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Ditkin sets in homogeneous spaces

by

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Abstract. Ditkin sets for the Fourier algebra $A(G/K)$, where K is a compact subgroup of a locally compact group G, are studied. The main results discussed are injection theorems, direct image theorems and the relation between Ditkin sets and operator Ditkin sets and, in the compact case, the inverse projection theorem for strong Ditkin sets and the relation between strong Ditkin sets for the Fourier algebra and the Varopoulos algebra. Results on unions of Ditkin sets and on tensor products are also given.

1. Introduction. Ditkin sets satisfy a strong form of spectral synthesis and have been studied for a long time, especially on abelian groups. The Fourier algebra $A(G)$ of a nonabelian locally compact group and its synthesis properties have been extensively studied since the fundamental work of Eymard [3]. Study of the Fourier algebra $A(G/K)$ on a homogeneous space G/K, where K is a compact (nonnormal) subgroup of G, was initiated more recently by Forrest [4]. The present paper studies Ditkin sets and operator Ditkin sets in the context of $A(G/K)$.

Assume G is abelian and H is a closed subgroup of G. A closed subset E of H is a Ditkin set for $A(H)$ if and only if it is a Ditkin set for $A(G)$. This theorem is known as the injection theorem for Ditkin sets and is a classical result of Reiter (see [17]). A nonabelian version of this theorem is due to Derighetti [1], who proved that for normal subgroups the injection theorem holds for local Ditkin sets. Theorem 3.4 of our paper gives an injection theorem for local Ditkin sets but the subgroups here are compact, not necessarily normal. We develop the required preliminary results for these, among them a *lemme à la Derighetti* in our context. We also present direct image theorems for coset spaces and double coset spaces of K.

Another classical result of Reiter (see [17]), the inverse projection theorem for sets of synthesis, states that if H is a closed subgroup of the abelian group G, then $\widetilde{E} \subset G/H$ is a set of synthesis for $A(G/H)$ if and only if its

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pull back is a set of synthesis for $A(G)$. No such result seems to be known for Ditkin sets, even in the abelian case. A partial, nonabelian version for local Ditkin sets is due to Derighetti [1]. Section 4 of the present paper contains an inverse projection theorem for strong Ditkin sets in the case of compact groups. That section also presents a relation between strong Ditkin sets in the Fourier algebra and the corresponding Varopoulous algebra in the context of homogenous spaces.

Shulman and Turowska [19] introduced the concept of operator Ditkin sets. Ludwig and Turowska [11] obtained a relation between these and local Ditkin sets. Extending their results, we present, in Section 5, the relation between local Ditkin sets for $A(G/K)$ (for noncompact G) and operator Ditkin sets with respect to $m_{K\backslash G} \times m_G$.

In the last section we make some simple observations on unions of Ditkin sets and on Ditkin sets for tensor products of Fourier algebras.

We begin with some of the required preliminaries in the next section.

2. Preliminaries. The basic reference for Fourier algebras is the fundamental paper of Eymard [3]. Let G be a locally compact group with a fixed left Haar measure m_G. The Fourier algebra $A(G)$ of G consists of the coefficient functions of the left regular representation λ of G. A function in $A(G)$ has a representation of the form $u(\cdot) = \langle \lambda(\cdot)f, g\rangle$, $f, g \in L^2(G)$, with the norm $\|u\|_{A(G)} = \|f\|_2\|g\|_2$. The dual of $A(G)$ is the group von Neumann algebra $\mathrm{VN}(G)$, the von Neumann subalgebra of $\mathscr{B}(L^2(G))$ generated by $\{\lambda(x) : x \in G\}$.

For a compact subgroup K of G, let G/K and $K\backslash G$ denote the homogeneous spaces of left and right cosets of K, respectively. We write $\mathsf{q} : G \to G/K$, $t \mapsto \dot{t}$, and $\mathsf{p} : G \to K\backslash G$, $t \mapsto \bar{t}$, for the corresponding quotient maps. There are unique G-invariant Radon measures $m_{G/K}$ and $m_{K\backslash G}$ in G/K and $K\backslash G$, respectively, satisfying the respective Weil formulas.

The range $A(G : K)$ of the map P_K defined on $A(G)$ by $P_K u(t) = \int_K u(tk)\, dk$, $t \in G$, consists of those functions in $A(G)$ that are constant on left cosets of K. It is a closed subalgebra of $A(G)$. A function u in $A(G : K)$ can be considered as a continuous function $\widetilde{u}$ on G/K given by $\widetilde{u}(\dot{t}) = u(t)$. We write $A(G/K) := \{\widetilde{u} : u \in A(G)\}$. This was introduced by Forrest [4] and is a regular, commutative, semisimple Banach algebra with Gelfand structure space G/K. We usually identify $A(G/K)$ with $A(G : K)$ and call it the *Fourier algebra* of G/K. (Whenever meaningful, we shall use similar notation and convention for other function spaces also; thus, for example, we shall consider $L^1(G : K)$ and $L^1(G/K)$. We shall also adopt a similar convention for function spaces on the right coset space $K\backslash G$.) Any function in $A(G/K)$ has a representation of the form $u(\cdot) = \langle \lambda(\cdot)f, g\rangle$ with

$f \in L^2(K\backslash G)$, $g \in L^2(G)$. The dual of $A(G/K)$ is $\mathrm{VN}(G/K)$, the weak-$*$ closure of $\{\lambda(f) : f \in L^1(G/K)\}$ in $\mathrm{VN}(G)$. Observe that, for $x \in G$ and $k \in K$, the restriction of $\lambda(xk)$ to $L^2(K\backslash G)$ is independent of k. We write $\lambda(\dot{x}) \in \mathscr{B}(L^2(K\backslash G), L^2(G))$ for this restriction. Thus $\mathrm{VN}(G/K)$ can be considered as the smallest subspace of $\mathscr{B}(L^2(K\backslash G), L^2(G))$ containing all the $\lambda(\dot{x})$ that is closed in the weak operator topology. As a linear functional on $A(G/K)$, $\lambda(\dot{x})$ acts as point evaluation.

The *Varopoulos algebra* $V(G, K\backslash G)$, considered in [16], is the algebra of continuous functions on $G \times K\backslash G$ which have representations of the form $w = \sum \varphi_i \otimes \widetilde{\psi}_i$, $\varphi_i \in C(G)$, $\widetilde{\psi}_i \in C(K\backslash G)$, with $\sum |\varphi_i|^2$ and $\sum |\widetilde{\psi}_i|^2$ uniformly convergent. It is the Haagerup tensor product $V(G, K\backslash G) = C(G) \otimes^h C(K\backslash G)$ and its norm is given by

$$\|w\|_V = \inf\Big\{\Big\|\sum |\varphi_i|^2\Big\|_\infty^{1/2}\Big\|\sum |\widetilde{\psi}_i|^2\Big\|_\infty^{1/2} : w = \sum \varphi_i \otimes \widetilde{\psi}_i\Big\}.$$

It is a commutative, semisimple, regular Banach algebra with Gelfand maximal ideal space $G \times K\backslash G$. If

$$V_{\mathrm{inv}}(G, K\backslash G) := \{\widetilde{w} \in V(G, K\backslash G) : \widetilde{w}(sx, \bar{t}.x) = \widetilde{w}(s, \bar{t}),\ s, t, x \in G\},$$

then Theorem 3.10 of [16] says that the map N_K on $G \times K\backslash G$ defined by $N_K\widetilde{u}(s, \bar{t}) = \widetilde{u}(s.\dot{t}^{-1})$ is an isometric isomorphism of $A(G/K)$ onto the invariant algebra $V_{\mathrm{inv}}(G, K\backslash G)$. (The proof of this heavily uses the operator space structure of the Fourier algebra. The monograph [2] of Effros and Ruan is a basic reference for operator spaces.)

For a discussion of operator Ditkin sets we need two more spaces which we now describe. The space $V^\infty(K\backslash G, G)$ is the weak-$*$ Haagerup tensor product $L^\infty(K\backslash G) \otimes^{w*h} L^\infty(G)$, and consists of functions (up to marginally null sets) on $K\backslash G \times G$ of the form $\widetilde{w} = \sum_{n=1}^\infty \widetilde{\varphi}_n \otimes \psi_n$ where $\widetilde{\varphi}_n, \psi_n$ are in $L^\infty(K\backslash G), L^\infty(G)$, respectively, and the series is weak-$*$ convergent. Also,

$$\|w\|_{V^\infty} = \inf\Big\{\Big\|\sum |\widetilde{\varphi}_n|^2\Big\|_\infty^{1/2}\Big\|\sum |\psi_n|^2\Big\|_\infty^{1/2} : w = \sum \widetilde{\varphi}_n \otimes \psi_n\Big\}$$

with the series $\sum |\widetilde{\varphi}_n|^2$ and $\sum |\psi_n|^2$ converging in the weak-$*$ topology. Here a *marginally null set* is a subset of a set of the form $\widetilde{E} \times G \cup K\backslash G \times F$ where $F, \widetilde{E}$ are locally null in $G, K\backslash G$ respectively. The invariant part $V^\infty_{\mathrm{inv}}(K\backslash G, G)$ $= \{\widetilde{w} \in V^\infty(K\backslash G, G) : \widetilde{w}(\bar{s}.x, tx) = \widetilde{w}(\bar{s}, t),\ s, t, x \in G\}$ is a closed subalgebra of $V^\infty(K\backslash G, G)$.

Let $T(K\backslash G, G)$ denote the projective tensor product $L^2(K\backslash G) \otimes^\gamma L^2(G)$. An element $\omega = \sum_{n=1}^\infty \widetilde{f}_n \otimes g_n \in T(K\backslash G, G)$ is considered as the function given by $\omega(\bar{y}, x) = \sum \widetilde{f}_n(\bar{y}) g_n(x)$ for marginally almost all $(\bar{y}, x) \in K\backslash G \times G$. For such an ω, $\operatorname{supp}\omega = \{(\bar{y}, x) \in G \times K\backslash G : \omega(\bar{y}, x) \neq 0\}$ is defined up to marginally null sets. The dual of $T(K\backslash G, G)$ is identified with $\mathscr{B}(L^2(K\backslash G), L^2(G))$ via the pairing given, for $f \in L^2(K\backslash G), g \in L^2(G)$

and $S \in \mathscr{B}(L^2(K\backslash G), L^2(G))$, by $\langle S, f \otimes g\rangle = \langle Sf, \bar{g}\rangle$, where on the right we have the $L^2(G)$-inner product. Moreover, $V^\infty(K\backslash G, G)$ is the algebra of multipliers of $T(K\backslash G, G)$:

$$V^\infty(K\backslash G, G) = \{w : w \text{ is a complex function on } K\backslash G \times G \\ \text{and } m_w : \omega \mapsto w.\omega \text{ is a bounded linear map on } T(K\backslash G, G)\}$$

with $\|w\|_{V^\infty(K\backslash G,G)} = \|m_w\|$. Here, m_w is defined by $m_w(\widetilde{f} \otimes g)(\bar{t}, s) = w(\bar{t}, s)\widetilde{f}(\bar{t})g(s)$ on elementary tensors. Two functions in $V^\infty(K\backslash G, G)$ are identified if they differ on a marginally null set. For a function $\widetilde{u}$ on G/K define a function on $K\backslash G \times G$ by $N_K^\dagger\widetilde{u}(\bar{t}, s) = \widetilde{u}(s.\dot{t}^{-1})$. Then Theorem 4.7 of [16] says that $N_K^\dagger$ is a complete isometry from the algebra $M_{cb}(A(G/K))$ of completely bounded multipliers onto $V^\infty_{\mathrm{inv}}(K\backslash G \times G)$. Essential use is made of this in our discussion on the relation between Ditkin sets and operator Ditkin sets in the last section.

Let $\mathcal{A}$ be a commutative, semisimple, regular Banach algebra with Gelfand space $\Delta(\mathcal{A})$. For a closed set E in $\Delta(\mathcal{A})$, let

$$j_{\mathcal{A}}(E) = \{a \in \mathcal{A} : \widehat{a} \text{ has compact support disjoint from E}\},$$
$$J_{\mathcal{A}}(E) = \overline{j_{\mathcal{A}}(E)},$$
$$I_{\mathcal{A}}(E) = \{a \in \mathcal{A} : \widehat{a} = 0 \text{ on } E\}.$$

The set E is said to be a *set of weak spectral synthesis* if there is an $n \in \mathbb{N}$ such that $u^n \in J_{\mathcal{A}}(E)$ whenever $u \in I_{\mathcal{A}}(E)$, and in this case we write $\xi_{\mathcal{A}}(E)$ for the least such n. Furthermore, E is called a *set of spectral synthesis* if $\xi_{\mathcal{A}}(E) = 1$, that is, if $I_{\mathcal{A}}(E) = J_{\mathcal{A}}(E)$. We call E a *Ditkin set* if for every $u \in I_{\mathcal{A}}(E)$, there exists a sequence $\{u_n\} \subset j_{\mathcal{A}}(E)$ such that $u.u_n$ converges in norm to u; if the condition holds for every $u \in I^c_{\mathcal{A}}(E) := \mathcal{A}^c \cap I_{\mathcal{A}}(E)$, where $\mathcal{A}^c$ stands for elements in $\mathcal{A}$ with compactly supported Gelfand transforms, then E is called a *local Ditkin set*. If the sequence can be chosen in such a way that it is bounded and is the same for all $u \in I_{\mathcal{A}}(E)$, then we say that E is a *strong Ditkin set*.

3. Ditkin sets. In [1], Derighetti proved the injection theorem for local Ditkin sets for Fourier algebras on quotients under the assumption that the subgroups are normal. We now prove the injection theorem assuming that the subgroups are compact, but not necessarily normal. We first prove the injection theorem for local Ditkin sets for groups. We begin with a lemma that captures the essence of what is needed to utilise Derighetti's arguments in our context.

LEMMA 3.1. *Let G be a locally compact group containing a compact subgroup K and let U be a neighbourhood of K.*

(a) *There is a neighbourhood* $U_1 \subset U$ *of* K *such that* $U_1K \subset U$.
(b) *There are two neighbourhoods* U_1 *and* U_2 *of* K *contained in* U *such that* $U_1U_2 \subseteq U$.
(c) *There is a neighbourhood* U_3 *of* K *contained in* U *such that* $U_3U_3 \subseteq U$.
(d) *There is a neighbourhood* U_4 *of* K *contained in* U *such that* $U_4 = U_4^{-1}$.

Proof. (a) It is an elementary fact that, since K is compact, there is a neighbourhood V of the identity element such that $VK \subset U$. Then $U_1 = VK$ is a neighbourhood of K and $U_1K = VKK = VK \subset U$.

(b) Choose a neighbourhood U_1 of K as in (a). Then $U_1 \cdot k \subset U$ for each $k \in K$ and so there is a neighbourhood U_k of k such that $U_1U_k \subseteq U$. Since $\{U_k\}_{k\in K}$ is an open cover of K and K is compact, there is a finite subset $\{k_1, \ldots, k_n\}$ such that $K \subseteq \bigcup_{i=1}^n U_{k_i}$. Let $U_2 = \bigcup_{i=1}^n U_{k_i}$.

(c) Choose neighbourhoods U_1 and U_2 as in (b). Let $U_3 = U_1 \cap U_2$.

(d) Choose $U_4 = U \cap U^{-1}$. ∎

The following lemma was inspired by a personal communication from Derighetti regarding a result in his paper. We thank Prof. A. Derighetti for his kind response to our query.

LEMMA 3.2. *Given a neighbourhood* U *of* K *and an* $\varepsilon > 0$ *there are a neighbourhood* V *of* K *and a* $u \in A(G)$ *such that* (a) $V \subseteq U$, (b) $u(x) = 1$ *for all* $x \in V$, (c) $\operatorname{supp} u \subseteq U$, *and* (d) $\|u\|_{A(G)} < 1 + \varepsilon$.

Proof. There are seven neighbourhoods $U_1, \ldots, U_6, V$ of K with the following properties:

(1) $\bar{U}_1$ is compact and $\bar{U}_1 \subseteq U$,
(2) $U_2U_2 \subseteq U_1$,
(3) $U_3 \subseteq U_2$ and $U_3 = U_3^{-1}$,
(4) $\bar{U}_3 \subseteq U_4$ and $m(U_4) < (1+\varepsilon)m(\bar{U}_3)$,
(5) $U_5 = U_2 \cap U_4$,
(6) $\bar{U}_3U_6 \subseteq U_5$ and $U_6\bar{U}_3 \subseteq U_5$,
(7) $V \subseteq U_6$ and $V = V^{-1}$.

Choose U_1 as in (1) and then apply the previous lemma to U_1 to get a U_2 as in (2). Regularity of the measure gives a U_4 as in (4). The remaining ones are easy. Now let

$$u = \frac{1}{m(\bar{U}_3)}\chi_{\bar{U}_3} * \chi_{V\bar{U}_3}$$

Simple computations show that this V and u satisfy the requirements of the lemma. ∎

This lemma enables us to adapt Derighetti's arguments to get the next proposition and the theorem following it.

Proposition 3.3. *Given* $u \in A(G)$ *and* $\varepsilon, \eta > 0$, *there exist a* $v \in A(G)$ *and a neighbourhood* W *of* K *such that* (a) $\operatorname{supp} u \subseteq W$, (b) $\|v\|_{A(G)} < 1+\eta$, *and* (c) $\|uv\|_{A(G)} < \varepsilon + \|u|_K\|_{A(K)}$.

Proof. First consider the case when u is identically 0 on K. Since K is a set of spectral synthesis for $A(G)$ (Herz [7], Takesaki–Tatsuuma [22]), there exists a compactly supported function $u' \in A(G)$ such that u' is identically 0 on a neighbourhood of K and $\|u - u'\| < \varepsilon/(1+\eta)$. Let U and V be neighbourhoods of K and $\operatorname{supp} u'$ with $U \cap V = \emptyset$. By the previous lemma, there exist a $v \in A(G)$ and a neighbourhood W of K such that $W \subseteq U$, v is 1 on W, $\operatorname{supp} v \subseteq U$ and $\|v\| < 1+\eta$. Also $\|uv\|_{A(G)} = \|(u-u')v\|_{A(G)} < \varepsilon$.

Next consider a general $u \in A(G)$. Our proof of the general case is slightly different from the one used in [1]. As in [1], we begin by invoking Theorem 1(b) of Herz [7] to get a $v \in A(G)$ such that $\|v\|_{A(G)} = \|u|_K\|_{A(K)}$ and $v|_K = u|_K$. Now, the function $u_1 = u - v$ is 0 on K and hence, by the case considered previously, we get a neighbourhood W of K and a function $v \in A(G)$ such that $v = 1$ on W, $\|v\|_{A(G)} < 1+\rho$ and $\|u_1.v\|_{A(G)} < \varepsilon/2$; here ρ is a positive real number, to be appropriately chosen below. Then

$$\begin{aligned}\|u.v\|_{A(G)} &= \|(u_1+v)v\|_{A(G)} \le \|u_1.v\|_{A(G)} + \|v.v\|_{A(G)} \\ &< \varepsilon/2 + (1+\rho)\|u|_K\|_{A(K)}.\end{aligned}$$

Choose $\rho > 0$ such that $\rho\|u|_K\|_{A(K)} < \varepsilon/2$ to complete the proof. ∎

The next result is known even without the compactness condition (see [11], where the methods are entirely different).

Theorem 3.4. *Let* K *be a compact subgroup* G. *Then* K *is a local Ditkin set for* $A(G)$.

Proof. In view of Proposition 3.3, the proof is essentially the same as the proof of Théorème 11 of [1]. ∎

Theorem 3.5 (Injection theorem). *Let* K *be a compact subgroup of* G *and let* E *be a closed subset of* K. *Then* E *is a local Ditkin set for* $A(K)$ *if and only if it is so for* $A(G)$.

Proof. Even without the compactness condition on the subgroup K, it is an easy fact to note that if E is locally Ditkin for $A(G)$ then it is so for $A(K)$. So we need only prove the other part of the statement. Using Proposition 3.3 and the injection theorem for sets of local synthesis [1, Proposition 8], we can repeat the proof of Théorème 12 in [1]. The details may safely be omitted. ∎

A direct image theorem for sets of spectral synthesis is given in [16]. We can now present the analogous result for Ditkin sets, first for $A(G/K)$ and then for $A(K\backslash G/K)$.

PROPOSITION 3.6 (Direct image theorem). *Let $E \subset G$ be a (local) Ditkin set for $A(G)$. Then $\mathsf{q}(E)$ is also a (local) Ditkin set for $A(G/K)$.*

Proof. We write the proof for Ditkin sets. Suppose E is a Ditkin set for $A(G)$ and let $\widetilde{u} \in I_{A(G/K)}(\mathsf{q}(E))$. Then the corresponding $u \in A(G)$ belongs to $I_{A(G)}(E)$ by Lemma 3.1 of [16]. By hypothesis, given $\varepsilon > 0$ there is a $v \in j_{A(G)}(E)$ such that $\|u - uv\|_{A(G)} < \varepsilon$. If $\widetilde{v} = P_K v$, then $\|\widetilde{u} - \widetilde{u}\widetilde{v}\|_{A(G/K)} < \varepsilon$ and $\widetilde{v} \in j_{A(G/K)}(\mathsf{q}(E))$. Hence the result follows. ∎

COROLLARY 3.7. *If H is any compact subgroup of G containing K, then $H/K \subset G/K$ is a local Ditkin set for $A(G/K)$.*

Proof. Apply Theorem 3.4 and Proposition 3.6. ∎

A function $u \in A(G)$ that is constant on double cosets of K can be considered as a function on the double coset space $K\backslash G/K$ in the natural way and this gives the Fourier algebra $A(K\backslash G/K)$. We write $\pi = {}_K\pi_K$ for the canonical quotient map $G \to K\backslash G/K$ and write $\underline{t}$ for the double coset $\pi(t) = KtK$. Observe also that there is a natural projection ${}_KP_K$ of $A(G)$ onto $A(K\backslash G/K)$. It is well known that $A(K\backslash G/K)$ is a commutative, regular, tauberian Banach algebra with Gelfand maximal ideal space identified with $K\backslash G/K$ via pointwise evaluation (in the more general context of ultraspherical hypergroups, the result is found in Theorem 3.13 of [13]).

LEMMA 3.8. *Let K be a compact subgroup of a locally compact group G. Let $\widetilde{E} \subseteq K\backslash G/K$ be closed and let $E = \pi^{-1}(\widetilde{E}) \subseteq G$. For a function $\widetilde{u}$ on $K\backslash G/K$ let u be the corresponding function on G that is constant on double cosets of K: $u(t) = \widetilde{u}(\pi(t))$. Then*

(i) $u \in j_{A(G)}(E)$ *if and only if* $\widetilde{u} \in j_{A(K\backslash G/K)}(\widetilde{E})$;
(ii) $u \in J_{A(G)}(E)$ *if and only if* $\widetilde{u} \in J_{A(K\backslash G/K)}(\widetilde{E})$;
(iii) $u \in I_{A(G)}(E)$ *if and only if* $\widetilde{u} \in I_{A(K\backslash G/K)}(\widetilde{E})$.

Proof. (i) Observe that if $u(t) = 0$ for all t in an open set $V \supseteq E$ in G, then $\widetilde{u}(\underline{t}) = 0$ for all $\underline{t}$ in the open set $\pi(V) \supseteq \widetilde{E}$, and if $\widetilde{u}(\underline{t}) = 0$ for all $\underline{t}$ in an open set $\widetilde{V} \supseteq \widetilde{E}$ in $K\backslash G/K$, then $u(t) = 0$ for all $t \in \pi^{-1}(\widetilde{V}) \supseteq E$.

(ii) is a consequence of (i) and the fact that $u \leftrightarrow \widetilde{u}$ is bicontinuous, and (iii) is clear. ∎

THEOREM 3.9 (Direct image theorem for double coset spaces). *Let K be a compact subgroup of a locally compact group G and let $E \subseteq G$ be closed.*

(i) *If E is a set of weak synthesis for $A(G)$, then the closed set $\pi(E)$ is a set of weak synthesis for $A(K\backslash G/K)$ with $\xi_{A(K\backslash G/K)}(\pi(E)) \le \xi_{A(G)}(E)$. In particular, if E is of synthesis for $A(G)$, then so is $\pi(E)$ for $A(K\backslash G/K)$.*

(ii) *Let $E \subset G$ be a (local) Ditkin set for $A(G)$. Then $\pi(E)$ is also a (local) Ditkin set for $A(K\backslash G/K)$.*

Proof. (i) Suppose E is a set of weak synthesis for $A(G)$ with $\xi_{A(G)}(E)=n$. Let $\widetilde{u} \in I_{A(K\backslash G/K)}(\pi(E))$. Then $u \in I_{A(G)}(E)$ by Lemma 3.8 and, by the assumption on E, $u^n \in J_{A(G)}(E)$. By Lemma 3.8 again, this in turn gives $\widetilde{u}^n \in J_{A(K\backslash G/K)}(\pi(E))$.

(ii) Using Lemma 3.8 (in place of Lemma 3.1 of [16]), the proof is quite similar to that of Proposition 3.6. ■

Corollary 3.10. *Singletons are sets of synthesis and local Ditkin sets for $A(K\backslash G/K)$.*

Proof. Choose E to be a singleton set in the previous theorem. ■

The next result is (one part of) the inverse projection theorem for synthesis in the context of $A(K\backslash G/K)$. Even for $A(G/K)$, only one part is known and is found in [4]. Our proof is similar to that of Forrest.

Proposition 3.11. *Let $\widetilde{E}$ be a closed subset of $K\backslash G/K$. If $E := \pi^{-1}(\widetilde{E})$ is a set of (weak) synthesis for $A(G)$, then $\widetilde{E}$ is a set of (weak) synthesis for $A(K\backslash G/K)$.*

Proof. We only consider the case of sets of synthesis and prove the contrapositive statement. Suppose that there is a function $\widetilde{u} \in I_{A(K\backslash G/K)}(\widetilde{E})$ with $\widetilde{u} \notin J_{A(K\backslash G/K)}(\widetilde{E})$. If $u = \widetilde{u} \circ \pi$, then $u \in I_{A(G)}(E)$. We claim that $u \notin J_{A(G)}(E)$. Otherwise there is a sequence $\{u_n\}$ in $j_{A(G)}(E)$ such that $\|u_n - u\|_{A(G)} \to 0$ and so, writing $u_n = \widetilde{u}_n \circ {}_KP_K$ with $\widetilde{u}_n \in A(K\backslash G/K)$, we have $\|\widetilde{u}_n - \widetilde{u}\|_{A(K\backslash G/K)} \to 0$. Since $\operatorname{supp} \widetilde{u}_n \subset \pi(\operatorname{supp} u_n)$, as is easy to see, it follows that $\widetilde{u}_n \in j_{A(K\backslash G/K)}(\widetilde{E})$ and we run into a contradiction, as $\widetilde{u} \notin J_{A(K\backslash G/K)}(\widetilde{E})$. This completes the proof. ■

Corollary 3.12. *The singleton $\{\underline{e}\}$, the double coset of the identity element, is a set of synthesis for $A(K\backslash G/K)$.*

Proof. If $\widetilde{E} = \{\underline{e}\}$, then $E = KeK = K$ is a set of synthesis for $A(G)$ by [7] or [22]. ■

Observe that Corollary 3.12 is a special case of Corollary 3.10. We also remark that Corollary 3.10 cannot be deduced from Proposition 3.11 since a double coset KxK may not be of synthesis for $A(G)$. In fact, it has been proved by Meaney [12] that if G is a noncompact, connected, real semisimple Lie group with finite centre, then, with the standard notation, KaK is not a set of synthesis for $A(G)$ when $a \in A$, $a \neq e$, provided that G/K has rank one and dimension at least two. This also shows that the converse of Proposition 3.10 is not true. Thus the full inverse projection theorem for sets of synthesis is not valid for $A(K\backslash G/K)$.

REMARK. To prove the injection theorem for coset spaces, we need the analogues of Lemma 3.2 and Proposition 3.3 with G/K in place of G. If the analogue of Lemma 3.2 for $A(G/K)$ could be proved, then invoking the injection theorem for sets of local synthesis on homogeneous spaces, the result that H/K is a set of synthesis for $A(G/K)$ (for a closed subgroup H of G containing K) and other results of [16] and proceeding as in Proposition 3.3 and Theorem 3.5, we could prove the injection theorem for G/K. But we are unable to quite get the needed lemma.

4. Strong Ditkin sets. Throughout this section we assume that G is compact. We prove the inverse projection theorem for strong Ditkin sets, and also obtain a relation between strong Ditkin sets in the Fourier algebra and the Varopoulos algebra.

We begin with what appears to be the first complete inverse projection theorem in the literature in the context of Ditkin sets.

THEOREM 4.1 (Inverse projection theorem). *Assume that G is compact. A closed set $\widetilde{E} \subset G/K$ is a strong Ditkin set for $A(G/K)$ if and only if $\mathsf{q}^{-1}(\widetilde{E})$ is a strong Ditkin set for $A(G)$.*

Proof. Assume $\mathsf{q}^{-1}(\widetilde{E})$ is a strong Ditkin set for $A(G)$ and let $\widetilde{u} \in I_{A(G/K)}(\widetilde{E})$. Then $u \in I_{A(G)}(\mathsf{q}^{-1}(\widetilde{E}))$ and there is a bounded sequence $u_n \in j_{A(G)}(\mathsf{q}^{-1}(\widetilde{E}))$ such that $\|u_n u - u\|_{A(G)} \to 0$. Observing that $P_K u_n \in j_{A(G/K)}(\widetilde{E})$ and that $\|P_K u_n.\widetilde{u} - \widetilde{u}\|_{A(G/K)} \to 0$, we conclude that $\widetilde{E}$ is a strong Ditkin set for $A(G/K)$.

Conversely, suppose that $\widetilde{E}$ is a strong Ditkin set for $A(G/K)$ and let $v \in I_{A(G)}(\mathsf{q}^{-1}(\widetilde{E}))$. For each irreducible, unitary representation π of G, consider the matrix-valued function v^π defined by $v^\pi(s) = \int_K v(sk)\pi(k)\,dk$ and let $\widetilde{v}^\pi(s) = \pi(s)v^\pi(s)$. It is clear that $v^\pi(sk) = v^\pi(s)$ for $s \in G$, $k \in K$. The same property holds for $\widetilde{v}^\pi$ as well:

$$\begin{aligned}\widetilde{v}^\pi(sk) &= \pi(sk)\int_K v(skk')\pi(k')\,dk' = \pi(sk)\int_K v(sk')\pi(k^{-1}k')\,dk' \\ &= \pi(sk)\int_K v(sk')\pi(k^{-1})\pi(k')\,dk' = \pi(s)\int_K v(sk')\pi(k')\,dk' = \widetilde{v}^\pi(s).\end{aligned}$$

Writing u^π_{ij} for the matrix coefficients of π let $v^\pi_{ij} := \int_K u^\pi_{ij}(k)v.k\,dk$ where $v.k(s) = v(sk)$. Note that $v \in I_{A(G)}(\mathsf{q}^{-1}(\widetilde{E}))$ implies that $v.k$ belongs to $I_{A(G)}(\mathsf{q}^{-1}(\widetilde{E}))$ for all $k \in K$ and hence $v^\pi_{ij} \in I_{A(G)}(\mathsf{q}^{-1}(\widetilde{E}))$. Moreover, $\widetilde{v}^\pi_{ij} = \sum u^\pi_{il}v^\pi_{lj} \in I_{A(G)}(\mathsf{q}^{-1}(\widetilde{E}))$. But $\widetilde{v}^\pi_{ij}(sk) = \widetilde{v}^\pi_{ij}(s)$, $s \in G$, $k \in K$. Thus $\widetilde{v}^\pi_{ij}$ may be considered as a function in $A(G/K)$ and then belongs to $I_{A(G/K)}(\widetilde{E})$. As $\widetilde{E}$ is a strong Ditkin set for $A(G/K)$, there is a sequence $(\widetilde{u}_n)$ in $j_{A(G/K)}(\widetilde{E})$

such that $\|\widetilde{u}_n\widetilde{v}^\pi_{ij}-\widetilde{v}^\pi_{ij}\|_{A(G/K)} \to 0$ and $\|\widetilde{u}_n\|_{A(G/K)}$ is bounded. Let u_n denote the function in $A(G)$ corresponding to $\widetilde{u}_n$.

Choose a bounded approximate identity $\{e_\alpha\}$ in $L^1(G)$ as in [21], [14]. Then, for each α, $e_\alpha.v = \lim_n u_n(e_\alpha.v)$ in $A(G)$. But $v = \lim_\alpha e_\alpha.v$. Now, writing $\|\cdot\|_A$ for the norm in $A(G)$,

$$\begin{aligned}\|u_n v - v\|_A &\le \|u_n v - u_n e_\alpha.v\|_A + \|u_n e_\alpha v - e_\alpha.v\|_A + \|e_\alpha.v - v\|_A \\ &\le c\|e_\alpha.v - v\|_A + \|u_n e_\alpha v - e_\alpha.v\|_A + \|e_\alpha.v - v\|_A,\end{aligned}$$

since (u_n) is bounded. Given $\varepsilon > 0$, fix an α such that each of the first and last terms is less than $\varepsilon/3$. For this α, there is an n_0 such that the middle term is less than $\varepsilon/3$ for $n \ge n_0$. Thus $u_n v \to v$ and hence $\mathsf{q}^{-1}(\widetilde{E})$ is a strong Ditkin set for $A(G)$ and the proof is complete. ∎

For abelian groups, Varopoulous [23] proved the classical theorem of Malliavin on the failure of spectral synthesis using the relation he obtained between synthesis in the Fourier algebra and the Varopoulous algebra. For such a relation on nonabelian groups, see [21] and [14]. The corresponding relation for Ditkin sets seems to be unavailable even for abelian groups. Here is the analogous result for strong Ditkin sets in homogeneous spaces. Define $\widetilde{E}^\natural = \{(\bar{t}, s) \in K\backslash G \times G : s.\dot{t}^{-1} \in \widetilde{E}\}$ for a subset $\widetilde{E}$ of G/K. Let Q_K be the contractive projection from $V(G, K\backslash G)$ onto $V_{\mathrm{inv}}(G, K\backslash G)$ defined by $Q_K w(s, \bar{t}) = \int_G w(sx, \bar{t}.x)\,dx$ considered in [16] (to which we refer for more details).

Theorem 4.2. *A closed subset $\widetilde{E}$ of G/K is a strong Ditkin set for $A(G/K)$ if and only if $\widetilde{E}^\natural$ is a strong Ditkin set for $V(G, K\backslash G)$.*

Proof. One part easily follows from Lemma 3.11 of [16]. Suppose, first, that $\widetilde{E}^\natural$ is a strong Ditkin set for $V(G, K\backslash G)$ and let $(\widetilde{v}_n)$ be a bounded sequence in $j_{V(G,K\backslash G)}(\widetilde{E}^\natural)$ such that $\|\widetilde{v}_n\widetilde{v} - \widetilde{v}\|_{V(G,K\backslash G)} \to 0$ as $n \to \infty$ for all $\widetilde{v} \in I_{V(G,K\backslash G)}(\widetilde{E}^\natural)$. If $\widetilde{u} \in I_{A(G/K)}(\widetilde{E})$, then $N_K\widetilde{u} \in I_{V(G,K\backslash G)}(\widetilde{E}^\natural)$ and so $\|\widetilde{v}_n N_K\widetilde{u} - N_K\widetilde{u}\|_{V(G,K\backslash G)} \to 0$. It follows that $Q_K\widetilde{v}_n.N_K\widetilde{u} \to N_K\widetilde{u}$ in $V(G, K\backslash G)$ and hence that $\widetilde{u}_n := N_K^{-1}(Q_K\widetilde{v}_n)\widetilde{u} \to \widetilde{u}$ in $A(G/K)$. Observe that $(\widetilde{u}_n)$ is a bounded sequence and that $\widetilde{v}_n \in j_{V(G,K\backslash G)}(\widetilde{E}^\natural)$ implies $Q_K\widetilde{v}_n \in j_{V(G,K\backslash G)}(\widetilde{E}^\natural)$. Thus, by Lemma 3.11 of [16], $\widetilde{u}_n \in j_{A(G/K)}(\widetilde{E})$. This shows that $\widetilde{E}$ is a strong Ditkin set for $A(G/K)$.

For the converse, we make use of the ideas in [14] and [16]. Assume that $\widetilde{E}$ is a strong Ditkin set for $A(G/K)$ and let $(\widetilde{u}_n)$ be a bounded sequence in $j_{A(G/K)}(\widetilde{E})$ as in the definition. Let $\widetilde{w} \in I_{V(G,K\backslash G)}(\widetilde{E}^\natural)$. For each $\pi \in \widehat{G}$, consider, with the notation of [16], $\widetilde{w}^\pi_{ij}, \widetilde{w}^{*\pi}_{ij} \in I_{V(G,K\backslash G)}(\widetilde{E}^\natural) \cap V_{\mathrm{inv}}(G, K\backslash G)$. Then $N_K^{-1}\widetilde{w}^{*\pi}_{ij} \in I_{A(G/K)}(\widetilde{E})$ and so $\widetilde{u}_n N_K^{-1}\widetilde{w}^{*\pi}_{ij} \to N_K^{-1}\widetilde{w}^{*\pi}_{ij}$. It follows that

$N_K\widetilde{u}_n\widetilde{w}_{ij}^{*\pi} \to \widetilde{w}_{ij}^{*\pi}$. Choosing an approximate identity (e_α) for $L^1(G)$ as in [16], we obtain $N_K\widetilde{u}_n(e_\alpha.\widetilde{w}) \to e_\alpha.\widetilde{w}$ for each α. Now

$$\begin{aligned}\|N_K\widetilde{u}_n\widetilde{w} - \widetilde{w}\|_{V(G,K\backslash G)} &\le \|N_K\widetilde{u}_n\widetilde{w} - N_K\widetilde{u}_n(e_\alpha.\widetilde{w})\|_{V(G,K\backslash G)} \\ &\quad + \|N_K\widetilde{u}_n(e_\alpha.\widetilde{w} - e_\alpha.\widetilde{w}\|_{V(G,K\backslash G)} \\ &\quad + \|e_\alpha.\widetilde{w} - \widetilde{w}\|_{V(G,K\backslash G)} \\ &\le C\|e_\alpha.\widetilde{w} - \widetilde{w}\|_{V(G,K\backslash G)} \\ &\quad + \|N_K\widetilde{u}_n(e_\alpha.\widetilde{w}) - e_\alpha.\widetilde{w}\|_{V(G,K\backslash G)}.\end{aligned}$$

Now fix an α so that the first term is small and then, for this α, the second term is small for all sufficiently large n. Since $N_K\widetilde{u}_n$ is a bounded sequence in $j_{V(G,K\backslash G)}(\widetilde{E}^\natural)$, this completes the proof. ∎

5. Operator Ditkin sets. In this penultimate section of the paper, we look for relations between Ditkin sets and operator Ditkin sets. (For a recent study of relations between sets of spectral synthesis and sets of operator synthesis, see [15].) In the case when K is the trivial one-element subgroup, the main result of this section, Theorem 5.2, is due to Ludwig and Turowska [11]. We use techniques from [11] (extended to our settings) and [16] to prove this result. The next lemma encapsulates much of the technicalities involved in the proof of Theorem 5.2. We write $\Phi(F) = \{\omega \in T(K\backslash G, G) : \omega = 0 \text{ on } F\}$ for any closed set F in $K\backslash G \times G$.

Lemma 5.1. *Let $\widetilde{E} \subseteq G/K$ be closed. Suppose (u_n) is a sequence in $j_{A(G/K)}(\widetilde{E})$ such that $\|u_n u - u\|_{A(G/K)} \to 0$ as $n \to \infty$ for all $u \in I_{A(G/K)}(\widetilde{E})$. Then $N_K^\dagger u_n.\omega \to \omega$ weakly for any $\omega \in \Phi(\widetilde{E}^\sharp)$.*

Proof. Let us begin by observing that as a consequence of the assumption on (u_n) we have

$$\|N_K^\dagger u_n N_K^\dagger u - N_K^\dagger u\|_{V^\infty(K\backslash G,G)} \to 0.$$

Let $w \in V^\infty(K\backslash G, G)$ vanish on $\widetilde{E}^\sharp$ and have compact support, say support $w \subseteq \mathsf{p}(C) \times C$ where $C \subseteq G$ is compact. For an irreducible unitary representation π of G, let u_{ij}^π, w_{ij}^π, and $\widetilde{w}_{ij}^\pi$ be as in the previous section. We have $\sum_k |\langle (N_K^\dagger u_n - 1)\widetilde{w}_{ik}^\pi.T, u_{jk}^\pi N_K^\dagger u.\omega\rangle| < \infty$ by an application of the Schwarz inequality, for $u \in A(G/K)$, $T \in \mathscr{B}(L^2(K\backslash G), L^2(G))$ and $\omega \in T(K\backslash G, G)$. So, given $\varepsilon > 0$, we have $\sum_{k\notin F} |\langle (N_K^\dagger u_n - 1)\widetilde{w}_{ik}^\pi.T, u_{jk}^\pi N_K^\dagger u.\omega\rangle| < \varepsilon$ for a suitably chosen finite set F. The finite sum $\sum_{k\in F} |\langle (N_K^\dagger u_n - 1)\widetilde{w}_{ik}^\pi.T, u_{jk}^\pi N_K^\dagger u.\omega\rangle|$ is estimated by applying Theorem 4.7 of [16] to get a $v_{ij}^\pi \in M_{cb}(A(G/K))$ such that $\widetilde{w}_{ij}^\pi = N_K^\dagger v_{ij}^\pi$. Then

$$\begin{aligned}\sum_{k\in F}|\langle (N_K^\dagger u_n-1)\widetilde{w}_{ik}^\pi.T, u_{jk}^\pi N_K^\dagger u.\omega\rangle| &= \sum_{k\in F}|\langle T, u_{jk}^\pi (N_K^\dagger u_n-1)N_K^\dagger(v_{ik}^\pi u).\omega\rangle|\\ &\le \sum_{k\in F}\|T\|\|(N_K^\dagger u_n-1)N_K^\dagger(v_{ik}^\pi u)\|\|\omega\|,\end{aligned}$$

the norms being, respectively, the operator norm, the norm in $V^\infty(K\backslash G,G)$ and the norm in $T(K\backslash G,G)$. The middle term in the sum tends to zero, by the opening line of this proof. Thus the finite sum is less than ε for sufficiently large n. So it follows that $\langle (N_K^\dagger u_n-1)w_{ij}^\pi.T, N_K^\dagger u.\omega\rangle \to 0$ as $n\to\infty$ because $w_{ij}^\pi = \sum_k u_{kj}^\pi.\widetilde{w}_{ik}^\pi$. An application of [19, Corollary 4.3], shows that $\mathrm{span}\{N_K^\dagger u.\omega : u\in A(G/K),\ \omega\in T(K\backslash G,G)\}$ is dense in $T(K\backslash G,G)$, and since $(N_K^\dagger u_n)$ is a bounded sequence, we conclude that $\langle (N_K^\dagger u_n-1)w_{ij}^\pi.T,\omega\rangle\to 0$ as $n\to\infty$ for all $\omega\in T(K\backslash G,G)$.

Choosing an approximate identity (e_α) such that each e_α is a continuous function supported in CC^{-1} and approximating these by sums of the form $u=\sum c_i u_i \chi_{CC^{-1}}$, where u_i are matrix coefficients, we see that w can be approximated in $T(K\backslash G,G)$ by $u.w$. For $\omega\in T(K\backslash G,G)$,

$$\begin{aligned}&|\langle (N_K^\dagger u_n-1)w.T,\omega\rangle|\\ &\qquad\le |\langle (N_K^\dagger u_n-1).T,(w-u.w)\omega\rangle| + |\langle (N_K^\dagger u_n-1).T, u.w\omega\rangle|.\end{aligned}$$

The preceding observation and the boundedness of $(N_K^\dagger u_n)$ imply that the first term is small for all n. On the other hand,

$$\langle (N_K^\dagger u_n-1).T, u_{ij}^\pi w\omega\rangle = \langle (N_K^\dagger u_n-1)w_{ij}^\pi.T,\omega\rangle \to 0$$

and hence the second term above tends to zero. Thus

$$\langle (N_K^\dagger u_n-1)w.T,\omega\rangle\to 0,\quad \text{i.e.}\quad \langle N_K^\dagger u_n.T, w.\omega\rangle \to \langle T, w.\omega\rangle$$

for all $\omega\in T(K\backslash G,G)$. But

$$\mathrm{span}\{w\omega : w\in V^\infty(K\backslash G,G),\ w=0 \text{ on } \widetilde{E}^\sharp,\ \omega\in T(K\backslash G,G)\}$$

is dense in $\Phi(\widetilde{E}^\sharp)$ by Proposition 5.3 of [20]. Thus $\langle T, N_K^\dagger u_n.\omega\rangle\to\langle T,\omega\rangle$ for all $\omega\in\Phi(\widetilde{E}^\sharp)$ and the lemma is proved. ∎

Before presenting the next result, we need a definition. A closed set $F\subseteq K\backslash G\times G$ is called an *operator Ditkin set with respect to* $m_{K\backslash G}\times m_G$, or an $m_{K\backslash G}\times m_G$*-Ditkin set*, if for each $\omega\in\Phi(F)$ there is a sequence $(\widetilde{w}_n)$ in

$$\psi_{00}(F) := \{\widetilde{w}\in V^\infty(K\backslash G,G) : \widetilde{w}=0 \text{ on a neighbourhood of } F\}$$

such that $\|\widetilde{w}_n.\omega-\omega\|_{T(K\backslash G,G)}\to 0$. If a bounded sequence $(\widetilde{w}_n)$ with this property can be found independent of $\omega\in\Phi(F)$, then the set F is said to be a *strong* $m_{K\backslash G}\times m_G$*-Ditkin set.*

THEOREM 5.2. *Let G be a second countable locally compact group with K a compact subgroup. Let $\widetilde{E}\subseteq G/K$ be a closed set.*

(i) *If $\widetilde{E}^\sharp$ is a strong $m_{K\backslash G}\times m_G$-Ditkin set, then $\widetilde{E}$ is a local Ditkin set for $A(G/K)$.*

(ii) *If $\widetilde{E}$ is a strong Ditkin set for $A(G/K)$, then $\widetilde{E}^\sharp$ is an $m_{K\backslash G}\times m_G$-Ditkin set.*

Proof. (i) Suppose that $\widetilde{E}^\sharp$ is a strong $m_{K\backslash G}\times m_G$-Ditkin set and let $(\widetilde{w}_n)$ be a sequence in $\psi_{00}(\widetilde{E}^\sharp)$ such that $\|\widetilde{w}_n.\omega-\omega\|_{T(K\backslash G,G)}\to 0$ for all ω in $\Phi(\widetilde{E}^\sharp)$. Let $0\neq u\in I^c_{A(G/K)}(\widetilde{E})$ and let $C'=\operatorname{supp} u$. Choose a compact set $C\subseteq G$ such that $\mathsf{q}(C)=C'$. Define $N_K^\dagger u^{C'}(\bar t,s)=N_K^\dagger u(\bar t,s)\chi_{C'}(\dot t^{-1})=u(s.\dot t^{-1})\chi_{C'}(\dot t^{-1})$, $s\in G$, $\bar t\in K\backslash G$. Note that if $\bar t=\bar{t_1}$, then $\dot t^{-1}=\dot{t_1}^{-1}$ and the function is well defined. If $s.\dot t^{-1}\in C'$ and $\dot t^{-1}$, then $st^{-1}\in CK$, $t\in KC^{-1}$ and so $s\in CKC^{-1}$. Thus $N_K^\dagger u^{C'}(\bar t,s)=\chi_{CKC^{-1}}(s)u(s.\dot t^{-1})\chi_{C'}(\dot t^{-1})$, and hence $N_K^\dagger u^{C'}\in V^\infty(K\backslash G,G)$ vanishes outside a compact set and can be considered as a function in $T(K\backslash G,G)$. Observe also that $N_K^\dagger u^{C'}$ vanishes on $\widetilde{E}^*$ and so belongs to $\Phi(\widetilde{E}^\sharp)$. Hence $\|\widetilde{w}_nN_K^\dagger u^{C'}-N_K^\dagger u^{C'}\|_{T(K\backslash G,G)}\to 0$ by assumption. If $Q_K^\dagger:T(K\backslash G,G)\to A(G/K)$ is the contraction defined by $Q_K^\dagger w(\dot s)=\int w(\overline{s^{-1}}.t,t)\,dt$, then $\|Q_K^\dagger(\widetilde{w}_nN_K^\dagger u^{C'})-Q_K^\dagger(N_K^\dagger u^{C'})\|_{A(G/K)}\to 0$. On the other hand

$$\begin{aligned}
Q_K^\dagger(\widetilde{w}_nN_Ku^{C'})(\dot s)&=\int_G \widetilde{w}_n(\overline{s^{-1}}.t,t)N_Ku^{C'}(\overline{s^{-1}}.t,t)\,dt\\
&=\int_G \widetilde{w}_n(\overline{s^{-1}}.t,t)\chi_{CKC^{-1}}(t)u(\dot s)\chi_{C'}(t^{-1}.\dot s)\,dt\\
&=u(\dot s)\int_G \widetilde{w}_n(\overline{s^{-1}}.t,t)\chi_{CKC^{-1}}(t)\chi_{C'}(t^{-1}.\dot s)\,dt\\
&=u(\dot s)\int_G \widetilde{w}_n(\overline{s^{-1}}.t,t)\chi_{C'}\otimes\chi_{CKC^{-1}}(t,t^{-1}.\dot s)\,dt\\
&=u(\dot s)\int_G \widetilde{w}_n(\overline{s^{-1}}.t,t)\chi_{C''}\otimes\chi_{CKC^{-1}}(\overline{s^{-1}}.t,t)\,dt\\
&=u(\dot s)Q_K^\dagger(\widetilde{w}_n\chi_{C''}\otimes\chi_{CKC^{-1}})(\dot s)
\end{aligned}$$

where $C''=\{\bar t\in K\backslash G:\dot t^{-1}\in C'\}$, whereas

$$\begin{aligned}
Q_K^\dagger(N_K^\dagger u^{C'})(\dot s)&=\int_G N_K^\dagger u^{C'}(\overline{s^{-1}}.t,t)\,dt=u(\dot s)\int_G \chi_{C'}(t^{-1}.\dot s)\,dt\\
&=u(\dot s)\int_G \chi_{C'}(\dot{t^{-1}})\,dt=\alpha_{C'}u(\dot s),\qquad\text{say.}
\end{aligned}$$

Taking $\alpha_{C'}\,u_n=Q_K^\dagger(\widetilde{w}_n\chi_{C''}\otimes\chi_{CKC^{-1}})$ we see that $\|u_nu-u\|_{A(G/K)}\to 0$ and, moreover, $u_n\in j_{A(G/K)}(\widetilde{E})$. The proof of (i) is thus complete.

(ii) If $\widetilde{E}$ is a strong Ditkin set, there is a sequence (u_n) as in Lemma 5.1. Therefore, the lemma implies that $N_K^\dagger u_n.\omega \to \omega$ weakly for all $\omega \in \Phi(\widetilde{E}^\sharp)$. Then $\|\widetilde{w}_n\omega - \omega\|_{T(K\backslash G,G)} \to 0$ where each $\widetilde{w}_n$ is a convex combination of the $N_K^\dagger u_j$ (see, for example, Rudin [18, Theorem 3.13]). This sequence $(\widetilde{w}_n)$ satisfies the requirements of the definition of an $m_{K\backslash G} \times m_G$-Ditkin set and the proof is complete. ∎

6. Miscellaneous remarks. Before we conclude, we would like to mention some easy results on unions of Ditkin sets and on Ditkin sets for tensor products.

In [24], Warner proved that the union of two sets of synthesis is again a set of synthesis provided their intersection is a Ditkin set. Following the methods in [10], where a nonabelian analogue was obtained, we can prove the corresponding results for Ditkin sets and sets of synthesis in the context of homogenous spaces. For the sake of completeness, we record these results in this section. We begin by recasting the definition of Ditkin sets as well as those of the local and strong variants. We let $\mathrm{UC}_c(\widehat{G/K})$ be the space of operators in $\mathrm{VN}(G/K)$ with compact support.

Proposition 6.1. *Let $\widetilde{E}$ be a closed set in G/K. Then the following are equivalent:*

(i) *$\widetilde{E}$ is a local Ditkin set for $A(G/K)$.*
(ii) *For $T \in \mathrm{UC}_c(\widehat{G/K})$ and $u \in I^c_{A(G/K)}(\widetilde{E})$ there is a $v \in j_{A(G/K)}(\widetilde{E})$ such that $\langle T,u\rangle = \langle T,uv\rangle$.*
(iii) *For $T \in \mathrm{UC}_c(\widehat{G/K})$ and $u \in I^c_{A(G/K)}(\widetilde{E})$ there is a net $\{v_\alpha\}$ in $j_{A(G/K)}(\widetilde{E})$ such that $\langle T,u\rangle = \lim \langle T,uv_\alpha\rangle$.*

Similar characterisations of Ditkin sets (*and strong Ditkin sets*) *are obtained by replacing* $\mathrm{UC}_c(\widehat{G/K})$ *by* $\mathrm{VN}(G/K)$ (*and prescribing that, for strong Ditkin sets, v in* (ii) *and v_α in* (iii) *come from a bounded set depending only on $\widetilde{E}$*).

Proof. We shall write the proof only for local Ditkin sets. To see that (i) implies (ii), we just have to observe that $\{\langle T,uv\rangle : v \in j_{A(G/K)}(\widetilde{E})\} = \mathbb{C}$ if $\langle T,u\rangle \neq 0$. That (ii) implies (iii) is trivial. So assume that (iii) holds and let $u \in I^c_{A(G/K)}(\widetilde{E})$. We have to show that if $T \in \mathrm{VN}(G/K)$ annihilates $uj_{A(G/K)}(\widetilde{E})$ then $\langle T,u\rangle = 0$. Choose $v \in A^c(G/K)$ with $v = 1$ on the support of u so that $u = uv$. Then $v.T \in \mathrm{UC}_c(\widehat{G/K})$ for any $T \in \mathrm{VN}(G/K)$ and, by assumption, $\langle T,u\rangle = \lim \langle T,uv_\alpha\rangle$ and the last quantity is zero if T annihilates $uj_{A(G/K)}(\widetilde{E})$. The proof is complete for the case of local Ditkin sets. ∎

THEOREM 6.2. *Let G be a locally compact group and K a compact subgroup of G. Suppose that E_1 and E_2 are closed subsets of G/K such that $E_1 \cap E_2$ are Ditkin sets for $A(G/K)$. Then $E_1 \cup E_2$ is a set of spectral synthesis (respectively, set of local synthesis, local Ditkin set, Ditkin set, strong Ditkin set) for $A(G/K)$ if and only if both E_1 and E_2 are so.*

Proof. We shall indicate the proof in the case of sets of synthesis and Ditkin sets. By the previous proposition, the other cases are similar.

Suppose first that E_1 and E_2 are sets of spectral synthesis for $A(G/K)$. Assume $T \in \mathrm{VN}(G/K)$, $\operatorname{supp} T \subset E_1 \cup E_2$ and $u \in I_{A(G/K)}(E_1 \cup E_2)$. Since $E_1 \cap E_2$ is a Ditkin set, there is a $v \in j_{A(G/K)}(E_1 \cap E_2)$ such that $\langle T, u\rangle = \langle v.T, u\rangle$. Then we construct, as in [10], $v_1, v_2 \in A(G/K)$ with disjoint compact supports such that $(v_1 + v_2)vT = vT$ and $\operatorname{supp} v_jT \subset E_j$. If E_1 and E_2 are sets of synthesis, it follows that $\langle (v_jv).T, u\rangle = 0$ for $j = 1, 2$ and hence $\langle v.T, u\rangle = 0$, which is what we required. The converse is also proved as for the case of $A(G)$ (see [10]).

Next consider Ditkin sets. The union of two Ditkin sets is again a Ditkin set for any commutative, regular, semisimple Banach algebra, without any assumption on their intersection. The proof of the converse is similar to the one given in [10] for $A(G)$ and we omit it. ∎

REMARK. The theorem holds for any regular, tauberian, commutative, semisimple Banach algebra.

COROLLARY 6.3. *Suppose that $A(G/K)$ has an approximate identity and let E be an open-closed set in G/K. Then E is a Ditkin set.*

We now come to tensor products. Let G be a locally compact group and H a compact group. Let K and L be compact subgroups of G and H, respectively. A result of [8] implies that $I(E \times H/L) \cong I(E) \otimes^{\gamma} A(H/L)$ if $E \subset G/K$ is a closed set. Assuming this result, we can easily prove that *if $E \subset G/K$ is a strong Ditkin set for $A(G/K)$, then $E \times H/L \subset G/K \times H/L$ is a strong Ditkin set for $A(G/K) \otimes^{\gamma} A(H/L)$*. But unfortunately, the author of [8] has withdrawn the result needed, having found a mistake in the proof (see [9]). So we do not know how to get the result now.

We can, however, prove a simple result on operator space projective tensor products. Note that, in view of Proposition 1.2 of [5], $A(G/K) \hat{\otimes} A(H/L)$, the operator space projective tensor product, can be identified, completely isometrically, with $A(G/K \times H/L)$.

Let $\dot{y} \in H/L$ and define

$$\chi_{\dot{y}} : A(G/K) \otimes A(H/L) \to A(G/K) \quad \text{by} \quad \chi_{\dot{y}}(f \otimes g) = g(\dot{y})f.$$

Then $\chi_{\dot{y}}$ extends to a homomorphism from $A(G/K) \hat{\otimes} A(H/L)$ to $A(G/K)$ that is completely contractive.

LEMMA 6.4. *Let E and F be closed subsets of G/K and H/L, respectively, and let $\dot{y} \in F$. Then*

$$\chi_{\dot{y}}(j(E \times F)) \subseteq j(E).$$

Proof. If $\dot{x}$ belongs to the support of $\chi_{\dot{y}}u$, then $(\dot{x}, \dot{y})$ belongs to the support of u. So if u has compact support disjoint from $E \times F$, then $\chi_{\dot{y}}u$ has compact support disjoint from E. ∎

We conclude with the following simple result.

THEOREM 6.5. *Let G and H be locally compact groups and K and L be compact subgroups of G and H, respectively. Let $E \subset G/K$ and $F \subset H/L$ be closed sets.*

(i) *If $E \times F$ is a set of synthesis for $A(G/K \times H/L)$, then E and F are both sets of synthesis, for $A(G/K)$ and $A(H/L)$, respectively.*
(ii) *If $E \times F$ is a Ditkin set for $A(G/K \times H/L)$, then E and F are both Ditkin sets, for $A(G/K)$ and $A(H/L)$, respectively.*

Proof. (i) Let $f \in I(E)$. Let $\dot{y} \in F$ and choose $g \in A(H/L)$ such that $g(\dot{y}) = 1$. For any positive ε, if $u \in j(E \times F)$ is such that $\|u - (f \otimes g)\| < \varepsilon$, then

$$\|f - \chi_{\dot{y}}(u)\| = \|\chi_{\dot{y}}(f \otimes g) - \chi_{\dot{y}}(u)\| \leq \|f \otimes g - u\| < \varepsilon.$$

Also, $\chi_{\dot{y}}(u) \in j(E)$ by the previous lemma. Thus (i) is proved.

(ii) Let $f \in I(E)$, let $\dot{y} \in F$ and choose $g \in A(H/L)$ as above. If $\{u_\alpha\}$ is a net in $j(E \times F)$ such that $\|f \otimes g - u_\alpha(f \otimes g)\|$ converges to zero, then so does $\|f - \chi_{\dot{y}}(u_\alpha)f\|$ with $\chi_{\dot{y}}(u_\alpha) \in j(E)$. Hence (ii) follows. ∎

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