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If the $[T, \mathrm{Id}]$ automorphism is Bernoulli then the $[T, \mathrm{Id}]$ endomorphism is standard

by

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Abstract. For any 1-1 measure preserving map T of a probability space we can form the $[T, \mathrm{Id}]$ and $[T, T^{-1}]$ automorphisms as well as the corresponding endomorphisms and decreasing sequence of σ-algebras. In this paper we show that if T has zero entropy and the $[T, \mathrm{Id}]$ automorphism is isomorphic to a Bernoulli shift then the decreasing sequence of σ-algebras generated by the $[T, \mathrm{Id}]$ endomorphism is standard. We also show that if T has zero entropy and the $[T^2, \mathrm{Id}]$ automorphism is isomorphic to a Bernoulli shift then the decreasing sequence of σ-algebras generated by the $[T, T^{-1}]$ endomorphism is standard.

1. Introduction. A *decreasing sequence of σ-algebras* is a measure space $(X, \mathcal{F}_0, \mu)$, together with a sequence of σ-algebras $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots$ A natural example of this arises from a sequence $\{X_i\}_{i \geq 0}$ of independent identically distributed random variables. Namely, set $\mathcal{F}_i = \sigma(X_i, X_{i+1}, \ldots)$. If the X_i take on the values 1 and -1 with probability $1/2$, then this sequence has the property that $\mathcal{F}_i | \mathcal{F}_{i+1}$ has 2-point fibers of equal mass for every i. A decreasing sequence of σ-algebras with this property is called *dyadic*. This example has the property that $\bigcap \mathcal{F}_i$ is trivial. A. Vershik, who began the modern study of such decreasing sequences of σ-algebras [14], refers to this example as the *standard dyadic* example.

Two decreasing sequences $(X, \mathcal{F}_i, \mu)$ and $(Y, \mathcal{G}_i, \nu)$ of σ-algebras are called *isomorphic* if there exists a 1-1 measure preserving map $\Phi : X \to Y$ such that $\Phi(\mathcal{F}_i) = \mathcal{G}_i$ for all i. A decreasing sequence of σ-algebras that is isomorphic to the standard dyadic example is said to be *standard*. A dyadic endomorphism is *standard* if it generates a standard decreasing sequence of σ-algebras. In [13] Vershik showed that there exist dyadic sequences of σ-algebras with trivial intersection that are not isomorphic to the standard dyadic example. He also gave a necessary and sufficient condition for a dyadic decreasing sequence of σ-algebras to be standard. An equivalent description

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of standardness for dyadic sequences is that there exists a sequence $\{P_i\}$ of partitions of X into two sets, each of measure $1/2$, such that

(i) the partitions P_i are mutually independent, and
(ii) for each i, $\mathcal{F}_i = \bigvee_{n=i}^{\infty} P_n$.

In this paper we will be working with decreasing sequences of σ-algebras arising from two classes of endomorphisms. These are the $[T, T^{-1}]$ *endomorphisms* and the $[T, \mathrm{Id}]$ *endomorphisms.* These are often referred to as random walks on a random scenery. Let T, the scenery process, be any 1-1 measure preserving map on a probability space $(Y, \mathcal{C}, \nu)$ such that T^2 is ergodic. Let σ be the shift on $(X, \mathcal{B}, \mu)$ where $X = \{-1, 1\}^{\mathbb{N}}$, $\mathcal{B}$ is the Borel σ-algebra, and μ is product measure $(1/2, 1/2)$. Let $\mathcal{F} = \mathcal{B} \otimes \mathcal{C}$. Define $[T, T^{-1}]$ on $(X \times Y, \mathcal{F}, \mu \times \nu)$ by

$$[T, T^{-1}](x, y) = (\sigma x, T^{x_0} y).$$

The $[T, \mathrm{Id}]$ endomorphism has $X = \{0, 1\}^{\mathbb{N}}$, but is otherwise defined the same as the $[T, T^{-1}]$ endomorphism.

The $[T, T^{-1}]$ endomorphism is 2-1, since any point (x, y) has the preimages $(-1x, Ty)$ and $(1x, T^{-1}y)$. Since each preimage has equal relative measure, the $[T, T^{-1}]$ endomorphism generates a dyadic decreasing sequence of σ-algebras. Furthermore, since T^2 is ergodic, $\bigcap \mathcal{F}_n$ is trivial [11]. Notice that if T is the trivial 1-point transformation then $[T, T^{-1}]$ reduces to the shift on $(X, \mathcal{B}, \mu)$, and the corresponding decreasing sequence of σ-algebras is the standard dyadic example.

The above construction, when carried out for $\overline{X} = \{-1, 1\}^{\mathbb{Z}}$ (or $\overline{X} = \{0, 1\}^{\mathbb{Z}}$), yields a 1-1 map we refer to as the $[T, T^{-1}]$ *automorphism* (or the $[T, \mathrm{Id}]$ *automorphism.*) Kalikow proved in [10] that if T has positive entropy then the $[T, T^{-1}]$ automorphism is not isomorphic to a Bernoulli shift. There are zero entropy transformations T such that the $[T, T^{-1}]$ automorphism is isomorphic to a Bernoulli shift, and other T such that the $[T, T^{-1}]$ automorphism is not isomorphic to a Bernoulli shift [2], [3], [10].

Building on Kalikow's techniques, Heicklen and Hoffman proved that if T has positive entropy then the decreasing sequence of σ-algebras generated by the $[T, T^{-1}]$ endomorphism is not standard [7]. When T has zero entropy the picture appears to be significantly more complicated. Feldman and Rudolph proved that if T is rank 1 then the $[T, T^{-1}]$ endomorphism generates a standard sequence of σ-algebras [5]. On the other hand Hoffman has given an example of a zero entropy T such that the $[T, T^{-1}]$ endomorphism is not standard [8].

Feldman and Rudolph's result, combined with the work of Burton [1], provides an example of an endomorphism which is standard, but is not isomorphic to a Bernoulli shift. The following two theorems are the main results of the present paper.

THEOREM 1.1. *If T is zero entropy and the $[T, \mathrm{Id}]$ automorphism is isomorphic to a Bernoulli shift then the $[T, \mathrm{Id}]$ endomorphism generates a standard decreasing sequence of σ-algebras.*

THEOREM 1.2. *If T is zero entropy and the $[T^2, \mathrm{Id}]$ automorphism is isomorphic to a Bernoulli shift then the $[T, T^{-1}]$ endomorphism generates a standard decreasing sequence of σ-algebras.*

In [9] there is an example of a dyadic endomorphism whose two-sided extensions are isomorphic to a Bernoulli shift, but the endomorphism does not generate a standard decreasing sequence of σ-algebras. Thus the results in this paper cannot be extended to general dyadic endomorphisms.

The rest of this paper is organized as follows. In Section 2 we lay out the necessary notation and background. At the start of Section 3 we sketch the proof of Theorem 1.2. Then we prove the two theorems.

2. Notation. In this section we set up the notation that we will be using in this paper. We also describe the very weak Bernoulli condition for automorphisms which Ornstein proved is equivalent to the automorphism being isomorphic to a Bernoulli shift, as well as the criteria for endomorphisms which Vershik proved are equivalent to the endomorphism being standard.

There is a natural relation between the 2^n preimages of (x, y) under $[T, T^{-1}]^n$, $\{-1, 1\}^n$, and the branches of the rooted binary tree of height n. In this relation a sequence $b = (b_1, \ldots, b_n) \in \{-1, 1\}^n$ is associated with the preimage $(b_n \ldots b_1 x, T^{-\sum_{i=1}^n b_i} y)$. We refer to a path from the root of the binary tree of height n to the edge of the tree as a *branch*. Let M be any map from $\{-1, 1\}^n$ to the branches of the binary tree of height n such that if $b_i = b'_i$ for all $i \le j$, then the branches agree on the j edges closest to the root.

We now introduce some notation corresponding to the relationships defined above. An *n-branch* is an element $b \in \{-1, 1\}^n$. (For the $[T, \mathrm{Id}]$ endomorphisms an *n-branch* is an element $b \in \{0, 1\}^n$.) A *labeled n-tree* W is a function $W : \{-1, 1\}^n \to P$, where P is some finite set. The *labeled n-tree* for a partition P over a point $y \in Y$ assigns to each branch b the label $P(T^{-\sum b_i} y)$.

The primary tool used for studying standardness of decreasing sequences of σ algebras is Vershik's metric on labeled n-trees. In order to define it we must first define the Hamming metric between two labeled n-trees. This is given by

$$d_n(W, W') = \frac{|\{b : W(b) \neq W'(b)\}|}{2^n}.$$

Fix a partition P and let W and W' be the labeled n-trees over y and y' respectively. Let $\mathcal{A}_n$ be the set of all invertible maps $a : \{-1, 1\}^n \to \{-1, 1\}^n$

such that if $b_i = b'_i$ for all $i \leq j$ then $a(b)_i = a(b')_i$ for all $i \leq j$. By the association of $\{-1,1\}^n$ with the branches of the binary tree of height n given above, any map $a : \{-1,1\}^n \to \{-1,1\}^n$ generates a map from the branches of the binary tree to themselves. We call such a map $a \in \mathcal{A}_n$ a *tree automorphism*, because the induced map on the binary tree is a graph homomorphism. Define

$$v_n^P(y,y') = \inf_{a \in \mathcal{A}_n} d_n(W \circ a, W').$$

In the case that $\{\mathcal{F}_n\}$ comes from a $[T, T^{-1}]$ endomorphism (or a $[T, \mathrm{Id}]$ endomorphism), Vershik's standardness criterion is the following.

THEOREM 2.1 ([13]). *Let P be any generating partition of Y. Then $\{\mathcal{F}_n\}$ is standard iff*

$$\int v_n^P(y,y')\, d\nu \times \nu \to 0.$$

REMARK 2.1. A proof of this can also be found in [6].

Now we describe Ornstein's very weak Bernoulli condition for a transformation $(\widetilde{X}, \widetilde{\mu}, T)$. For any partition $\widetilde{P}$ of $\widetilde{X}$ and any $a, b \in \widetilde{X}$ let

$$\overline{d}^{\widetilde{P}}_{[0,N]}(a,b) = |\{i \ : \ i \in [0,N] \text{ and } \widetilde{P}(T^i(a)) \neq \widetilde{P}(T^i(b))\}|/(N+1).$$

For any two measures μ_1 and μ_2 on $\widetilde{X}$ define

$$\overline{d}^{\widetilde{P}}_{[0,N]}(\mu_1, \mu_2) = \inf_m \int d^{\widetilde{P}}_{[0,N]}(a,b)\, dm$$

where the inf is taken over all joinings of μ_1 and μ_2. (A *joining* of μ_1 and μ_2 is a measure on $\widetilde{X} \times \widetilde{X}$ which has μ_1 and μ_2 as its marginals.)

Also define $\widetilde{\mu}_a$ by

$$\widetilde{\mu}_a(A) = \widetilde{\mu}\{a' \in A : \widetilde{P}(T^i(a)) = \widetilde{P}(T^i(a')) \text{ for all } i < 0\}.$$

DEFINITION 2.1. An invertible transformation $(\widetilde{X}, T, \widetilde{\mu})$ with partition $\widetilde{P}$ is *very weak Bernoulli* with respect to $\widetilde{P}$ if for every $\varepsilon > 0$ there exists an N and a set G such that

(i) $\widetilde{\mu}(G) > 1 - \varepsilon$, and
(ii) for any $a, b \in G$,

$$\overline{d}^{\widetilde{P}}_{[0,N]}(\widetilde{\mu}_a, \widetilde{\mu}_b) < \varepsilon.$$

THEOREM 2.2 ([12]). *A transformation is isomorphic to a Bernoulli shift if and only if it is very weak Bernoulli with respect to every finite generating partition.*

For most of the proof we will be working with the $[T^2, \mathrm{Id}]$ automorphism on the space $(\overline{X} \times Y, \mathcal{F}, \mu \times \nu)$ and the $[T, T^{-1}]$ endomorphism on the space $(X \times Y, \mathcal{F}, \mu \times \nu)$. We will fix a finite generating partition P of Y under the map T^2. Then P is also a finite generating partition of Y under the map T.

The partition $\widetilde{P}(x,y) = (x_0, P(y))$ is a finite generating partition of both the $[T, T^{-1}]$ endomorphism and the $[T^2, \mathrm{Id}]$ automorphism.

3. Proof. First we give an outline of the proof of Theorem 1.2. In this section we examine the very weak Bernoulli condition in the context of a $[T^2, \mathrm{Id}]$ automorphism. We fix a $T : Y \to Y$ and a generating partition P of Y. We will show that if the $[T^2, \mathrm{Id}]$ automorphism is very weak Bernoulli then for large n and most points $y, y' \in Y$ there exists a map $M : [-n, n] \to [-n, n]$ and a "large" subset $T \subset [-n, n]$ with the following properties:

(i) $M|_T$ is "approximately linear", and
(ii) for any $t \in T$ we have $P(T^t(y)) = P(T^{M(t)}(y'))$.

Lemma 3.1 is a precise version of the above statement. Then in Lemmas 3.2 to 3.5 we show that since M is "approximately linear" we can construct a tree automorphism $a \in \mathcal{A}_n$ such that for most branches

$$M\Big(\sum_{i=1}^{n} b_i\Big) = \sum_{i=1}^{n} a(b_i). \tag{1}$$

When we combine condition (ii) with line (1) we will have shown that $v_n^P(y, y')$ is small. Since the very weak Bernoulli condition implies we can do this for most y and y', Theorem 2.1 will show that the $[T, T^{-1}]$ endomorphism is standard.

The proofs of Theorems 1.2 and 1.1 are very similar. We give the proof of Theorem 1.2 first because it is notationally simpler. Then we indicate how to modify that proof to prove Theorem 1.1.

We need to make one last definition that will play an important role in the proof. For $M : [-n, \ldots, n] \to [-n, \ldots, n]$ and Q a subset of $[-n, \ldots, n]$ define the function $S(k)$ by

$$S(k) = S(k, M, Q) = \sup |M(i) - M(j) - (i - j)|$$

where the sup is taken over all i and j such that $|i - j| = k$ and i and j are in Q. This is a measure of how far M restricted to Q is from being linear with slope 1.

Lemma 3.1. *If $[T^2, \mathrm{Id}]$ is very weak Bernoulli with respect to $\widetilde{P}$ then for any $\varepsilon > 0$ and any c there exist N, n_0 and $G \subset Y$ with the following property. For any $n > N$ and $y, y' \in G$ there exist $M : [-n, n] \to [-n, n]$ and $Q \subset [-c\sqrt{n}, c\sqrt{n}\,]$ such that*

(i) $y_t = y'_{M(t)}$ *for all* $t \in Q$,
(ii) $|Q| > (2c - \varepsilon)\sqrt{n}$,
(iii) $S(k) = S(k, M, Q) < k^{.51}$ *for all* k,
(iv) $i + M(i) = 0 \bmod 2$,

(v) $M(j) - j < n_0$ *for all* $j \in [-c\sqrt{n}, c\sqrt{n}]$, *and*
(vi) $\nu(G) > 1 - \varepsilon$.

Before we begin the proof we will comment on the sense in which the set Q which satisfies condition (ii) is a "large" subset of $[-n, n]$. If c is large and ε is small then by the central limit theorem the fraction

$$\left|\left\{b \in \{-1,1\}^n : \sum_{i=1}^{n} b_i \in Q\right\}\right| / 2^n$$

is close to one. It is in this sense that Q is a "large" subset of $[-n, n]$.

Proof of Lemma 3.1. First we show how to get G that satisfies the sixth condition, and M and Q that satisfy the first four conditions. Then we show how to adapt it so that they satisfy all six. Let Q_0 be a constant which will be defined later. If $[T^2, \mathrm{Id}]$ is very weak Bernoulli with respect to $\widetilde{P}$ then for any $\varepsilon > 0$ there exists an n and a set $G \subset Y$ with the following property. If $y, y' \in G$ there exist sets $W, W' \subset \{-1,1\}^{\mathbb{Z}}$ and a measure preserving map $m : W \to W'$ such that for any $x \in W$ if $m(x) = x'$ then

$$(2) \quad |\{i \in [-c\sqrt{n}/2, c\sqrt{n}/2] : \widetilde{P}([T^2, \mathrm{Id}]^i(x, y)) \neq \widetilde{P}([T^2, \mathrm{Id}]^i(x', y'))\}| \\ < \varepsilon\sqrt{n}/(2Q_0 + 1).$$

Restrict W to x such that the fraction of $t \in [-c\sqrt{n}/2, c\sqrt{n}/2]$ such that there exist $-c\sqrt{n} \le t_1 \le t \le t_2 \le c\sqrt{n}$ with $t_2 - t_1 > Q_0$ and

$$\left|\sum_{i=t_1}^{t_2} (x_1)_i - (t_2 - t_1)/2\right| \ge .5(t_2 - t_1)^{.51} \quad \text{or} \quad \left|\sum_{i=t_1}^{t_2} (x_2)_i - (t_2 - t_1)/2\right| \ge .5(t_2 - t_1)^{.51}$$

is less than $\varepsilon/(2c)$. By the law of iterated logarithms and the ergodic theorem we can choose Q_0 and n large enough so that $\mu(W) > 1 - \varepsilon$.

Now pick any $x \in W$. In order to define Q we first define Q' by letting Q'^C be the set of all t satisfying either

(a) there exist $-c\sqrt{n}/2 \le t_1 \le t \le t_2 \le c\sqrt{n}/2$ such that $t_2 - t_1 > Q_0$ and $|\sum_{i=t_1}^{t_2} (x_1)_i - (t_2 - t_1)/2| \ge .5(t_2 - t_1)^{.51}$, or

(b) there exist $-c\sqrt{n}/2 \le t_1 \le t \le t_2 \le c\sqrt{n}/2$ such that $t_2 - t_1 > Q_0$ and $|\sum_{i=t_1}^{t_2} (x_2)_i - (t_2 - t_1)/2| \ge .5(t_2 - t_1)^{.51}$, or

(c) there exists t' such that $|t' - t| < Q_0$ and $\widetilde{P}([T^2, \mathrm{Id}]^{t'}(x, y)) \neq \widetilde{P}([T^2, \mathrm{Id}]^{t'}(x', y'))$.

If $i \in [-c\sqrt{n}, c\sqrt{n}\,]$, even, and there is a $t \in Q'$ so that $i = 2\sum_{j=0}^{t-1} x_j$ then $i \in Q$. Then set $M(i)$ to be $2\sum_{j=0}^{t-1} x'_j$. Also let $i \in Q$ if $i+1, i-1 \in Q$. In this case set $M(i) = M(i-1) + 1$. Condition (i) of the lemma is satisfied by condition (c) in the definition of Q'^C. Conditions (ii) and (iii) of the lemma are satisfied by conditions (a) and (b) in the definition of Q'^C. Condition (iv) of the lemma is satisfied by the definition of M.

Now we show how to use the above approach to get a map that satisfies condition (v). Find n_0 and G_0 so that M satisfies the first three conditions with $\varepsilon/10$. Set

$$G = \{y : \text{the density of } j \in [-c\sqrt{n}/n_0, c\sqrt{n}/n_0] \text{ such that } T^{jn_0}(y) \notin G_0 \\ \text{is less than } \varepsilon/5\},$$

where n is large enough so that $m(G) > 1 - \varepsilon$. This is possible because of the ergodic theorem. Define Q by the previous paragraphs on intervals of the form $[jn_0, (j+1)n_0]$. ■

Thus for most points we get a map $M : [-n, n] \to [-n, n]$. Now we want to construct the homomorphism A so that for most branches $\sum A(b)_m = M(\sum b_m)$. This will show that $v_n^P(y, y')$ is small. Ideally we would like to construct a tree automorphism A such that for each $j \le n$ and each $b' \in \{-1, 1\}^j$,

$$\sum_{m=1}^{j} A(b')_m = \frac{1}{2^{n-j}} \sum M\Big(\sum_{m=1}^{n} b_m\Big) \tag{3}$$

where the first sum on the right hand side is taken over all $b \in \{-1, 1\}^n$ with $b_m = b'_m$ for all $m = 1, \dots, j$. If we could construct such an A then by setting $j = n$ we see that

$$\sum_{m=1}^{n} A(b)_m = M\Big(\sum_{m=1}^{m} b_m\Big)$$

for all $b \in \{-1, 1\}^n$. No such A can exist as the left hand side of (3) is an integer, but the right hand side need not be. What we will do is construct the tree automorphism A' such that for certain values of j (which will be labeled h_i) and for elements of $\{-1, 1\}^j$ the two sides of line (3) are close to equal. We will do this in such a way that

$$\sum A(b)_m = M\Big(\sum b_m\Big)$$

for most $b \in \{-1, 1\}^j$.

Assume $\varepsilon > 0$ and $c > 0$ are fixed and the values N and n_0 and set G are as found in Lemma 3.1. We now proceed to approximate M by a product of tree automorphisms. Let L be a large integer to be defined later. Define a sequence of heights $h_i = \lfloor (3^i L)^{1.5} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. We partition the interval $[-n, n]$ into intervals of the form $I_{i,j} = [(j - .5)3^i L, (j + .5)3^i L)$ for each i. These partitions are nested.

Set $d_{i,j} = 0$ if $h_{i-1} \ge (n_0)^3$. For the interval $I_{i,j}$, where i is the smallest integer so that $h_i \ge (n_0)^3$, set

$$d_{i,j} = 2\left\lfloor \frac{.5}{|I_{i,j} \cap Q|} \sum_{I_{i,j} \cap Q} (M(k) - k) \right\rfloor.$$

So $d_{i,j}$ is the average amount that M moves an element of $I_{i,j}$. Assuming that $d_{i+1,j}$ has been defined for all j we now define $d_{i,j}$. For an interval $I_{i,j}$ which is contained in $I_{i+1,j'}$ we set

$$d_{i,j} = 2\left\lfloor \frac{.5}{|I_{i,j} \cap Q|} \sum_{I_{i,j}\cap Q} (M(k) - k)\right\rfloor - d_{i+1,j'}.$$

In this case $d_{i,j}$ is the average amount that M moves an element of $I_{i,j} \cap Q$ minus the average amount that M moves an element of $I_{i+1,j'} \cap Q$. For each i and $k \in I_{i,j}$ define $d_i(k) = d_{i,j}$.

Now we want to construct the homomorphism A so that $\sum A(b)_m = M(\sum b_m)$ for most branches. Given a pair i, j and $d_{i,j}$ we construct a tree homomorphism $A_{i,j}$ that is the identity on branches b such that $\sum_{m=h_i+1}^{n} b_m \notin I_{i,j}$. We also construct $A_{i,j}$ such that most of the branches that satisfy $\sum_{m=h_i+1}^{n} b_m \in I_{i,j}$ also satisfy $\sum_{m=h_{i-1}+1}^{h_i} (A(b)_m - b_m) = d_{i,j}$. We will then form A by the composition of the $A_{i,j}$ so that $\sum A(b)_m = M(b_m)$ for most b.

LEMMA 3.2. *For any i, j, and even d there exists a tree homomorphism $A_{i,j}$ such that*

(i) $A_{i,j}(b)_m = b_m$ *for all b and all m such that* $\sum_{m=h_i+1}^{n} b_m \notin I_{i,j}$,
(ii) $A_{i,j}(b)_m = b_m$ *for all b and all $m > h_i$ and $m \le h_{i-1}$, and*
(iii) $$\frac{|\{b : \sum_{m=h_i+1}^{n} b_m \in I_{i,j} \text{ and } \sum_{m=h_{i-1}+1}^{h_i} A_{i,j}(b)_m - b_m \neq d\}|}{|\{b : \sum_{m=h_i+1}^{n} b_m \in I_{i,j}\}|} < d/(h_i - h_{i-1})^{1/2}.$$

Proof. The homomorphism A will be defined inductively. First define $A(b)_m = b_m$ for all $m > h_i$ and all b. For all b such that $\sum_{m=h_i+1}^{n} b_m \notin I_{i,j}$ define $A(b)_m = b_m$ for all m. For branches b such that $\sum_{m=h_i+1}^{n} b_m \in I_{i,j}$ and j such that $h_{i-1} \le j < h_i$ define $A(b)_j$ as follows. If $\sum_{m=j+1}^{h_i} A(b)_m - b_m \neq d$ then define $A(b)_j = -b_j$. Otherwise set $A(b)_j = b_j$. For $j \le h_{i-1}$ let $A(b)_j = b_j$.

For any j the fraction of branches b such that $\sum_{m=h_i+1}^{n} b_m \in I_{i,j}$ where $\sum_{m=j}^{h_i} A(b)_m - b_m \neq d$ is the same as the fraction of simple random walks on the integers that have not reached d in $h_i - j$ steps. This probability is bounded by $d/(h_i - j)^{1/2}$. ∎

Now we can use the $d_{i,j}$ to define homomorphisms $A_{i,j}$. We define A by first applying the $A_{i,j}$ with $i = 0$, then applying the $A_{i,j}$ with $i = 1$, etc. Now we want to show that for most branches $\sum A(b)_m = M(\sum b_m)$.

LEMMA 3.3. $d_{i,j} < 2(3^{i+1}L)^{.51}$.

Proof. Pick any $k' \in I_{i,j} \cap Q$. We can rewrite $M(k) - k$ as $((M(k) - k) - (M(k') - k')) + (M(k') - k')$. For the smallest i such that $h_i \ge (n_0)^3$ we can bound $M(k') - k'$ by $n_0 < (3^{i+1}L)^{.51}$. For all other i the $M(k') - k'$ terms in the two sums cancel. The first four terms are bounded by $(M(k) - k) - (M(k') - k') < S(k - k') < (3^{i+1}L)^{.51}$. Since an average cannot be greater than the largest term, $d_i < 2(3^{i+1}L)^{.51}$. ∎

LEMMA 3.4. *For any* i,

$$\sum_j \Big|\Big\{b : \sum_{m=1}^{n} b_m \in I_{i,j} \ \textit{and} \ \sum_{m=h_i+1}^{n} b_m \notin I_{i,j}\Big\}\Big| < \frac{2 \cdot 2^n}{(3^i L)^{.1}}.$$

Proof. Suppose there exists a j such that

$$\sum_{m=h_i+1}^{n} b_m \in [(j - .5)3^i L + (3^i L)^{.8}, (j + .5)3^i L - (3^i L)^{.8}] \tag{4}$$

and

$$\Big|\sum_{m=1}^{h_i} b_m\Big| < (3^i L)^{.8}. \tag{5}$$

Then $\sum_{m=1}^{n} b_m, \sum_{m=h_i+1}^{n} b_m \in I_{i,j}$. The fraction of branches that do not satisfy line (4) is less than $(3^i L)^{.8}/.5(3^i L) = 2(3^i L)^{-.2}$. The branches that do not satisfy line (5) are at least $(3^i L)^{.8}/(3^i L)^{.75}$ standard deviations away from the mean. Thus by Chebyshev's inequality they make up a fraction at most $1/(3^i L)^{.1}$ of the branches. ∎

LEMMA 3.5. *For any* $\varepsilon > 0$ *there exist* δ *and* c *with the following property. For any* n, n_0, $M : [-c\sqrt{n}, c\sqrt{n}\,] \to [-c\sqrt{n}, c\sqrt{n}\,]$, *and* $Q \subset [-c\sqrt{n}, c\sqrt{n}\,]$ *which satisfies*

(i) $|Q| > (2c - \delta)\sqrt{n}$,
(ii) $S(k) = S(k, M, Q) < k^{.51}$ *for all* k,
(iii) $M(j) - j < n_0$ *for all* $j \in [-c\sqrt{n}, c\sqrt{n}]$, *and*
(iv) $i + M(i) = 0 \bmod 2$,

there exists a tree homomorphism A *such that*

$$\Big|\Big\{b : \sum_{m=1}^{n} A(b)_m \neq M\Big(\sum_{m=1}^{n} b_m\Big)\Big\}\Big| < \varepsilon 2^n.$$

Proof. Given M define $d_{i,j}$ and from them $A_{i,j}$. Let A be the composition of all the $A_{i,j}$. Consider the set of branches that satisfy

(a) $M(\sum_{m=1}^{n} b_m) - \sum_{m=1}^{n} b_m = \sum d_i(\sum_{m=1}^{n} b_m)$, and
(b) $\sum_{m=h_{i-1}+1}^{h_i} (A(b)_m - b_m) = d_i(\sum_{m=1}^{n} b_m)$ for all i.

These two conditions imply

$$\sum_{m=1}^{n} A(b)_m = M\Big(\sum_{m=1}^{n} b_m\Big).$$

To bound the fraction of branches that do not satisfy the first condition notice that if $I_{j,0} \cap Q = I_{j,0}$ then $M(k) - k$ is constant over $I_{j,0}$. This implies that $M(k) - k = \sum d_i(k)$ for any $k \in I_{j,0}$. Since $|Q| > (2c - \delta)\sqrt{n}$, the number of $k \in [-c\sqrt{n}, c\sqrt{n}]$ such that $I_0(k) \cap Q = I_0(k)$ is at least $(2c - L\delta)\sqrt{n}$. With any choice of δ and of c sufficiently large the number of branches that do not land on such a k is less than $2L\delta 2^n$.

If

$$\sum A(b)_m - b_m = d_i\Big(\sum_{m=h_i+1}^{n} b_m\Big) \quad \text{for all } i \tag{6}$$

and

$$\sum_{h_i+1}^{n} b_m \in I_i\Big(\sum_{m=1}^{n} b_m\Big) \quad \text{for all } i \tag{7}$$

then b satisfies (b). The fraction of branches that do not satisfy (6) can be bounded by Lemmas 3.2 and 3.3. By Lemma 3.3 we have $d_{i,j} < 2(3^{i+1}L)^{.51}$. Thus the fraction of branches that do not satisfy (6) for a given i is bounded by

$$\begin{aligned} d_{i,j}(h_i - h_{i-1})^{-.5} &\le 2(3^{i+1}L)^{.51}((3^iL)^{1.5} - (3^{i-1}L)^{1.5})^{-.5} \\ &\le 2(3^{i+1}L)^{.51}(2L)^{-.75}(3^{i-1})^{-.75} \\ &\le CL^{-.24}(3^i)^{-.24}. \end{aligned}$$

Thus the number of branches that do not satisfy condition (b) for any i is less than $\sum_i CL^{-.24}3^{-.24i}2^n < 30L^{-.24}C2^n$.

The fraction of branches that do not satisfy (7) can be bounded by Lemma 3.4. This says that for a given i the number of branches that do not satisfy (7) is less than $2^n(L^{-.1}3^{-.1i})$. Thus the number of branches that do not satisfy (7) for any i is less than $\sum_i 2^n(L^{-.1}3^{-.1i}) < 10C_1(L^{-.1})2^n$. Thus by choosing L large enough and then c large enough and δ small enough the fraction of branches that do not satisfy conditions (a) and (b) can be made arbitrarily small. ∎

Now we just need to combine these lemmas to prove our two results.

Proof of Theorem 1.2. Combining Lemmas 3.1 and 3.5 we deduce that for any $\varepsilon > 0$ there is a set $G \subset Y$ such that $\mu(G) > 1 - \varepsilon$ and $v_n^P(y, y') < \varepsilon$ for any $y, y' \in G$. Thus

$$\int v_n^P(y, y')\, d\nu \times \nu \to 0.$$

By Vershik's standardness criteria the decreasing sequence of σ-algebras generated by the $[T, \mathrm{Id}]$ endomorphism is standard. ∎

Proof of Theorem 1.1. We can construct a map M' by the methods of Lemma 3.1. To get M that satisfies the hypothesis of Lemma 3.5 define $M(n - 2i) = M'(n - i)$ for all $i \in [0, n]$. Then Lemma 3.5 gives us a tree homomorphism A. It is naturally associated with a tree homomorphism A' that has the property that $\sum A'(b)_m = M'(\sum b_m)$ for most b. Thus for most y and y' this shows that $v_n^P(y, y') < \varepsilon$. Thus by Vershik's standardness criteria the decreasing sequence of σ-algebras generated by the $[T, \mathrm{Id}]$ endomorphism is standard. ∎

4. Open questions. We conclude by posing two open questions related to our results. It is easy to check that condition (ii) in Lemma 3.5 could be changed to $S(k) < k^{1-\alpha}$ for any $\alpha > 0$. The proof would only require changing the definition of h_i to $h_i = (3L^i)^{2-\alpha/2}$. It is possible to weaken the condition to $S(k) < k/(\log k)^{1+\alpha}$ for any $\alpha > 0$. Since every zero entropy loosely Bernoulli transformation that we know of has this property for some $\alpha > 0$ it is natural to ask the following question.

QUESTION 4.1. *Does there exist a zero entropy loosely Bernoulli transformation T that does not satisfy Lemma* 3.1 *with condition* (iii) *replaced by* $S(k) = S(k, M, Q) < k/(\log k)$*?*

If there is no such T then the methods of this paper should be able to be extended to show that if T is zero entropy loosely Bernoulli then the $[T, T^{-1}]$ and $[T, \mathrm{Id}]$ endomorphisms generate a standard decreasing sequence of σ-algebras. If there is such a T then we can ask the following question.

QUESTION 4.2. *Does there exist a zero entropy loosely Bernoulli transformation T such that the* $[T, \mathrm{Id}]$ *or* $[T, T^{-1}]$ *endomorphisms do not generate a standard decreasing sequence of σ-algebras?*

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