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Extensions of weak type multipliers

by

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Abstract. We prove that if $\Lambda \in M_p(\mathbb{R}^N)$ and has compact support then Λ is a weak summability kernel for $1 < p < \infty$, where $M_p(\mathbb{R}^N)$ is the space of multipliers of $L^p(\mathbb{R}^N)$.

1. Introduction. Let G be a locally compact abelian group with Haar measure μ, and let $\widehat{G}$ be its dual. We call an operator $T : L^p(G) \to L^{p,\infty}(G)$, $1 \le p < \infty$, a *multiplier of weak type* (p,p) if it is translation invariant, i.e. $\tau_x T = T\tau_x$ for all $x \in G$, and there exists a constant $C > 0$ such that

$$\mu\{x \in G : |Tf(x)| > t\} \le \frac{C^p}{t^p}\, \|f\|_p^p \tag{1.1}$$

for all $f \in L^p(G)$ and $t > 0$. (Here $L^{p,\infty}$ denotes the standard weak L^p space.) Asmar, Berkson and Gillespie in [3] proved that for all such operators T there exists a $\phi \in L^\infty(\widehat{G})$ such that $(Tf)^\wedge = \phi\widehat{f}$ for all $f \in L^2 \cap L^p(G)$. We will also call such ϕ's *multipliers of weak type* (p,p). Let $M_p^{(w)}(\widehat{G})$ denote the space of multipliers of weak type (p,p) for $1 \le p < \infty$, and let $N_p^{(w)}(\phi)$ be the smallest constant C such that inequality (1.1) holds.

In this paper we are concerned with extensions of weak type multipliers from $\mathbb{Z}^N$ to $\mathbb{R}^N$ through summability kernels. For similar results on strong type multipliers, see [6] and [4]. Here we identify $\mathbb{T}^N$ with $[0,1)^N$ and for $f \in L^1(\mathbb{R}^N)$ we define its Fourier transform as $\widehat{f}(\xi) = \int_{\mathbb{R}^N} f(x)e^{-2\pi i\xi.x}\,dx$ for $\xi \in \mathbb{R}^N$. Let us define summability kernels for weak type multipliers as follows.

Definition 1.1. A bounded measurable function $\Lambda : \mathbb{R}^N \to \mathbb{C}$ is called a *weak summability kernel for* $M_p^{(w)}(\mathbb{R}^N)$ if for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ the function $W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda(\xi - n)$ is defined and belongs to $M_p^{(w)}(\mathbb{R}^N)$.

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This definition is just the weak type analogue of summability kernels for strong type multipliers [4]. We first cite two important results regarding the summability kernels of strong type multipliers from the work of Jodeit [6] and of Berkson, Paluszyński, and Weiss [4]:

THEOREM 1.1 ([6]). *Let* $S \in L^1(\mathbb{R}^N)$ *and* $\operatorname{supp} S \subseteq [1/4, 3/4]^N$ *with* $\tau = \sum_{n\in\mathbb{Z}^N} |\widehat{s}(n)| < \infty$, *where* s *is the* 1*-periodic extension of* S. *Then the function defined by* $W_{\phi,\widehat{S}}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi(n)\widehat{S}(\xi - n)$ *belongs to* $M_p(\mathbb{R}^N)$ *for* $1 \le p < \infty$, *with* $\|W_{\phi,\widehat{S}}\|_{M_p(\mathbb{R}^N)} \le C_p\tau\|\phi\|_{M_p(\mathbb{Z}^N)}$.

THEOREM 1.2 ([4]). *For* $1 \le p < \infty$, *let* $\Lambda \in M_p(\mathbb{R}^N)$ *and* $\operatorname{supp} \Lambda \subseteq [1/4, 3/4]^N$. *For* $\phi \in M_p(\mathbb{Z}^N)$ *define* $W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda(\xi - n)$ *on* $\mathbb{R}^N$. *Then* $W_{\phi,\Lambda} \in M_p(\mathbb{R}^N)$ *and* $\|W_{\phi,\Lambda}\|_{M_p(\mathbb{R}^N)} \le C_p\|\Lambda\|_{M_p(\mathbb{R}^N)}\|\phi\|_{M_p(\mathbb{Z}^N)}$ *where* C_p *is a constant.* (*Further*, *if* Λ *has an arbitrary compact support the same result holds except that the constant* C_p *necessarily depends on the support of* Λ, *as shown in* [4].)

Asmar, Berkson and Gillespie proved a weak type analogue of Theorem 1.1 in [2]. In the same paper they also proved that Λ defined by $\Lambda(\xi) = \prod_{j=1}^N \max(1-|\xi_j|, 0)$ for $\xi = (\xi_1, \ldots, \xi_N)$ is a weak type summability kernel. In this paper, we prove the weak type analogue of Theorem 1.2 in §2, for $1 < p < \infty$. In §3 we relax the hypothesis that $\operatorname{supp} \Lambda \subseteq [1/4, 3/4]^N$. For the proof of our main result, as in [4], we will obtain the weak type inequalities by applying the technique of transference couples due to Berkson, Paluszyński, and Weiss [4].

DEFINITION 1.2. For a locally compact group G, a *transference couple* is a pair $(S, T) = (\{S_u\}, \{T_u\})$, $u \in G$, of strongly continuous mappings defined on G with values in $\mathcal{B}(X)$, where X is a Banach space, satisfying

(i) $C_S = \sup\{\|S_u\| : u \in G\} < \infty$,
(ii) $C_T = \sup\{\|T_u\| : u \in G\} < \infty$,
(iii) $S_vT_u = T_{vu}$ for all $u, v \in G$.

In §4, as an application of our result, we prove a weak type analogue of an extension theorem by de Leeuw.

2. Weak type inequality for transference couples and the main theorem. Let $\Lambda \in L^\infty(\mathbb{R}^N)$ and $\operatorname{supp} \Lambda \subseteq [1/4, 3/4]^N$. Consider the following transference couple (S, T) used by Berkson, Paluszyński, and Weiss in [4]. For $u \in \mathbb{T}^N$ the family $T = \{T_u\}$ is given by

$$(2.2) \qquad (T_uf)^\wedge(\xi) = \sum_{n\in\mathbb{Z}^N} \Lambda(\xi - n)e^{2\pi iu.n}\widehat{f}(\xi) \quad \text{for } f \in L^p \cap L^1(\mathbb{R}^N)$$

and the family $S = \{S_u\}$ is defined by

$$(2.3)\qquad (S_u f)^\wedge(\xi) = \sum_{n\in\mathbb{Z}^N} b(\xi - n)e^{2\pi i u.n}\widehat{f}(\xi) \quad \text{for } f \in L^p \cap L^1(\mathbb{R}^N),$$

where $b(\xi) = \prod_{i=1}^N b_i(\xi_i)$ for $\xi = (\xi_1, \dots, \xi_N)$, and for each i, b_i is the continuous function defined on $\mathbb{R}$ as $b_i(x) = 1$ if $x \in [1/4, 3/4]$, 0 outside $[0, 1]$ and linear in $[0, 1/4) \cup (3/4, 1]$. It is easy to see that

$$(2.4)\qquad S_u f(x) = \sum_{l\in\mathbb{Z}^N} \check{\beta}_u(l) f(x + u - l) \quad \text{a.e.},$$

where $\check{\beta}_u$ is the inverse Fourier transform of the function $\beta_u(\xi) = b(\xi)e^{2\pi i \xi.u}$, given explicitly by

$$\check{\beta}_u(\xi) = \prod_{i=1}^N \check{\beta}_{u_i}(\xi_i),$$

where

$$(2.5)\qquad \check{\beta}_{u_i}(\xi_i) = \begin{cases} \dfrac{2e^{2\pi i(\xi_i+u_i)/2}}{\pi^2(\xi_i - u_i)^2}\left(\cos\dfrac{\pi}{2}(\xi_i - u_i) - \cos\pi(\xi_i - u_i)\right) & \text{if } \xi_i \neq u_i, \\ \dfrac{3e^{2\pi i(\xi_i+u_i)/2}}{4} & \text{if } \xi_i = u_i. \end{cases}$$

Then by a straightforward calculation using (2.5) we have

$$(2.6)\qquad \sum_{l\in\mathbb{Z}^N} |\check{\beta}_u(l)| \le \sum_{l\in\mathbb{Z}^N} \beta(l) = C < \infty,$$

where $\beta(l) = \prod_{i=1}^N \beta_i(l_i)$ and

$$\beta_i(l_i) = \begin{cases} 1/(l_i - 1)^2 & \text{if } l_i > 1, \\ 1/(l_i + 1)^2 & \text{if } l_i < 1, \\ \|b_i\|_1 & \text{otherwise.} \end{cases}$$

In the following theorem we shall show that the operator transferred by T (of the transference couple (S, T) defined in (2.2) and (2.3)) given by

$$H_k f(\cdot) = \int_{\mathbb{T}^N} k(u) T_{u^{-1}} f(\cdot)\, du,$$

where $k \in L^1(\mathbb{T}^N)$ and $f \in L^p(\mathbb{R}^N)$, satisfies a weak (p, p) inequality.

THEOREM 2.1. *Let* (S, T) *be the transference couple as defined in* (2.2) *and* (2.3). *Then for* $1 < p < \infty$, $t > 0$ *and* $f \in \mathcal{S}$,

$$\lambda\{x \in \mathbb{R}^N : |H_k f(x)| > t\} \le \left(\frac{CC_p}{t} C_T N_p^{(w)}(k)\|f\|_p\right)^p,$$

where λ *denotes the Lebesgue measure of* $\mathbb{R}^N$, $C = \sum_{l\in\mathbb{Z}^N} \beta(l)$ *as in* (2.6), C_T *is the uniform bound for the family* $T = \{T_u\}$, *and* $C_p = p/(p-1)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^N)$. For $t > 0$ define $E_t = \{x : |H_k f(x)| > t\}$. Notice that

$$H_k f(x) = S_{v^{-1}} S_v H_k f(x) = \sum_{l\in\mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x - v - l)\, du.$$

Let

$$\mathcal{F}_t = \Big\{(v, x) \in \mathbb{T}^N \times \mathbb{R}^N : \Big| \sum_{l\in\mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x - l)\, du\Big| > t\Big\}.$$

Then, using translation invariance of Lebesgue measure, we obtain

$$\begin{aligned}
\lambda(E_t) &= \lambda\Big\{x \in \mathbb{R}^N : \Big|S_{v^{-1}} \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x)\, du\Big| > t\Big\} \\
&= \lambda\Big\{x \in \mathbb{R}^N : \Big| \sum_{l\in\mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x - l)\, du\Big| > t\Big\} \\
&= \int_{\mathbb{T}^N} \int_{\mathbb{R}^N} \chi_{\mathcal{F}_t}(v, x)\, dx\, dv \\
&= \int_{\mathbb{R}^N} \Big|\Big\{v : \Big| \sum_{l\in\mathbb{Z}^N} \check{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x - l)\, du\Big| > t\Big\}\Big|\, dx,
\end{aligned}$$

where $|E|$ denotes the measure of the subset $E \subseteq \mathbb{T}^N$. Thus

$$\begin{aligned}
\lambda(E_t) &\le \int_{\mathbb{R}^N} \Big|\Big\{v : \sum_{l\in\mathbb{Z}^N} \beta(l) \Big| \int_{\mathbb{T}^N} k(u) T_{u^{-1}v} f(x - l)\Big|\, du > t\Big\}\Big|\, dx \\
&= \int_{\mathbb{R}^N} \Big|\Big\{v : \sum_{l\in\mathbb{Z}^N} \beta(l) |k * F(\cdot, x - l)(v)| > t\Big\}\Big|\, dx,
\end{aligned}$$

where $F(v, x) = T_v f(x)$ a.e.

We know that $\sup_{t>0} t\lambda_f(t)^{1/p} = \|f\|_{L^{p,\infty}}$ for $f \in L^{p,\infty}$. Also, since $p > 1$, $\|\ \|_{p,\infty}$ is equivalent to a norm $\|\ \|^*_{p,\infty}$ ([8]), using the triangle inequality for norms we have

$$\begin{aligned}
\lambda(E_t) &\le \int_{\mathbb{R}^N} \frac{1}{t^p} \Big\| \sum_{l\in\mathbb{Z}^N} \beta(l) k * F(\cdot, x - l)\Big\|^p_{L^{p,\infty}(\mathbb{T}^N)} dx \\
&\le C_p^p \int_{\mathbb{R}^N} \frac{1}{t^p} \Big(\sum_{l\in\mathbb{Z}^N} \beta(l) \|k * F(\cdot, x - l)\|^*_{L^{p,\infty}(\mathbb{T}^N)}\Big)^p dx, \\
&\le C_p^p \int_{\mathbb{R}^N} \frac{1}{t^p} \Big(\sum_{l\in\mathbb{Z}^N} \beta(l) N_p^{(w)}(k) \|F(\cdot, x - l)\|_{L^p(\mathbb{T}^N)}\Big)^p dx,
\end{aligned}$$

where $N_p^{(w)}(k)$ is the weak-type norm of the convolution operator $f \mapsto k * f$

for $f \in L^p(\mathbb{T}^N)$. Thus,

$$\begin{aligned}\lambda(E_t) &\le C_p^p \frac{1}{t^p}\Big(\sum_{l\in\mathbb{Z}^N}\beta(l)N_p^{(w)}(k)\Big(\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{T}^N}|T_vf(x-l)|^p\,dx\,dv\Big)^{1/p}\Big)^p\\ &= C_p^p \frac{1}{t^p}\Big(\sum_{l\in\mathbb{Z}^N}\beta(l)N_p^{(w)}(k)\Big(\int\limits_{\mathbb{T}^N}\int\limits_{\mathbb{R}^N}|T_vf(x-l)|^p\,dx\,dv\Big)^{1/p}\Big)^p\\ &\le \left(\frac{CC_pC_T}{t^p}\,N_p^{(w)}(k)\|f\|_p\right)^p.\end{aligned}$$

Hence, H_kf satisfies a weak (p,p) inequality.

In order to prove the weak-type analogue of Theorem 1.2 we need the following lemma proved by Asmar, Berkson, and Gillespie in [1].

LEMMA 2.1 ([1]). *Suppose that* $1 \le p < \infty$, $\{\phi_j\} \subseteq M_p^{(w)}(\widehat{G})$ *with* $\sup\{|\phi_j(\gamma)| : j \in \mathbb{N},\ \gamma \in \widehat{G}\} < \infty$ *and suppose* ϕ_j *converges pointwise a.e. on* $\widehat{G}$ *to a function* ϕ. *If* $\liminf_j N_p^{(w)}(\phi_j) < \infty$ *then* $\phi \in M_p^{(w)}(\widehat{G})$ *and* $N_p^{(w)}(\phi) \le \liminf_j N_p^{(w)}(\phi_j)$.

In the following theorem, we use the family of operators $\{T_u\}$ defined in (2.2) with $\Lambda \in M_p(\mathbb{R}^N)$ and $\operatorname{supp}\Lambda \subseteq [1/4,3/4]^N$. In this case, by [4] we have $C_T \le c_p\|\Lambda\|_{M_p(\mathbb{R}^N)}$, where c_p is a constant.

THEOREM 2.2. *Suppose* $1 < p < \infty$ *and* $\Lambda \in M_p(\mathbb{R}^N)$ *is supported in the set* $[1/4,3/4]^N$. *For* $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ *define*

$$W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N}\phi(n)\Lambda(\xi-n) \qquad \text{on } \mathbb{R}^N.$$

Then $W_{\phi,\Lambda} \in M_p^{(w)}(\mathbb{R}^N)$ *and* $N_p^{(w)}(W_{\phi,\Lambda}) \le CN_p^{(w)}(\phi)\|\Lambda\|_{M_p(\mathbb{R}^N)}$.

Proof. Using Lemma 2.1 we first show that it is enough to prove the theorem for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ having finite support. Suppose the theorem is true for finitely supported ϕ. Then, for arbitrary $\phi \in M_p^{(w)}(\mathbb{Z}^N)$, define $\phi_j = \widehat{k}_j\phi$, where k_j is the jth Fejér kernel. Then for each j, ϕ_j's have finite support and $(T_{\phi_j}f)^\wedge(n) = \phi_j(n)\widehat{f}(n) = (T_\phi(k_j * f))^\wedge(n)$. So $\phi_j \in M_p^{(w)}(\mathbb{Z}^N)$ for each j and $N_p^{(w)}(\phi_j) \le N_p^{(w)}(\phi)$. Define $W_{\phi_j,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N}\phi_j(n)\Lambda(\xi-n)$. Now $\liminf_j W_{\phi_j,\Lambda}(\xi) = W_{\phi,\Lambda}(\xi)$. Also, by our assumption,

$$N_p^{(w)}(W_{\phi_j,\Lambda}) \le CN_p^{(w)}(\phi_j)\|\Lambda\|_{M_p(\mathbb{R}^N)} \le CN_p^{(w)}(\phi)\|\Lambda\|_{M_p(\mathbb{R}^N)}$$

and $|W_{\phi_j,\Lambda}| \le 2\|\Lambda\|_\infty\|\phi_j\|_\infty \le 2\|\Lambda\|_\infty\|\phi\|_\infty$. Thus by Lemma 2.1, applied to $W_{\phi_j,\Lambda}$'s, we conclude that $W_{\phi,\Lambda} \in M_p^{(w)}(\mathbb{R}^N)$. Hence it is enough to assume that $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ has finite support.

Now let $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ be finitely supported. Define

$$k(u) = \sum_{n\in\mathbb{Z}^N} \phi(n)e^{-2\pi iu.n}.$$

Then $k \in L^1(\mathbb{T}^N)$ and $\widehat{k}(n) = \phi(n)$. For this particular k and the transference couple (S,T) defined above, we have

$$(H_k f)^\wedge(\xi) = (T_{W_{\phi,\Lambda}} f)^\wedge(\xi).$$

Thus $T_{W_{\phi,\Lambda}} f = H_k f$. Hence from Theorem 2.1 and since $C_T \le c_p\|\Lambda\|_{M_p(\mathbb{R}^N)}$, we have

$$\lambda\{x \in \mathbb{R}^N : |T_{W_{\phi,\Lambda}} f(x)| > t\} \le \left(\frac{C}{t} N_p^{(w)}(\phi)\|\Lambda\|_{M_p(\mathbb{R}^N)}\|f\|_p\right)^p.$$

3. Lattice preserving linear transformations and multipliers. We shall now relax the hypothesis that $\operatorname{supp}\Lambda \subseteq [1/4, 3/4]^N$ to allow Λ to have arbitrary compact support. In fact this can be done by a partition of identity argument as in [4]. Here we give a different method by proving Lemma 3.2 below. Particular cases of this lemma occur in [6] and in [2]. Suppose $\operatorname{supp}\Lambda \subseteq [-M, M]^N$; define $\Lambda_M(\xi) = \Lambda_1(4M\xi)$, where $\Lambda_1(\xi) = \Lambda(\xi - 1/2)$. So $\operatorname{supp}\Lambda_M \subseteq [1/4, 3/4]^N$. Thus if we define a non-singular transformation $A : \mathbb{R}^N \to \mathbb{R}^N$ such that $Ax = 4Mx$ then $\Lambda_M = \Lambda_1 \circ A$. In order to replace the support condition we need to prove $\Lambda_M \circ A^{-1}$ is a summability kernel. In the work of Jodeit and of Asmar, Berkson, and Gillespie they assume A in Lemma 3.2 to be multiplication by 2. We have combined some of the results proved by Gröchenig and Madych [5] in the following lemma which will help us to prove Lemma 3.2. In the proof of Theorem 3.1, we only use the case of a diagonal linear transform, but the more general results proved below are of some interest in their own right.

Lemma 3.1 ([5]). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a non-singular linear transformation which preserves the lattice $\mathbb{Z}^N$ (i.e. $A(\mathbb{Z}^N) \subseteq \mathbb{Z}^N$). Then the following are true.*

(i) *The number of distinct coset representatives of $\mathbb{Z}^N/A\mathbb{Z}^N$ is equal to $q = |\det A|$.*

(ii) *If $Q_0 = [0, 1)^N$ and $k_1, \dots, k_q$ are the distinct coset representatives of $\mathbb{Z}^N/A\mathbb{Z}^N$ then the sets $\{A^{-1}(Q_0 + k_i)\}$ are mutually disjoint.*

(iii) *Let $Q = \bigcup_{i=1}^q A^{-1}(Q_0+k_i)$. Then $\lambda(Q) = 1$ and $\bigcup_{k\in\mathbb{Z}^N}(Q+k) \simeq \mathbb{R}^N$.*

(iv) *$AQ \simeq \bigcup_{i=1}^q (Q_0 + k_i)$.*

Here $E \simeq F$ if $\lambda(F \triangle E) = 0$.

Using this lemma, we prove

LEMMA 3.2. *Let A be as in Lemma* 3.1. *Define $A^t = B$, where A^t is the transpose of A. For $\phi \in l_\infty(\mathbb{Z}^N)$ define*

$$\psi(n) = \phi(Bn), \qquad \eta(n) = \begin{cases} \phi(B^{-1}n) & \text{if } n \in B\mathbb{Z}^N, \\ 0 & \text{otherwise.} \end{cases}$$

(i) *If $\phi \in M_p(\mathbb{Z}^N)$ then $\psi, \eta \in M_p(\mathbb{Z}^N)$ with multiplier norms not exceeding the multiplier norm of ϕ.*

(ii) *If $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ then $\psi, \eta \in M_p^{(w)}(\mathbb{Z}^N)$ with weak multiplier norms not exceeding the weak multiplier norm of ϕ.*

Proof. (i) For $f \in L^p(Q_0)$, we let f again denote its periodic extension to $\mathbb{R}^N$. Define $Sf(x) = f(Ax)$. Then Sf is also periodic and

$$\begin{aligned} \int_{Q_0} |Sf(x)|^p\,dx &= \int_{Q_0} |Sf(x)|^p \sum_j \chi_Q(x-j)\,dx = \sum_j \int_{Q_0+j} |Sf(x)|^p \chi_Q(x)\,dx \\ &= \int_Q |Sf(x)|^p\,dx = \frac{1}{|\det A|} \int_{AQ} |f(x)|^p\,dx \\ &= \frac{1}{q} \sum_{i=1}^q \int_{Q_0+k_i} |f(x)|^p\,dx \qquad \text{(Lemma 3.1(iv))} \\ &= \int_{Q_0} |f(x)|^p\,dx. \end{aligned}$$

Thus S is an isometry, i.e., $\|Sf\|_{L^p(Q_0)} = \|f\|_{L^p(Q_0)}$. Further, from the orthogonality relations for characters (Lemma 1 of [7]) we have

$$(Sf)^\wedge(n) = \begin{cases} \widehat{f}(B^{-1}n) & \text{if } n \in B\mathbb{Z}^N, \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in L^p(Q_0)$ we define an operator W on $L^p(Q_0)$ by

$$Wf(x) = \frac{1}{q} \sum_{i=1}^q f(A^{-1}(x+k_i)),$$

where $k_1, \ldots, k_q$ are distinct coset representatives of $\mathbb{Z}^N/A\mathbb{Z}^N$. Then for a trigonometric polynomial f,

$$(Wf)^\wedge(n) = \widehat{f}(Bn),$$

and so

$$\begin{aligned} \Big(\int_{Q_0} |Wf(x)|^p\,dx\Big)^{1/p} &= \Big(\int_{Q_0} \Big|\frac{1}{q}\sum_{i=1}^q f(A^{-1}(x+k_i))\Big|^p\,dx\Big)^{1/p} \\ &\le \frac{1}{q}\sum_{i=1}^q \Big(\int_{Q_0} |f(A^{-1}(x+k_i))|^p\,dx\Big)^{1/p} \\ &= \frac{q^{1/p}}{q}\sum_{i=1}^q \Big(\int_{A^{-1}(Q_0+k_i)} |f(x)|^p\,dx\Big)^{1/p}. \end{aligned}$$

Therefore $\|Wf\|_{L^p(Q_0)} \le q^{(1-p)/p}\|f\|_{L^p(Q_0)}$, since $\int_{Q_0}|f(x)|^p\,dx = \int_Q |f(x)|^p\,dx$ as above. It is easy to see that

$$ST_\phi W = T_\eta, \tag{3.7}$$

$$WT_\phi S = T_\psi. \tag{3.8}$$

It follows that if $\phi \in M_p(\mathbb{Z}^N)$, then $\|T_\psi f\| \le C_p\|\phi\|_{M_p(\mathbb{Z}^N)}\|f\|_{L^p(Q_0)}$. Also $\|T_\eta f\|_{L^p(Q_0)} \le C_p\|\phi\|_{M_p(\mathbb{Z}^N)}\|f\|_{L^p(Q_0)}$. Hence $\psi, \eta \in M_p(\mathbb{Z}^N)$.

(ii) For $\phi \in M_p^{(w)}(\mathbb{Z}^N)$, we need to calculate the distribution functions of Sf and Wf. Define $E_t = \{x \in Q_0 : |Sf(x)| > t > 0\}$. Then

$$\begin{aligned} |E_t| &= \int_{Q_0} \chi_{E_t}(x)\,dx = \int_{Q_0} \chi_{\mathbb{R}_+}(|f(Ax)| - t)\,dx = \frac{1}{q}\int_{AQ} \chi_{\mathbb{R}_+}(|f(x)| - t)\,dx \\ &= \frac{1}{q}\sum_{i=1}^{q}\int_{Q_0+k_i} \chi_{\mathbb{R}_+}(|f(x)| - t)\,dx = |\{x : |f(x)| > t\}|. \end{aligned}$$

Therefore,

$$|\{x \in Q_0 : |Sf(x)| > t\}| = |\{x \in Q_0 : |f(x)| > t\}|. \tag{3.9}$$

Also

$$\begin{aligned} |\{x \in Q_0 : |Wf(x)| > t\}| &= \Big|\Big\{x \in Q_0 : \Big|\sum_{i=1}^{q} f(A^{-1}(x + k_i))\Big| > tq\Big\}\Big| \\ &\le \Big|\Big\{x \in Q_0 : \sum_{i=1}^{q} |f(A^{-1}(x + k_i))| > tq\Big\}\Big| \\ &= \sum_{i=1}^{q}\int_{Q_0} \chi_{\mathbb{R}_+}(|f(A^{-1}(x + k_i))| - t)\,dx \\ &= q\sum_{i=1}^{q}\int_{A^{-1}(Q_0+k_i)} \chi_{\mathbb{R}_+}(|f(x)| - t)\,dx. \end{aligned}$$

Thus

$$|\{x \in Q_0 : |Wf(x)| > t\}| \le q|\{x \in Q_0 : |f(x)| > t\}|. \tag{3.10}$$

From the relations (3.7)–(3.10), we conclude that $\psi, \eta \in M_p^{(w)}(\mathbb{Z}^N)$ whenever $\phi \in M_p^{(w)}(\mathbb{Z}^N)$. Also $N_p^{(w)}(\psi) \le CN_p^{(w)}(\phi)$ and $N_p^{(w)}(\eta) \le CN_p^{(w)}(\phi)$.

As an application of this lemma we get the following result regarding weak summability kernels.

LEMMA 3.3. *Let A be as in Lemma* 3.1. *Suppose Λ is a weak* (*strong*) *summability kernel. Then $\Lambda \circ B$ and $\Lambda \circ B^{-1}$ are also weak* (*strong*) *summability kernels.*

Proof. Define $W_{\phi,\Lambda\circ B}$ on $\mathbb{R}^N$ for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ by

$$W_{\phi,\Lambda\circ B}(x) = \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda\circ B(x-n) = \sum_{n\in\mathbb{Z}^N} \eta(n)\Lambda(Bx-n) = W_{\eta,\Lambda}(Bx).$$

As $\eta \in M_p^{(w)}(\mathbb{Z}^N)$ (by Lemma 3.2) and since Λ is a summability kernel we have $W_{\eta,\Lambda} \in M_p^{(w)}(\mathbb{R}^N)$. Hence $W_{\phi,\Lambda\circ B} \in M_p^{(w)}(\mathbb{R}^N)$. Similarly

$$\begin{aligned} W_{\phi,\Lambda\circ B^{-1}}(x) &= \sum_{n\in\mathbb{Z}^N} \phi(n)\Lambda(B^{-1}x - B^{-1}n) \\ &= \sum_{j=1}^{q} \sum_{n\in B\mathbb{Z}^N+p_j} \phi(n)\Lambda(B^{-1}x - B^{-1}n), \end{aligned}$$

$p_1,\ldots,p_q$ being distinct coset representatives of $B\mathbb{Z}^N/\mathbb{Z}^N$ $(p_1 = 0)$. We have

$$\begin{aligned} W_{\phi,\Lambda\circ B^{-1}}(x) &= \sum_{j=1}^{q} \sum_{n\in\mathbb{Z}^N} \phi(Bn+p_j)\Lambda(B^{-1}x + B^{-1}p_j - n) \\ &= W_{\psi,\Lambda}(B^{-1}x) + \ldots + W_{\psi_{p_{q-1}},\Lambda}(B^{-1}x - B^{-1}p_q) \end{aligned}$$

where $\psi_{p_i}(l) = \phi(Bl+p_j)$, $i = 1,\ldots,q$. As $\psi \in M_p^{(w)}(\mathbb{Z}^N)$ and Λ is a summability kernel we conclude that $W_{\phi,\Lambda\circ B^{-1}} \in M_p^{(w)}(\mathbb{R}^N)$.

Hence from Lemma 3.3 and the discusssion preceding Lemma 3.1 we obtain the following theorem.

THEOREM 3.1. *Suppose* $\Lambda \in M_p(\mathbb{R}^N)$ *and* $\operatorname{supp}\Lambda \subseteq [-M,M]$; *for* $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ *define* $W_{\phi,\Lambda}(\xi) = \sum_{n\in\mathbb{Z}^N}\phi(n)\Lambda(\xi-n)$ *on* $\mathbb{R}^N$. *Then* $W_{\phi,\Lambda} \in M_p^{(w)}(\mathbb{R}^N)$ *and* $N_p^{(w)}(W_{\phi,\Lambda}) \le C_\Lambda N_p^{(w)}(\phi)\|\Lambda\|_{M_p(\mathbb{R}^N)}$, *where* C_Λ *is a constant depending on* Λ.

4. An application. As an application of Theorem 3.1, we prove a weak-type version of a result proved by de Leeuw [8].

THEOREM 4.1. *For* $1 < p < \infty$ *and* $\varepsilon > 0$, *let* $\{\phi_\varepsilon\} \subseteq M_p^{(w)}(\mathbb{Z})$ *satisfy*

(i) $\lim_{\varepsilon\to 0}\phi_\varepsilon([x/\varepsilon]) = \phi(x)$ *a.e.*,

(ii) $\sup_\varepsilon N_p^{(w)}(\phi_\varepsilon) = K < \infty$.

Then $\phi \in M_p^{(w)}(\mathbb{R})$ *and* $N_p^{(w)}(\phi) \le \sup_\varepsilon N_p^{(w)}(\phi_\varepsilon)$.

Proof. For each $\varepsilon > 0$, define W_{ϕ_ε} on $\mathbb{R}$ by

$$W_{\phi_\varepsilon}(x) = \sum_{n\in\mathbb{Z}} \phi_\varepsilon(n)\chi_{[0,1)}(x-n). \tag{4.11}$$

As $\chi_{[0,1)} \in M_p(\mathbb{R})$ for $1 < p < \infty$, from Theorem 3.1 we have $W_{\phi_\varepsilon} \in M_p^{(w)}(\mathbb{R})$ and $N_p^{(w)}(W_{\phi_\varepsilon}) \le CN_p^{(w)}(\phi_\varepsilon) \le CK$. We define another function ψ_ε, for each

$\varepsilon > 0$, by $\psi_\varepsilon(x) = W_{\phi_\varepsilon}(x/\varepsilon)$. Then $\psi_\varepsilon \in M_p^{(w)}(\mathbb{R})$ and

$$N_p^{(w)}(\psi_\varepsilon) \le N_p^{(w)}(W_{\phi_\varepsilon}) \le CK. \tag{4.12}$$

From (4.11) we have

$$\psi_\varepsilon(x) = W_{\phi_\varepsilon}(x/\varepsilon) = \sum_{n\in\mathbb{Z}} \phi_\varepsilon(n)\chi_{[0,1)}(x/\varepsilon - n) = \phi_\varepsilon([x/\varepsilon]).$$

So from our hypothesis

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(x) = \phi(x) \quad \text{a.e.} \tag{4.13}$$

Also we have $|\psi_\varepsilon(x)| < \infty$ (as $\sup_{\varepsilon,n} |\phi_\varepsilon(n)| < \infty$).

Hence from (4.11)–(4.13) along with Lemma 2.1 we have $\phi \in M_p^{(w)}(\mathbb{R})$ and $N_p^{(w)}(\phi) \le \lim_\varepsilon N_p^{(w)}(\phi_\varepsilon) \le CK$.

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