathcal{A}$. Thus $x' = \sum_{k=0}^{\infty} \sum_{|a|=k} \lambda_a^{(x)} y'_a$ and by virtue of Lemma 2.3, $(\lambda_a^{(x)})_{a\in\mathcal{A}}$ is a non-atomic martingale coordinatization.

As before, $y = \sum_{k=0}^{\infty} \sum_{|a|=k} \lambda_a^{(x)} y_a$ is well defined and $y \in K$. It remains to show that $y = x$.

Towards a contradiction, suppose that $x_n \nrightarrow y$. By passing to an appropriate subsequence, there exists $\varepsilon > 0$ such that $\|x_n - y\| > \varepsilon$ for all $n \in \mathbb{N}$. There also exists $\ell_0 \in \mathbb{N}$ such that $\sum_{|a|\geq\ell_0} \|y_a - y'_a\| < \varepsilon/10$, and there exists $n_1 \in \mathbb{N}$ such that $\|\sum_{k\leq\ell_0} \sum_{|a|=k} (\lambda_a^n - \lambda_a^{(x)}) y_a\| < \varepsilon/10$ for all $n \geq n_1$. Hence for $n \geq n_1$ we have

$$\Big\|\sum_{k>\ell_0}\sum_{|a|=k}(\lambda_a^n-\lambda_a^{(x)})y_a\Big\| = \Big\|x_n-y-\sum_{k\le\ell_0}\sum_{|a|=k}(\lambda_a^n-\lambda_a^{(x)})y_a\Big\|$$
$$\ge \|x_n-y\|-\Big\|\sum_{k\le\ell_0}\sum_{|a|=k}(\lambda_a^n-\lambda_a^{(x)})y_a\Big\|$$
$$> \varepsilon-\frac{\varepsilon}{10}=\frac{9\varepsilon}{10}.$$

Also,

$$\Big\|\sum_{k>\ell_0}\sum_{|a|=k}(\lambda_a^n-\lambda_a^{(x)})y_a\Big\|$$
$$= \Big\|\sum_{k>\ell_0}\sum_{|a|=k}(\lambda_a^n-\lambda_a^{(x)})(y_a-y'_a)+\sum_{k>\ell_0}\sum_{|a|=k}(\lambda_a^n-\lambda_a^{(x)})y'_a\Big\|$$
$$\le \Big\|\sum_{k>\ell_0}\sum_{|a|=k}(\lambda_a^n-\lambda_a^{(x)})(y_a-y'_a)\Big\|+\|P_{|a|>\ell_0}(x'_n-x')\|$$
$$\le 2\frac{\varepsilon}{10}+2C\|x'_n-x'\|.$$

Here $P_{|a|>\ell_0}(x) = x-\sum_{|a|\le\ell_0}\tilde P_\alpha(x)$, where $\tilde P_a(x)=\sum_{i\in I_a}P_i(x)$, $P_i : X\to X_i$ are the natural projections of the decomposition, and C is the constant of the decomposition.

By choosing n sufficiently large, we have $\|\sum_{k>\ell_0}\sum_{|a|=k}(\lambda_a^n-\lambda_a^{(x)})y_a\| < 9\varepsilon/10$, a contradiction that concludes our proof. ■

Lemma 2.5. *Let X be a Banach space, $\mathcal{A}$ a finitely branching tree, $(y_a)_{a\in\mathcal{A}}$, $(y'_a)_{a\in\mathcal{A}}$ subsets of X, and $(\varepsilon_n)_{n=0}^\infty$ a sequence of positive reals with $\sum_{n=0}^\infty\varepsilon_n<\infty$ and $\|y_a-y'_a\|<\varepsilon_{|a|}$ for all $a\in\mathcal{A}$. Define*

$$K=\Big\{x\in X: x=\sum_{k=0}^\infty\sum_{|a|=k}\lambda_a^{(x)}y_a,\ \lambda_\emptyset^{(x)}=1,\ \lambda_a^{(x)}\ge 0,\ \lambda_a^{(x)}=\sum_{b\in S_a}\lambda_b^{(x)},\ a\in\mathcal{A}\Big\}.$$

Suppose L is a subset of K and that on the set

$$L'=\Big\{x\in X: x=\sum_{k=0}^\infty\sum_{|a|=k}\lambda_a^{(x)}y'_a \text{ with } \sum_{k=0}^\infty\sum_{|a|=k}\lambda_a^{(x)}y_a\in L\Big\}$$

the weak and norm topologies coincide. Then on L the weak and norm topologies also coincide.

Proof. Define $(r_a)_{a\in\mathcal{A}}$ by $r_a=\sum_{\gamma\le a}(y_\gamma-y'_\gamma)$. It will be shown that the set $(r_a)_{a\in\mathcal{A}}$ is totally bounded.

Let $\varepsilon>0$. There exists $n_0\in\mathbb{N}$ such that $\sum_{n\ge n_0}\varepsilon_n<\varepsilon$. Let $\gamma\in\mathcal{A}$ with $|\gamma|\ge n_0$. Then there exists $a\in\mathcal{A}$ with $|a|=n_0$ and $a\le\gamma$. We have

$$\|r_\gamma - r_a\| = \Big\| \sum_{a<\delta\le\gamma} (r_\delta - r_{\delta-}) \Big\| = \Big\| \sum_{a<\delta\le\gamma} (y_\delta - y'_\delta) \Big\| \le \sum_{a<\delta\le\gamma} \|y_\delta - y'_\delta\| < \varepsilon.$$

Thus the set $(r_a)_{a\in\mathcal{A}}$ is totally bounded and this means that $\overline{\mathrm{co}}\,(r_a)_{a\in\mathcal{A}}$ is norm compact.

Let $x \in L$, $x = \sum_{k=0}^\infty \sum_{|a|=k} \lambda_a^{(x)} y_a$. Since $\|y_a - y'_a\| < \varepsilon_{|a|}$, we conclude that $\sum_{k=0}^\infty \sum_{|a|=k} \lambda_a^{(x)} y'_a \in L'$ and $x = \sum_{k=0}^\infty \sum_{|a|=k} \lambda_a^{(x)} (y_a - y'_a) + \sum_{k=0}^\infty \sum_{|a|=k} \lambda_a^{(x)} y'_a$. Then

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{|a|=k} \lambda_a^{(x)} (y_a - y'_a) &= \lim_{n\to\infty} \sum_{k=0}^{n} \sum_{|a|=k} \lambda_a^{(x)} (y_a - y'_a)\\ &= \lim_{n\to\infty} \sum_{|a|=n} \lambda_a^{(x)} \Big(\sum_{\gamma\le a} (y_\gamma - y'_\gamma) \Big)\\ &= \lim_{n\to\infty} \sum_{|a|=n} \lambda_a^{(x)} r_a \in \overline{\mathrm{co}}\,(r_a)_{a\in\mathcal{A}}.\end{aligned}$$

This means that $L \subset \overline{\mathrm{co}}\,(r_a)_{a\in\mathcal{A}} + L'$. Since $\overline{\mathrm{co}}\,(r_a)_{a\in\mathcal{A}}$ is norm compact and on L' the weak and norm topologies coincide, it can be easily seen that on $\overline{\mathrm{co}}\,(r_a)_{a\in\mathcal{A}} + L'$ the weak and norm topologies coincide, which of course means that the same is true for L. ∎

DEFINITION 2.6. Let X be a Banach space with a Schauder decomposition $(X_n)_{n\in\mathbb{N}}$, $\mathcal{A}$ a finitely branching tree and $(y_a)_{a\in\mathcal{A}}$ a subset of X. Then $(y_a)_{a\in\mathcal{A}}$ is called *eventually block* if there exists $n_0 \in \mathbb{N}$ and a family $(I_a)_{|a|\ge n_0}$ of disjoint intervals of $\mathbb{N}$ such that if $a <_{\mathrm{lex}} b$, then $I_a < I_b$, and for every $a \in \mathcal{A}$, $\operatorname{supp} y_a \subset I_a$.

REMARK. For some $a \in \mathcal{A}$ with $|a| \ge n_0$ it may occur that $y_a = 0$.

LEMMA 2.7. *Let X, X_k, $k \in \mathbb{N}$, be Banach spaces with $X = (\sum_{k=1}^\infty \oplus X_k)_0 = \{(x_k)_{k\in\mathbb{N}} : x_k \in X_k,$ and $\lim_{k\to\infty} \|x_k\| = 0\}$, and let $(y_a)_{a\in\mathcal{A}}$ be bounded and eventually block. Set*

$$L = \Big\{ x \in X : x = \sum_{k=0}^{\infty} \sum_{|a|=k} \lambda_a^{(x)} y_a \ \textit{with}\ \lambda_\emptyset^{(x)} = 1,\ \lambda_a^{(x)} \ge 0,\ \lambda_a^{(x)} = \sum_{b\in S_a} \lambda_b^{(x)}$$
$$\textit{for all}\ a \in \mathcal{A}\ \textit{and}\ \lim_{k\to\infty} \max\{\lambda_a^{(x)} : |a| = k\} = 0 \Big\}.$$

Then on L the weak and norm topologies coincide.

Proof. We shall first prove the lemma with the additional assumption that each $y_a \ne 0$. Let $x \in L$, $x = \sum_{k=0}^\infty \sum_{|a|=k} \lambda_a^{(x)} y_a$, $\varepsilon > 0$. It will be shown that there exists a relative weak neighbourhood U of x in L such that $\operatorname{diam}(U) < \varepsilon$, hence x will be a point of continuity.

Since $\langle\{y_a : |a| < n_0\}\rangle$ is finite-dimensional, there exists $n_1 \geq n_0$ such that $\langle\{y_a : |a| < n_0\}\rangle \cap \overline{\langle\{y_a : |a| \geq n_1\}\rangle} = \{0\}$.

Indeed, if $\{x_1, \ldots, x_j\}$ is a Hamel basis of

$$\langle\{y_a : |a| < n_0\}\rangle \cap \overline{\langle\{y_a : |a| \geq n_0\}\rangle}$$

then $x_i = \sum_{k=n_0}^{\infty} \sum_{|a|=k} \mu_a^i y_a$ for $i = 1, \ldots, j$. Pick $a_1, \ldots, a_j \in \mathcal{A}$ with $|a_i| \geq n_0$ and $\mu_{a_i}^i \neq 0$ for $i = 1, \ldots, j$. Set $n_1 = \max\{|a_i| : i = 1, \ldots, j\} + 1$.

If $M = \sup\{\|y_a\| : a \in \mathcal{A}\}$, there exists $n_2 \geq n_1$ such that $\max\{\lambda_a^{(x)} : |a| = k\} < \varepsilon/16M$ for all $k \geq n_2$. Then

$$x = x_1 + x_2, \quad \text{where} \quad x_1 = \sum_{k=0}^{n_2-1} \sum_{|a|=k} \lambda_a^{(x)} y_a, \quad x_2 = \sum_{k=n_2}^{\infty} \sum_{|a|=k} \lambda_a^{(x)} y_a.$$

Set

$$\ell = \#\{a \in \mathcal{A} : |a| \leq n_2\}, \qquad \varepsilon' = \frac{\varepsilon}{8(\ell^2 M + 2M)}.$$

Define the biorthogonal functionals $(y_a^*)_{|a|=n_2}$ on $\overline{\langle\{y_a : a \in \mathcal{A}\}\rangle}$ by

$$y_a^*(y_\gamma) = \begin{cases} 1 & \text{if } a = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

This is possible since $n_2 \geq n_1$ and since we assume that $y_a \neq 0$ for all $a \in \mathcal{A}$.

Define $U = \{y \in L : |y_a^*(y - x)| < \varepsilon',\ |a| = n_2\}$ and let $y \in U$ be such that $y = \sum_{k=0}^{\infty} \sum_{|a|=k} \lambda_a^{(y)} y_a$. Then

$$y = y_1 + y_2, \quad \text{where} \quad y_1 = \sum_{k=0}^{n_2-1} \sum_{|a|=k} \lambda_a^{(y)} y_a, \quad y_2 = \sum_{k=n_2}^{\infty} \sum_{|a|=k} \lambda_a^{(y)} y_a.$$

For $a \in \mathcal{A}$ with $|a| = n_2$, we have $|\lambda_a^{(y)} - \lambda_a^{(x)}| = |y_a^*(y - x)| < \varepsilon'$, so $\lambda_a^{(y)} < \varepsilon' + \varepsilon/16M$ for all $a \in \mathcal{A}$ with $|a| \geq n_2$.

For $a \in \mathcal{A}$ with $|a| < n_2$, we have

$$|\lambda_a^{(y)} - \lambda_a^{(x)}| = \Big| \sum_{\substack{|b|=n_2 \\ a<b}} (\lambda_b^{(y)} - \lambda_b^{(x)}) \Big| < \varepsilon' \ell,$$

so

$$\|y_1 - x_1\| = \Big\| \sum_{k=0}^{n_2-1} \sum_{|a|=k} (\lambda_a^{(y)} - \lambda_a^{(x)}) y_a \Big\| \leq \varepsilon' \ell^2 M.$$

Also we have

$$\|y_2 - x_2\| = \Big\|\sum_{k=n_2}^{\infty}\sum_{|a|=k}(\lambda_a^{(y)} - \lambda_a^{(x)})y_a\Big\| = \sup\{\|(\lambda_a^{(y)} - \lambda_a^{(x)})y_a\| : |a| \ge n_2\}$$
$$\le \sup\{(|\lambda_a^{(y)}| + |\lambda_a^{(x)}|)M : |a| \ge n_2\}$$
$$\le \left(\frac{\varepsilon}{16M} + \varepsilon' + \frac{\varepsilon}{16M}\right)M = \left(\varepsilon' + \frac{\varepsilon}{8M}\right)M.$$

Then

$$\|y - x\| \le \|y_1 - x_1\| + \|y_2 - x_2\| \le \varepsilon'\ell^2 M + \varepsilon' M + \frac{\varepsilon}{8}$$
$$= \frac{\varepsilon}{8(\ell^2 M + 2M)}(\ell^2 M + 2M) + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}.$$

Thus $\operatorname{diam}(U) \le \varepsilon/2 < \varepsilon$.

This completes the proof if each $y_a \ne 0$. For the general case, we reduce the proof to the previous one as follows. Choose a sequence $(\varepsilon_n)_{n=0}^{\infty}$ of positive reals with $\sum_{n=0}^{\infty}\varepsilon_n < \infty$ and define $(y'_a)_{a\in\mathcal{A}}$ by

$$y'_a = \begin{cases} y_a & \text{if } y_a \ne 0,\\ y''_a & \text{otherwise,}\end{cases}$$

where $\operatorname{supp} y''_a \subset I_a$ and $0 < \|y''_a\| \le \varepsilon_{|a|}$. We observe that $\|y_a - y'_a\| \le \varepsilon_{|a|}$, and by the previous case and Lemma 2.5 the result follows. ■

3. The main theorems. This section contains the main results of the paper. Among other things we show that the KMP is equivalent to the RNP on subsets of $C(\omega^{\omega^k})$. In fact we show something stronger: every non-dentable subset of $C(\omega^{\omega^k})$ contains a convex closed subset L such that L has the PCP and fails the RNP.

PROPOSITION 3.1. *Let $Y, Y_k, X, Z, Z_n, Z_{n,k}$, $n, k \in \mathbb{N}$, be Banach spaces such that $Y = \sum_{k=1}^{\infty} \oplus Y_k$, $Z_n = (\sum_{k=1}^{\infty} \oplus Z_{n,k})_0$, $X \hookrightarrow Y$ and X contains no copy of $\ell_1(\mathbb{N})$. Let $Q_n : X \to Z_n$, $n \in \mathbb{N}$, be bounded linear operators, K a closed, convex, bounded, non-PCP subset of X, and suppose that on $P_k(K)$ and $R_{n,k}Q_n(K)$ the weak and norm topologies coincide for all $n, k \in \mathbb{N}$ (where $P_k : Y \to Y_k$ and $R_{n,k} : Z_n \to Z_{n,k}$ denote the projections). Then there exists a closed, convex, non-dentable subset L of K such that on $Q_n(L)$ the weak and norm topologies coincide for all $n \in \mathbb{N}$.*

Proof. Since K is non-PCP there exists a 2δ-non-PCP subset W of K, for some $\delta > 0$. We will inductively construct:

(a) a δ-bush $(x_a)_{a\in\mathcal{A}} \subset W$,
(b) a family $(I_a)_{a\in\mathcal{A}}$ of disjoint intervals of $\mathbb{N}$ such that if $a <_{\text{lex}} b$, then $I_a < I_b$,

such that:

(1) if $x'_a = \sum_{\gamma \le a} P_{I_\gamma}(y_\gamma)$, then $(x'_a)_{a\in\mathcal{A}}$ is a block $\delta/2$-approximate bush in Y and $\sum_{a\in\mathcal{A}} \|y_a - y'_a\| < \infty$,
(2) if $R_{n,I_a} Q_n(y_a) = y^n_a$, then $(y^n_a)_{a\in\mathcal{A}}$ is eventually block and

$$\sum_{a\in\mathcal{A}} \|Q_n(y_a) - y^n_a\| < \infty \quad \text{ for all } n \in \mathbb{N}.$$

Granting this construction, it will now be shown that $L := \overline{\mathrm{co}}\,(\tilde{x}_a)_{a\in\mathcal{A}}$ is as desired.

By (1) and Lemma 2.4, L has the non-atomic martingale coordinatization property. Set

$$L'_n = \Big\{x \in Z_n : x = \sum_{k=0}^{\infty} \sum_{|a|=k} \lambda^{(x)}_a y^n_a,\ \lambda^{(x)}_{\emptyset} = 1,\ \lambda^{(x)}_a \ge 0,\ \lambda^{(x)}_a = \sum_{b\in S_a} \lambda^{(x)}_b \\ \text{for all } a \in \mathcal{A},\ \lim_{k\to\infty} \max\{\lambda^{(x)}_a : |a| = k\} = 0\Big\}.$$

Then by (2) and Lemma 2.4, on L'_n the weak and norm topologies coincide. Now define

$$L_n = \Big\{x \in Z_n : x = \sum_{k=0}^{\infty} \sum_{|a|=k} \lambda^{(x)}_a Q_n(y_a),\ \lambda^{(x)}_{\emptyset} = 1,\ \lambda^{(x)}_a \ge 0,\ \lambda^{(x)}_a = \sum_{b\in S_a} \lambda^{(x)}_b \\ \text{for all } a \in \mathcal{A},\ \lim_{k\to\infty} \max\{\lambda^{(x)}_a : |a| = k\} = 0\Big\}.$$

By (2) and Lemma 2.5, on L_n the weak and norm topologies coincide. But L has the non-atomic martingale coordinatization property, thus $Q_n(L) \subset L_n$, hence on $Q_n(L)$ the weak and norm topologies coincide, for all $n \in \mathbb{N}$.

In order to complete the proof, we shall now proceed to the previously mentioned construction.

An important ingredient is the following fact: If X contains no copy of ℓ_1, K is a bounded subset of X and $x \in \overline{K}^w$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that $w\text{-}\lim_{n\to\infty} x_n = x$ (see [11], [20]).

Choose $x_\emptyset = x \in W$. Since X contains no copy of $\ell_1(\mathbb{N})$ and W is 2δ-non-PCP, there exists a sequence $(x_m)_{m\in\mathbb{N}} \subset W$ such that $x_m \xrightarrow{w} x$ and $\|x_m - x\| > \delta$ for all $m \in \mathbb{N}$.

For $\varepsilon_0 > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\|P_{[1,k_0]}(x) - x\| < \varepsilon_0.$$

Define $I_\emptyset = [1, k_0]$.

For $\varepsilon_1 > 0$, since $\|P_k(x_m - x)\|, \|R_{1,k} Q_1(x_m - x)\| \xrightarrow{m\to\infty} 0$ for all $k \in \mathbb{N}$, there exists $m_1 \in \mathbb{N}$ such that $\|P_{[1,k_0]}(x_{m_1} - x)\|, \|R_{1,[1,k_0]} Q_1(x_{m_1} - x)\| < \frac{\varepsilon_1}{2\cdot 2}$.

There exists $k_1 > k_0$ such that

$$\|P_{[1,k_1]}(x_{m_1}-x)-(x_{m_1}-x)\|, \|R_{1,[1,k_1]}Q_1(x_{m_1}-x)-Q_1(x_{m_1}-x)\| < \frac{\varepsilon_1}{2\cdot 2}.$$

Then

$$\|P_{(k_0,k_1]}(x_{m_1}-x)-(x_{m_1}-x)\|, \|R_{1,(k_0,k_1]}Q_1(x_{m_1}-x)-Q_1(x_{m_1}-x)\| < \frac{\varepsilon_1}{2}.$$

Define $I_1^1 = (k_0, k_1]$.

Inductively choose a subsequence $(x_{m_i})_{i\in\mathbb{N}}$ of $(x_m)_{m\in\mathbb{N}}$ and successive intervals $(I_i^1)_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\|P_{I_i^1}(x_{m_i}-x)-(x_{m_i}-x)\|, \|R_{1,I_i^1}Q_1(x_{m_i}-x)-Q_1(x_{m_i}-x)\| < \frac{\varepsilon_1}{2^i}.$$

For $\delta_0 > 0$, by Mazur's theorem, there exists a finite set $(x_b)_{b\in S_\emptyset} \subset (x_{m_i})_{i\in\mathbb{N}}$ and positive reals $(\lambda_b)_{b\in S_\emptyset}$ with $\sum_{b\in S_\emptyset}\lambda_b = 1$ such that

$$\Big\|x_\emptyset - \sum_{b\in S_\emptyset}\lambda_b x_b\Big\| < \delta_0.$$

Denote by $(I_b)_{b\in S_a}$ the corresponding intervals. Then

$$\sum_{b\in S_\emptyset}\|P_{I_b}(y_b)-y_b\|, \sum_{b\in S_\emptyset}\|R_{1,I_b}Q_1(y_b)-Q_1(y_b)\| < \sum_{i=1}^{\infty}\frac{\varepsilon_1}{2^i} = \varepsilon_1.$$

Suppose that $(x_a)_{|a|\le j}$ and $(I_a)_{|a|\le j}$ have been chosen such that if $|a|,|b|\le j$ and $a <_{\mathrm{lex}} b$, then $I_a < I_b$, $\|x_a - \sum_{b\in S_a}\lambda_b x_b\| < \delta_{|a|}$, $\|x_a - x_b\| > \delta$ for $|a| < j$, $b\in S_a$, and also

$$\sum_{|a|=i}\|P_{I_a}(y_a)-y_a\|, \sum_{|a|=i}\|R_{\ell,I_a}Q_\ell(y_a)-Q_\ell(y_a)\| < \varepsilon_i \quad \text{for } 1\le \ell\le i\le j.$$

Enumerate the set $\{a : |a| = j\}$ in lexicographic order and for a_1, if $N = \#\{a : |a| = j\}$, for $\varepsilon_{j+1}, \delta_j$ as before choose $(x_b)_{b\in S_{a_1}}$, $(I_b)_{b\in S_{a_1}}$ such that $(I_b)_{|b|\le j} < (I_b)_{b\in S_{a_1}}$, $\|x_{a_1} - x_b\| > \delta$, $\|x_{a_1} - \sum_{b\in S_{a_1}}\lambda_b x_b\| < \delta_j$ and

$$\sum_{b\in S_{a_1}}\|P_{I_b}(y_b)-y_b\|, \sum_{b\in S_{a_1}}\|R_{\ell,I_b}Q_\ell(y_b)-Q_\ell(y_b)\| < \frac{\varepsilon_{j+1}}{N} \quad \text{for } 1\le \ell\le j+1.$$

Continue in the same manner for the rest of the set $\{a : |a| = j\}$. Then

$$\sum_{|a|=j+1}\|P_{I_a}(y_a)-y_a\|, \sum_{|a|=j+1}\|R_{\ell,I_a}Q_\ell(y_a)-Q_\ell(y_a)\| < \varepsilon_{j+1} \quad \text{for } 1\le \ell\le j+1.$$

The inductive construction is complete. If the sequences $(\varepsilon_j)_{j=0}^{\infty}$ and $(\delta_j)_{j=0}^{\infty}$ have been suitably chosen, then the conclusion of the theorem holds. In fact they need to be chosen in such a way that $\sum_{j=0}^{\infty}\varepsilon_j, \sum_{j=0}^{\infty}\delta_j < \delta/16$.

Then it is easy to see that:

(i) $(x_a)_{a\in\mathcal{A}}$ is a δ-approximate bush,
(ii) $(x'_a)_{a\in\mathcal{A}}$ is a block $\delta/2$-approximate bush and $\sum_{a\in\mathcal{A}}\|y_a - y'_a\| < \infty$,
(iii) $(y^n_a)_{|a|\geq n}$ is block and $\sum_{a\in\mathcal{A}}\|y^n_a - Q_n(y_a)\| < \infty$ for all $n \in \mathbb{N}$. ∎

Remark. The proof of Proposition 3.1 shows that if X contains no copy of ℓ_1 and X fails the PCP then there exists a δ-approximate bush $(x_a)_{a\in\mathcal{A}}$ whose nodes form a basic sequence. Therefore X contains a subspace with a basis that fails the RNP. This result is known to experts but we have been unable to trace a reference. In [3] a Banach space X is constructed such that X^* is separable and the PCP is equivalent to the RNP on subsets of X. It follows that if a subspace Y of X fails the RNP then Y contains a space Z with a basis that fails the RNP.

Theorem 3.2. *Let X be a separable Banach space that contains no copy of $\ell_1(\mathbb{N})$ and $Q_n : X \to C(\omega^{\omega^k})$, $n \in \mathbb{N}$, be bounded linear operators. Suppose K is a closed, convex, bounded, non-PCP subset of X such that the PCP and RNP are equivalent on subsets of K. Then there exists a closed, convex, bounded, non-dentable subset L of K such that on $Q_n(L)$ the weak and norm topologies coincide for all $n \in \mathbb{N}$.*

Proof. We prove the theorem by induction. For $k = 0$ we have $C(\omega^{\omega^0}) = C(\omega) \cong c_0(\mathbb{N})$.

In Proposition 3.1, consider $Y = C[0,1]$, $Y_k = \langle e_k\rangle$, $Z_n = c_0(\mathbb{N})$, $Z_{n,k} = \mathbb{R}$, where $(e_k)_{k\in\mathbb{N}}$ is a Schauder basis of $C[0,1]$. Since Y_k and $Z_{n,k}$ are finite-dimensional, the requirements of Proposition 3.1 are fulfilled, and thus the desired set L exists.

Suppose that this is true for $k = m \geq 0$; it will be shown that it is true for $k = m + 1$.

It is well known that $C(\omega^{\omega^{m+1}}) = (\sum_{k=1}^{\infty} \oplus (C(\omega^{\omega^m}), \|\cdot\|_k))_0$, where $\|\cdot\|_k$ is an equivalent norm on $C(\omega^{\omega^m})$. Then the family $R_{n,k}Q_n : X \to C(\omega^{\omega^m})$ is countable and by the inductive assumption, there exists a closed, convex, non-dentable subset L' of K such that on $R_{n,k}Q_n(L')$ the weak and norm topologies coincide. Since the PCP and RNP are equivalent on subsets of K, L' is non-PCP. Applying once more Proposition 3.1 for the set L' and the family of operators $(Q_n)_{n\in\mathbb{N}}$, we conclude that there exists a closed, convex, bounded, non-dentable subset L of L' such that on $Q_n(L)$ the weak and norm topologies coincide for all $n \in \mathbb{N}$. ∎

Theorem 3.3. *Let K be a closed, convex, bounded, non-dentable subset of $C(\omega^{\omega^k})$. Then there exists a convex closed subset L of K such that L has the PCP and fails the RNP. Therefore the KMP and RNP are equivalent on subsets of $C(\omega^{\omega^k})$.*

Proof. Towards a contradiction, suppose that K is a closed, convex, bounded non-dentable subset of $C(\omega^{\omega^k})$ such that the PCP and RNP are equivalent on subsets of K. We apply Theorem 3.2 for $Q = I : C(\omega^{\omega^k}) \to C(\omega^{\omega^k})$, the identity map. Then there exists a closed, convex, bounded, non-dentable subset L of K such that on $I(L) = L$ the weak and norm topologies coincide. But this means that the PCP and RNP are not equivalent on subsets of K, a contradiction completing the proof. ∎

PROBLEM. The problem of equivalence of the Radon–Nikodym Property and the Krein–Milman Property remains open on subsets of $C(\omega^{\omega^a})$ for ordinals $a \geq \omega$.

Acknowledgements. We want to express our gratitude to Professor S. Argyros for many important discussions and invaluable help. We also thank the referee for suggestions that greatly simplified the proofs of the theorems.

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Received May 10, 2010
Revised version March 1, 2011 (6879)