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Addendum to: "Sequences of 0's and 1's"

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by

Johann Boos (Hagen) and Toivo Leiger (Tartu)

Abstract. There is a nontrivial gap in the proof of Theorem 5.2 of [2] which is one of the main results of that paper and has been applied three times (cf. [2, Theorem 5.3, (G) in Section 6, Theorem 6.4]). Till now neither the gap has been closed nor a counterexample found. The aim of this paper is to give, by means of some general results, a better understanding of the gap. The proofs that the applications hold will be given elsewhere.

Concerning notations and preliminary results we refer to the original paper [2] and to [9], [10] and [3]. Let χ be the set of all sequences of 0's and 1's, and, if E is any sequence space, let $\chi(E)$ denote the linear hull of $\chi \cap E$.

In [2] (cf. also [11, 4]) the authors considered sequence spaces E with the property that

$$\chi(E) \subset F \;\Rightarrow\; E \subset F$$

whenever F is an arbitrary FK-space, a separable FK-space, and a matrix domain c_B, respectively. Then E is said to have the *Hahn property*, the *separable Hahn property*, and the *matrix Hahn property*, respectively. A sequence space having any Hahn property is necessarily a subspace of ℓ^∞ (cf. [2, Theorem 5.1]). Obviously, the Hahn property implies the separable Hahn property, and the latter implies the matrix Hahn property. None of the converse implications holds in general (cf. [2, Theorem 5.3] and [11, Theorem 1.3]).

In Theorem 5.2 of [2] the authors stated that *for monotone sequence spaces E containing φ the following properties are equivalent*:

(i) *E has the matrix Hahn property*;
(ii) *E has the separable Hahn property*;
(iii) $\chi(E)^\beta = E^\beta$.

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However, in the proof of (iii)⇒(ii) it was argued that $\tau(E,E^\beta)|_{\chi(E)} = \tau(\chi(E),E^\beta)$, which is false in general for dense subspaces of Mackey spaces (cf. [6, Theorem 5.2.1] or consider c_0 as a subspace of $(c,\tau(c,\ell))$ with the natural bilinear map). Till now neither the gap has been closed nor a counterexample found. Nevertheless, the fact that the applications of the theorem in doubt hold (which will be proved elsewhere) gives a little hope that Theorem 5.2 is true.

In this addendum we examine the situation around the gap, aiming at a better understanding of it.

We start with a simple, but useful characterization of the matrix Hahn property which is essentially due to Webb.

Proposition 1. *If E is any sequence space containing φ and satisfying $E^\beta = \chi(E)^\beta$, then the following statements are equivalent*:

(a) *E has the matrix Hahn property.*
(b) *$\sigma(E^\beta,\chi(E))$ and $\sigma(E^\beta,E)$ have the same Cauchy sequences in E^β.*
(c) *$\sigma(E^\beta,\chi(E))$ and $\sigma(E^\beta,E)$ have the same convergent sequences in E^β.*
(d) *$\sigma(E^\beta,\chi(E))$ and $\sigma(E^\beta,E)$ have the same compact subsets in E^β.*
(e) *$\chi(E)\subset c_{0A}$ implies $E\subset c_{0A}$ for any matrix A.*

Proof. The equivalence of (a)–(c) follows immediately from a result of Webb (cf. [8, Proposition 1·4]); (e) is equivalent to the condition that $\sigma(E^\beta,\chi(E))$ and $\sigma(E^\beta,E)$ have the same sequences converging to 0 in E^β, thus it is equivalent to (c). Theorem 11.4.5 in [3], essentially due to Köthe, tells us that in a K-space (X,τ) a subset K is compact if and only if K is $\tau_\omega|_X$-compact, and τ and $\tau_\omega|_X$ give rise to the same convergent sequences in X. So (d) and (c) are equivalent. ∎

Theorem 2 (cf. [2, Theorem 5.2]). *Let E be a sequence space with $\varphi\subset E$ and $E^\beta = \chi(E)^\beta$. If $(\chi(E)^\beta,\sigma(\chi(E)^\beta,\chi(E)))$ is sequentially complete* (*for instance, if $\chi(E)$ is monotone* (cf. [1, Proposition 3])), *then the following statements are equivalent*:

(a) *E has the matrix Hahn property.*
(b) *$\tau(\chi(E),\chi(E)^\beta) = \tau(E,E^\beta)|_{\chi(E)}$.*
(c) *E has the separable Hahn property.*

Proof. We prove (a)⇒(b)⇒(c)⇒(a) where the last implication is obvious.

Condition (b) holds if and only if $\sigma(E^\beta,\chi(E))$ and $\sigma(E^\beta,E)$ have the same absolutely convex compact subsets in E^β. Thus (a) implies (b) by Proposition 1, (a)⇒(d).

We suppose that (b) holds and a separable FK-space F with $\chi(E)\subset F$ is given, and deduce $E\subset F$, that is, (b)⇒(c). Since $(\chi(E)^\beta,\sigma(\chi(E)^\beta,\chi(E)))$

is assumed to be sequentially complete, it follows from Kalton's closed graph theorem (cf. [5, Theorem 2.4]) that the natural injection

$$i : (\chi(E), \tau(\chi(E), \chi(E)^\beta)) \to F$$

is continuous. We find that $\chi(E)$ is dense in $(E, \tau(E, E^\beta))$ since φ is contained in $\chi(E)$ and obviously dense in $(E, \sigma(E, E^\beta))$. Now, by (b), we have

$$\tau(E, E^\beta)|_{\chi(E)} = \tau(\chi(E), E^\beta) = \tau(\chi(E), \chi(E)^\beta),$$

so that i extends to E, forcing $E \subset F$. ■

EXAMPLE 3. Let E be the sequence space in [11, Theorem 2.5]. Then E has the matrix Hahn property, thus $E^\beta = \chi(E)^\beta$ (cf. [2, Theorem 5.1]), but E does not enjoy the separable Hahn property. Consequently, $(\chi(E)^\beta, \sigma(\chi(E)^\beta, \chi(E)))$ is not sequentially complete by Theorem 2.

PROPOSITION 4. *Let E be a sequence space with $\varphi \subset E$ such that $\chi(E)^\beta = \ell^1 = E^\beta$ and $\chi(E)$ is monotone. Then $\chi(E) \subset \ell^\infty_A$ implies $\|A\| := \sup_n \sum_k |a_{nk}| < \infty$, thus $E \subset \ell^\infty_A$ for any matrix A, where $\ell^\infty_A := \{x \in \omega_A \mid Ax \in \ell^\infty\}$.*

Proof. Since $\chi(E)$ is monotone and $\chi(E)^\beta = \ell^1$, the set $\mathcal{F}$ of subsets of $\mathbb{N}$ corresponding to $\chi \cap E$ is full in the sense of [7, Definition 1]. So $\chi(E) \subset \ell^\infty_A$ implies $\|A\| < \infty$ by [7, Proposition 1, (i)⇒(iv)]. Now, $\|A\| < \infty$ implies obviously $\ell^\infty \subset \ell^\infty_A$, thus $E \subset \ell^\infty_A$. (Note that $E \subset \ell^\infty$ since $\ell^1 = E^\beta$.) ■

THEOREM 5. *Let E be a solid subspace of ℓ^∞ containing φ. Then E has the separable Hahn property if the following conditions are satisfied*:

(a) *$\chi(E)^\beta = E^\beta$.*
(b) *$\sigma(E^\beta, \chi(E))$ and $\sigma(E^\beta, E)$ have the same bounded sequences (sets) in E^β.*
(c) *$\chi(E)$ is dense in $(E, \beta(E, E^\beta))$.*

Proof. Let F be a separable FK-space containing $\chi(E)$. We show $F \supseteq E$. Now, $\chi(E)$ is a monotone sequence space (E is solid) so that $(\chi(E)^\beta, \sigma(\chi(E)^\beta, \chi(E)))$ is sequentially complete. It follows from Kalton's closed graph theorem that the injection $i : (\chi(E), \tau(\chi(E), \chi(E)^\beta)) \to F$ is continuous. Since $\chi(E)^\beta = E^\beta$ and because (b) holds, we have

$$\tau(\chi(E), \chi(E)^\beta) \subset \beta(\chi(E), \chi(E)^\beta) = \beta(E, E^\beta)|_{\chi(E)}.$$

But $\chi(E)$ is assumed to be $\beta(E, E^\beta)$-dense in E, thus for every $x \in E$ there exists a net $(x^{(\alpha)})_\alpha$ in $\chi(E)$ which is $\beta(E, E^\beta)$-convergent to x. In particular, $(x^{(\alpha)})$ is a $\tau(\chi(E), \chi(E)^\beta)$-Cauchy net, and since i is continuous, $(x^{(\alpha)})$ converges in the FK-space F to a $y \in F$. Because $(x^{(\alpha)})$ converges coordinatewise to x and y, we get $x = y$. Altogether we have proved $E \subset F$. ■

THEOREM 6. *Let E be a solid sequence space with $\varphi \subset E$ and $\chi(E)^\beta = E^\beta = \ell^1$. Then E has the separable Hahn property.*

Proof. We apply Theorem 5. Condition (a) holds by the assumptions whereas (b) is satisfied by Proposition 4. Moreover, (c) holds, since $\chi(E)$ is $\|\ \|_\infty$-dense in E, thus $\beta(E, \ell^1)$-dense because $\tau_{\|\ \|_\infty} \supset \beta(E, \ell^1)$. The last inclusion may be verified as follows: If Y is a $\sigma(\ell^1, E)$-bounded subset of $\ell^1 = E^\beta$, then $M := \sup_{y \in Y} \|y\|_1 < \infty$ by the same argument as at the beginning of the proof of Proposition 4. Consequently,

$$q_Y(x) := \sup_{y \in Y} \Big| \sum_k y_k x_k \Big| \leq M \|x\|_\infty \quad (x \in E),$$

proving the $\|\ \|_\infty$-continuity of the seminorm q_Y. ∎

Some problems

1. Does (iii)⇒(ii) in [2, Theorem 5.2] hold?
2. Is the sequential completeness of $(\chi(E)^\beta, \sigma(\chi(E)^\beta, \chi(E)))$ in Theorem 2 also necessary for the validity of the implication (a)⇒(c)?
3. Let E be a solid sequence space containing φ and satisfying $\chi(E)^\beta = E^\beta = \ell^1$. Then E has the separable Hahn property by Theorem 6. Does it have the Hahn property?

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Fachbereich Mathematik
FernUniversität in Hagen
58084 Hagen, Germany
E-mail: Johann.Boos@FernUni-Hagen.de

Puhta Matemaatika Instituut
Tartu Ülikool
EE 50090 Tartu, Estonia
E-mail: Toivo.Leiger@ut.ee

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