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Asymptotic behavior of a steady flow in a two-dimensional pipe

by

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Abstract. The paper investigates the asymptotic behavior of a steady flow of an incompressible viscous fluid in a two-dimensional infinite pipe with slip boundary conditions and large flux. The convergence of the solutions to data at infinities is examined. The technique enables computing optimal factors of exponential decay at the outlet and inlet of the pipe which are unsymmetric for nonzero fluxes of the flow. As a corollary, the asymptotic structure of the solutions is obtained. The results show strong dependence on the magnitude of the Reynolds number.

1. Introduction. In this paper we study a steady flow of a viscous incompressible Newtonian fluid governed by the steady Navier–Stokes equations in a two-dimensional pipe-like domain with slip boundary conditions. The motion is described by the following system:

$$\begin{aligned} &v\cdot\nabla v-\nu\Delta v+\nabla p=f && \text{in } \Omega,\\ &\operatorname{div} v=0 && \text{in } \Omega,\\ &n\cdot v=0,\ \ n\cdot\mathbb{T}(v,p)\cdot\tau=0 && \text{on } \partial\Omega,\\ &v\to(v_\infty,0)\quad \text{as } |x|\to\infty, \end{aligned}\tag{1.1}$$

where $v=(v^1,v^2)$ is the velocity of the fluid, p the pressure, f the external force, n and τ the normal and tangent vectors to the boundary $\partial\Omega$, ν the constant positive viscous coefficient, $(v_\infty,0)$ the constant velocity at the inlet and outlet of the pipe, and $\mathbb{T}$ the stress tensor for Newtonian fluids, i.e.

$$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-p\,\mathrm{Id}=\{\nu(v^i_{,j}+v^j_{,i})-p\delta_{ij}\}_{i,j=1,2}.\tag{1.2}$$

The domain Ω is a straight pipe with a local obstacle inside—see the picture on the next page.

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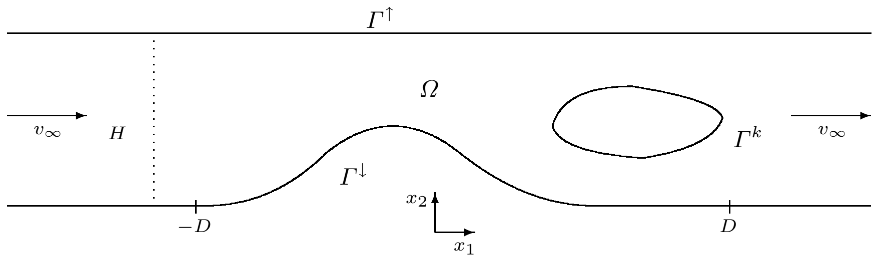

We assume that

$$\Omega = W \setminus \mathcal{O}, \tag{1.3}$$

where $W = \mathbb{R} \times (0, H)$ and $\mathcal{O}$ is a closed set consisting of obstacles and satisfying

$$\mathcal{O} \subset [-D, D] \times [0, H]. \tag{1.4}$$

The aim of the paper is to analyze the asymptotic behavior of the solutions. We want to get precise information about the velocity as $x_1 \to \infty$ and as $x_1 \to -\infty$, and also about the quantities which define the flow. We concentrate on the vorticity which, as we will see, is very sensitive to the magnitude of the flux of the flow. The convergence is connected with the properties of fundamental solutions of problems arising from the Oseen system for the straight pipe.

Since we are interested only in behavior for large $|x_1|$ we may modify the boundary conditions $(1.1)_3$. Our method does not require simple connectedness of the domain, hence in general we may split the boundary into connected elements as follows:

$$\partial\Omega = \Gamma^{\uparrow} \cup \Gamma^{\downarrow} \cup \bigcup_{k=1}^{K_0} \Gamma^k, \tag{1.5}$$

where $\Gamma^{\uparrow}$ and $\Gamma^{\downarrow}$ are the unbounded parts (connected), and Γ^k are the boundaries of holes in the pipe, K_0 being the total number of holes. Thus, instead of $(1.1)_3$, we may put the following conditions:

$$(1.1)'_3 \qquad \begin{gathered} n \cdot v = 0, \quad n \cdot \mathbb{T}(v,p) \cdot \tau = 0 \quad \text{on } \Gamma^{\uparrow} \cup \Gamma^{\downarrow}, \\ n \cdot v = 0, \quad n \cdot \mathbb{T}(v,p) \cdot \tau + f_k v \cdot \tau = 0 \quad \text{on } \Gamma^k \text{ for } k = 1, \ldots, K_0, \end{gathered}$$

where f_k is the friction coefficient on the boundary of the ith hole; f_k may be equal to infinity, then $((1.1)'_3)_2$ becomes the zero Dirichlet condition.

The slip boundary condition $(1.1)_3$ describes phenomena when the friction between the fluid and the boundary is negligible. This type of problem can also be treated as an approximation of an external problem—a flow around an obstacle or an approximation of an Eulerian flow, if the viscous coefficient is small and so is the action of the fluid on the boundary (see [8]).

The key element of our approach is a reformulation of the original problem. We use here a special feature of the two-dimensional case as well as an interesting property of the slip boundary condition. From problem (1.1) we obtain a system for the vorticity of the velocity:

$$\begin{aligned} &v \cdot \nabla \alpha - \nu \Delta \alpha = \operatorname{rot} f && \text{in } \Omega, \\ &\alpha = 2v \cdot \tau \chi && \text{on } \partial\Omega, \\ &\alpha \to 0 \quad \text{as } |x| \to \infty, \end{aligned} \tag{1.6}$$

where χ is the curvature of $\partial\Omega$ and

$$\alpha = \operatorname{rot} v = v^2_{,1} - v^1_{,2} \tag{1.7}$$

is the vorticity of the velocity of the fluid. Equation $(1.6)_1$ has the above form only for 2D, in 3D there appears an extra term $\alpha \cdot \nabla v$ which causes worse properties of this equation: we lose the maximum principle. The boundary datum $(1.6)_2$ is calculated from condition $(1.1)_3$. It is worth pointing out that this interesting feature also holds for the 3D case (not exactly the same, because α in 3D is a vector; this property has been noted by Zajączkowski [10]).

To complete the reformulation we add to (1.6) the following problem for the velocity:

$$\begin{aligned} &\operatorname{rot} v = \alpha && \text{in } \Omega, \\ &\operatorname{div} v = 0 && \text{in } \Omega, \\ &n \cdot v = 0 && \text{on } \partial\Omega, \\ &v^1 \to v_\infty && \text{as } x_1 \to \pm\infty. \end{aligned} \tag{1.8}$$

Thus, instead of (1.1), we investigate (1.6) and (1.8).

Since we plan to examine the behavior of solutions to problem (1.1), we need to ensure their existence. We assume the following properties.

THE EXISTENCE HYPOTHESIS. *Let $f \in H^2(\Omega)$, $\partial\Omega \in C^2$ and $v_\infty \geq 0$. Then there exists at least one regular solution of problem* (1.1) *such that*

$$\|v - (v_\infty, 0)\|_{H^4(\Omega)} \leq S_0(\|f\|_{H^2(\Omega)}, v_\infty). \tag{1.9}$$

The above assumption is reasonable, because a similar existence theorem for solutions to problem (1.1) has been proved in [7]. The essential information conveyed by the Existence Hypothesis is the sufficient smoothness of the solutions and vanishing of the perturbation, here in the L^2-norm (we will need the boundedness of the gradient of the vorticity, hence we need $v \in H^4(\Omega)$). There is no restriction on the size of data. The assumption about nonnegativity of v_∞ determines only the system of coordinates. Having such a standard existence assumption we give a precise spatial asymptotics, which is the main result of our paper.

THEOREM 1.1. *Suppose that*

$$\nabla f(x) e^{(\sigma_1 + a_f)|x_1|} \in L^2(\Omega \cap \{x : x_1 \le 0\}), \quad \nabla f(x) e^{(\pi_1 + a_f)|x_1|} \in L^2(\Omega \cap \{x : x_1 > 0\}) \tag{1.10}$$

for some number $a_f > 0$. *Then the solutions to problem* (1.1) *given by the Existence Hypothesis satisfy*

$$\alpha(x) e^{\sigma_1 |x_1|} \in C(\Omega \cap \{x : x_1 \le 0\}), \quad \alpha(x) e^{\pi_1 |x_1|} \in C(\Omega \cap \{x : x_1 > 0\}). \tag{1.11}$$

Furthermore, if $v_\infty > 0$ *then*

$$(v(x) - (v_\infty, 0)) e^{\lambda_1 |x_1|} \in C(\Omega \cap \{x : x_1 \le 0\}), \quad (v(x) - (v_\infty, 0)) e^{\pi_1 |x_1|} \in C(\Omega \cap \{x : x_1 > 0\}), \tag{1.12}$$

and if $v_\infty = 0$ *then*

$$v(x) \frac{e^{\lambda_1 |x_1|}}{1 + |x_1|} \in C(\Omega), \tag{1.13}$$

where

$$\pi_1 = \sqrt{\left(\frac{v_\infty}{2\nu}\right)^2 + \left(\frac{\pi}{H}\right)^2} - \frac{v_\infty}{2\nu} \le \lambda_1 = \frac{\pi}{H} \le \sigma_1 = \sqrt{\left(\frac{v_\infty}{2\nu}\right)^2 + \left(\frac{\pi}{H}\right)^2} + \frac{v_\infty}{2\nu}. \tag{1.14}$$

Moreover,

$$|\alpha(x)| \le \begin{cases} Z_\alpha^- e^{-\sigma_1 |x_1|} & \text{for } x_1 < 0, \\ Z_\alpha^+ e^{-\pi_1 |x_1|} & \text{for } x_1 > 0, \end{cases} \tag{1.15}$$

if $v_\infty > 0$ *then*

$$|v(x) - (v_\infty, 0)| \le \begin{cases} Z_v^- e^{-\lambda_1 |x_1|} & \text{for } x_1 < 0, \\ Z_v^+ e^{-\pi_1 |x_1|} & \text{for } x_1 > 0, \end{cases} \tag{1.16}$$

and if $v_\infty = 0$ *then*

$$|v(x)| \le Z_v^0 (1 + |x_1|) e^{-\lambda_1 |x_1|} \quad \text{for } x_1 \in \mathbb{R}, \tag{1.17}$$

where the constants Z_α^-, Z_α^+, Z_v^-, Z_v^+ *and* Z_v^0 *depend only on* S_0 *and norms of the function* f.

The result shows the crucial role of the vorticity. Since in 2D it is a scalar, it describes precisely the behavior of the flow under the influence of an obstacle. The information about the velocity is just a consequence of the analysis of problem (1.8). This is the reason why decay rates in front of the obstacle are different for the velocity and vorticity. A quantity which describes the decay factors of the perturbation is v_∞/ν. If $v_\infty/\nu \to \infty$, then

$\pi_1 \to 0$ and $\sigma_1 \to \infty$. Hence for large v_∞/ν the perturbation behind the obstacle will be vanishing more slowly, but in front of the obstacle the very vorticity will be less perturbed, but the velocity will be vanishing with factor λ_1 independently of v_∞/ν. Note that v_∞/ν is proportional to the Reynolds number of our flow: $\mathrm{Re} = v_\infty H/\nu$.

In the special case of $v_\infty = 0$, we have $\lambda_1 = \pi_1 = \sigma_1$. This symmetry leads to a slightly weaker convergence for the velocity. It is worth noting that for the external problem for the same system, if $v_\infty \neq 0$, then the decay is proportional to a negative power of $|x|$ ($O(|x|^{-1/2})$, see [4]), and if $v_\infty = 0$, then the velocity may vanish more slowly than any negative power of $|x|$ [3, Chap. X].

Similar studies in the literature concentrate on Leray's problem. This classical system describes the flow in a pipe with no slip boundary data. The present knowledge regarding existence concerns only small data [2, 9]. The results about spatial asymptotic behavior give symmetric exponential decay [1; 3, Chap. XI]. See also [5, 6].

In our case, the slip boundary conditions $(1.1)_3$ together with the reformulation enable us to analyze precisely the structure of solutions of (1.1) for large $|x_1|$. The main tools to examine the behavior of solutions of problems (1.6) and (1.8) are the Fourier series expansions with respect to x_2 and fundamental solutions for ordinary differential equations arising from the systems after the transformation. The estimates are found in nonstandard weighted Banach spaces connected with a representation of functions given by the Fourier series, such as $l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))$ defined by (2.6) in Section 2. This approach makes it possible to compute optimal factors of decay, (1.14), as well as the asymptotic structure of the solutions, which is the second main result of this note.

THEOREM 1.2. *Let a_f, defined as in Theorem* 1.1, *be greater than λ_1. If $v_\infty > 0$, then*

$$\alpha(x) = \Sigma^+ e^{-\pi_1 x_1} \sin\lambda_1 x_2 + O(e^{-2\pi_1 x_1}) \quad \text{as } x_1 \to \infty, \tag{1.18}$$

$$\begin{aligned}\alpha(x) = {} & \Sigma_1^- e^{-\sigma_1|x_1|} \sin\lambda_1 x_2 + \Sigma_2^- e^{-\sigma_2|x_1|}\sin\lambda_2 x_2 \\ & + \ldots + \Sigma_{B_\alpha}^- e^{-\sigma_{B_\alpha}|x_1|}\sin\lambda_{B_\alpha} x_2 \\ & + O(e^{-(\lambda_1+\sigma_1)|x_1|}) \quad \text{as } x_1 \to -\infty,\end{aligned} \tag{1.19}$$

where

$$\lambda_k = \frac{k\pi}{H}, \qquad \sigma_k = \sqrt{\left(\frac{v_\infty}{2\nu}\right)^2 + \left(\frac{k\pi}{2H}\right)^2} + \frac{v_\infty}{2\nu},$$

and B_α is defined as follows: $\sigma_{B_\alpha} < \lambda_1 + \sigma_1$ *and* $\sigma_{B_\alpha+1} \geq \lambda_1 + \sigma_1$.

If $v_\infty = 0$, then

$$\alpha(x) = \Sigma e^{-\lambda_1|x_1|}\sin\lambda_1 x_2 + O(|x_1| e^{-2\lambda_1|x_1|}) \quad \text{as } |x_1| \to \infty. \tag{1.20}$$

Moreover, if $v_\infty > 0$ *then*

$$v(x) = (v_\infty, 0) + V^+ e^{-\pi_1 x_1}(\lambda_1 \cos\lambda_1 x_2, \pi_1 \sin\lambda_1 x_2) + O(e^{-2\pi_1 x_1}) \quad \text{as } x_1 \to \infty, \tag{1.21}$$

and

$$\begin{aligned} v(x) = (v_\infty, 0) &+ V_1^- e^{-\lambda_1 |x_1|}(\cos\lambda_1 x_2, \sin\lambda_1 x_2) + \ldots + \\ &+ V_{B_v}^- e^{-\lambda_{B_v}|x_1|}(\cos\lambda_{B_v} x_2, \sin\lambda_{B_v} x_2) \\ &+ V_{B_v+1}^- e^{-\sigma_1 |x_1|}(\lambda_1 \cos\lambda_1 x_2, \sigma_1 \sin\lambda_1 x_2) \\ &+ \ldots + V_{B_v+B_\alpha}^- e^{-\sigma_{B_\alpha}|x_1|}(\lambda_{B_\alpha} \cos\lambda_{B_\alpha} x_2, \sigma_{B_\alpha} \sin\lambda_{B_\alpha} x_2) \\ &+ O(e^{-(\lambda_1+\sigma_1)|x_1|}) \quad \text{as } x_1 \to -\infty, \end{aligned} \tag{1.22}$$

where B_v *is defined as follows*: $\lambda_{B_v} < \lambda_1 + \sigma_1$ *and* $\lambda_{B_v+1} \geq \lambda_1 + \sigma_1$; *and if* $v_\infty = 0$ *then*

$$v(x) = V|x_1|e^{-\lambda_1|x_1|}(\cos\lambda_1 x_2, \sin\lambda_1 x_2) + O(|x_1|e^{-2\lambda_1|x_1|}) \quad \text{as } |x_1| \to \infty. \tag{1.23}$$

The constants $\Sigma^+, \Sigma_k^-, \Sigma, V^+, V_k^-, V$ *depend on the solution.*

The above result describes precisely the form of the solutions for large $|x_1|$ in dependence on the flux of the flow. The structure of the solutions is connected with the magnitude of the quantities B_α and B_v which are increasing functions of the Reynolds number $\mathrm{Re} = v_\infty H/\nu$, since π_k and σ_k can be written as follows:

$$\pi_k = \frac{H}{2}\,(\sqrt{\mathrm{Re}^2 + 4\pi^2 k^2} - \mathrm{Re}), \qquad \sigma_k = \frac{H}{2}\,(\sqrt{\mathrm{Re}^2 + 4\pi^2 k^2} + \mathrm{Re}).$$

Note that the information is more precise in front of the obstacle and shows a laminar character of the flow on this side, even for large v_∞. And behind the obstacle the information we obtain is poorer.

We underline that our results are obtained under assumptions (1.10) and $a_f > \lambda_1$, which, in some sense, neglects the influence of the external force. This way the coefficients describing the asymptotic behavior of the solutions depend only on the Reynolds number and the height of the pipe. Such a restriction enables us to show the natural structure of the flow. Otherwise, if the gradient of the force f did not decay sufficiently fast, the rates would depend on the behavior of the force at infinity. It would add only technical difficulties with no new interesting features. That is why we omit this case.

The paper is organized as follows. In Section 2 the necessary notations and definitions of function spaces with elementary properties are introduced. Next, we reformulate the problem and prove a basic result about decay of solutions. In Section 4 we prove Theorem 1.1, and at the end Theorem 1.2.

2. Notation. We denote by $L^p(\Omega)$ the standard Lebesgue space of p-integrable functions. Our technique requires introducing weighted Banach spaces with different behavior at ∞ and $-\infty$. We use exponential weights guaranteeing fast decay of functions at infinity.

For $\sigma, \pi \geq 0$, let

$$\begin{aligned} m_{\sigma,\pi}(\mathbb{R}) := \{f \in L^\infty(\mathbb{R}) : \text{there is } M > 0 \text{ such that}\\ |f(t)| \leq Me^{-\sigma|t|} \text{ for } t < 0 \text{ and } |f(t)| \leq Me^{-\pi|t|} \text{ for } t > 0\}. \end{aligned} \tag{2.1}$$

This is a Banach space with the norm $\|f\|_{m_{\sigma,\pi}(\mathbb{R})} = \inf\{M : M \text{ as in (2.1)}\}$.

Similarly, for $\Omega \subset \mathbb{R}^2$ we define

$$\begin{aligned} M_{\sigma,\pi}(\Omega) := \{f \in L^\infty(\Omega) : \text{there is } M > 0 \text{ such that}\\ |f(x)| \leq Me^{-\sigma|x_1|} \text{ for } x_1 < 0 \text{ and } |f(x)| \leq Me^{-\pi|x_1|} \text{ for } x_1 > 0\}, \end{aligned} \tag{2.2}$$

with the norm $\|f\|_{M_{\sigma,\pi}(\Omega)} = \inf\{M : M \text{ as in (2.2)}\}$.

Also we need the weighted Hilbert spaces

$$L^2_\sigma(\Omega) = \{f \in L^2(\Omega) : \|fe^{\sigma|x_1|}\|_{L^2(\Omega)} < \infty\} \tag{2.3}$$

with the norm $\|f\|_{L^2_\sigma(\Omega)} = \|fe^{\sigma|x_1|}\|_{L^2(\Omega)}$, and

$$\begin{aligned} L^2_{\sigma,\pi}(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : f(t)e^{\sigma|t|} \in L^2(-\infty, 0)\\ \text{and } f(t)e^{\pi|t|} \in L^2(0,\infty)\} \end{aligned} \tag{2.4}$$

with the norm

$$\|f\|_{L^2_{\sigma,\pi}(\mathbb{R})} = (\|f(t)e^{\sigma|t|}\|^2_{L^2(-\infty,0)} + \|f(t)e^{\pi|t|}\|^2_{L^2(0,\infty)})^{1/2}.$$

Our main considerations concern a straight pipe $W = \mathbb{R} \times (0, H)$. We need a special type of L^2 spaces which arise from the Fourier transform with respect to x_2.

Let $f \in L^2(0, H; L^2(\mathbb{R}))$. We introduce

$$f_k(x_1) = \langle f(x), v_k\rangle = \int_0^H f(x)v_k(x_2)\,dx_2, \tag{2.5}$$

where $v_k = \sqrt{2/H}\sin\frac{k\pi}{H}x_2$ for $k = 1, 2, \ldots$ Then we define the Banach space

$$l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R})) := \Big\{f \in L^2(W) : \sum_{k=1}^\infty \|f_k\|^2_{m_{\sigma,\pi}(\mathbb{R})} < \infty\Big\} \tag{2.6}$$

with the norm

$$\|f\|_{l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))} = \Big(\sum_{k=1}^\infty \|f_k\|^2_{m_{\sigma,\pi}(\mathbb{R})}\Big)^{1/2}. \tag{2.7}$$

It is worth noting that $l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R})) \neq L^2(0,H;m_{\sigma,\pi}(\mathbb{R}))$, since one easily sees that $\sin(x_1x_2) \notin l^2_{(0,H)}(m_{0,0}(\mathbb{R}))$.

PROPOSITION 2.1. *The following relation holds*:

$$\|\nabla f\|_{l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))} \equiv \Big(\sum_{k=1}^{\infty}(\|\dot{f}_k\|^2_{m_{\sigma,\pi}(\mathbb{R})} + \|kf_k\|^2_{m_{\sigma,\pi}(\mathbb{R})})\Big)^{1/2}, \tag{2.8}$$

where the dot denotes differentiation with respect to x_1.

PROPOSITION 2.2. *If* $\nabla f \in l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))$, *then* $f \in M_{\sigma,\pi}(W)$ *with the estimate*

$$\|f\|_{M_{\sigma,\pi}(W)} \le c\|\nabla f\|_{l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))}. \tag{2.9}$$

PROPOSITION 2.3. *Let* $0 < \tau < \min\{\sigma,\pi\}$. *Then* $l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R})) \subset L^2_{\sigma-\tau,\pi-\tau}(W)$ *with the estimate*

$$\|f\|_{L^2_{\sigma-\tau,\pi-\tau}(W)} \le c(\tau)\|f\|_{l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))}, \tag{2.10}$$

where $c(\tau) \to \infty$ *as* $\tau \to 0$.

In this paper, the symbol π is used as a parameter of various spaces, but also sometimes as the 3.14... constant. The author hopes that this will cause no misunderstanding.

By c we denote a generic constant. Also in the proofs of lemmas we use S for constants which depend only on S_0 from (1.9) and norms of the force f.

3. Preliminaries. We consider the velocity as the sum of a constant flow and a perturbation:

$$v = (v_\infty, 0) + u. \tag{3.1}$$

Since $\operatorname{rot}(v_\infty,0) = 0$, the vorticity stays the same. By (3.1), problem (1.6) takes the following form:

$$\begin{aligned} &v_\infty\partial_{x_1}\alpha - \nu\Delta\alpha = -u\cdot\nabla\alpha + \operatorname{rot} f && \text{in } \Omega,\\ &\alpha = 2(u + (v_\infty,0))\cdot\tau\chi && \text{on } \partial\Omega,\\ &\alpha \to 0 \quad \text{as } |x|\to\infty, \end{aligned} \tag{3.2}$$

where χ is the curvature of the boundary and by (1.8) we have

$$\begin{aligned} &\operatorname{rot} u = \alpha && \text{in } \Omega,\\ &\operatorname{div} u = 0 && \text{in } \Omega,\\ &u\cdot n = -(v_\infty,0)\cdot n && \text{on } \partial\Omega,\\ &u^1 \to 0 \quad \text{as } |x|\to\infty. \end{aligned} \tag{3.3}$$

Although the domain Ω may not be simply connected, the boundary condition $v\cdot n = 0$ (see $(1.8)_3$) and (3.1) guarantee the existence of a potential (stream function) for the velocity:

$$u = (-\partial_{x_2}\varphi, \partial_{x_1}\varphi). \tag{3.4}$$

If Ω is not simply connected the kernel of the elliptic operator (rot-div) generated by problem (3.3) is nontrivial. We take a potential which is orthogonal to the kernel. This way from (3.3) we get the Dirichlet problem for the system

$$\begin{aligned} &\Delta\varphi = \alpha && \text{in } \Omega,\\ &\varphi = b && \text{on } \partial\Omega,\\ &\varphi \to 0 \quad \text{as } |x| \to \infty, \end{aligned} \tag{3.5}$$

where b can be obtained from condition $(3.3)_3$ such that $b = 0$ for $|x_1| > D$. This feature of the Dirichlet data follows from $(1.8)_3$ and the fact that the potential of $(v_\infty, 0)$ is constant on the unbounded parts $\Gamma^\uparrow$ and $\Gamma^\downarrow$ (recall that $(v_\infty, 0) = (-\partial_{x_2}(-v_\infty x_2), \partial_{x_1}(-v_\infty x_2)))$.

Introduce a smooth function $\eta : \mathbb{R} \to [0, 1]$ such that

$$\eta(x_1) = \begin{cases} 1 & \text{for } |x_1| > D + 2,\\ 0 & \text{for } |x_1| < D, \end{cases} \tag{3.6}$$

and $|\nabla\eta| \le 1$. Put

$$\beta = \eta\alpha, \qquad \psi = \eta\varphi. \tag{3.7}$$

By the properties of η and Ω, the new functions β and ψ are well defined in $W = \mathbb{R} \times (0, H)$, since they vanish for $|x| < D$. By (3.2), β satisfies the following problem:

$$\begin{aligned} &v_\infty\partial_{x_1}\beta - \nu\Delta\beta = -\nabla^\perp\psi \cdot \nabla\beta + G && \text{in } W,\\ &\beta = 0 && \text{on } \partial W,\\ &\beta \to 0 \quad \text{as } |x| \to \infty, \end{aligned} \tag{3.8}$$

where $G = G_1 + G_2$ and

$$\begin{aligned} G_1 &= -(\varphi\nabla^\perp(1-\eta) + (1-\eta)\nabla^\perp\varphi) \cdot \nabla(\eta\alpha) - 2\nu\nabla\eta \cdot \nabla\alpha - \nu(\Delta\eta)\alpha,\\ G_2 &= \eta \operatorname{rot} g. \end{aligned}$$

Note that $\operatorname{supp} G_1 \subset [-D, D] \times [0, H]$ and since the L^∞-bound for α is given, $G_1 \in M_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{R}_+$.

From problem (3.5), the function ψ satisfies the following system:

$$\begin{aligned} &\Delta\psi = \beta + G_3 && \text{in } W,\\ &\psi = 0 && \text{on } \partial W,\\ &\psi \to 0 \quad \text{as } |x| \to \infty, \end{aligned} \tag{3.9}$$

where $G_3 = 2\nabla\eta \cdot \nabla\varphi + (\Delta\eta)\varphi$. Just as for G_1, $\operatorname{supp} G_3 \subset [-D, D] \times [0, H]$ and by the boundedness of φ, we also have $G_3 \in M_{\alpha,\beta}$ for any $\alpha, \beta \in \mathbb{R}_+$.

The structure of the domain together with the boundary data $(3.8)_2$ and $(3.9)_2$ allows us to apply the Fourier series expansion with respect to x_2. Introduce

$$\beta = \sum_{k=1}^{\infty} b_k(x_1)v_k(x_2), \qquad \psi = \sum_{k=1}^{\infty} q_k(x_1)v_k(x_2), \tag{3.10}$$

where

$$v_k(x_2) = \sqrt{\frac{2}{H}} \sin \frac{k\pi}{H} x_2 \quad \text{for } k = 1, 2, \ldots \tag{3.11}$$

The functions v_k are eigenvectors of the following problem:

$$-\partial^2_{x_2} v_k = \lambda_k^2 v_k, \quad v_k(0) = v_k(H) = 0, \quad \text{with} \quad \lambda_k = \frac{k\pi}{H}. \tag{3.12}$$

By the Existence Hypothesis, the series are well defined, since $\beta, \psi \in L^2(W)$.

Since, by assumption, the r.h.s. of (3.8) and (3.9) are known we rewrite these problems as follows:

$$\begin{aligned} &v_\infty \dot{b}_k - \nu \ddot{b}_k + \nu \lambda_k^2 b_k = \langle -\nabla^\perp \psi \cdot \nabla \beta, v_k \rangle + \langle G, v_k \rangle \quad &&\text{on } \mathbb{R}, \\ &\ddot{q}_k - \lambda_k^2 q_k = b_k + \langle G_3, v_k \rangle &&\text{on } \mathbb{R}, \end{aligned} \tag{3.13}$$

for $k = 1, 2, \ldots$, remembering that b_k and q_k vanish at infinity.

Equations (3.13) are one-dimensional, which simplifies our considerations. The mail tools for examining the spatial asymptotics are the fundamental solutions for the operators on the l.h.s. of (3.13).

Proposition 3.1. *The fundamental solution to problem* $(3.13)_1$ *satisfies*

$$v_\infty \dot{E}_k - \nu \ddot{E}_k + \nu \lambda_k^2 E_k = \delta \quad \text{in } \mathbb{R}, \tag{3.14}$$

where δ *is the Dirac delta and*

$$E_k(t) = n_k \begin{cases} e^{\sigma_k t} & \text{for } t < 0, \\ e^{-\pi_k t} & \text{for } t > 0, \end{cases} \tag{3.15}$$

with

$$\begin{aligned} \sigma_k &= \frac{\sqrt{v_\infty^2 + 4\nu^2 \lambda_k^2} + v_\infty}{2\nu}, \qquad n_k = \frac{1}{\sqrt{v_\infty^2 + 4\nu^2 \lambda_k}}, \\ \pi_k &= \frac{\sqrt{v_\infty^2 + 4\nu^2 \lambda_k^2} - v_\infty}{2\nu}. \end{aligned} \tag{3.16}$$

Proposition 3.2. *The fundamental solution to problem* $(3.13)_2$ *satisfies*

$$\ddot{D}_k - \lambda_k^2 D_k = \delta \quad \text{in } \mathbb{R} \tag{3.17}$$

and

$$D_k(t) = -\frac{1}{2\lambda_k} \begin{cases} e^{\lambda_k t} & \text{for } t < 0, \\ e^{-\lambda_k t} & \text{for } t > 0. \end{cases} \tag{3.18}$$

REMARK. Since the quantities $\lambda_k, \pi_k, \sigma_k$ and n_k are important in our considerations, note that for all k,

$$\pi_k \le \lambda_k \le \sigma_k, \tag{3.19}$$

$$n_k\lambda_k + n_k\pi_k + n_k\sigma_k \le c(v_\infty, \nu), \tag{3.20}$$

where the constant in (3.20) is independent of k.

First we show some basic estimates in weighted spaces which follow from the energy approach.

LEMMA 3.1. *The solutions given by the Existence Hypothesis satisfy*

$$\|\nabla\beta|x_1|\,\|_{L^2(W)} + \|\nabla\psi|x_1|\,\|_{L^2(W)} \le S. \tag{3.21}$$

Moreover, *for any* $L > 1$,

$$\|\nabla\psi\|_{L^\infty((-\infty,-L)\cup(L,\infty))\times(0,H))} \le L^{-a}S \tag{3.22}$$

for some $a > 0$.

Proof. Multiplying $(3.8)_1$ by βx_1^2, integrating over W, and remembering that $\beta \in L^2(W)$ and $\nabla f \in L^2_{\sigma,\pi}(W)$ we obtain

$$\|\nabla\beta|x_1|\,\|_{L^2(W)} \le S, \tag{3.23}$$

since

$$\Big|\int\limits_W \nabla^\perp\psi\cdot\nabla\beta\beta x_1^2\,dx\Big| \le \varepsilon\|\nabla\beta|x_1|\,\|^2_{L^2(W)} + S. \tag{3.24}$$

Similarly from (3.9) we get

$$\|\nabla\psi|x_1|\,\|_{L^2(W)} \le S. \tag{3.25}$$

Hence by the interpolation theorem from (3.25) and the C^1-bound of u we get estimate (3.22). Lemma 3.1 is proved.

LEMMA 3.2. *The vorticity satisfies the following bound*:

$$\|\nabla\beta\|_{L^2_{\sigma_0,\pi_0}(W)} \le S, \tag{3.26}$$

where

$$\sigma_0 = \sqrt{\left(\frac{v_\infty}{4\nu}\right)^2 + \frac{\lambda_1^2}{8}} + \frac{v_\infty}{4\nu}, \qquad \pi_0 = \sqrt{\left(\frac{v_\infty}{4\nu}\right)^2 + \frac{\lambda_1^2}{8}} - \frac{v_\infty}{4\nu}. \tag{3.27}$$

Proof. From $(3.13)_1$,

$$\frac{v_\infty}{\nu}\,\dot b_k - \ddot b_k + \lambda_k^2 b_k = \frac{1}{\nu}\,N_k, \tag{3.28}$$

where $N_k = \langle -\nabla^\perp\psi\cdot\nabla\beta + G, v_k\rangle$. Since $\operatorname{supp} b_k \subset (-\infty,-L]\cup[L,\infty)$ we consider the problem for $x_1 > 0$; the case of $x_1 < 0$ can be treated similarly.

Multiply (3.28) by $b_k e^{2\pi_0 x_1}$ and integrate over $\mathbb{R}_+$ to get

$$\int_0^\infty \left(\dot b_k^2 e^{2\pi_0 x_1} + \left(\lambda_k^2 - \frac{v_\infty}{\nu}\,\pi_0 - 2\pi_0^2\right) b_k^2 e^{2\pi_0 x_1}\right) dx_1 = \int_0^\infty N_k b_k e^{2\pi_0 x_1}\, dx_1. \tag{3.29}$$

To control the whole norm of $\nabla\beta$ (see Proposition 2.1), we split the coefficient as

$$\lambda_k^2 - \frac{v_\infty}{\nu}\,\pi_0 - 2\pi_0^2 = \frac{\lambda_k^2}{2} + \frac{\lambda_1^2}{4} + \left(\frac{\lambda_k^2}{2} - \frac{\lambda_1^2}{4} - \frac{v_\infty}{\nu}\,\pi_0 - 2\pi_0^2\right). \tag{3.30}$$

Now, we find π_0. We require that

$$\frac{\lambda_k^2}{2} - \frac{\lambda_1^2}{4} - \frac{v_\infty}{\nu}\,\pi_0 - 2\pi_0^2 \ge 0 \tag{3.31}$$

for all $k \ge 1$. Since $\lambda_k \ge \lambda_1$ it is enough to take for π_0 the positive root of the equation $\frac{\lambda_1^2}{4} - \frac{v_\infty}{\nu}\pi_0 - 2\pi_0^2 = 0$, i.e. π_0 as in (3.27). Then, applying the Schwarz inequality to the r.h.s. of (3.29) we get

$$\int_0^\infty \left(\dot b_k^2 + \frac{\lambda_k^2}{2}\, b_k^2\right) e^{2\pi_0 x_1}\, dx_1 + \int_0^\infty \frac{\lambda_1^2}{4}\, b_k^2 e^{2\pi_0 x_1}\, dx_1 \le c\int_0^\infty N_k^2 e^{2\pi_0 x_1}\, dx_1 + \int_0^\infty \frac{\lambda_1^2}{4}\, b_k^2 e^{2\pi_0 x_1}\, dx_1. \tag{3.32}$$

This way we obtain

$$\|\nabla\beta\|_{L^2_{\pi_0}(\mathbb{R}_+\times(0,H))} \le c\|-\nabla^\perp\psi\cdot\nabla\beta + G\|_{L^2_{\pi_0}(\mathbb{R}_+\times(0,H))}. \tag{3.33}$$

By Lemma 3.1 we choose L so large that

$$c\|\nabla\psi\|_{L^\infty((L,\infty)\times(0,H))} \le 1/2, \tag{3.34}$$

where the constant c is the same as in (3.33); and by (3.33) we get

$$\|\nabla\beta\|_{L^2_{\pi_0}(\mathbb{R}_+\times(0,H))} \le S. \tag{3.35}$$

We repeat the same procedure for $(-\infty,0)$ with weight $e^{\sigma|x_1|}$ to obtain

$$\|\nabla\beta\|_{L^2_{\sigma_0}(\mathbb{R}_-\times(0,H))} \le S \tag{3.36}$$

with σ_0 as in (3.27). Estimates (3.35) and (3.36) give (3.26). Lemma 3.2 is proved.

LEMMA 3.3. *The velocity satisfies the following bounds*:

$$\|\nabla^2\psi\|_{l^2_{(0,H)}(m_{\sigma_0/2,\pi_0/2}(\mathbb{R}))} \le S, \qquad \|\nabla\psi\|_{M_{\sigma_0/2,\pi_0/2}} \le S. \tag{3.37}$$

Proof. By $(3.13)_2$,

$$\ddot q_k - \lambda_k^2 q_k = B_k, \tag{3.38}$$

where $B_k = b_k + \langle G_3, v_k\rangle$. By Lemma 3.2 and the definition of G_3 we get $\sum_{k=1}^{\infty} B_k v_k \in L^2_{\sigma_0,\pi_0}(W)$. Hence, using Proposition 3.2 we have

$$\dot{q}_k(t) = \int_{\mathbb{R}} \dot{D}_k(t-y) B_k(y)\,dy = \int_{-\infty}^{0} + \int_0^t + \int_t^{\infty} = K_1 + K_2 + K_3. \tag{3.39}$$

If $t > 0$, then

$$\begin{aligned} K_1^2(t) &\le c\Big(\int_{-\infty}^{0} e^{-\lambda_k(t-y)} B_k(y)\,dy\Big)^2 \\ &\le c \int_{-\infty}^{0} e^{-2\lambda_k(t-y)} e^{2\sigma_0 y}\,dy\,\|B_k\|^2_{L^2_{\sigma_0,\pi_0}(\mathbb{R})} \le ce^{-2\lambda_k t}\|B_k\|^2_{L^2_{\sigma_0,\pi_0}(\mathbb{R})}, \\ K_2^2(t) &\le c\Big(\int_0^t e^{-2\lambda_k(t-y)} e^{-2\pi_0 y}\,dy\Big)\|B_k\|^2_{L^2_{\sigma_0,\pi_0}(\mathbb{R})} \le cte^{-2\pi_0 t}\|B_k\|^2_{L^2_{\sigma_0,\pi_0}(\mathbb{R})}, \end{aligned}$$

where we used $\pi_0 \le \lambda_1, \sigma_0$; hence by the boundedness of $te^{-\pi_0 t}$ we obtain

$$K_2^2(t) \le ce^{-\pi_0 t}\|B_k\|^2_{L^2_{\sigma_0,\pi_0}(\mathbb{R})}.$$

Finally,

$$K_3^2(t) \le c\Big(\int_t^{\infty} e^{-2\lambda_k(y-t)} e^{-2\pi_0 y}\,dy\Big)\|B_k\|^2_{L^2_{\sigma_0,\pi_0}(\mathbb{R})} \le ce^{-2\pi_0 t}\|B_k\|^2_{L^2_{\sigma_0,\pi_0}(\mathbb{R})}.$$

By (3.39) and the estimates for K_i, we conclude that

$$\dot{q}_k^2(t) \le ce^{-\pi_0 t}\|B_k\|^2_{L^2_{\sigma_0,\pi_0}(\mathbb{R})}. \tag{3.40}$$

For $x_1 < 0$ with weight $e^{\sigma_0|x_1|}$ as well as for the derivative with respect to x_2 analogous estimates can be obtained. Thus we have proved

$$\|\nabla\psi\|_{l^2_{(0,H)}(m_{\sigma_0/2,\pi_0/2}(\mathbb{R}))} \le S. \tag{3.41}$$

Since, by Lemma 3.2, $\nabla\beta \in L^2_{\sigma_0,\pi_0}(W)$, we can also obtain

$$\|\nabla^2\psi\|_{l^2_{(0,H)}(m_{\sigma_0/2,\pi_0/2}(\mathbb{R}))} \le S, \tag{3.42}$$

which, by Proposition 2.2, guarantees $\nabla\psi \in M_{\sigma_0/2,\pi_0/2}$ with the bound

$$\|\nabla\psi\|_{M_{\sigma_0/2,\pi_0/2}} \le S. \tag{3.43}$$

Lemma 3.3 is proved.

4. Proof of Theorem 1.1. The goal of this section is to show that

$$\nabla\beta \in l^2_{(0,H)}(m_{\pi_1,\sigma_1}(\mathbb{R})).$$

If we have this information the rest of Theorem 1.1 will be a corollary from Proposition 2.2. At the beginning we assume $\nabla\beta \in l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))$

for some $\sigma, \pi > 0$. Next, we examine $\nabla^{\perp}\psi \cdot \nabla\beta$. By Lemma 3.3 we have $\nabla^{\perp}\psi \in M_{\sigma_0/2,\pi_0/2}(W)$, hence by Proposition 2.3 for $0 < \tau < \pi_0/2$ it follows that

$$\nabla^{\perp}\psi \cdot \nabla\beta \in L^2(0, H; L^2_{\sigma+\tau,\pi+\tau}(\mathbb{R}))$$

with the estimate

$$\|\nabla^{\perp}\psi \cdot \nabla\beta\|_{L^2(0,H;L^2_{\sigma+\tau,\pi+\tau}(\mathbb{R}))} \leq c(\pi_0/2 - \tau)\|\nabla\psi\|_{M_{\sigma_0/2,\pi_0/2}}\|\nabla\beta\|_{l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))}, \tag{4.1}$$

where $c(\pi_0/2 - \tau)$ is well defined for $0 < \tau < \pi_0/2$.

Since $G \in L^2(0, H; L^2_{\sigma+\tau,\pi+\tau}(\mathbb{R}))$, assuming that $\tau < a_f$, where a_f is as in (1.10), we introduce the coefficients

$$\langle -\nabla^{\perp}\psi \cdot \nabla\beta + G, v_k\rangle \cdot \begin{cases} e^{(\sigma+\tau)|x_1|} & \text{for } x_1 \leq 0 \\ e^{(\pi+\tau)|x_1|} & \text{for } x_1 > 0 \end{cases} = A_k(x_1) \tag{4.2}$$

such that $\{\|A_k\|_{L^2(\mathbb{R})}\}_{k=1}^{\infty} = \{a_k\}_{k=1}^{\infty} \in l^2$ with

$$\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} = \|-\nabla^{\perp}\psi \cdot \nabla\beta + G\|_{L^2(0,H;L^2_{\sigma+\tau,\pi+\tau}(\mathbb{R}))}. \tag{4.3}$$

Moreover, by (4.1) we have

$$\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \leq S\|\nabla\beta\|_{l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))} + S, \tag{4.4}$$

where the constants depend on the quantities already given by Lemmas 3.2 and 3.3.

Take $t > L$ (the parameter L will be defined later). Then

$$\dot{b}_k(t) = \int_{\mathbb{R}} \dot{E}_k(t-y)N_k(y)\,dy = \int_t^{\infty} + \int_{-\infty}^{t} = I_1(t) + I_2(t). \tag{4.5}$$

For I_1, we have $t - y < 0$, hence by (3.15),

$$I_1^2(t) \leq \Big(n_k \int_t^{\infty} \sigma_k e^{-\sigma_k(y-t)} e^{-(\pi+\tau)y}|A_k(y)|\,dy\Big)^2$$
$$\leq ce^{2\sigma_k t}\Big(\int_t^{\infty} e^{-2(\sigma_k+\pi+\tau)y}\,dy\Big)a_k^2 \leq \frac{ca_k^2}{\sigma_k+\pi+\tau}\,e^{-2\tau t}e^{-2\pi t}.$$

For I_2, we have $t - y > 0$, hence by (3.15),

$$I_2(t) = \int_{-\infty}^{0} + \int_0^t = I_{21}(t) + I_{22}(t), \tag{4.6}$$

and

$$I_{21}^2(t) \le (n_k\pi_k)^2\Big(\int_{-\infty}^{0} e^{-\pi_k(t-y)}|N_k(y)|\,dy\Big)^2$$
$$\le ce^{-2\pi_k t}\Big(\int_{-\infty}^{0} e^{2\pi_k y}\,dy\Big)\Big(\int_{-\infty}^{0} N_k^2(y)\,dy\Big) \le \frac{c}{\pi_k}\,e^{-2\pi_k t}\int_{-\infty}^{0} N_k^2(y)\,dy.$$

To get more precise information we divide $I_{22}(t)$ into two terms

$$I_{22}(t) = \int_0^L + \int_L^t = I_{221}(t) + I_{222}(t). \tag{4.7}$$

For $I_{221}(t)$ we have

$$I_{221}^2(t) \le c\Big(\int_0^L e^{-\pi_k(t-y)}N_k(y)\,dy\Big)^2 \le ce^{-2\pi_k t}\Big(\int_0^L e^{2\pi_k y}\,dy\Big)\Big(\int_0^L N_k^2(y)\,dy\Big)$$
$$\le \frac{c}{\pi_k}\,e^{-2\pi_1 t}e^{2\pi_1 L}\int_0^L N_k^2(y)\,dy.$$

Finally,

$$I_{222}^2(t) \le c\Big(\int_L^t e^{-\pi_k(t-y)}e^{-(\pi+\tau)y}|A_k(y)|\,dy\Big)^2$$
$$\le ce^{-2\pi_k t}\Big(\int_L^t e^{2(\pi_k-\pi-\tau)y}\,dy\Big)a_k^2 \le ca_k^2e^{-2\pi t}e^{-2(\pi_k-\pi)t}\int_L^t e^{2(\pi_k-\pi)y}e^{-2\tau y}\,dy;$$

but we remember that $\pi_k - \pi \ge 0$, hence

$$I_{222}^2(t) \le ca_k^2e^{-2\pi t}\int_L^t e^{2(\pi_k-\pi)(y-t)}e^{-2\tau y}\,dy;$$

but $2(\pi_k-\pi)(y-t) \le 0$, which implies $e^{2(\pi_k-\pi)(y-t)} \le 1$, and so

$$I_{222}^2(t) \le ca_k^2e^{-2\pi t}\int_L^t e^{-2\tau y}\,dy \le \frac{ca_k^2}{\tau}\,e^{-2\tau L}e^{-2\pi t}.$$

Since $|\dot b_k(t)|^2 \le c(I_1^2(t) + I_{21}^2(t) + I_{221}^2(t) + I_{222}^2(t))$, from the above estimations we conclude that

$$|\dot b_k(t)|^2 \le \frac{ca_k^2}{\sigma_k+\pi+\tau}\,e^{-2\tau t}\,e^{-2\pi t} + \frac{c}{\pi_k}\,e^{-2\pi_k t}\int_{-\infty}^{0} N_k^2(y)\,dy$$
$$+ \frac{c}{\pi_k}\,e^{-2\pi_1 t}\,e^{2\pi_1 L}\int_0^L N_k^2(y)\,dy + \frac{ca_k^2}{\tau}\,e^{-2\tau L}e^{-2\pi t}. \tag{4.8}$$

Therefore

$$\sum_{k=1}^{\infty}|\dot{b}_k(t)|^2e^{2\pi t} \le ce^{-2\tau L}\sum_{k=1}^{\infty}a_k^2 + c(1+e^{2\pi_1 L})\sum_{k=1}^{\infty}\Big(\int_{-\infty}^{\infty}N_k^2(y)\,dy\Big)e^{-2(\pi_1-\pi)t}. \tag{4.9}$$

Thus, we see that the optimal factor of decay for positive x_1 is $\pi = \pi_1$. Next, note that

$$\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}N_k^2(y)\,dy = \|-\nabla^{\perp}\psi\cdot\nabla\beta + G\|_{L^2(W)}^2 \le S. \tag{4.10}$$

Similar considerations for the derivative with respect to x_2 show that for $x_1 > L$,

$$\sum_{k=1}^{\infty}(|\dot{b}_k(x_1)|^2 + \lambda_k|b_k(t)|^2)e^{2\pi_1 x_1} \le Ce^{-2\tau L}\sum_{k=1}^{\infty}(|\dot{b}_k(x_1)|^2 + \lambda_k|b_k(t)|^2)e^{2\pi_1 x_1} + S. \tag{4.11}$$

Taking L so large that $Ce^{-2\tau L} < 1/2$, we get

$$\sum_{k=1}^{\infty}(|\dot{b}_k(t)|^2 + |\lambda_k b_k(t)|^2)e^{2\pi_1 x_1} \le S. \tag{4.12}$$

We repeat the same analysis for $x_1 < -L$ (maybe for larger L) to obtain

$$\sum_{k=1}^{\infty}(|\dot{b}_k(t)|^2 + |\lambda_k b_k(t)|^2)e^{2\sigma_1|x_1|} \le S. \tag{4.13}$$

From (4.12) and (4.13) we conclude

$$\|\nabla\beta\|_{l^2_{(0,H)}(m_{\sigma_1,\pi_1}(\mathbb{R}))} \le S. \tag{4.14}$$

By Proposition 2.2, this estimate gives us the inclusion

$$\beta \in M_{\sigma_1,\pi_1}. \tag{4.15}$$

But (4.15) is valid only for $x_1 \in (-\infty,-L)\cup(L,\infty)$. To fill the gap it is enough to note that by the Existence Hypothesis $\nabla\alpha$ is bounded in the L^{∞}-norm in the whole domain, so in particular for $x_1 \in [-L,L]$. By the definition of β we conclude that $\alpha \in M_{\sigma_1,\pi_1}$ with a suitable estimate (1.15) as in the statement of Theorem 1.1.

Next, we prove the second part of Theorem 1.1 concerning the behavior of the velocity. Since $u = \nabla^{\perp}\varphi$, we analyze the equation for ψ. By Proposition 3.2, as in Lemma 3.3, we recall

$$\dot{q}_k(t) = \int_{\mathbb{R}} \dot{D}_k(t-y)(b_k + \langle G_3, v_k\rangle)\, dy = J_1(t) + J_2(t), \tag{4.16}$$

where J_1 and J_2 are connected with b_k and G_3, respectively. Since, by the definition, G_3 belongs to $L^2_{\sigma,\pi}(W)$ together with its derivatives for any σ, π, but not to $l^2_{(0,H)}(m_{\sigma,\pi}(\mathbb{R}))$, we analyze these terms separately.

Take $t > 1$. Then

$$J_1(t) = \int_{-\infty}^{0} + \int_{0}^{t} + \int_{t}^{\infty} = J_{11}(t) + J_{12}(t) + J_{13}(t). \tag{4.17}$$

We estimate

$$|J_{11}(t)| \le c \int_{-\infty}^{0} e^{-\lambda_k(t-y)} e^{-\sigma_1|y|} \|b_k\|_{m_{\sigma_1,\pi_1}(\mathbb{R})}\, dy \le c e^{-\lambda_k t} \|b_k\|_{m_{\sigma_1,\pi_1}(\mathbb{R})},$$

$$|J_{13}(t)| \le c \int_{t}^{\infty} e^{-\lambda_k(y-t)} e^{-\pi_1 y} \|b_k\|_{m_{\sigma_1,\pi_1}(\mathbb{R})}\, dy \le c e^{-\pi_1 t} \|b_k\|_{m_{\sigma_1,\pi_1}(\mathbb{R})},$$

$$\begin{aligned} |J_{12}(t)| &\le c \|b_k\|_{m_{\sigma_1,\pi_1}(\mathbb{R})} e^{-\lambda_k t} \int_0^t e^{-(\pi_1-\lambda_k)y}\, dy \\ &\le c \|b_k\|_{m_{\sigma_1,\pi_1}(\mathbb{R})} \begin{cases} |\lambda_k - \pi_1|^{-1} e^{-\min\{\pi_1,\lambda_k\}t} & \text{for } \pi_1 \ne \lambda_k, \\ t e^{-\pi_1 t} & \text{for } \pi_1 = \lambda_k. \end{cases} \end{aligned}$$

But the condition $\pi_1 \ne \lambda_k$, by the definition of these quantities, is equivalent to $v_\infty > 0$. Hence for $t > 1$,

$$|J_1(t)| \le c \|b_k\|_{m_{\sigma_1,\pi_1}} \begin{cases} e^{-\pi_1 t} & \text{if } v_\infty > 0, \\ t e^{-\lambda_1 t} & \text{if } v_\infty = 0. \end{cases} \tag{4.18}$$

For J_2, just as for the vorticity we obtain

$$|J_2(t)| \le c \|\langle G_3, v_k\rangle\|_{L^2_{2\lambda_1,2\lambda_1}(\mathbb{R})} e^{-\lambda_1 t}. \tag{4.19}$$

The considerations for $t < -1$ lead to the estimate

$$|J_1(t)| \le c \|b_k\|_{m_{\sigma_1,\pi_1}} \begin{cases} e^{-\lambda_1|t|} & \text{if } v_\infty > 0, \\ |t| e^{-\lambda_1|t|} & \text{if } v_\infty = 0. \end{cases} \tag{4.20}$$

To see (4.20), it is enough to note that $\sigma_1 > \lambda_1$ for $v_\infty > 0$.

Hence for $v_\infty > 0$ we obtain

$$\|\nabla\psi\|_{l^2_{(0,H)}(m_{\lambda_1,\pi_1}(\mathbb{R}))} \le c \|\beta\|_{l^2_{(0,H)}(m_{\sigma_1,\pi_1}(\mathbb{R}))} + S. \tag{4.21}$$

To finish the proof of Theorem 1.1 we recall that (4.14) gives a bound on $\nabla\beta$, hence we can also get

$$\|\nabla^2\psi\|_{l^2_{(0,H)}(m_{\lambda_1,\pi_1}(\mathbb{R}))} \le S. \tag{4.22}$$

And by Proposition 2.2 and the boundedness of φ we obtain

$$v - (v_\infty, 0) \in M_{\lambda_1,\pi_1} \tag{4.23}$$

with estimates (1.16).

If $v_\infty = 0$, from (4.18) and (4.19) we conclude that

$$\left\| \frac{\nabla^2 \psi}{1+|x_1|} \right\|_{l^2_{(0,H)}(m_{\lambda_1,\lambda_1}(\mathbb{R}))} \le S, \tag{4.24}$$

which together with Proposition 2.2 for $|x_1| > 1$ gives

$$|\nabla \psi| \le c|x_1| e^{-\lambda_1 |x_1|}. \tag{4.25}$$

Estimate (4.25) implies (1.17). Theorem 1.1 is proved.

5. Asymptotic structure. We now prove Theorem 1.2. We assume that $a_f > \lambda_1$ to neglect the influence of the external force. To analyze the asymptotic structure we apply the forms (3.10). First, we consider the behavior of the vorticity. By Proposition 3.1,

$$b_k(t) = \int_{-\infty}^{\infty} E_k(t-y) N_k(y)\, dy, \tag{5.1}$$

where $N_k(\cdot)$ is as in (3.28). Consider the case $x_1 \to \infty$. Then

$$\begin{aligned} \lim_{t\to\infty} b_k(t) &= \lim_{t\to\infty} \int_{-\infty}^{t} E_k(t-y) N_k(y)\, dy \\ &= \lim_{t\to\infty} \Big(\int_{-\infty}^{0} + \int_{0}^{t} \Big) = \lim_{t\to\infty} (H_1^+(t) + H_2^+(t)). \end{aligned} \tag{5.2}$$

We see that

$$H_1^+(t) = n_k e^{-\pi_k t} \int_{-\infty}^{0} e^{\pi_k y} N_y(y)\, dy. \tag{5.3}$$

This leads to the first term of the expansion for $k = 1$; we get a nonzero element with decay $e^{-\pi_1 x_1}$. The terms with $k \ge 2$ decay at least with factor π_2.

By the results of Theorem 1.1 we get the following behavior of N_k:

$$N_k(y) \sim e^{-2\pi_1 y} \quad \text{for } y \gg 1. \tag{5.4}$$

Hence

$$H_2^+(t) = n_k e^{-\pi_k t} \int_{0}^{t} e^{\pi_k y} e^{-2\pi_1 y} \theta(y)\, dy, \tag{5.5}$$

where $\theta(\cdot)$ is in L^∞. And again a term with decay $e^{-\pi_1 t}$ appears only for $k = 1$; for $k \ge 2$ we get (just as for J_1 in Section 4) the behavior

$$H_2^+(t) \sim \begin{cases} e^{-2\pi_1 t} & \text{if } v_\infty > 0, \\ t e^{-\lambda_2 t} & \text{if } v_\infty = 0. \end{cases} \tag{5.6}$$

Note that $\pi_2 > 2\pi_1$ for $v_\infty > 0$ and $\pi_2 = \lambda_2 = 2\pi_1 = 2\lambda_1$ if $v_\infty = 0$. Relation (5.6) yields suitable estimates which guarantee the convergence of the series β, by Theorem 1.1. Finally, one can easily check that

$$e^{2\pi_1 t}\int_t^\infty E_k(t-y)N_k(y)\,dy = O(1).$$

Thus for $x_1 \to \infty$ we have

$$\alpha(x) = \Sigma^+ e^{-\pi_1 x_1}\sin\lambda_1 x_1 + \begin{cases} O(e^{-2\pi_1 x_1}) & \text{if } v_\infty > 0, \\ O(x_1 e^{-2\lambda_1 x_1}) & \text{if } v_\infty = 0. \end{cases} \tag{5.7}$$

For $x_1 < 0$ we investigate

$$\begin{aligned} \lim_{t\to-\infty} b_k(t) &= \lim_{t\to-\infty}\int_t^\infty E_k(t-y)N_k(y)\,dy \\ &= \lim_{t\to-\infty}\Big(\int_t^0 + \int_0^\infty\Big) = \lim_{t\to-\infty}(H_1^-(t) + H_2^-(t)). \end{aligned} \tag{5.8}$$

The analysis of

$$H_2^-(t) = n_k e^{\sigma_k t}\int_0^\infty e^{-\sigma_k y}N_k(y)\,dy \tag{5.9}$$

gives elements with decay $e^{-\sigma_k|x_1|}$. Next, by Theorem 1.1,

$$H_1^-(t) = n_k e^{\sigma_k t}\int_t^0 e^{-\sigma_k y}e^{(\lambda_1+\sigma_1)y}\theta(y)\,dy \tag{5.10}$$

decays with speed $e^{-\sigma_k|t|}$ for $k \le B_\alpha$, where

$$\sigma_{B_\alpha} < \lambda_1 + \sigma_1, \qquad \sigma_{B_\alpha+1} \ge \lambda_1 + \sigma_1. \tag{5.11}$$

And for $k > B_\alpha$,

$$H_2^-(t) \sim \begin{cases} e^{-(\lambda_1+\sigma_1)|t|} & \text{if } v_\infty > 0, \\ |t|e^{-\lambda_2|t|} & \text{if } v_\infty = 0. \end{cases} \tag{5.12}$$

Note that $\sigma_2 < \lambda_1 + \sigma_1$ if $v_\infty > 0$. And in general $B_\alpha \to \infty$ as $v_\infty/\nu \to \infty$. Then we write the asymptotic behavior for $x_1 \to -\infty$ as follows: if $v_\infty > 0$ then

$$\begin{aligned} \alpha(x) = {}& \Sigma_1^- e^{-\sigma_1|x_1|}v_1 + \Sigma_2^- e^{-\sigma_2|x_1|}v_2 + \ldots + \Sigma_{B_\alpha}^- e^{-\sigma_{B_\alpha}|x_1|}v_{B_\alpha} \\ & + O(e^{-(\lambda_1+\sigma_1)|x_1|}), \end{aligned} \tag{5.13}$$

and if $v_\infty = 0$ then

$$\alpha(x) = \Sigma e^{-\lambda_1|x_1|}v_1 + O(|x_1|e^{-2\lambda_1|x_1|}), \tag{5.14}$$

where v_k are the eigenvectors defined as in (3.10). The behavior of the rest $\int_{-\infty}^t E_k(t-y)N_k(y)\,dy$ is as desired.

We omit the proof of the part of Theorem 1.2 concerning the velocity, since it follows from calculations similar to those for the vorticity. To get the expansions (1.21), (1.22) and (1.23) it is enough to repeat the analysis from the second part of the proof of Theorem 1.1, for the velocity given as the solution of $(3.13)_2$, applying the obtained structure of the vorticity.

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