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Normed "upper interval" algebras without nontrivial closed subalgebras

by

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Abstract. It is a long standing open problem whether there is any infinite-dimensional commutative Banach algebra without nontrivial closed ideals. This is in some sense the Banach algebraists' counterpart to the invariant subspace problem for Banach spaces. We do not here solve this famous problem, but solve a related problem, that of finding (necessarily commutative) infinite-dimensional normed algebras which do not even have nontrivial closed subalgebras. Our examples are incomplete normed algebras rather than Banach algebras. The problem of finding such algebras, posed by W. Żelazko, was until now open not only for normed algebras but for more general topological algebras. Our construction here is kept short because it uses a key lemma involved in the construction of the "LRRW algebra" of Loy, Read, Runde and Willis. The algebras we find are dense subalgebras of certain commutative Banach algebras with compact multiplication.

1. Introduction. The author would first like to express his gratitude to W. Żelazko for much kind hospitality, and for introducing him to this particular problem. Żelazko [13] produced examples of infinite-dimensional semitopological complex algebras without nontrivial closed subalgebras, and he asked if one could have infinite-dimensional topological algebras without nontrivial closed subalgebras. Here we produce normed algebras with that property, making fundamental use of ideas found in [10].

The normed algebras we use are dense subalgebras of certain singly generated radical Banach algebras, the subalgebra consisting of polynomials in the generator S. For such an algebra to solve our problem, it is clear that the standard ideals (that is, the closed principal ideals generated by S^k, $k \in \mathbb{N}$) of the Banach algebras in question must all be trivial. This is a property that our Banach algebras share with the original "LRRW algebra" described in [10]. The LRRW algebra is very similar to the ones described here; but in the LRRW algebra, the dense subalgebra generated by S does have nontrivial relatively closed subalgebras, for a reason we will explain in §6 below.

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So although our present construction can be very similar to that of [10], it cannot be quite the same. The matter of standard versus nonstandard ideals in singly generated radical Banach algebras has been extensively discussed (see e.g. Domar [4], Grabiner and Thomas [8], Dales and McClure [3] and the general exposition in §4.6 of Dales [2]), the high point in the investigation probably being Marc Thomas' discovery [12] of nonstandard ideals in the Banach algebra $l^1(\omega)$, for certain radical weights ω.

Concerning the invariant subspace problem, counterexamples were first found independently by Enflo [5] and Read [11]. Read further found counterxamples on a number of well known Banach spaces, but never yet on any reflexive Banach space. Lomonosov's conjecture that every nontrivial weakly compact operator has hyperinvariant subspaces therefore remains open. If true, it would have the corollary that any nontrivial commutative Banach algebra with weakly compact multiplication has nontrivial closed ideals.

Now the completions of the normed algebras in our present construction are commutative Banach algebras with compact multiplication, which must therefore have nontrivial closed ideals by Lomonosov's theorem [9]. (There is a proof in §4 of [10] that the LRRW algebra has compact multiplication; we do not repeat it here, but nonetheless our algebras will have compact multiplication for similar reasons.) Indeed, since the algebras are singly generated the earlier theorem of Aronszajn and Smith [1] would suffice to show that such closed ideals must exist. Therefore, we are fairly confident that our example cannot, by any small perturbation, be turned into a counterexample for the closed ideals problem for commutative Banach algebras. When one takes the completion of our algebras, one introduces not only nontrivial closed subalgebras, but nontrivial closed ideals as well. In a forthcoming paper with F. Ghahramani and G. A. Willis [7], we will characterise all the ideals of the LRRW algebra, and investigate its cohomology more thoroughly.

2. "Upper interval" functions. As with many mathematical constructions, our present work has an underlying sequence which must satisfy growth conditions—it must "increase sufficiently rapidly". For this paper it is convenient for us to emphasise that we are not really constructing just one normed algebra; we are constructing a function which takes a sequence $\mathbf{d} \subset \mathbb{N}$ satisfying growth conditions and returns a normed algebra with interesting properties.

2.1. Definition. Let $D = \mathbb{N}^{\mathbb{N}}$ be the collection of all sequences of positive integers. Let $f_0 \in \mathbb{N}$ and $f_i : \mathbb{N}^i \to \mathbb{N}$ be arbitrary functions. Write $\mathbf{f}$ for the sequence $(f_i)_{i=0}^{\infty}$. We define the *upper interval* $D_{\mathbf{f}}$ to be the collection of all sequences $\mathbf{d} \in D$ satisfying the growth conditions $d_1 \geq f_0$ and

$d_{i+1} \geq f_i(d_1, \ldots, d_i)$ for all $i \in \mathbb{N}$. An *upper interval function* is a function from an upper interval into some set X; an *upper interval algebra* is an upper interval function $\mathscr{A} : D_{\mathbf{f}} \to X$ where X is a set of algebras in the category of complex normed algebras. So for each $\mathbf{d} \in D_{\mathbf{f}}$ we get a complex normed algebra $\mathscr{A}(\mathbf{d})$.

In this paper our algebras $\mathscr{A}(\mathbf{d})$ will be of countable dimension, generated by an element S, in such a way that the linear span $\operatorname{lin}\{S^i : 1 \leq i \leq n\}$ is isometrically isomorphic to l_1^n for all n.

Now if $D_{\mathbf{f}} \subset D$ is an upper interval, and $P(\mathbf{d})$ a proposition depending on the element $\mathbf{d} \in D_{\mathbf{f}}$, we say $P(\mathbf{d})$ is true "provided $\mathbf{d}$ increases sufficiently rapidly" if there is a subinterval $D_{\mathbf{g}} \subset D_{\mathbf{f}}$ such that $P(\mathbf{d})$ is true for all $\mathbf{d} \in D_{\mathbf{g}}$.

If $M : D_{\mathbf{f}} \to \mathbb{N}$ and $N : D_{\mathbf{f}} \to \mathbb{N}$ are positive-integer-valued interval functions, we say $M \gg N$ (with constant $m \in \mathbb{N}$) if $N(\mathbf{d})$ is a function of $d_1, \ldots, d_{m-1}$ alone, $M(\mathbf{d})$ is a function of $d_1, \ldots, d_m$ alone, and for fixed $d_1, \ldots, d_{m-1}$ one has $M(d_1, \ldots, d_m) \to \infty$ as $d_m \to \infty$. If so, it is easily seen that for any fixed function $F : \mathbb{N} \to \mathbb{N}$, one has

$$M(\mathbf{d}) > F(N(\mathbf{d}))$$

provided $\mathbf{d}$ increases sufficiently rapidly; for the only extra growth condition we need is that

$$M(d_1, \ldots, d_m) > F(N(d_1, \ldots, d_{m-1})), \tag{2.1}$$

which (for fixed $d_1, \ldots, d_{m-1}$) is true for large enough d_m.

Likewise if $M : D_{\mathbf{f}} \to D$ and $N : D_{\mathbf{f}} \to D$ are interval functions we say that $M \gg N$ if, writing M_i (resp. N_i) for the ith element of the sequence $M(\mathbf{d})$ (resp. $N(\mathbf{d})$) one has $M_i \gg N_i$ for all i, with constants m_i tending to infinity as $i \to \infty$. If $M \gg N$, and $F : \mathbb{N}^2 \to \mathbb{N}$ is any function, then provided $\mathbf{d}$ increases sufficiently rapidly one has

$$M_i(\mathbf{d}) > F(i, N_i(\mathbf{d})) \tag{2.2}$$

for all $i \in \mathbb{N}$.

3. The method of proof

3.1. Definition. Now let $\mathscr{A} = \mathscr{A}(\mathbf{d})$ be an upper interval algebra defined for sequences $\mathbf{d}$ in the upper interval $D_{\mathbf{f}} \subset D$. We say $\mathscr{A}$ has *property* A if for all $\mathbf{d} \in D_{\mathbf{f}}$, $\mathscr{A}(\mathbf{d})$ consists of all polynomials $Sp(S)$ ($p \in \mathbb{C}[S]$) in a generator S, and has the further properties:

(i) $\|S\| = 1$, and there is a bounded approximate identity for S consisting of normalised powers $S^{1+n_i}/\|S^{1+n_i}\|$ of the generator.

(ii) This bounded approximate identity is "good enough" that, writing $\eta_i = \|S^{2+n_i}/\|S^{1+n_i}\| - S\|$, the sequences $(n_i)_{i=1}^{\infty}$ and $(\eta_i^{-1})_{i=1}^{\infty}$, both depending on $\mathbf{d}$, satisfy

$$(\eta_i^{-1})_{i=1}^{\infty} \gg (n_i)_{i=1}^{\infty}. \tag{3.1}$$

(iii) Writing $\zeta_i = \|S^{1+n_i}\|$, we have

$$(\zeta_i^{-1})_{i=1}^{\infty} \gg (n_i)_{i=1}^{\infty}. \tag{3.2}$$

(iv) The sequence n_i is strictly increasing; and for all $k \in \mathbb{N}$, the number of i such that $1 + n_i$ is not coprime to k is finite.

Now it is a fairly straightforward consequence of [10] that there is an upper interval algebra with property A. The main result of this paper is the following:

3.2. Theorem. *Let $\mathscr{A}$ be an upper interval algebra with property* A. *Then provided* $\mathbf{d}$ *increases sufficiently rapidly, the normed algebra* $\mathscr{A}(\mathbf{d})$ *has no closed subalgebra except* $\{0\}$ *and* $\mathscr{A}(\mathbf{d})$.

Let us note in passing that (3.2) ensures that, provided $\mathbf{d}$ increases sufficiently rapidly, the normed algebras obtained from our construction will be radical (for provided $\mathbf{d}$ increases sufficiently rapidly, we can assume that (say) $\|S^{1+n_i}\| \le n_i^{-n_i}$ for all i).

4. Proof of Theorem 3.2. Let $x \in \mathscr{A}(\mathbf{d})$ be nonzero; we claim the closed subalgebra generated by x is $\mathscr{A}$ itself. We may assume that

$$x = q(S) = S^k(1 + p(S))$$

where $k > 0$ and t divides the nonzero polynomial $p(t)$. Write $M = 1 + |p|$, where $|p|$ denotes the sum of the absolute values of the coefficients of the polynomial p; and write $d = d(p)$, the degree of p. Let I be large enough that $1 + n_i$ is coprime to k for all $i \ge I$. For each $i \ge I$, pick integers $(l_{i,j})_{j=1}^{n_i}$, $1 \le l_{i,j} \le n_i$, with the property that $kl_{i,j}$ is congruent to j mod $1 + n_i$ $(j = 1, \ldots, n_i)$. Write

$$kl_{i,j} = j + q_{i,j}(1 + n_i),$$

where $q_{i,j} \in \mathbb{N}_0$ and, obviously, $q_{i,j} \le k$. For each $i \le I$ and $1 \le j \le n_i$, let us define polynomials $p_{i,j}$ by $p_{i,j}(t) = (1 + p(t))^{l_{i,j}} - 1$, so that t divides $p_{i,j}(t)$, and the degree $d(p_{i,j}) \le dn_i$. We will have

$$x^{l_{i,j}} = S^{kl_{i,j}}(1 + p(S))^{l_{i,j}} = S^{j + q_{i,j}(1+n_i)}(1 + p_{i,j}(S)). \tag{4.1}$$

For $n \in \mathbb{N}$, let $\tau_n : \mathscr{A} \to \mathscr{A}$ be the (necessarily discontinuous) truncation operator with $\tau_n(S^i) = S^i$ $(i \le n)$ or zero $(i > n)$. Let us define

$$\pi_{i,j}(S) = \tau_{n_i}(S^j(1 + p_{i,j}(S))).$$

Let $p_{i,j}(t) = \sum_{r=1}^{dn_i} \lambda_{i,j,r} t^r$.

Now the vectors $(\pi_{i,j}(S))_{j=1}^{n_i}$ are linearly independent and span the subspace $\operatorname{lin}\{S^j : 1 \leq j \leq n_i\}$. So for each $i \geq I$, there are complex numbers $(\mu_1, \ldots, \mu_{n_i}) \in \mathbb{C}^{n_i}$ such that

$$\sum_{j=1}^{n_i} \mu_j \pi_{i,j}(S) = S.$$

In fact the vector $\boldsymbol{\mu}$ is equal to $(1, 0, \ldots, 0)(I + L)^{-1}$ where L is the n_i by n_i square matrix with entries $L_{j,r} = \lambda_{i,j,r}$ $(r > j)$ or zero $(r \leq j)$. Since L is nilpotent, $(I + L)^{-1} = \sum_{j=0}^{n_i-1} L^j$; if $\|\cdot\|_2$ denotes the l_2 norm on vectors in (or operators on) $\mathbb{C}^{n_i}$, we have

$$\|\boldsymbol{\mu}\|_2 \leq \sum_{j=0}^{n_i-1} \|L^j\|_2 \leq \sum_{j=0}^{n_i-1} \|L\|_2^j. \tag{4.2}$$

The jth row of L has entries $L_{j,r}$ with $\sum_r |L_{j,r}| \leq |p_{i,j}| \leq (1 + |p|)^{l_{i,j}} - 1 = M^{l_{i,j}} - 1$. So the Hilbert–Schmidt norm of L is at most $\sqrt{n_i} \cdot (M^{l_{i,j}} - 1) \leq \sqrt{n_i} \cdot (M^{n_i} - 1)$. Hence,

$$\|\boldsymbol{\mu}\|_2 \leq \sum_{j=0}^{n_i-1} \|L\|_{\mathrm{HS}}^j \leq \sum_{j=0}^{n_i-1} (\sqrt{n_i} \cdot (M^{n_i} - 1))^j \leq M^{n_i^2} n_i^{(1+n_i)/2}. \tag{4.3}$$

We claim the vector

$$y_i = \sum_{j=1}^{n_i} \mu_j \frac{x^{l_{i,j}}}{\|S^{1+n_i}\|^{q_{i,j}}}$$

is a good approximation to S. For by (4.1), the norm $\|y_i - S\|$ is equal to

$$\begin{aligned} \left\| \sum_{j=1}^{n_i} \mu_j \frac{x^{j+q_{i,j}(1+n_i)}}{\|S^{1+n_i}\|^{q_{i,j}}} - S \right\| &= \left\| \sum_{j=1}^{n_i} \mu_j \frac{S^{j+q_{i,j}(1+n_i)}(1 + p_{i,j}(S))}{\|S^{1+n_i}\|^{q_{i,j}}} - S \right\| \\ &\leq \varepsilon_1 + \varepsilon_2 \end{aligned} \tag{4.4}$$

where

$$\varepsilon_1 = \left\| \sum_{j=1}^{n_i} \mu_j \left(\frac{S^{j+q_{i,j}(1+n_i)}}{\|S^{1+n_i}\|^{q_{i,j}}} - S^j \right) (1 + p_{i,j}(S)) \right\|,$$

$$\varepsilon_2 = \left\| \sum_{j=1}^{n_i} \mu_j S^j (1 + p_{i,j}(S)) - S \right\|.$$

Now $\|S\| = 1$ and $\|S^{2+n_i}/\|S^{1+n_i}\| - S\| = \eta_i$, hence

$$\left\| \frac{S^{j+q_{i,j}(1+n_i)}}{\|S^{1+n_i}\|^{q_{i,j}}} - S^j \right\| \leq q_{i,j}\eta_i \leq k\eta_i \tag{4.5}$$

for all i, j. Hence

$$\varepsilon_1 \le k\eta_i \cdot \sum_{j=1}^{n_i} |\mu_j| \cdot |1+p_{i,j}| \le k\eta_i \cdot n_i \cdot \|\boldsymbol{\mu}\|_2 \cdot \sum_{j=1}^{n_i} |1+p_{i,j}| \tag{4.6}$$
$$\le k\eta_i \cdot M^{n_i^2} n_i^{2+n_i/2} \cdot \sum_{j=1}^{n_i} M^{l_{i,j}} \le k\eta_i \cdot M^{2n_i^2} n_i^{3+n_i/2}.$$

Now we have chosen the vector $\boldsymbol{\mu}$ so that $\sum_{j=1}^{n_i} \mu_j \tau_{n_i}(S^j(1+p_{i,j}(S))) = S$. Accordingly

$$\varepsilon_2 = \Big\| \sum_{j=1}^{n_i} \mu_j (1-\tau_{n_i})(S^j(1+p_{i,j}(S))) \Big\| \le \sum_{j=1}^{n_i} |\mu_j| \cdot |1+p_{i,j}| \cdot \|S^{1+n_i}\| \tag{4.7}$$

because $\|S\| = 1$, hence $\|(1-\tau_{n_i})q(S)\| \le \|S^{1+n_i}\| \cdot |q|$ for any polynomial q. Now we have $\|S^{1+n_i}\| = \zeta_i$ and $|1+p_{i,j}| \le M^{l_{i,j}} \le M^{n_i}$ and $|\mu_i| \le M^{n_i^2} n_i^{(1+n_i)/2}$. So,

$$\varepsilon_2 \le \zeta_i \cdot M^{2n_i^2} n_i^{(3+n_i)/2}. \tag{4.8}$$

Putting (4.6) and (4.8) into (4.4), we can be sure that the vectors y_i tend to S as $i \to \infty$ (and so the closed algebra generated by x is indeed the whole of $\mathscr{A}(\mathbf{d})$) provided that for each fixed $R > 0$, we have

$$R^{n_i^2}(\eta_i + \zeta_i) \to 0$$

as $i \to \infty$. It is enough, for example, that $\min(\eta_i^{-1}, \zeta_i^{-1}) \ge i^{n_i^2}$ for all $i \in \mathbb{N}$; and since we know $(\eta_i^{-1})_{i=1}^\infty \gg (n_i)_{i=1}^\infty$ and $(\zeta_i^{-1})_{i=1}^\infty \gg (n_i)_{i=1}^\infty$, that will certainly happen provided $\mathbf{d}$ increases sufficiently rapidly. So provided $\mathbf{d}$ increases sufficiently rapidly, the normed algebra $\mathscr{A}(\mathbf{d})$ has no nontrivial closed subalgebra.

5. Obtaining property A from [10]. In this section we make Theorem 3.2 useful by using the methods of [10] to obtain an upper interval algebra $\mathscr{A}$ with property A. We can then use the theorem to deduce that (provided $\mathbf{d}$ increases sufficiently rapidly) the normed algebra $\mathscr{A}(\mathbf{d})$ has no nontrivial closed subalgebra.

We begin by recalling some notation from [10] (more specifically, from §4 of [10]). Recall that a *valid partial basis* for the algebra $S\mathbb{C}[S]$ is a finite sequence of polynomials $(p_i(S))_{i=1}^n$ such that $\deg p_i = i$, and one obtains an algebra seminorm $\|\cdot\|_n$ on $S\mathbb{C}[S]$ if one defines

$$\Big\| \sum_{i=1}^{n} \lambda_i p_i(S) \Big\|_n = \sum_{i=1}^{n} |\lambda_i|, \tag{5.1}$$

and for a general polynomial $p \in S\mathbb{C}[S]$,

$$\|p(S)\|_n = \|\tau_n p(S)\|_n, \tag{5.2}$$

where τ_n is the truncation operator. The partial basis is considered "valid" only if the seminorm is submultiplicative with "room to spare", that is, $\|p_i(S)p_j(S)\|_n < 1$ (strict inequality) for all $1 \le i, j \le n$.

Obviously the use of notation $\|\cdot\|_n$ for the norm hints that we are going to extend things to a larger value of n; and indeed it is a lemma of [10] (Lemma 4.1) that if $(p_i(S))_{i=1}^n$ is a valid partial basis with $p_1(S) = S$, and if $\eta > 0$, then one can find an extended valid partial basis $(p_i(S))_{i=1}^N$, for some $N > n$, with good properties.

The main good property is that for some $m \in (n, N)$ the element $S^m/\|S^m\|_N$ is a "good" approximate identity: one has

$$\|S^{m+1}/\|S^m\|_N - S\|_N < \eta. \tag{5.3}$$

Another good property for the longer valid partial basis is that $\|S^{n+1}\|_N < \eta\|S^n\|_N$, in particular $\|S^{n+1}\|_N < \eta$ (for $p_1(S) = S$ gives us $\|S\|_N = 1$). The lemma of [10] gives other properties, but these two are the ones we shall use in this paper. In [10], the statement of the lemma does not specify what the value of m is; but in the proof given there, one in fact has $m = n + 1$. This turns out to be important for our purposes, so let us state our own lemma, in the form we are going to use in this paper.

5.1. LEMMA. *Let $(p_i(S))_{i=1}^n$ be a valid partial basis of $S\mathbb{C}[S]$, and let $\eta > 0$. Then there is an $N > n$ and a valid partial basis $(p_i(S))_{i=1}^N$, extending the original one, such that*

$$\|S^{n+1}\|_N < \eta \quad \textit{and} \quad \|S^{n+2}/\|S^{n+1}\|_N - S\|_N < \eta.$$

As a trivial corollary we note that it is always possible to extend a valid partial basis $(p_i(S))_{i=1}^n$ to a larger one $(p_i(S))_{i=1}^N$ for arbitrarily large N; one can picture oneself repeatedly using Lemma 5.1 until a large enough N is obtained (or better still, convince oneself that given $(p_i(S))_{i=1}^n$, if one picks a large enough $C > 0$ and defines $p_i(S) = C^{i-1}S^i$ for $i > n$, then the sequence $(p_i(S))_{i=1}^N$ is "valid" for all $N \ge n$). If $(p_i(S))_{i=1}^N$ is "valid", so is $(p_i(S))_{i=1}^m$ for any $1 \le m < N$, so we have the following result:

5.2. LEMMA. *Let $(p_i(S))_{i=1}^n$ be a valid partial basis and let m be any integer greater than n. Then there is a valid partial basis $(p_i(S))_{i=1}^m$ extending the original one.*

Let us now construct an upper interval algebra with property A. We will define a normed algebra $\mathscr{A}(\mathbf{d})$ for each $\mathbf{d} \in D$.

Let $\mathbf{d} \in D$ be given. We recursively define sequences $(n_i)_{i=1}^\infty \subset \mathbb{N}$ and polynomials $(p_i)_{i=1}^\infty$ such that for each i, $(p_i(S))_{i=1}^{n_i}$ is a valid partial basis. We begin with $n_1 = 1$ and $p_1(S) = S$. Given $k \in \mathbb{N}$, the sequence $(n_i)_{i=1}^k$ and a valid partial basis $(p_i(S))_{i=1}^{n_k}$, we first find an $N \ge n_k$, as small as possible, and pick an extended valid partial basis $(p_i(S))_{i=1}^N$ with

$$\|S^{1+n_k}\|_N < 1/d_k \tag{5.4}$$

and

$$\|S^{2+n_k}/\|S^{1+n_k}\|_N - S\|_N < 1/d_k. \tag{5.5}$$

Such an extended valid partial basis exists by Lemma 5.1. We then pick the least integer $n_{k+1} > N$ with the property that n_{k+1} is divisible by $k!$; and we extend our valid partial basis in an arbitrary manner to a valid partial basis $(p_i(S))_{i=1}^{n_{k+1}}$. This can be done by Lemma 5.2.

Continuing in this manner we end up with a sequence $(p_i(S))_{i=1}^\infty$ with the property that for $i \le n_k$, the polynomial $p_i(S)$ is determined by the elements $(d_i)_{i=1}^{k-1}$ of the sequence $\mathbf{d}$, as also is the integer n_k itself. In view of (5.4) one has

$$\eta_k = \|S^{1+n_k}\|_{n_{k+1}} < 1/d_k,$$

so

$$(\eta_k^{-1})_{k=1}^\infty \gg (n_k)_{k=1}^\infty. \tag{5.6}$$

In view of (5.5) one has

$$\zeta_k = \|S^{2+n_k}/\|S^{1+n_k}\|_{n_{k+1}} - S\|_{n_{k+1}} < 1/d_k$$

and so

$$(\zeta_k^{-1})_{k=1}^\infty \gg (n_k)_{k=1}^\infty. \tag{5.7}$$

Furthermore, since $k!$ divides n_{k+1} for all k, for each fixed r the number of k such that $1 + n_k$ is not coprime to r is finite.

Define the upper interval algebra $\mathscr{A}(\mathbf{d})$ to be $S\mathbb{C}[S]$ equipped with the unique norm such that $\|\sum_{i=1}^N \lambda_i p_i(S)\| = \sum_{i=1}^N |\lambda_i|$ for all N. Plainly this norm agrees (for each k) with $\|\cdot\|_{n_k}$ on the subspace consisting of polynomials of degree less than or equal to n_k; and in view of (5.6) and (5.7), the normed algebra $\mathscr{A}(\mathbf{d})$ has property A.

6. Conclusion. It remains to point out the way in which our present construction differs from the LRRW algebra as defined in [10]. The difference is small but significant; in the LRRW algebra one defines sequences of integers $(n_i)_{i=1}^\infty$ and of polynomials $(p_i)_{i=1}^\infty$ very much along the lines of this construction, and one uses them, as here, to define an l_1 norm. But in the LRRW construction, the sequence $(n_i)_{i=1}^\infty$ does not have the property that for each fixed k, the number of i with $1 + n_i$ not coprime to k is finite. On the contrary, in the LRRW construction one has $1 + n_i \mid 1 + n_{i+1}$ for all i. Consequently, the proof given here that there are no nontrivial closed subalgebras (for the normed algebra of polynomials in S) would break down for the LRRW algebra. Indeed, one may check that in that algebra, the subalgebra consisting of polynomials in S^{1+n_i} is (for each i) relatively closed in the dense subalgebra consisting of polynomials in S.

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