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Weyl spectra and Weyl's theorem

by

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Abstract. "Weyl's theorem" for an operator on a Hilbert space is the statement that the complement in the spectrum of the Weyl spectrum coincides with the isolated eigenvalues of finite multiplicity. In this paper we consider how Weyl's theorem survives for polynomials of operators and under quasinilpotent or compact perturbations. First, we show that if T is reduced by each of its finite-dimensional eigenspaces then the Weyl spectrum obeys the spectral mapping theorem, and further if T is reduction-isoloid then for every polynomial p, Weyl's theorem holds for $p(T)$. The results on perturbations are as follows. If T is a "finite-isoloid" operator and if K commutes with T and is either compact or quasinilpotent then Weyl's theorem is transmitted from T to $T+K$. As a noncommutative perturbation theorem, we also show that if the spectrum of T has no holes and at most finitely many isolated points, and if K is a compact operator then Weyl's theorem holds for $T+K$ when it holds for T.

Introduction. H. Weyl [22] examined the spectra of all compact perturbations $T+K$ of a hermitian operator T and discovered that $\lambda \in \sigma(T+K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$. Today this result is known as Weyl's theorem, and it has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L. Coburn [7], to several classes of operators including seminormal operators by S. Berberian [2], [3], and to a few classes of Banach space operators [15], [17]. Weyl's theorem may fail for the square of T when it holds for T (see [18, Example 1]). In [14], it was shown that Weyl's theorem holds for polynomials of hyponormal operators. The first aim of this paper is to extend this result via Berberian spectra. On the other hand, Weyl's theorem is liable to fail under small perturbations if "small" is interpreted in the sense of compact or quasinilpotent. Recently Weyl's theorem under small perturbations has been considered in [11]–[13], and [18]. The second aim of this paper is to explore

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how Weyl's theorem survives under quasinilpotent or compact perturbations.

Throughout this paper, $\mathcal{H}$ denotes an infinite-dimensional separable Hilbert space. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ the closed ideal of compact operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ we write $\varrho(T)$ for the resolvent set of T; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T; $\pi_{0\mathrm{f}}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0\mathrm{i}}(T)$ for the eigenvalues of infinite multiplicity. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Fredholm* if $T^{-1}(0)$ and $T(\mathcal{H})^{\perp}$ are both finite-dimensional. The *index* of a Fredholm operator $T \in \mathcal{L}(\mathcal{H})$, denoted by $\mathrm{ind}(T)$, is given by

$$\mathrm{ind}(T) = \dim T^{-1}(0) - \dim T(\mathcal{H})^{\perp} \ (= \dim T^{-1}(0) - \dim T^{*-1}(0)).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm "of finite ascent and descent"; equivalently [9, Theorem 7.9.3] if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The *essential spectrum* $\sigma_{\mathrm{e}}(T)$, the *Weyl spectrum* $\omega(T)$ and the *Browder spectrum* $\sigma_{\mathrm{b}}(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by

$$\begin{aligned} \sigma_{\mathrm{e}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},\\ \omega(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},\\ \sigma_{\mathrm{b}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}; \end{aligned}$$

then (cf. [9])

$$(0.1)\quad \sigma_{\mathrm{e}}(T) \subseteq \omega(T) \subseteq \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{e}}(T) \cup \mathrm{acc}\,\sigma(T) \quad \text{and} \quad \omega(T) \subseteq \eta\,\sigma_{\mathrm{e}}(T),$$

where we write $\mathrm{acc}\,K$ and $\eta\,K$ for the *accumulation points* and the *polynomially-convex hull*, respectively, of $K \subseteq \mathbb{C}$. If we write $\mathrm{iso}\,K = K \setminus \mathrm{acc}\,K$, and ∂K for the topological boundary of K, and

$$(0.2)\qquad \pi_{00}(T) := \{\lambda \in \mathrm{iso}\,\sigma(T) : 0 < \dim (T - \lambda I)^{-1}(0) < \infty\}$$

for the isolated eigenvalues of finite multiplicity, and ([9])

$$(0.3)\qquad p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$$

for the *Riesz points* of $\sigma(T)$, then by the punctured neighborhood theorem, i.e., $\partial\sigma(T) \setminus \sigma_{\mathrm{e}}(T) \subseteq \mathrm{iso}\,\sigma(T)$ (cf. [9], [10]),

$$(0.4)\qquad \mathrm{iso}\,\sigma(T) \setminus \sigma_{\mathrm{e}}(T) = \mathrm{iso}\,\sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T).$$

We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if there is equality

$$(0.5)\qquad \sigma(T) \setminus \omega(T) = \pi_{00}(T).$$

If $T \in \mathcal{L}(\mathcal{H})$, we write $r(T)$ for the spectral radius of T. It is familiar that $r(T) \leq \|T\|$. An operator T is called *normaloid* if $r(T) = \|T\|$, and *isoloid* if $\mathrm{iso}\,\sigma(T) \subseteq \pi_0(T)$. An operator T is called *reduction-isoloid* if the restriction of T to any reducing subspace is isoloid. It is well known [21, Theorem 2] that every hyponormal operator is reduction-isoloid.

In Section 1, we prove that Weyl spectra, Browder spectra, and Berberian spectra all coincide for operators reduced by each of their finite-dimensional eigenspaces. We also use this result to show that if T is reduced by each of its finite-dimensional eigenspaces then Weyl spectrum obeys the spectral mapping theorem, and further if T is reduction-isoloid then for every polynomial p, Weyl's theorem folds for $p(T)$.

In Section 2, we show that if T is "finite-isoloid" then Weyl's theorem is transmitted from T to $T+K$ when K is either compact or quasinilpotent and commutes with T, and that if T is a finite-isoloid operator whose spectrum has no holes and at most finitely many isolated points then Weyl's theorem is transmitted from T to $T+K$ when K is a compact operator. In addition we give applications to p-hyponormal operators, Toeplitz operators, and unilateral weighted shifts.

1. Berberian spectra and Weyl's theorem. Suppose that $T \in \mathcal{L}(\mathcal{H})$ is reduced by each of its finite-dimensional eigenspaces. If

$$\mathfrak{M} := \bigvee\{(T-\lambda I)^{-1}(0) : \lambda \in \pi_{0\mathrm{f}}(T)\},$$

then $\mathfrak{M}$ reduces T. Let $T_1 := T|\mathfrak{M}$ and $T_2 := T|\mathfrak{M}^{\perp}$. Then we have [3, Proposition 4.1]:

(i) T_1 is a normal operator with pure point spectrum;
(ii) $\pi_0(T_1) = \pi_{0\mathrm{f}}(T)$;
(iii) $\sigma(T_1) = \mathrm{cl}\, \pi_0(T_1)$;
(iv) $\pi_0(T_2) = \pi_0(T) \setminus \pi_{0\mathrm{f}}(T) = \pi_{0\mathrm{i}}(T)$.

In this case, Berberian [3, Definition 5.4] defined

$$\tau(T) := \sigma(T_2) \cup \mathrm{acc}\, \pi_{0\mathrm{f}}(T). \tag{1.0.1}$$

We shall call $\tau(T)$ the *Berberian spectrum* of T. Berberian used the notation $\tau'(T)$. He also showed that $\tau(T)$ is a nonempty compact subset of $\sigma(T)$. We can, however, show that Weyl spectra, Browder spectra, and Berberian spectra all coincide for operators reduced by each of their finite-dimensional eigenspaces:

THEOREM 1.1. *If $T \in \mathcal{L}(\mathcal{H})$ is reduced by each of its finite-dimensional eigenspaces then*

$$\tau(T) = \omega(T) = \sigma_{\mathrm{b}}(T). \tag{1.1.1}$$

Proof. Let $\mathfrak{M}$ be the closed linear span of the eigenspaces $(T-\lambda I)^{-1}(0)$ $(\lambda \in \pi_{0\mathrm{f}}(T))$ and write

$$T_1 := T|\mathfrak{M} \quad \text{and} \quad T_2 := T|\mathfrak{M}^{\perp}.$$

From the preceding arguments it follows that T_1 is normal, $\pi_0(T_1) = \pi_{0\mathrm{f}}(T)$ and $\pi_{0\mathrm{f}}(T_2) = \emptyset$. For (1.1.1) it will be shown that

$$\omega(T) \subseteq \tau(T) \subseteq \sigma_{\mathrm{b}}(T) \tag{1.1.2}$$

and

$$\sigma_{\mathrm{b}}(T) \subseteq \omega(T). \tag{1.1.3}$$

For the first inclusion of (1.1.2) suppose $\lambda \in \sigma(T) \setminus \tau(T)$. Then $T_2 - \lambda I$ is invertible and $\lambda \in \operatorname{iso} \pi_0(T_1)$. Since also $\pi_0(T_1) = \pi_{0\mathrm{f}}(T_1)$, we see that $\lambda \in \pi_{00}(T_1)$. But since T_1 is normal, it follows that $T_1 - \lambda I$ is Weyl and hence so is $T - \lambda I$. This proves the first inclusion. For the other inclusion of (1.1.2) suppose $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$. Thus $T - \lambda I$ is Browder but not invertible. Observe that the following equality holds with no other restriction on either R or S:

$$\sigma_{\mathrm{b}}(R \oplus S) = \sigma_{\mathrm{b}}(R) \cup \sigma_{\mathrm{b}}(S) \quad \text{for each } R \in \mathcal{L}(\mathcal{H}_1) \text{ and } S \in \mathcal{L}(\mathcal{H}_2). \tag{1.1.4}$$

Indeed, if $\lambda \in \operatorname{iso} \sigma(R \oplus S)$ then λ is either an isolated point of the spectra of direct summands or a resolvent element of direct summands, so that if $R - \lambda I$ and $S - \lambda I$ are Fredholm then by (0.4), λ is either a Riesz point or a resolvent element of direct summands, which implies that $\sigma_{\mathrm{b}}(R) \cup \sigma_{\mathrm{b}}(S) \subseteq \sigma_{\mathrm{b}}(R \oplus S)$, and the reverse inclusion is evident. From this we can see that $T_1 - \lambda I$ and $T_2 - \lambda I$ are both Browder. But since $\pi_{0\mathrm{f}}(T_2) = \emptyset$, it follows that $T_2 - \lambda I$ is one-one and hence invertible. Therefore $\lambda \in \pi_{00}(T_1) \setminus \sigma(T_2)$, which implies that $\lambda \notin \tau(T)$. This proves the second inclusion of (1.1.2).

For (1.1.3) suppose $\lambda \in \sigma(T) \setminus \omega(T)$ and hence $T - \lambda I$ is Weyl but not invertible. Observe that if $\mathcal{H}_1$ is a Hilbert space and an operator $R \in \mathcal{L}(\mathcal{H}_1)$ satisfies $\omega(R) = \sigma_{\mathrm{e}}(R)$, then (cf. [11, Theorem 5])

$$\omega(R \oplus S) = \omega(R) \cup \omega(S) \quad \text{for each Hilbert space } \mathcal{H}_2 \text{ and } S \in \mathcal{L}(\mathcal{H}_2). \tag{1.1.5}$$

Since T_1 is normal, applying (1.1.5) to T_1 in place of R shows that $T_1 - \lambda I$ and $T_2 - \lambda I$ are both Weyl. But since $\pi_{0\mathrm{f}}(T_2) = \emptyset$, $T_2 - \lambda I$ must be invertible and therefore $\lambda \in \sigma(T_1) \setminus \omega(T_1)$. Thus from Weyl's theorem for normal operators we can see that $\lambda \in \pi_{00}(T_1)$ and hence $\lambda \in \operatorname{iso} \sigma(T_1) \cap \varrho(T_2)$, which by (0.4) implies that $\lambda \notin \sigma_{\mathrm{b}}(T)$. This proves (1.1.3) and completes the proof. ∎

As applications of Theorem 1.1 we will give several corollaries below.

COROLLARY 1.2. *If $T \in \mathcal{L}(\mathcal{H})$ is reduced by each of its finite-dimensional eigenspaces then $\sigma(T) \setminus \omega(T) \subseteq \pi_{00}(T)$.*

Proof. This follows at once from Theorem 1.1. ∎

Weyl's theorem is not transmitted to dual operators: for example if $T : \ell_2 \to \ell_2$ is the unilateral weighted shift defined by

$$Te_n = \frac{1}{n+1} e_{n+1} \quad (n \geq 0), \tag{1.2.1}$$

then $\sigma(T) = \omega(T) = \{0\}$ and $\pi_{00}(T) = \emptyset$, and therefore Weyl's theorem holds for T, but fails for its adjoint T^*. We however have:

COROLLARY 1.3. *If $T \in \mathcal{L}(\mathcal{H})$ is reduced by each of its finite-dimensional eigenspaces and* $\operatorname{iso}\sigma(T) = \emptyset$, *then Weyl's theorem holds for T and T^*. In this case*, $\sigma(T) = \omega(T)$.

Proof. If $\operatorname{iso}\sigma(T) = \emptyset$, then it follows from Corollary 1.2 that $\sigma(T) = \omega(T)$, which says that Weyl's theorem holds for T. The assertion that Weyl's theorem holds for T^* follows by noting that $\sigma(T)^* = (\sigma(T))^-$, $\omega(T^*) = (\omega(T))^-$ and $\pi_{00}(T^*) = (\pi_{00}(T))^- = \emptyset$. ∎

In Corollary 1.3, "$\operatorname{iso}\sigma(T) = \emptyset$" cannot be replaced by "$\pi_{00}(T) = \emptyset$": for example consider the operator T defined by (1.2.1).

COROLLARY 1.4 ([2, Theorem]). *If $T \in \mathcal{L}(\mathcal{H})$ is reduction-isoloid and is reduced by each of its finite-dimensional eigenspaces then Weyl's theorem holds for T.*

Proof. In view of Corollary 1.2, it is sufficient to show that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. Suppose $\lambda \in \pi_{00}(T)$. Then with the preceding notations, $\lambda \in \pi_{00}(T_1) \cap [\operatorname{iso}\sigma(T_2) \cup \varrho(T_2)]$. If $\lambda \in \operatorname{iso}\sigma(T_2)$, then since by assumption T_2 is isoloid we have $\lambda \in \pi_0(T_2)$ and hence $\lambda \in \pi_{0\mathrm{f}}(T_2)$. But since $\pi_{0\mathrm{f}}(T_2) = \emptyset$, we should have $\lambda \notin \operatorname{iso}\sigma(T_2)$. Thus $\lambda \in \pi_{00}(T_1) \cap \varrho(T_2)$. Since T_1 is normal it follows that $T_1 - \lambda I$ is Weyl and so is $T - \lambda I$; therefore $\lambda \in \sigma(T) \setminus \omega(T)$. ∎

If "reduction-isoloid" is replaced by "isoloid" then Corollary 1.4 may fail (see Examples (1) of [2]).

COROLLARY 1.5. *If $T \in \mathcal{L}(\mathcal{H})$ is reduced by each of its finite-dimensional eigenspaces then*

$$p(\omega(T)) = \omega(p(T)) \quad \textit{for every polynomial } p. \tag{1.5.1}$$

Further if T is reduction-isoloid then for every polynomial p, Weyl's theorem holds for $p(T)$.

Proof. We first claim that if T is reduced by each of its finite-dimensional eigenspaces then so is $p(T)$ for any polynomial p: indeed, if we write $T = T_1 \oplus T_2$ as in the proof of Theorem 1.1, then $p(T) = p(T_1) \oplus p(T_2)$ shows that $p(T_1)$ is normal and $\pi_{0\mathrm{f}}(p(T_2)) \subseteq p(\pi_{0\mathrm{f}}(T_2)) = p(\emptyset) = \emptyset$, which implies that $p(T)$ is reduced by each of its finite-dimensional eigenspaces because any normal operator is reduced by each of its finite-dimensional eigenspaces. Therefore the first assertion follows from Theorem 1.1 together with the fact that the Browder spectrum obeys the spectral mapping theorem. The second assertion follows from Theorem 1 of [18] and Corollary 1.4. ∎

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ (cf. [1], [4]). If $p = 1$, then T is *hyponormal* and if $p = \frac{1}{2}$, then T is *semi-hyponormal*. In [14, Theorem 2], it was shown that if T is hyponormal then for every polynomial p, Weyl's theorem holds for $p(T)$. We can prove more:

COROLLARY 1.6. *If $T \in \mathcal{L}(\mathcal{H})$ is p-hyponormal then for every polynomial p, Weyl's theorem holds for $p(T)$.*

Proof. This follows from Corollary 1.5 and the fact that every p-hyponormal operator is isoloid ([6, Theorem 1]) and is reduced by each of its eigenspaces ([5, Theorem 4]). ∎

L. Coburn [7, Corollary 3.2] has shown that if $T \in \mathcal{L}(\mathcal{H})$ is hyponormal and $\pi_{00}(T) = \emptyset$, then T is *extremally noncompact*, in the sense that

$$\|T\| = \|\pi(T)\|,$$

where π is the canonical map of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. His proof relies upon the fact that Weyl's theorem holds for hyponormal operators, and hence $\sigma(T) = \omega(T)$ since $\pi_{00}(T) = \emptyset$. Now we can strengthen Coburn's argument slightly:

COROLLARY 1.7. *If $T \in \mathcal{L}(\mathcal{H})$ is normaloid and $\pi_{00}(T) = \emptyset$, then T is extremally noncompact.*

Proof. Since $\sigma(T) \subseteq \eta\,\omega(T) \cup p_{00}(T)$ for any $T \in \mathcal{L}(\mathcal{H})$, we deduce that $\eta\,\sigma(T) \setminus \eta\,\omega(T) \subseteq \pi_{00}(T)$. Thus by our assumption, $\eta\,\sigma(T) = \eta\,\omega(T)$. Therefore we can argue that for each compact operator $K \in \mathcal{L}(\mathcal{H})$,

$$\|T\| = r(T) = r_\omega(T) = r_\omega(T+K) \leq r(T+K) \leq \|T+K\|,$$

where $r_\omega(T)$ denotes the "Weyl spectral radius". This completes the proof. ∎

Note that if $T \in \mathcal{L}(\mathcal{H})$ is normaloid and $\pi_{00}(T) = \emptyset$, then Weyl's theorem may fail for T; for example, take $\mathcal{H} = \ell_2 \oplus \ell_2$ and $T = U \oplus U^*$, where U is the unilateral shift.

2. Weyl's theorem under small perturbations. In general Weyl's theorem for $T \in \mathcal{L}(\mathcal{H})$ is not sufficient for Weyl's theorem for $T + K$ with compact (even finite rank) $K \in \mathcal{L}(\mathcal{H})$ commuting with T (cf. [13, Example 2.3]). But if $T \in \mathcal{L}(\mathcal{H})$ is isoloid then Weyl's theorem is transmitted from T to $T + K$ for commuting finite rank operators $K \in \mathcal{L}(\mathcal{H})$ (cf. [13, Theorem 2.2]). But this may fail if "finite rank" is replaced by "compact". In fact Weyl's theorem may fail even if K is both compact and quasinilpotent: for example, take $T = 0$ and K the operator on ℓ_2 defined by $K(x_1, x_2, \ldots) = (x_2/2, x_3/3, x_4/4, \ldots)$. We will however show that if the "isoloid" condition is slightly strengthened then Weyl's theorem is transmitted from T to $T+K$ if K is either a compact or a quasinilpotent operator commuting with T. We begin with:

PROPOSITION 2.1. *If $K \in \mathcal{L}(\mathcal{H})$ is a compact operator commuting with $T \in \mathcal{L}(\mathcal{H})$ then*

$$\pi_{00}(T+K) \subseteq \operatorname{iso}\sigma(T) \cup \varrho(T).$$

Proof. Suppose $\lambda \in \pi_{00}(T+K)$. Assume to the contrary that $\lambda \in \operatorname{acc}\sigma(T)$. Observe that $\sigma(T) = \sigma_{\mathrm{b}}(T) \cup p_{00}(T)$ for every $T \in \mathcal{L}(\mathcal{H})$. Since the Browder spectrum is invariant under commuting compact perturbations [9, Theorem 7.7.5], it follows that the difference between $\sigma(T)$ and $\sigma(T+K)$ consists of the difference between $p_{00}(T)$ and $p_{00}(T+K)$. Since by our assumption, $\lambda \in \operatorname{iso}\sigma(T+K) \cap \operatorname{acc}\sigma(T)$, we can find a sequence $\{\lambda_n\}$ of distinct numbers in $p_{00}(T) \setminus p_{00}(T+K)$ satisfying

(i) $\lim_n \lambda_n = \lambda$;

(ii) $\sigma := \{\lambda_n\}_{n=1}^{\infty} \cup \{\lambda\}$ is an isolated part of $\sigma(T)$.

If $\mathcal{N}$ is a neighborhood of σ which contains no other points of $\sigma(T)$, then by using the spectral projection $P = (2\pi i)^{-1}\int_{\partial\mathcal{N}}(\mu I - T)^{-1}\,d\mu$ corresponding to σ, we can write T as

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \sigma$ and $\sigma(T_2) = \sigma(T) \setminus \sigma$. Since $TK = KT$ and $\sigma(T_1) \cap \sigma(T_2) = \emptyset$, it follows from a corollary of Rosenblum's Theorem (cf. [19, Corollary 0.14]) that K admits a matrix representation

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \quad \text{with} \quad T_iK_i = K_iT_i \ (i = 1, 2).$$

Since $\lambda \in \pi_{00}(T+K)$ we can suppose that $\dim\,(T+K-\lambda I)^{-1}(0) =: m < \infty$ and hence $\dim\,(T_1 + K_1 - \lambda I)^{-1}(0) \le m$. Observe that $\sigma_{\mathrm{b}}(T_1 + K_1) = \sigma_{\mathrm{b}}(T_1) = \{\lambda\}$. But since $\lambda \in \operatorname{iso}\sigma(T_1 + K_1)$, it follows that $\sigma(T_1 + K_1) \setminus \{\lambda\}$ consists of finitely many elements which are its Riesz points. Therefore $\sigma(T_1 + K_1) = p_{00}(T_1 + K_1) \cup \{\lambda\}$. Write

$$s := \sum_{z \in p_{00}(T_1+K_1)} \dim\,(T_1 + K_1 - z\,I)^{-1}(0).$$

Then evidently $s < \infty$. On the other hand, since $\lambda_n \in p_{00}(T)$ for every $n = 1, 2, \ldots$, using the spectral projections corresponding to the set $\{\lambda_j\}$ for $j = 1, \ldots, s+m+1$, we can write T_1 as

$$T_1 = \begin{pmatrix} T_{11} & & & & \\ & T_{12} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & T_{1,s+m+1} \end{pmatrix} \oplus T_3,$$

where $\sigma(T_{1j}) = \{\lambda_j\}$ for $j = 1, \ldots, s+m+1$, and $\sigma(T_3) = \{\lambda_n\}_{n=s+m+2}^{\infty} \cup \{\lambda\}$. Note that $\sigma_{\mathrm{b}}(T_{1j}) = \emptyset$ for $j = 1, \ldots, s+m+1$. Therefore each T_{1j} $(j = 1, \ldots, s+m+1)$ is a finite-dimensional operator because $\sigma_{\mathrm{b}}(S) \neq \emptyset$ for every bounded linear operator S on an infinite-dimensional Hilbert space. Since $T_1K_1 = K_1T_1$ and the λ_j $(j = 1, \ldots, s+m+1)$ are mutually distinct, it again follows from [19, Corollary 0.14] that K_1 can be written in the form

$$K_1 = \begin{pmatrix} K_{11} & & & & \\ & K_{12} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & K_{1,s+m+1} \end{pmatrix} \oplus K_3.$$

Observe that $T_{1j} + K_{1j}$ is a finite-dimensional operator for every $j = 1, \ldots, s+m+1$ and $\sigma(T_{1j} + K_{1j}) \subseteq \sigma(T_1 + K_1) = p_{00}(T_1 + K_1) \cup \{\lambda\}$ for every $j = 1, \ldots, s+m+1$. But since

$$\sum_{j=1}^{s+m+1} \sum_{z \in \sigma(T_{1j}+K_{1j})} \dim\,(T_{1j} + K_{1j} - zI)^{-1}(0) \geq s+m+1,$$

it follows that $\lambda \in \sigma(T_{1j} + K_{1j})$ for at least $(m+1)$ j's, which implies that $\dim\,(T_1 + K_1 - \lambda I)^{-1}(0) \geq m+1$, a contradiction. This completes the proof. ∎

An operator $T \in \mathcal{L}(\mathcal{H})$ will be said to be *finite-isoloid* if $\operatorname{iso}\sigma(T) \subseteq \pi_{0\mathrm{f}}(T)$. Evidently finite-isoloid ⇒ isoloid. The converse is not true in general: for example, take $T = 0$. In particular if $\sigma(T)$ has no isolated points then T is finite-isoloid. We now have:

THEOREM 2.2. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is finite-isoloid. If Weyl's theorem holds for T then it holds for $T + K$ if $K \in \mathcal{L}(\mathcal{H})$ commutes with T and is either compact or quasinilpotent.*

Proof. First we assume that K is a compact operator commuting with T. Suppose Weyl's theorem holds for T. We first claim that with no restriction on T,

$$\sigma(T+K) \setminus \omega(T+K) \subseteq \pi_{00}(T+K). \tag{2.2.1}$$

It suffices to show that if $\lambda \in \sigma(T+K) \setminus \omega(T+K)$ then $\lambda \in \operatorname{iso}\sigma(T+K)$. Assume to the contrary that $\lambda \in \operatorname{acc}\sigma(T+K)$. Then $\lambda \in \sigma_{\mathrm{b}}(T+K) = \sigma_{\mathrm{b}}(T)$, so that $\lambda \in \sigma_{\mathrm{e}}(T)$ or $\lambda \in \operatorname{acc}\sigma(T)$. Remember that the essential spectrum and the Weyl spectrum are invariant under compact perturbations. Thus if $\lambda \in \sigma_{\mathrm{e}}(T)$ then $\lambda \in \sigma_{\mathrm{e}}(T+K) \subseteq \omega(T+K)$, a contradiction. Therefore we should have $\lambda \in \operatorname{acc}\sigma(T)$. But since Weyl's theorem holds for T and $\lambda \notin \omega(T+K) = \omega(T)$, it follows that $\lambda \in \pi_{00}(T)$, a contradiction. This proves (2.2.1).

For the reverse inclusion suppose $\lambda \in \pi_{00}(T+K)$. Then by Proposition 2.1, either $\lambda \in \operatorname{iso}\sigma(T)$ or $\lambda \in \varrho(T)$. If $\lambda \in \varrho(T)$ then evidently $T+K-\lambda I$ is Weyl, i.e., $\lambda \notin \omega(T+K)$. If instead $\lambda \in \operatorname{iso}\sigma(T)$ then $\lambda \in \pi_{00}(T)$ whenever T is finite-isoloid. Since Weyl's theorem holds for T, it follows that $\lambda \notin \omega(T)$ and hence $\lambda \notin \omega(T+K)$. Therefore Weyl's theorem holds for $T+K$.

Next we assume that K is a quasinilpotent operator commuting with T. Then it is known [18, Lemma 2] that $\varpi(T) = \varpi(T+Q)$ with $\varpi = \sigma, \omega$. Suppose Weyl's theorem holds for T. Then

$$\begin{aligned}\sigma(T+K) \setminus \omega(T+K) = \sigma(T) \setminus \omega(T) = \pi_{00}(T)\\ \subseteq \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T+K),\end{aligned}$$

which implies that $\sigma(T+K) \setminus \omega(T+K) \subseteq \pi_{00}(T+K)$. Conversely, suppose $\lambda \in \pi_{00}(T+K)$. If T is finite-isoloid then $\lambda \in \operatorname{iso}\sigma(T+K) = \operatorname{iso}\sigma(T) \subseteq \pi_{0\mathrm{f}}(T)$. Thus $\lambda \in \pi_{00}(T) = \sigma(T) \setminus \omega(T) = \sigma(T+K) \setminus \omega(T+K)$. This completes the proof. ∎

COROLLARY 2.3. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is p-hyponormal. If either*

(i) $\operatorname{iso}\sigma(T) = \emptyset$, *or*

(ii) *T has finite-dimensional eigenspaces*,

then Weyl's theorem holds for $T+K$ if $K \in \mathcal{L}(\mathcal{H})$ is either compact or quasinilpotent and commutes with T.

Proof. Observe that each of the conditions (i) and (ii) forces p-hyponormal operators to be finite-isoloid. Since Weyl's theorem holds for p-hyponormal operators ([6]), the result follows at once from Theorem 2.2. ∎

It is known [18, Theorem 3] that Weyl's theorem is transmitted from $T \in \mathcal{L}(\mathcal{H})$ to $T+K$ for commuting nilpotent operators $K \in \mathcal{L}(\mathcal{H})$. This however does not extend to commuting quasinilpotent operators (see the remark above Proposition 2.1). But if K is an *injective* quasinilpotent operator commuting with T then Weyl's theorem is transmitted from T to $T+K$.

THEOREM 2.4. *If Weyl's theorem holds for $T \in \mathcal{L}(\mathcal{H})$ then it holds for $T+K$ if $K \in \mathcal{L}(\mathcal{H})$ is an injective quasinilpotent operator commuting with T.*

Proof. First of all we prove that if there exists an injective quasinilpotent operator commuting with T, then

$$T \text{ Weyl } \Rightarrow T \text{ injective.} \tag{2.4.1}$$

To show this suppose K is an injective quasinilpotent operator commuting with T. Assume to the contrary that T is Weyl but not injective. Then there exists a nonzero vector $x \in \mathcal{H}$ such that $Tx = 0$. Then by commutativity, $TK^n x = K^n Tx = 0$ for every $n = 0, 1, 2, \ldots$, so that $K^n x \in T^{-1}(0)$ for every $n = 0, 1, 2, \ldots$ We now claim that $\{K^n x\}_{n=0}^{\infty}$ is a sequence of linearly independent vectors in $\mathcal{H}$. To see this suppose $c_0 x + c_1 Kx + \ldots + c_n K^n x = 0$. We may then write $c_n(K - \lambda_1 I) \ldots (K - \lambda_n I)x = 0$. Since K is an injective quasinilpotent operator it follows that $(K - \lambda_1 I) \ldots (K - \lambda_n I)$ is injective. But since $x \neq 0$ we have $c_n = 0$. By induction we also have $c_{n-1} = \ldots = c_1 = c_0 = 0$. This shows that $\{K^n x\}_{n=0}^{\infty}$ is a sequence of linearly independent vectors in $\mathcal{H}$. From this we can see that $T^{-1}(0)$ is infinite-dimensional, which contradicts the fact that T is Weyl. This proves (2.4.1).

From (2.4.1) we see that if Weyl's theorem holds for T then $\pi_{00}(T) = \emptyset$. We now claim that $\pi_{00}(T + K) = \emptyset$. Indeed, if $\lambda \in \pi_{00}(T + K)$, then $0 < \dim (T + K - \lambda I)^{-1}(0) < \infty$, so that there exists a nonzero vector $x \in \mathcal{H}$ such that $(T + K - \lambda I)x = 0$. But since K commutes with $T + K - \lambda I$, the same argument as in the proof of (2.4.1) with $T + K - \lambda I$ in place of T shows that $(T + K - \lambda I)^{-1}(0)$ is infinite-dimensional, a contradiction. Therefore $\pi_{00}(T+K) = \emptyset$ and hence Weyl's theorem holds for $T+K$ because $\varpi(T) = \varpi(T + K)$ with $\varpi = \sigma, \omega$. ■

In Theorem 2.4, "quasinilpotent" cannot be replaced by "compact". For example consider the following operators on $\ell_2 \oplus \ell_2$:

$$T = \begin{pmatrix} 0 & & & \\ & 1/2 & & 0 \\ & & 1/3 & \\ & 0 & & 1/4 \\ & & & & \ddots \end{pmatrix} \oplus I \quad \text{and} \quad K = \begin{pmatrix} 1 & & & \\ & -1/2 & & 0 \\ & & -1/3 & \\ & 0 & & -1/4 \\ & & & & \ddots \end{pmatrix} \oplus Q,$$

where Q is an injective compact quasinilpotent operator on ℓ_2. Observe that Weyl's theorem holds for T, K is an injective compact operator, and $TK = KT$. But

$$\sigma(T + K) = \{0, 1\} = \omega(T + K) \quad \text{and} \quad \pi_{00}(T + K) = \{1\},$$

which says that Weyl's theorem does not hold for $T + K$.

In perturbation theory the commutativity condition looks so rigid. Without it, however, the spectrum can undergo a substantial change even under rank one perturbations. In spite of it, Weyl's theorem may hold for (noncommutative) compact perturbations of "good" operators. We now give such a perturbation theorem. To do this we need:

LEMMA 2.5. *If $N \in \mathcal{L}(\mathcal{H})$ is a quasinilpotent operator commuting with $T \in \mathcal{L}(\mathcal{H})$ modulo compact operators (i.e., $TN-NT \in \mathcal{K}(\mathcal{H})$) then $\sigma_{\mathrm{e}}(T+N) = \sigma_{\mathrm{e}}(T)$ and $\omega(T+N) = \omega(T)$.*

Proof. Straightforward from [18, Lemma 2]. ∎

THEOREM 2.6. *Suppose $T \in \mathcal{L}(\mathcal{H})$ satisfies the following:*

(i) *T is finite-isoloid;*

(ii) *$\sigma(T)$ has no "holes" (bounded components of the complement), i.e., $\sigma(T) = \eta\,\sigma(T)$;*

(iii) *$\sigma(T)$ has at most finitely many isolated points;*

(iv) *Weyl's theorem holds for T.*

If $K \in \mathcal{L}(\mathcal{H})$ is either compact or quasinilpotent and commutes with T modulo compact operators then Weyl's theorem holds for $T+K$.

Proof. By Lemma 2.5, we have $\sigma_{\mathrm{e}}(T+K) = \sigma_{\mathrm{e}}(T)$ and $\omega(T+K) = \omega(T)$. Suppose Weyl's theorem holds for T and $\lambda \in \sigma(T+K) \setminus \omega(T+K)$. We now claim that $\lambda \in \operatorname{iso}\sigma(T+K)$. Assume to the contrary that $\lambda \in \operatorname{acc}\sigma(T+K)$. Since $\lambda \notin \omega(T+K) = \omega(T)$, it follows from the punctured neighborhood theorem that $\lambda \notin \partial\sigma(T+K)$. Also since the set of all Weyl operators forms an open subset of $\mathcal{L}(\mathcal{H})$, we have $\lambda \in \operatorname{int}(\sigma(T+K) \setminus \omega(T+K))$. Then there exists $\varepsilon > 0$ such that $\{\mu \in \mathbb{C} : |\mu-\lambda| < \varepsilon\} \subseteq \operatorname{int}(\sigma(T+K) \setminus \omega(T+K))$, and hence $\{\mu \in \mathbb{C} : |\mu-\lambda| < \varepsilon\} \cap \omega(T) = \emptyset$. But since

$$\partial\sigma(T+K) \setminus \operatorname{iso}\sigma(T+K) \subseteq \sigma_{\mathrm{e}}(T+K) = \sigma_{\mathrm{e}}(T),$$

it follows from our assumption that

$$\begin{aligned}\{\mu \in \mathbb{C} : |\mu-\lambda| < \varepsilon\} &\subseteq \operatorname{int}(\sigma(T+K) \setminus \omega(T+K))\\ &\subseteq \eta\,(\partial\sigma(T+K) \setminus \operatorname{iso}\sigma(T+K))\\ &\subseteq \eta\,\sigma_{\mathrm{e}}(T) \subseteq \eta\,\sigma(T) = \sigma(T),\end{aligned}$$

which implies that $\{\mu \in \mathbb{C} : |\mu-\lambda| < \varepsilon\} \subseteq \sigma(T) \setminus \omega(T)$. This contradicts Weyl's theorem for T. Therefore $\lambda \in \operatorname{iso}\sigma(T+K)$ and hence $\sigma(T+K) \setminus \omega(T+K) \subseteq \pi_{00}(T+K)$. For the reverse inclusion suppose $\lambda \in \pi_{00}(T+K)$. Assume to the contrary that $\lambda \in \omega(T+K)$ and hence $\lambda \in \omega(T)$. Then we claim $\lambda \notin \partial\sigma(T)$. Indeed if $\lambda \in \operatorname{iso}\sigma(T)$ then by assumption $\lambda \in \pi_{00}(T)$, which contradicts Weyl's theorem for T. If instead $\lambda \in \operatorname{acc}\sigma(T) \cap \partial\sigma(T)$ then since $\operatorname{iso}\sigma(T)$ is finite it follows that

$$\lambda \in \operatorname{acc}(\partial\sigma(T)) \subseteq \operatorname{acc}\sigma_{\mathrm{e}}(T) = \operatorname{acc}\sigma_{\mathrm{e}}(T+K),$$

which contradicts the fact that $\lambda \in \operatorname{iso}\sigma(T+K)$. Therefore $\lambda \notin \partial\sigma(T)$. Also since $\lambda \in \operatorname{iso}\sigma(T+K)$, there exists $\varepsilon > 0$ such that

$$\{\mu \in \mathbb{C} : 0 < |\mu - \lambda| < \varepsilon\} \subseteq \sigma(T) \cap \varrho(T+K),$$

so that $\{\mu \in \mathbb{C} : 0 < |\mu - \lambda| < \varepsilon\} \cap \omega(T) = \emptyset$, which contradicts Weyl's theorem for T. Thus $\lambda \in \sigma(T+K) \setminus \omega(T+K)$ and therefore Weyl's theorem holds for $T+K$. ∎

If, in Theorem 2.6, the condition "$\sigma(T)$ has no holes" is dropped then Theorem 2.6 may fail even if T is normal. For example, if on $\ell_2 \oplus \ell_2$,

$$T = \begin{pmatrix} U & I - UU^* \\ 0 & U^* \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & I - UU^* \\ 0 & 0 \end{pmatrix},$$

where U is the unilateral shift on ℓ_2, then T is unitary (essentially the bilateral shift) with $\sigma(T) = \mathbb{T}$ (the unit circle), K is a rank one nilpotent, and Weyl's theorem does not hold for $T - K$.

Also in Theorem 2.6, the condition "$\operatorname{iso}\sigma(T)$ is finite" is essential in the cases where K is compact. For example, if on ℓ_2,

$$T(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/3, \ldots),$$
$$Q(x_1, x_2, \ldots) = (x_2/2, x_3/3, x_4/4, \ldots),$$

we define $K := -(T+Q)$. Then: (i) T is finite-isoloid; (ii) $\sigma(T)$ has no holes; (iii) Weyl's theorem holds for T; (iv) $\operatorname{iso}\sigma(T)$ is infinite; (v) K is compact because T and Q are both compact; (vi) Weyl's theorem does not hold for $T + K$ $(= -Q)$.

Corollary 2.7. *If $\sigma(T)$ has no holes and at most finitely many isolated points and if K is a compact operator then Weyl's theorem is transmitted from T to $T+K$.*

Proof. Straightforward from Theorem 2.6. ∎

Corollary 2.7 shows that if Weyl's theorem holds for T whose spectrum has no holes and at most finitely many isolated points then for every compact operator K, the passage from $\sigma(T)$ to $\sigma(T+K)$ adds at most countably many isolated points outside $\sigma(T)$ which are Riesz points of $\sigma(T+K)$. Here we should note that this holds even if T is quasinilpotent because for every quasinilpotent operator T (more generally, "Riesz operator"), we have

$$\begin{aligned}\sigma(T+K) &\subseteq \eta\,\sigma_{\mathrm{e}}(T+K) \cup p_{00}(T+K) = \eta\,\sigma_{\mathrm{e}}(T) \cup p_{00}(T+K)\\ &= \{0\} \cup p_{00}(T+K).\end{aligned}$$

Corollary 2.7 can easily be applied for Toeplitz operators and unilateral weighted shifts. Below we give two corollaries on those operators. Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T} = \partial\mathbb{D}$ in the complex plane. Recall ([16]) that given $\varphi \in L^\infty(\mathbb{T})$, the *Toeplitz operator* with symbol φ is the operator T_φ on $H^2(\mathbb{T})$ defined by $T_\varphi f = P(\varphi \cdot f)$, where $f \in H^2(\mathbb{T})$

and P denotes the projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. For example, if $\varphi(z) = z$ then T_φ represents the unilateral shift on ℓ_2. Write $C(\mathbb{T})$ for the algebra of all continuous complex-valued functions on $\mathbb{T}$.

COROLLARY 2.8. *Suppose T_φ is a Toeplitz operator with nonconstant continuous symbol $\varphi \in C(\mathbb{T})$ whose winding number with respect to each hole of $\varphi(\mathbb{T})$ is nonzero. If $K \in \mathcal{L}(H^2)$ is a compact operator then Weyl's theorem holds for $T_\varphi + K$. Hence, in particular, if U is the unilateral shift on ℓ_2 then Weyl's theorem holds for $U + K$ with every compact operator $K \in \mathcal{L}(\ell_2)$.*

Proof. Remember ([7]) that Weyl's theorem holds for every Toeplitz operator and $\sigma(T_\varphi)$ has no isolated points for nonconstant symbols φ. The spectral theory for Toeplitz operators with continuous symbols shows that our assumption implies $\sigma(T_\varphi)$ has no holes (cf. [16]). Therefore the result follows at once from Corollary 2.7. The second assertion is immediate from the first. ∎

COROLLARY 2.9. *If T is a unilateral weighted shift with positive weights and is not quasinilpotent, and if $K \in \mathcal{L}(\ell_2)$ is a compact operator then Weyl's theorem holds for $T + K$.*

Proof. If T is a unilateral weighted shift which is not quasinilpotent then $\sigma(T)$ is a nondegenerate disk [20, Theorem 4]. Moreover since the weights are positive it follows that $\pi_0(T) = \emptyset$, and hence $\sigma(T) = \omega(T)$, which implies that Weyl's theorem holds for T. Therefore the result follows at once from Corollary 2.7. ∎

Corollary 2.9 may fail if T is quasinilpotent: for example consider the operators T and K defined by $T(x_1, x_2, \ldots) = (0, x_1, x_2/2, x_3/3, \ldots)$ and $K(x_1, x_2, \ldots) = (0, -x_1, 0, 0, \ldots)$.

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