

A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix

by

FUAD KITTANEH (Amman)

Abstract. It is shown that if A is a bounded linear operator on a complex Hilbert space, then

$$w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{1/2}),$$

where $w(A)$ and $\|A\|$ are the numerical radius and the usual operator norm of A , respectively. An application of this inequality is given to obtain a new estimate for the numerical radius of the Frobenius companion matrix. Bounds for the zeros of polynomials are also given.

1. Introduction. Let $B(H)$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. For $A \in B(H)$, the *numerical range* of A is defined as the set of complex numbers given by

$$(1) \quad W(A) = \{ \langle Ax, x \rangle : x \in H, \|x\| = 1 \}.$$

The most important properties of the numerical range are that it is convex and that its closure includes the spectrum of the operator.

The *numerical radius* of A is given by

$$(2) \quad w(A) = \sup\{ |\langle Ax, x \rangle| : x \in H, \|x\| = 1 \}.$$

It is well known that $w(\cdot)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\| \cdot \|$. In fact, for every $A \in B(H)$,

$$(3) \quad \frac{1}{2}\|A\| \leq w(A) \leq \|A\|.$$

The inequalities in (3) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. For a comprehensive account of the theory of the numerical range, the reader is referred to [5], [6], and [8].

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In Section 2 of this paper, we establish an inequality that refines the second inequality in (3). This inequality will be used in Section 3 to obtain a new estimate for the numerical radius of the Frobenius companion matrix. Our computations enable us to derive new bounds for the zeros of polynomials.

2. A numerical radius inequality. In order to prove our desired numerical radius inequality, we need the following lemmas. The first lemma, which contains a mixed Schwarz inequality, can be found in [6, pp. 75–76]. For generalizations of this inequality, we refer to [9] and references therein.

LEMMA 1. *If $A \in B(H)$, then*

$$(4) \quad |\langle Ax, y \rangle| \leq \langle |A|x, x \rangle^{1/2} \langle |A^*|y, y \rangle^{1/2}$$

for all $x, y \in H$. Here $|T|$ stands for the positive (semidefinite) operator $(T^*T)^{1/2}$.

The second lemma contains a special case of a more general norm inequality that is equivalent to some Löwner–Heinz type inequalities. See [4], [11], and references therein.

LEMMA 2. *If $A, B \in B(H)$ are positive operators, then*

$$(5) \quad \|A^{1/2}B^{1/2}\| \leq \|AB\|^{1/2}.$$

The third lemma contains a recent norm inequality for sums of positive operators that is sharper than the triangle inequality. See [12].

LEMMA 3. *If $A, B \in B(H)$ are positive operators, then*

$$(6) \quad \|A + B\| \leq \frac{1}{2}(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|A^{1/2}B^{1/2}\|^2}).$$

Now we are in a position to present our refined numerical radius inequality.

THEOREM 1. *If $A \in B(H)$, then*

$$(7) \quad w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{1/2}).$$

Proof. By Lemma 1 and by the arithmetic-geometric mean inequality, we have for every $x \in H$,

$$\begin{aligned} |\langle Ax, x \rangle| &\leq \langle |A|x, x \rangle^{1/2} \langle |A^*|x, x \rangle^{1/2} \leq \frac{1}{2}(\langle |A|x, x \rangle + \langle |A^*|x, x \rangle) \\ &= \frac{1}{2} \langle (|A| + |A^*|)x, x \rangle. \end{aligned}$$

Thus

$$(8) \quad \begin{aligned} w(A) &= \sup\{|\langle Ax, x \rangle| : x \in H, \|x\| = 1\} \\ &\leq \frac{1}{2} \sup\{\langle (|A| + |A^*|)x, x \rangle : x \in H, \|x\| = 1\} \\ &= \frac{1}{2} \| |A| + |A^*| \|. \end{aligned}$$

Applying Lemmas 2 and 3 to the positive operators $|A|$ and $|A^*|$, and using the facts that $\||A|\| = \||A^*|\| = \|A\|$ and $\||A||A^*|\| = \|A^2\|$, we have

$$(9) \quad \||A| + |A^*|\| \leq \|A\| + \|A^2\|^{1/2}.$$

The desired inequality (7) now follows from (8) and (9). ■

To see that (7) is a refinement of the second inequality in (3), one has to recall that $\|A^2\| \leq \|A\|^2$ for every $A \in B(H)$.

It has been mentioned in Section 1 that if $A \in B(H)$ is such that $A^2 = 0$, then $w(A) = \frac{1}{2}\|A\|$. This can be easily seen as an immediate consequence of the first inequality in (3) and the inequality (7).

COROLLARY 1. *If $A \in B(H)$ is such that $A^2 = 0$, then $w(A) = \frac{1}{2}\|A\|$.*

Proof. Combining the first inequality in (3) and the inequality (7), we have

$$(10) \quad \frac{1}{2}\|A\| \leq w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{1/2})$$

for every $A \in B(H)$. Thus, if $A^2 = 0$, then $w(A) = \frac{1}{2}\|A\|$, as required. ■

It should be mentioned here that the converse of Corollary 1 is not true if $\dim H > 2$. To see this, consider

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $w(A) = \frac{1}{2}\|A\| = 1$, but $A^2 \neq 0$.

The following result is another consequence of the inequality (7).

COROLLARY 2. *If $A \in B(H)$ is such that $w(A) = \|A\|$, then $\|A^2\| = \|A\|^2$.*

Proof. It follows from the inequality (7) that

$$(11) \quad 2w(A) \leq \|A\| + \|A^2\|^{1/2}$$

for every $A \in B(H)$. Thus, if $w(A) = \|A\|$, then $\|A\| \leq \|A^2\|^{1/2}$, and hence $\|A\|^2 \leq \|A^2\|$. But the reverse inequality is always true. Thus $\|A^2\| = \|A\|^2$, as required. ■

It should also be mentioned here that the converse of Corollary 2 is not true if $\dim H > 2$. To see this, consider

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $\|A^2\| = \|A\|^2 = 1$, but $w(A) = 1/\sqrt{2} < 1 = \|A\|$.

To end this section, we remark that some tedious computations show that if $\dim H = 2$, then the converses of Corollaries 1 and 2 are true. This can also be inferred by appealing to [5, p. 11] and [6, p. 110].

3. An estimate for the numerical radius of the Frobenius companion matrix. Several classical bounds for the zeros of monic polynomials have been obtained from matrix inequalities applied to various types of companion matrices of these polynomials. See, e.g., [1]–[3], [7], [10], [13]–[16]. Let

$$(12) \quad C(p) = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

be the *Frobenius companion matrix* of the monic polynomial

$$(13) \quad p(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1,$$

where $n \geq 3$, and the coefficients a_1, \dots, a_n are complex numbers, with $a_1 \neq 0$. It is well known that the zeros of p are exactly the eigenvalues of $C(p)$. See, e.g., [7, p. 316]. Since the eigenvalues of $C(p)$, regarded as an operator on the Hilbert space \mathbb{C}^n , are contained in its numerical range, it follows that if z is any zero of p , then

$$(14) \quad |z| \leq w(C(p)).$$

Several estimates for the numerical radii of some special companion matrices have been employed in [1], [3], [13], [15], and references therein to derive bounds for the zeros of polynomials.

A singular value computation has been utilized in [10] to show that

$$(15) \quad \|C(p)\| = \sqrt{\frac{\alpha + 1 + \sqrt{(\alpha + 1)^2 - 4|a_1|^2}}{2}},$$

$$(16) \quad \|C(p)^{-1}\|^{-1} = \sqrt{\frac{\alpha + 1 - \sqrt{(\alpha + 1)^2 - 4|a_1|^2}}{2}},$$

where $\alpha = \sum_{j=1}^n |a_j|^2$. Consequently, every zero of p lies in the annulus

$$(17) \quad \sqrt{\frac{\alpha + 1 - \sqrt{(\alpha + 1)^2 - 4|a_1|^2}}{2}} \leq |z| \leq \sqrt{\frac{\alpha + 1 + \sqrt{(\alpha + 1)^2 - 4|a_1|^2}}{2}}.$$

These bounds improve the classical bounds of Carmichael and Mason, which assert that every zero of p lies in the annulus

$$(18) \quad \frac{|a_1|}{\sqrt{\alpha+1}} \leq |z| \leq \sqrt{\alpha+1}.$$

See, e.g., [7, pp. 317–318].

Basing on our refined inequality (7) and the identity (15), we give a new estimate for $w(C(p))$.

THEOREM 2.

$$(19) \quad w(C(p)) \leq \frac{1}{2} \left(\sqrt{\frac{\alpha+1+\sqrt{(\alpha+1)^2-4|a_1|^2}}{2}} + \sqrt[4]{\frac{\delta+1+\sqrt{(\delta-1)^2+4\delta'}}{2}} \right),$$

where $\delta = \frac{1}{2}(\alpha+\beta+\sqrt{(\alpha-\beta)^2+4|\gamma|^2})$, $\delta' = \frac{1}{2}(\alpha'+\beta'+\sqrt{(\alpha'-\beta')^2+4|\gamma'|^2})$, $\alpha = \sum_{j=1}^n |a_j|^2$, $\beta = \sum_{j=1}^n |b_j|^2$, $\alpha' = \sum_{j=3}^n |a_j|^2$, $\beta' = \sum_{j=3}^n |b_j|^2$, $\gamma = -\sum_{j=1}^n \bar{a}_j b_j$, $\gamma' = -\sum_{j=3}^n \bar{a}_j b_j$, and $b_j = a_n a_j - a_{j-1}$ for $j = 1, \dots, n$, with $a_0 = 0$.

Proof. In view of the inequality (7) and the identity (15), it remains to give an upper bound estimate for $\|C(p)^2\|$. Let $K = C(p)^2$. Then

$$(20) \quad K = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}.$$

Now write K as $K = R + S$, where

$$(21) \quad R = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_3 & -a_2 & -a_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$(22) \quad S = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix},$$

and observe that $R^*S = S^*R = 0$. Thus $K^*K = (R + S)^*(R + S) = (R^* + S^*)(R + S) = R^*R + S^*S$. Applying Lemma 3 to the positive operators R^*R and S^*S , and using the fact that $\| |R| |S| \| = \|RS^*\|$, we have

$$(23) \quad \begin{aligned} \|K\|^2 &= \|K^*K\| = \|R^*R + S^*S\| \\ &\leq \frac{1}{2}(\|R\|^2 + \|S\|^2 + \sqrt{(\|R\|^2 - \|S\|^2)^2 + 4\|RS^*\|^2}). \end{aligned}$$

But, by simple computations, $\|R\|^2 = \frac{1}{2}(\alpha + \beta + \sqrt{(\alpha - \beta)^2 + 4|\gamma|^2}) = \delta$, $\|S\| = 1$, and $\|RS^*\|^2 = \frac{1}{2}(\alpha' + \beta' + \sqrt{(\alpha' - \beta')^2 + 4|\gamma'|^2}) = \delta'$. Consequently,

$$(24) \quad \|C(p)^2\|^{1/2} \leq \sqrt[4]{\frac{\delta + 1 + \sqrt{(\delta - 1)^2 + 4\delta'}}{2}}.$$

Now the desired inequality (19) follows from the inequalities (7) and (24) and the identity (15). ■

A bound for the zeros of p can be obtained from the inequalities (14) and (19). However, using the spectral mapping theorem, we conclude that if z is any zero of p , then $|z| \leq \|C(p)^2\|^{1/2}$. Thus, by the inequality (24) we have the following better bound for the zeros of p .

COROLLARY 3. *If z is any zero of p , then*

$$(25) \quad |z| \leq \sqrt[4]{\frac{\delta + 1 + \sqrt{(\delta - 1)^2 + 4\delta'}}{2}}.$$

It should be mentioned here that, except for the obvious comparison that the bounds (17) are better than the bounds (18), none of the bounds presented here, for the zeros of p , is uniformly better than the other bounds. If the exact value of $\|C(p)^2\|$ is computed, then bounds better than the bounds (17) and (25) would be obtained.

Finally, we remark that it is possible to complement the bound (25) by giving a lower bound estimate for the zeros of p . To see this, observe that the zeros of the polynomial

$$q(z) = \frac{z^n}{a_1} p\left(\frac{1}{z}\right)$$

are the reciprocals of those of p . Thus, applying the upper bound estimate (25) to the zeros of q yields the desired lower bound estimate for the zeros of p . This enables us to describe a new annulus containing the zeros of p in addition to those given in the bounds (17) and (18). Similar arguments have been invoked in [7, p. 318] to obtain lower bound counterparts of some classical upper bounds for the zeros of polynomials.

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Department of Mathematics
 University of Jordan
 Amman, Jordan
 E-mail: fkitt@ju.edu.jo