STUDIA MATHEMATICA 155 (3) (2003)

On the automorphisms of the spectral unit ball

by

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Abstract. Let Ω be the spectral unit ball of $M_n(\mathbb{C})$, that is, the set of $n \times n$ matrices with spectral radius less than 1. We are interested in classifying the automorphisms of Ω. We know that it is enough to consider the normalized automorphisms of Ω, that is, the automorphisms F satisfying $F(0) = 0$ and $F'(0) = I$, where I is the identity map on $M_n(\mathbb{C})$. The known normalized automorphisms are conjugations. Is every normalized automorphism a conjugation? We show that locally, in a neighborhood of a matrix with distinct eigenvalues, the answer is yes. We also prove that a normalized automorphism of Ω is a conjugation almost everywhere on Ω.

1. Introduction. Let $M_n(\mathbb{C})$ be the set of $n \times n$ square matrices with complex coefficients. When there is no ambiguity, we will simply write M. We denote by $\sigma(x)$ the *spectrum* of a matrix $x \in M_n(\mathbb{C})$ and by $\varrho(x)$ its *spectral radius*, that is,

$$\sigma(x) := \{\lambda \in \mathbb{C} : x - \lambda e \notin M^{-1}(\mathbb{C})\}, \qquad \varrho(x) := \max\{|\lambda| : \lambda \in \sigma(x)\},$$

where e is the identity matrix and where $M^{-1} := M_n^{-1}(\mathbb{C})$ is the subset of invertible matrices of $M_n(\mathbb{C})$. The *spectral unit ball* of $M_n(\mathbb{C})$ is the set

$$\Omega := \Omega_n := \{x \in M_n(\mathbb{C}) : \varrho(x) < 1\}.$$

The collection of all automorphisms of Ω_n will be denoted by $\operatorname{Aut} \Omega_n$. Recall that an *automorphism* of Ω_n is a holomorphic function from Ω_n onto Ω_n such that the inverse function exists and is also holomorphic on Ω_n.

The interest in classifying the automorphisms of the spectral unit ball Ω is justified for at least two reasons. Firstly, Ω is of interest in control theory. This arises from a reformulation of a robust-stability problem as a spectral Nevanlinna–Pick problem (see [14, 16, 3–9]). Also, from the point of view of a pure mathematician, the problem of classifying the automorphisms of Ω is interesting in itself. In order to get the best understanding of a mathematical object, it is desirable to know the transformations that preserve that object. For example, the automorphisms of the Euclidean unit ball B_n of $\mathbb{C}^n$ are

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2000 *Mathematics Subject Classification*: 32H02, 32A07, 32M05, 15A18.

Research supported by scholarships from NSERC (Canada), FCAR (Québec) and ISM (Québec).

well known (see for example [13, Chapter 2]). The spectral unit ball is a much more complicated set than B_n (for example, Ω_n is neither convex nor bounded) and it is harder to characterize its automorphisms. Some advances have been obtained in [11, 1], as we will now describe.

An important property of the automorphisms of the unit ball of $\mathbb{C}^n$ is that they are transitive: for each x and y in B_n, there exists an automorphism ϕ of B_n such that $\phi(x) = y$. This property is no longer satisfied by the automorphisms of Ω. Indeed, we have the following result.

THEOREM 1 ([11, Theorem 4]). *Let F be an automorphism of Ω and let $\Delta := B_1$ be the unit disk in the complex plane. Then there exists a Möbius map $\phi : \Delta \to \Delta$ of the form*

$$\phi(z) := \gamma \frac{z-\alpha}{1-\overline{\alpha}z}, \qquad \alpha \in \Delta,\ |\gamma| = 1,$$

such that

(a) *$\sigma(F(x)) = \phi(\sigma(x))$ for each $x \in \Omega$,*
(b) *$F(\lambda e) = \phi(\lambda)e$ for each $\lambda \in \Delta$.*

In particular, the set $\{\lambda e : \lambda \in \mathbb{C}\}$ is invariant under $\operatorname{Aut}\Omega$, and thus the automorphisms are not transitive. A more straightforward proof of this result is obtained in a more general setting in [10, Theorem 2].

The natural and fundamental question we are interested in is to classify the automorphisms of Ω. It is easy to see that among them there are at least the following three forms:

- *Transposition*: $\mathscr{T}(x) := x^t$.
- *Conjugations*: $\mathscr{C}(x) := u(x)^{-1}xu(x)$,

where $u : \Omega \to M^{-1}$ is a holomorphic map such that $u(q^{-1}xq) = u(x)$ for each $x \in \Omega$ and $q \in M^{-1}$.

- *Möbius maps*: $\mathscr{M}(x) := \gamma(x-\alpha e)(e-\overline{\alpha}x)^{-1}$, where $\alpha \in \Delta$ and $|\gamma| = 1$.

In the conjugation case, the condition on u is sufficient for the map $\mathscr{C}$ to be invertible on Ω. Indeed, $\mathscr{C}^{-1}(y) = u(y)yu(y)^{-1}$. For the Möbius maps, we have

$$\sigma(\mathscr{M}(x)) = \left\{\gamma \frac{\lambda-\alpha}{1-\overline{\alpha}\lambda} : \lambda \in \sigma(x)\right\} = \phi(\sigma(x)),$$

where ϕ is the function defined in Theorem 1. Since ϕ is an automorphism of Δ, we have $\mathscr{M}(\Omega) \subset \Omega$. On the other hand, it is clear that $\mathscr{M}$ is holomorphic and invertible on Ω. Ransford and White have asked the following question in [11]: do the compositions of the three preceding forms generate the whole of $\operatorname{Aut}\Omega$? The question is still open.

The problem of classifying the automorphisms of Ω can be reduced to the study of a subfamily of $\operatorname{Aut}\Omega$. If F is in $\operatorname{Aut}\Omega$, then by Theorem 1 we know that $F(0) = \lambda e$ for a certain $\lambda \in \Delta$. By composing F with a suitably chosen Möbius map, we find that

$$\widetilde{F}(x) := \mathscr{M}(F(x)) = (F(x) - \lambda e)(e - \overline{\lambda} F(x))^{-1}$$

is an automorphism of Ω such that $\widetilde{F}(0) = 0$. Therefore, from the point of view of classifying the automorphisms of Ω, one can assume without loss of generality that $F(0) = 0$.

Under the condition $F(0) = 0$, it is known that $F'(0)$ is a linear automorphism of Ω (see [11, p. 260]). Therefore, the map $\widetilde{F} := F'(0)^{-1} \circ F$ is an automorphism of Ω such that $\widetilde{F}(0) = 0$ and $\widetilde{F}'(0) = I$, where I is the identity map from $M_n(\mathbb{C})$ onto $M_n(\mathbb{C})$ ($I(x) := x$). Hence, it suffices to consider the automorphisms F of Ω *normalized* by the conditions $F(0) = 0$ and $F'(0) = I$.

The only automorphisms of this type that are known are the conjugations $\mathscr{C}(x) := u(x)^{-1} x u(x)$ where $u : \Omega \to M^{-1}$ is a holomorphic map satisfying $u(0) = \lambda e$ ($\lambda \in \mathbb{C} \setminus \{0\}$) and $u(q^{-1} x q) = u(x)$ for each $x \in \Omega$ and $q \in M^{-1}$. If we could show that these conjugations are the only automorphisms of Ω with $F(0) = 0$ and $F'(0) = I$, then we would have a complete characterization of $\operatorname{Aut} \Omega$.

The concept of conjugation will play a central role in what follows. We will say that two matrices x and y are *conjugate* if there exists a matrix $q \in M^{-1}$ such that $x = q^{-1} y q$. This equivalence relation on M will be denoted by $\sim$.

In 1998 Baribeau and Ransford proved a very interesting result: every normalized automorphism of Ω is a pointwise conjugation, i.e. x and $F(x)$ are conjugate. More precisely, we have the following theorem.

THEOREM 2. *Let F be an automorphism of Ω such that $F(0) = 0$ and $F'(0) = I$. Then, for each $x \in \Omega$, there exists an invertible matrix $u(x)$ such that $F(x) = u(x)^{-1} x u(x)$.*

Proof. See [1, Corollary 1.3]. One can find, in a subsequent paper of Baribeau and Roy [2], a more elementary proof of this theorem. ∎

In this paper, the question we are particularly interested in is **whether it is possible to make a *holomorphic* choice of u on Ω**. In a general manner, we will be interested in holomorphic functions F with the property that for each matrix x, the matrices x and $F(x)$ are conjugate. This class of functions includes, in view of the preceding theorem, the normalized automorphisms of Ω.

Let $\Gamma := \Gamma_n(\mathbb{C})$ be the set of matrices of $M_n(\mathbb{C})$ having n distinct eigenvalues. In the next section we will present a local solution on Γ to the question set in boldface above. It is always possible, in a neighborhood of a matrix having distinct eigenvalues, to express F as a holomorphic conjugation: $F(x) = u(x)^{-1}xu(x)$.

THEOREM 3. *Let $a \in \Gamma$ and let F be a holomorphic map defined in a neighborhood W of a and such that $F(x) \sim x$ for all $x \in W$. Then there exists a neighborhood $V \subset W$ of a and a holomorphic map $u : V \to M^{-1}$ such that $F(x) = u(x)^{-1}xu(x)$ for all $x \in V$.*

Clearly, this result gives us some additional information on the normalized automorphisms of Ω.

COROLLARY 1. *Let F be an automorphism of Ω such that $F(0) = 0$ and $F'(0) = I$. Then for each $a \in \Gamma \cap \Omega$, there exists a neighborhood V of a and a holomorphic map $u : V \to M^{-1}$ such that $F(x) = u(x)^{-1}xu(x)$ for each $x \in V$.*

Proof. By Theorem 1, we know that $x \sim F(x)$ for each $x \in \Gamma \cap \Omega$ (note that Theorem 2 reveals actually that $x \sim F(x)$ for each $x \in \Omega$). It suffices now to apply the preceding theorem. ∎

Next, we will prove a theorem about conjugation with matrices in a neighborhood of e. If two matrices x and y are conjugate and close to each other, then there exists an invertible matrix h close to e such that $y = hxh^{-1}$.

THEOREM 4. *Let $x \in M$. There exists a neighborhood V of x and a holomorphic map $h : V \to M^{-1}$ such that*

(a) $h(x) = e$,
(b) *if $y \in V$ and y is conjugate to x, then $y = h(y)xh(y)^{-1}$.*

Theorems 3 and 4 will be needed in Section 4 to obtain a global result about the normalized automorphisms of Ω. We will show that the following theorem holds.

THEOREM 5. *Let V be a neighborhood of 0 and let $F : V \to M$ be a holomorphic map such that $F'(0) = I$ and $F(x) \sim x$ for each $x \in V$. Then there exists a holomorphic map u defined on $V \cap \Gamma$ such that*

$$u(x)F(x) = xu(x), \qquad \forall x \in V \cap \Gamma.$$

Moreover, $u(x)$ is invertible for each $x \in V \cap \Gamma \setminus Z$, where Z is the zero-set of a non-constant holomorphic function on $V \cap \Gamma$.

This theorem and Theorem 2 yield the following result.

COROLLARY 2. *Let F be an automorphism of Ω such that $F(0) = 0$ and $F'(0) = I$. Then there exists a holomorphic map u defined on $\Omega \cap \Gamma$ such that*

$$u(x)F(x) = xu(x), \qquad \forall x \in \Omega \cap \Gamma.$$

Moreover, $u(x)$ *is invertible everywhere on* $\Omega\cap\Gamma\setminus Z$ *where* Z *is the zero-set of a non-constant holomorphic function on* $\Omega\cap\Gamma$.

In Section 5 we will look at some examples where the solution given by Theorem 5 is nice and can be extended to the whole of V, and others where this is not the case. Finally, in the last section, we will explicitly exhibit the set Z in the case $n=2$.

I would like to thank Thomas J. Ransford for his comments and suggestions about this paper.

2. Local holomorphic conjugation on Γ. We will show that in a neighborhood of a matrix in Γ, it is always possible, given a normalized automorphism F of Ω, to find a holomorphic map u such that $u(x)$ is invertible for each x in that neighborhood and $F(x)=u(x)^{-1}xu(x)$.

The core of the work will be to prove the following lemma.

Lemma 1. *For each* $a\in\Gamma$ *there exists a neighborhood* V *of* a *and holomorphic functions* $\pi:V\to M$ *and* $v:V\to M^{-1}$ *such that*

(a) $x=v(x)^{-1}\pi(x)v(x)$ *for each* $x\in V$,
(b) $\pi(x)=\pi(y)$ *for each* $x,y\in V$ *for which* $x\sim y$.

Once we have those functions in hand the proof of Theorem 3 is as follows.

Proof of Theorem 3. Let $p\in M^{-1}$ be such that $F(a)=p^{-1}ap$. We set

$$u(x):=v(x)^{-1}v(pF(x)p^{-1})p.$$

Since F and v are holomorphic on V the same is true for u. Moreover, v being M^{-1}-valued we have $u(x)\in M^{-1}$ for each $x\in V$. Now, a direct computation using the hypothesis $F(x)\sim x$ and the properties of π and v yields

$$\begin{aligned}u(x)^{-1}xu(x)&=[p^{-1}v(pF(x)p^{-1})^{-1}v(x)]x[v(x)^{-1}v(pF(x)p^{-1})p]\\&=p^{-1}v(pF(x)p^{-1})^{-1}[v(x)xv(x)^{-1}]v(pF(x)p^{-1})p\\&=p^{-1}v(pF(x)p^{-1})^{-1}\pi(x)v(pF(x)p^{-1})p\\&=p^{-1}[v(pF(x)p^{-1})^{-1}\pi(pF(x)p^{-1})v(pF(x)p^{-1})]p\\&=p^{-1}[pF(x)p^{-1}]p=F(x).\ \blacksquare\end{aligned}$$

The construction of the functions π and v of Lemma 1 will be done in two steps. First we focus on matrices of Γ that are diagonal and then we extend the results to arbitrary members of Γ. For the first part we will need the implicit function theorem.

THEOREM 6 (Implicit function theorem). *Let W be a domain in $\mathbb{C}^{n+m}$ and let f be a holomorphic map from W into $\mathbb{C}^n$. Suppose that*

(a) $f(\overline{x},\overline{y})=0$ *for some* $(\overline{x},\overline{y})\in W$,
(b) *the map* $T:\mathbb{C}^n\to\mathbb{C}^n$ *defined by* $T(h)=f'(\overline{x},\overline{y})(h,0)$ *is invertible.*

Then there exists an open neighborhood $V\subset\mathbb{C}^m$ of $\overline{y}$ and a holomorphic function $g:V\to\mathbb{C}^n$ such that $f(g(y),y)=0$ for each $y\in V$.

Proof. This theorem is classic. One can find a proof in [12, Theorem 9.28] for example. ∎

We will denote by $D:=D_n(\mathbb{C})$ the set of *diagonal matrices* of $M_n(\mathbb{C})$. We write $P_D(x)$ for the projection of $x\in M$ onto D, that is, the diagonal matrix obtained from x by keeping only its principal diagonal. Also, let $a_1,\ldots,a_k$ be square matrices of orders $n_1,\ldots,n_k$ respectively. The block diagonal matrix of order $n_1+\ldots+n_k$ obtained by taking the direct sum $a_1\oplus\ldots\oplus a_k$ will be denoted by $\operatorname{diag}(a_1,\ldots,a_k)$.

PROPOSITION 1. *Let $d\in\Gamma\cap D$. There exists a neighborhood W of d and holomorphic maps $\delta:W\to D$ and $w:W\to M^{-1}$ such that $\delta(d)=d$, $w(d)=e$ and $z=w(z)^{-1}\delta(z)w(z)$ for each $z\in W$.*

Proof. Let z and w be matrices of M and let $\delta=\operatorname{diag}(\delta_1,\ldots,\delta_n)$ be a diagonal matrix. We set

$$g(w,\delta,z):=wz-\delta w,\qquad h(w,\delta,z):=P_n(ww^t-e),$$

where $P_n(x)$ is a row matrix whose entries correspond to those of the diagonal of x. A solution to the system $g(w,\delta,z)=0$, $h(w,\delta,z)=0$ may be interpreted as follows: δ is the matrix of eigenvalues of z (and also of z^t) and w is the matrix whose rows are the eigenvectors of z^t. We will show that w and δ can be chosen to be holomorphic functions of z in a neighborhood of d. Note that the condition $h(w,\delta,z)=0$ is enough to ensure that each row of w is not identically zero, and thus that it is really an eigenvector of z^t.

We set

$$x:=(w_{11},w_{12},\ldots,w_{nn},\delta_1,\ldots,\delta_n)\in\mathbb{C}^{n^2+n},$$
$$y:=(z_{11},z_{12},\ldots,z_{nn})\in\mathbb{C}^{n^2}.$$

We now define $f:\mathbb{C}^{(n^2+n)+(n^2)}\to\mathbb{C}^{(n^2+n)}$ by

$$f(x,y):=(g_{11}(x,y),g_{12}(x,y),\ldots,g_{nn}(x,y),h_1(x,y),\ldots,h_n(x,y)).$$

Then f is a holomorphic map, since each of its components is a polynomial in x and y. Set $\overline{z}:=d$, $\overline{\delta}:=d$ and $\overline{w}:=e$ and let $\overline{x}$ and $\overline{y}$ be the corresponding values of x and y. Then $f(\overline{x},\overline{y})=0$.

We will now compute $f'(\overline{x},\overline{y})$. Let $\triangle z$ and $\triangle w$ be two matrices in M and let $\triangle\delta$ be a diagonal matrix. We have

$$\begin{aligned}(\overline{x}+\triangle x,\overline{y}+\triangle y)-g(\overline{x},\overline{y}) &= (e+\triangle w)(d+\triangle z)-(d+\triangle\delta)(e+\triangle w)\\ &= \triangle w\, d - d\triangle w + \triangle z - \triangle\delta + \triangle w\,\triangle z - \triangle\delta\,\triangle w\\ &= (d_i-d_j):\triangle w + \triangle z - \triangle\delta + (\triangle w\,\triangle z - \triangle\delta\,\triangle w),\end{aligned}$$

where $A:B := (a_{ij}b_{ij})$ denotes the Schur product of A and B. We also have

$$\begin{aligned}h(\overline{x}+\triangle x,\overline{y}+\triangle y)-h(\overline{x},\overline{y}) &= P_n((e+\triangle w)(e+\triangle w)^t-e)\\ &= P_n(\triangle w+(\triangle w)^t+\triangle w(\triangle w)^t) = 2P_n(\triangle w)+P_n(\triangle w(\triangle w)^t).\end{aligned}$$

Let $T:\mathbb{C}^{n^2}\times\mathbb{C}^n\to\mathbb{C}^{n^2}\times\mathbb{C}^n$ be the $\mathbb{C}$-linear operator defined by

$$T(\triangle w,\triangle\delta) := ((d_i-d_j):\triangle w-\triangle\delta, 2\,P_n(\triangle w)).$$

The preceding lines show that

$$T:(\triangle w,\triangle\delta)\mapsto f'(\overline{x},\overline{y})(\triangle w,\triangle\delta,0).$$

We now prove that T is invertible. Since T is a linear map of $\mathbb{C}^{n^2}\times\mathbb{C}^n$ into itself, it suffices to show that T is surjective. Let $b\in M$ and $c\in\mathbb{C}^n$. The system

$$(d_i-d_j):\triangle w-\triangle\delta = b,\qquad 2P_n(\triangle w)=c$$

has the unique solution

$$\triangle\delta := -P_D(b)\quad\text{and}\quad \triangle w := \frac{1}{2}\,\mathrm{diag}(c)+\beta:b,$$

where

$$\beta_{ij} := \begin{cases} 1/(d_i-d_j) & \text{if } i\neq j,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, by the implicit function theorem, there exists an open neighborhood W' of d on which $z\mapsto\delta(z)$ and $z\mapsto w(z)$ are holomorphic maps. Since $w(d)=e$, it is clear that w is invertible in a neighborhood $W\subset W'$ of d. ■

We are now ready to prove Lemma 1.

Proof of Lemma 1. Let $q\in M^{-1}$ be such that $a=q^{-1}dq$ for some $d\in D$. By Proposition 1, there exists a neighborhood W of d and holomorphic maps $\delta:W\to D$ and $w:W\to M^{-1}$ such that $\delta(d)=d$, $w(d)=e$ and $z=w(z)^{-1}\delta(z)w(z)$ for each $z\in W$. By reducing W if necessary, we can assume that, for each $z\in W$,

$$\max_i|\delta(z)_i-d_i| < \min_{i\neq j}|d_i-d_j|.$$

This reduction ensures that if z_1 and z_2 are conjugate matrices in W, then $\delta(z_1) = \delta(z_2)$.

Let V be a neighborhood of a such that $q^{-1}Vq \subset W$. For each $x \in V$ we set

$$\pi(x) := q^{-1}\delta(qxq^{-1})q, \quad v(x) := q^{-1}w(qxq^{-1})q.$$

Then π and v are holomorphic maps on V and v takes its values in M^{-1}. Moreover,

$$\begin{aligned} v(x)^{-1}\pi(x)v(x) &= [q^{-1}w(qxq^{-1})^{-1}q][q^{-1}\delta(qxq^{-1})q][q^{-1}w(qxq^{-1})q] \\ &= q^{-1}[w(qxq^{-1})^{-1}\delta(qxq^{-1})w(qxq^{-1})]q = x, \end{aligned}$$

and if $x \sim y$, we have

$$\pi(x) = q^{-1}\delta(qxq^{-1})q = q^{-1}\delta(qyq^{-1})q = \pi(y). \blacksquare$$

3. Conjugation with matrices in a neighborhood of e. When a matrix y is conjugate to x, there exists an invertible matrix q such that $y = qxq^{-1}$. If we add the hypothesis that y is close to x, is it possible to choose q close to the identity matrix e? Theorem 4 is an affirmative answer to this question.

Proof of Theorem 4. The proof is carried out in 5 steps.

(i) *Reduction to the case of Jordan matrices.* It is sufficient to prove the theorem in the case where x is a Jordan matrix. For suppose the theorem is true for each Jordan matrix. Let x be an arbitrary matrix and choose $q \in M^{-1}$ and a Jordan matrix j such that $x = qjq^{-1}$. By hypothesis, there exists a holomorphic map h_j defined in a neighborhood V_j of j such that $h_j(j) = e$ and $\widetilde{j} = h_j(\widetilde{j})jh_j(\widetilde{j})^{-1}$ for each $\widetilde{j} \in V_j$ conjugate to j. Set

$$h_x(y) := qh_j(q^{-1}yq)q^{-1}, \quad \forall y \in V_x := qV_jq^{-1}.$$

Then h_x satisfies the conclusions of the theorem.

(ii) *Reformulation of condition* (b). Let x be a Jordan matrix. Let f_n be the matrix of order n having 1s on the diagonal $j = i + 1$ and 0s elsewhere. There exist scalars λ_k and integers n_k ($k = 1, \ldots, N$) such that

$$x = \mathrm{diag}(B_1, \ldots, B_N),$$

where B_k is the matrix of order n_k defined by $B_k := \lambda_k e + f_{n_k}$. These matrices are the Jordan blocks of x. We set $t_k := n_1 + \ldots + n_k$ and $s_k := t_{k-1} + 1$ with $s_1 := 1$. The kth Jordan block B_k of x is the submatrix of x obtained by keeping only rows $s_k, \ldots, t_k$ and columns $s_k, \ldots, t_k$. The matrix x is of the following form:

$$x = \begin{array}{cc} \begin{pmatrix} \lambda_1 & 1 & & & & & & & & \\ & \ddots & 1 & & & & & & & \\ & & \lambda_1 & & & & & & & \\ & & & \lambda_2 & 1 & & & & & \\ & & & & \ddots & 1 & & & & \\ & & & & & \lambda_2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \lambda_N & 1 & \\ & & & & & & & & \ddots & 1 \\ & & & & & & & & & \lambda_N \end{pmatrix} & \begin{matrix} s_1 \\ \\ t_1 \\ s_2 \\ \\ t_2 \\ \\ s_N \\ \\ t_N \end{matrix} \\ \begin{matrix} s_1 & & t_1 & s_2 & & t_2 & & s_N & & t_N \end{matrix} & \end{array}$$

Let y be a matrix in M. When y is conjugate to x and when it is sufficiently close to x, condition (b) of the statement of the theorem requires

$$(\star) \qquad yh(y) = h(y)x.$$

In order to simplify the notation in the following computations, we will write $h := h(y)$. Looking at the jth column of each side of $(\star)$, we find

$$yh_j = hx_j \qquad (j = 1, \ldots, n),$$

where h_j and x_j stand for the jth columns of h and x respectively. Since the entries of x are 0 almost everywhere, the right-hand side is easily computed. We have, for each $j \in \{1, \ldots, n\}$,

$$hx_j = \begin{cases} \lambda_k h_j & \text{if } j = s_k, \\ \lambda_k h_j + h_{j-1} & \text{if } j = s_k + 1, \ldots, t_k, \end{cases}$$

Hence, $(\star)$ is satisfied if and only if for each j,

$$(y - \lambda_k e)h_j = \begin{cases} 0 & \text{if } j = s_k, \\ h_{j-1} & \text{if } j = s_k + 1, \ldots, t_k. \end{cases}$$

Since h_j is determined by h_{j-1} for each $j \notin \{s_1, \ldots, s_N\}$, it suffices to solve the equations

$$0 = (y - \lambda_k e)h_{s_k} = (y - \lambda_k e)^{n_k} h_{t_k} \qquad (k = 1, \ldots, N).$$

So, condition (b) is satisfied if and only if h is invertible and its columns $h_{t_1}, \ldots, h_{t_N}$ are solutions of

$$(\star\star) \qquad (y - \lambda_k e)^{n_k} h_{t_k} = 0 \qquad (k = 1, \ldots, N).$$

(iii) *Structure of* x. Fix $k \in \{1, \ldots, N\}$ and define

$$w := (x - \lambda_k e)^{n_k}.$$

The matrix w can be written in the form $w = \operatorname{diag}(w_1, \ldots, w_N)$, where $w_l = (B_l - \lambda_k e)^{n_k}$ $(l = 1, \ldots, N)$. Each block w_l is upper triangular. Also, if $\lambda_l \neq \lambda_k$, then w_l has no 0 on its principal diagonal. On the other hand, if $\lambda_l = \lambda_k$, then $B_l - \lambda_k e = f_{n_l}$ and so $w_l = f_{n_l}^{n_k}$. This is the zero matrix if $n_k \geq n_l$ and it has 1s on the diagonal $j = i + n_k$ and 0s elsewhere if $n_k < n_l$. For example,

$$f_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_4^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_4^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, in the case $\lambda_l = \lambda_k$, w_l has exactly $\min\{n_k, n_l\}$ zero rows and also $\min\{n_k, n_l\}$ zero columns. Define $I_k := \{i_1, \ldots, i_r\}$, the set of the indices of the $r := n - \operatorname{rank}(w)$ zero rows of w, and define $J_k := \{j_1, \ldots, j_r\}$, the set of the indices of the r zero columns of w. We note that $t_k \in I_k \cap J_k$ since $w_k = f_{n_k}^{n_k} = 0$.

Let A and B be sets of row indices and column indices respectively. We will write $m_{A,B}$ for the matrix obtained from a matrix m by deleting the rows and columns given by A and B. With this notation and in the case $w \neq 0$ (we then have $\operatorname{rank}(w) > 0$), the matrix w_{I_k,J_k} is a square upper-triangular matrix of order $n - r$ having no zero on its principal diagonal. In particular, w_{I_k,J_k} is invertible.

(iv) *Construction of h.* The preceding point gives us a set I_k of rows and J_k of columns for each $k \in \{1, \ldots, N\}$. These sets depend only on the structure of x. We will now use this information to define h.

Let y be a matrix in a neighborhood of x which is conjugate to x. To satisfy condition (b), we have seen in $(\star\star)$ that it suffices to find vectors h_{t_k} such that

$$(y - \lambda_k e)^{n_k} h_{t_k} = 0 \qquad (k = 1, \ldots, N)$$

and $h \in M^{-1}$. Fix a value of k and set

$$z := (y - \lambda_k e)^{n_k}, \qquad v := h_{t_k}, \qquad I := I_k, \qquad J := J_k.$$

The equation to solve can now be written as $zv = 0$. Considering the rows of this linear system indexed by I and $I^{\mathrm{c}} := \{1, \ldots, n\} \setminus I$, we can write

$$z_{I,\emptyset} v = 0, \tag{$\dagger$}$$

$$z_{I^{\mathrm{c}},\emptyset} v = 0. \tag{$\dagger\dagger$}$$

Let us focus on equation (†). Considering the J and J^{c} rows of $z_{I,\emptyset}$ we have

$$z_{I,J} v_J = -z_{I,J^{\mathrm{c}}} v_{J^{\mathrm{c}}},$$

where v_A is the matrix obtained from v by deleting the rows indexed by A. Since x and y are matrices close to each other, we see that $z_{I,J}$ is close to $w_{I,J}$.

On the other hand, $w_{I,J}$ is invertible. Hence, we deduce that $z_{I,J} \in M^{-1}$ for each y in a neighborhood of x. Consequently,

$$v_J = -z_{I,J}^{-1} z_{I,J^c} v_{J^c}.$$

This equation tells us that the J^c components of v can be defined in terms of the J components of v. With the aim of eventually satisfying condition (a), define

$$v_j := \begin{cases} 1 & \text{if } j = t_k, \\ 0 & \text{if } j \in J \setminus \{t_k\}. \end{cases}$$

We have thus defined v_{J^c} and also $v = h_{t_k}$.

In the case where $w = 0$ ($I = J = \{1, \dots, n\}$), we also have $z = 0$ since w and z are conjugate. In this case, every choice of v satisfies the equation $zv = 0$. We will set $v := e_{t_k}$.

Hence, for each k, we have constructed a function h_{t_k} of y satisfying equation (†). As a consequence of the preceding remarks, we have defined the map $y \mapsto h(y)$. This definition holds for all matrices y in a neighborhood of x, even for those which are not conjugate to x. The entries of h are rational functions of y. Therefore, h will be holomorphic and its values invertible in a neighborhood of x if we can verify that $h(x) = e$.

(v) *Verification of conditions* (a) *and* (b). It only remains to show that h satisfies conditions (a) and (b). First of all, when $y = x$, we have $z = w$ for each k. Then

$$v_J = -z_{I,J}^{-1} z_{I,J^c} v_{J^c} = -w_{I,J}^{-1} w_{I,J^c} v_{J^c} = 0,$$

since $w_{I,J^c} = 0$ by the choice of its J columns. Consequently, $v = h_{t_k} = e_{t_k}$. On the other hand, for each k and each $j = s_k, \dots, t_k - 1$, we have

$$\begin{aligned} h_j &= (y - \lambda_k e) h_{j+1} = (y - \lambda_k e)^{t_k - j} h_{t_k} \\ &= (x - \lambda_k e)^{t_k - j} e_{t_k} = [(x - \lambda_k e)^{t_k - j}]_{t_k}. \end{aligned}$$

In view of the block-diagonal structure of x, this vector has 0 entries everywhere, except possibly for the $s_k, \dots, t_k$ components. These are given by

$$[(B_k - \lambda_k e)^{t_k - j}]_{n_k} = [f_{n_k}^{t_k - j}]_{n_k} = e_{j - s_k + 1}.$$

So, $h_j = e_j$ for each $j \in \{1, \dots, n\}$ and then $h(x) = e$.

We now verify that (b) is satisfied. Let y be a matrix in a neighborhood of x that is conjugate to x. We have shown previously that h is a solution of (†). It remains to show that (††) is also satisfied, or equivalently that $v = h_{t_k}$ is a solution of $zv = 0$. Clearly, $\ker z := \{\xi : z\xi = 0\} \subset \ker z_{I,\emptyset}$. By the rank theorem, $\dim \ker z_{I,\emptyset} = n - \operatorname{rank}(z_{I,\emptyset})$. Since $z_{I,J}$ is invertible, $\operatorname{rank}(z_{I,\emptyset}) = n - r$ and so, $\dim \ker z_{I,\emptyset} = r$. Now, by using the hypothesis that y is conjugate to x, we have $\operatorname{rank}(z) = \operatorname{rank}(w)$ and since $\operatorname{rank}(w) = n - r$, we find $\dim \ker z = n - \operatorname{rank}(z) = r$.

Hence, as $\ker z \subset \ker z_{I,\emptyset}$ and since both these vector spaces have the same dimension, we have $\ker z = \ker z_{I,\emptyset}$, and so every solution v of (†) is also a solution of (††). Under the hypothesis $x \sim y$, $h(y)$ is therefore a solution of ($\star$). ∎

The problem solved in this theorem may be stated in a more general setting. Indeed, one can ask if for each element x of a general Banach algebra B with unity e, there exists a neighborhood V of x and a holomorphic map $h : V \to B$ such that

(a) $h(x) = e$,
(b) $h(y)$ is invertible for each $y \in V$,
(c) if $y \in V$ and y is conjugate to x, then $y = h(y)xh(y)^{-1}$.

The preceding proof is essentially based on the Jordan form of x. This argument cannot be directly adapted to the case of Banach algebras. In fact, M. White [personal communication] showed that the above statement is false by constructing a counter-example based on an idea of D. Voiculescu [15].

4. Almost global holomorphic conjugation. We are now going to look for a global solution u of the equation $F(x) = u(x)^{-1}xu(x)$. In the neighborhood of a matrix in the complement of Γ, the situation is more complicated. For example, it is not possible to choose a holomorphic branch that gives the eigenvalues of a matrix. As a consequence, the main tool of the preceding section becomes useless. However, it is possible to use our knowledge of the spectrum-preserving functions F to investigate the boundary of Γ which is the same as its complement. We will focus on the matrix 0. We will show that under suitable hypotheses, it is possible to find a solution u defined "almost everywhere" on the domain of F.

Since we will have to deal with diagonal representations on Γ, we first recall some basic results on this topic before we continue with our favorite equation.

4.1. *Diagonal representations on Γ.* A *permutation matrix* s in M is a matrix obtained by permuting the rows of the identity matrix of order n. The *permutation τ associated to s* is the permutation of the integers $\{1, \ldots, n\}$ such that row i of s is the same as row $\tau(i)$ of e. The next proposition shows some properties of permutation matrices. We omit the proof since it is easy and elementary.

PROPOSITION 2 (Properties of permutation matrices). *Let s be a permutation matrix and let τ be its associated permutation. Then*

(a) *s is invertible and $s^{-1} = s^t$,*
(b) *$sxs^{-1} = [x_{\tau(i)\tau(j)}]$ and $s^{-1}xs = [x_{\tau^{-1}(i)\tau^{-1}(j)}]$ for each $x \in M$,*

(c) *if* $st \in D$ *for some permutation matrix* t, *then* $s = t^{-1}$,
(d) $P_D(s^{-1}xs) = s^{-1}P_D(x)s$ *for each* $x \in M$.

It is clear that permutation matrices play an important role in different possible diagonalizations of a matrix in Γ. Two diagonal matrices conjugate to the same matrix $x \in \Gamma$ are necessarily linked by a permutation matrix. Indeed, we have the following proposition.

PROPOSITION 3. *Suppose that* $x \in \Gamma$, $q \in M^{-1}$ *and* $d \in D$ *are such that* $x = q^{-1}dq$. *Suppose also that* $\widetilde{q} \in M^{-1}$ *and* $\widetilde{d} \in D$. *Then* $x = \widetilde{q}^{-1}\widetilde{d}\widetilde{q}$ *if and only if there exists a permutation matrix* s *and an invertible diagonal matrix* Δ *such that* $\widetilde{d} = s^{-1}ds$ *and* $\widetilde{q} = s^{-1}\Delta q$.

Proof. First, suppose we have $\widetilde{d} = s^{-1}ds$ and $\widetilde{q} = s^{-1}\Delta q$ for a permutation matrix s and for an invertible diagonal matrix Δ. Then

$$\widetilde{q}^{-1}\widetilde{d}\widetilde{q} = [s^{-1}\Delta q]^{-1}s^{-1}ds[s^{-1}\Delta q] = q^{-1}\Delta^{-1}d\Delta q = q^{-1}dq = x.$$

The last equality but one is justified by the fact that the matrices Δ and d commute since both are diagonal.

Conversely, suppose $x = \widetilde{q}^{-1}\widetilde{d}\widetilde{q}$. Since d and $\widetilde{d}$ are diagonal matrices, they have the same set of entries, namely the eigenvalues of x. Proposition 2 shows that there exists a permutation matrix s such that $\widetilde{d} = s^{-1}ds$. Therefore,

$$q^{-1}dq = x = \widetilde{q}^{-1}\widetilde{d}\widetilde{q} = \widetilde{q}^{-1}s^{-1}ds\widetilde{q}.$$

As a consequence, we get

$$s\widetilde{q}q^{-1}d = ds\widetilde{q}q^{-1}.$$

It is easy to show that the only matrices that commute with a diagonal matrix in Γ are themselves diagonal. Using this fact, we deduce that $\Delta := s\widetilde{q}q^{-1}$ is a diagonal matrix and this implies the conclusion. ∎

4.2. *Definitions and properties of* w_f *and* $\widehat{w}_f$. We will construct two maps w_f and $\widehat{w}_f$ that depend on a holomorphic map f. The first will be helpful in the process of building a solution u to the equation $u(x)F(x) = xu(x)$ and the second will give us some information about the invertibility of $u(x)$.

PROPOSITION 4 (Definitions of w_f and $\widehat{w}_f$). *Let* V *be an open subset of* M *and let* $F : V \to M$ *be a holomorphic map such that* $F(x) \sim x$ *for each* $x \in V$. *For each holomorphic map* $f : V \to M$ *define* $w_f : V \cap \Gamma \to M$ *and* $\widehat{w}_f : V \cap \Gamma \to \mathbb{C}$ *as follows*:

$$w_f(x) := q^{-1}P_D(qf(x)r^{-1})r, \qquad \widehat{w}_f(x) := \det P_D(qf(x)r^{-1}),$$

where r *and* q *are invertible matrices and* d *is a diagonal matrix such that* $x = q^{-1}dq$, $F(x) = r^{-1}dr$ *and* $\det q = \det r$. *Then* $w_f(x)$ *and* $\widehat{w}_f(x)$ *are well defined*, *that is*, *they do not depend on the choice of* q, r *and* d.

Proof. Let $\widetilde{r}$ and $\widetilde{q}$ be invertible matrices and let $\widetilde{d}$ be a diagonal matrix such that $x = \widetilde{q}^{-1}\widetilde{d}\widetilde{q}$, $F(x) = \widetilde{r}^{-1}\widetilde{d}\widetilde{r}$ and $\det \widetilde{q} = \det \widetilde{r}$. By Proposition 3, there exist permutation matrices s_r and s_q and diagonal invertible matrices Δ_r and Δ_q such that

$$\widetilde{d} = s_r^{-1} d s_r = s_q^{-1} d s_q, \qquad \widetilde{q} = s_q^{-1}\Delta_q q, \qquad \widetilde{r} = s_r^{-1}\Delta_r r.$$

Since $s_r^{-1} d s_r = s_q^{-1} d s_q$ implies that $s_q s_r^{-1}$ commutes with a diagonal matrix of Γ, we know that $s_q s_r^{-1}$ is diagonal. By Proposition 2(c) we then have $s_r = s_q =: s$.

It remains to do some computations. Proposition 2 gives

$$\begin{aligned}
\widetilde{q}^{-1} P_D(\widetilde{q} f(x)\widetilde{r}^{-1})\widetilde{r} &= [s^{-1}\Delta_q q]^{-1} P_D([s^{-1}\Delta_q q] f(x)[s^{-1}\Delta_r r]^{-1})[s^{-1}\Delta_r r] \\
&= q^{-1}\Delta_q^{-1} s P_D(s^{-1}\Delta_q q f(x) r^{-1}\Delta_r^{-1} s) s^{-1}\Delta_r r \\
&= q^{-1}\Delta_q^{-1} P_D(\Delta_q q f(x) r^{-1}\Delta_r^{-1})\Delta_r r \\
&= q^{-1}\Delta_q^{-1}\Delta_q P_D(q f(x) r^{-1})\Delta_r^{-1}\Delta_r r \\
&= q^{-1} P_D(q f(x) r^{-1}) r = w_f(x).
\end{aligned}$$

Also, since $\det q = \det r$ and $\det \widetilde{q} = \det \widetilde{r}$, we have

$$\begin{aligned}
\det P_D(\widetilde{q} f(x)\widetilde{r}^{-1}) &= \det P_D([s^{-1}\Delta_q q] f(x)[s^{-1}\Delta_r r]^{-1}) \\
&= \det P_D(s^{-1}\Delta_q q f(x) r^{-1}\Delta_r^{-1} s) \\
&= \det[s^{-1}\Delta_q P_D(q f(x) r^{-1})\Delta_r^{-1} s] \\
&= \frac{\det \Delta_q}{\det \Delta_r} \det P_D(q f(x) r^{-1}) \\
&= \frac{\det \widetilde{q} \det s \det q^{-1}}{\det \widetilde{r} \det s \det r^{-1}}\, \widehat{w}_f(x) = \widehat{w}_f(x). \ \blacksquare
\end{aligned}$$

The functions w_f and $\widehat{w}_f$ enjoy some properties that are worth noting.

Proposition 5 (Properties of w_f and $\widehat{w}_f$). (a) *w_f and $\widehat{w}_f$ are holomorphic on $V \cap \Gamma$.*

(b) *For each $x \in V \cap \Gamma$, $w_f(x)F(x) = x w_f(x)$.*

(c) *For each $x \in V \cap \Gamma$, $w_f(x)$ is invertible if and only if $\widehat{w}_f(x) \neq 0$.*

Proof. (a) Let $a \in V \cap \Gamma$. Choose $\widetilde{q} \in M^{-1}$ and $\widetilde{d} \in \Gamma \cap D$ such that $a = \widetilde{q}^{-1}\widetilde{d}\widetilde{q}$. Now, define $d(x) := \delta(\widetilde{q} x \widetilde{q}^{-1})$ and $q(x) = w(\widetilde{q} x \widetilde{q}^{-1})\widetilde{q}$ where δ and w are the functions given by Proposition 1 (with $d = \widetilde{d}$). Then, by the same proposition, $x = q(x)^{-1} d(x) q(x)$ in a neighborhood of a. On the other hand, Theorem 3 gives us a holomorphic map u with invertible values such that $F(x) = u(x)^{-1} x u(x)$. Set $r(x) := \frac{1}{\det u(x)} q(x) u(x)$. Then q, r and d are holomorphic in a neighborhood of a and so are w_f and $\widehat{w}_f$.

(b) It suffices to calculate. Let r and q be invertible matrices and let d be a diagonal matrix such that $x = q^{-1} d q$, $F(x) = r^{-1} d r$ and $\det q = \det r$.

Then

$$\begin{aligned} w_f(x)F(x) &= q^{-1}P_D(qf(x)r^{-1})rF(x) = q^{-1}P_D(qf(x)r^{-1})dr \\ &= q^{-1}dP_D(qf(x)r^{-1})r = xq^{-1}P_D(qf(x)r^{-1})r = xw_f(x). \end{aligned}$$

(c) A careful look at the definitions of w_f and $\widehat{w}_f$ shows this is trivial. ∎

4.3. *Construction of an almost global solution*

LEMMA 2. *Let V be a neighborhood of 0 and let $F : V \to M$ be a holomorphic map such that $F'(0) = I$ and $F(x) \sim x$ for each $x \in V$. For each holomorphic function $f : V \to M$ and for each $a \in V \cap \Gamma$, we have*

$$\lim_{\varepsilon\to 0} \widehat{w}_f(\varepsilon a) = \det P_D(qf(0)q^{-1}),$$

where q is any invertible matrix such that $qaq^{-1} \in D$.

Proof. Fix $a \in V \cap \Gamma$. Let h be the function of Theorem 4 satisfying $h(a) = e$ and $x = h(x)ah(x)^{-1}$ for each x conjugate to a and sufficiently close to a. Let q be an invertible matrix and let d be a diagonal matrix such that $a = q^{-1}dq$. For all small ε, we have

$$\frac{1}{\varepsilon} F(\varepsilon a) = h\biggl(\frac{1}{\varepsilon} F(\varepsilon a)\biggr) ah\biggl(\frac{1}{\varepsilon} F(\varepsilon a)\biggr)^{-1}.$$

We can write

$$F(\varepsilon a) = h\biggl(\frac{1}{\varepsilon} F(\varepsilon a)\biggr) q^{-1}\varepsilon dqh\biggl(\frac{1}{\varepsilon} F(\varepsilon a)\biggr)^{-1}.$$

Set $d(\varepsilon) := \varepsilon d$, $q(\varepsilon) := q$ and

$$r(\varepsilon) := qh\biggl(\frac{1}{\varepsilon} F(\varepsilon a)\biggr)^{-1} \det h\biggl(\frac{1}{\varepsilon} F(\varepsilon a)\biggr).$$

Then we get $\varepsilon a = q(\varepsilon)^{-1}d(\varepsilon)q(\varepsilon)$, $F(\varepsilon a) = r(\varepsilon)^{-1}d(\varepsilon)r(\varepsilon)$ and $\det q(\varepsilon) = \det r(\varepsilon)$. Therefore,

$$\begin{aligned} \widehat{w}_f(\varepsilon a) &= \det P_D(q(\varepsilon)f(\varepsilon a)r(\varepsilon)^{-1}) \\ &= \det P_D\biggl(qf(\varepsilon a)h\biggl(\frac{1}{\varepsilon} F(\varepsilon a)\biggr)q^{-1} \det h\biggl(\frac{1}{\varepsilon} F(\varepsilon a)\biggr)^{-1}\biggr). \end{aligned}$$

Since $F'(0) = I$, the Taylor expansion of F around 0 in the direction a is of the form

$$F(\varepsilon a) = \varepsilon a + \mathrm{O}(\varepsilon^2).$$

Therefore, $\lim_{\varepsilon\to 0} \varepsilon^{-1}F(\varepsilon a) = a$ and since $\lim_{x\to a} h(x) = e$, we find

$$\lim_{\varepsilon\to 0} \widehat{w}_f(\varepsilon a) = \det P_D(qf(0)q^{-1}). \quad ∎$$

If f and a are such that $qf(0)q^{-1}$ has no 0 on its principal diagonal, then

$$\lim_{\varepsilon\to 0} \widehat{w}_f(\varepsilon a) \neq 0$$

and, consequently, $\widehat{w}_f$ is not identically 0 in a neighborhood of 0. We deduce from this fact that $w_f(x)$ is invertible for "almost every x" on the domain of definition of w_f, that is, everywhere but on the zero-set of a non-identically-zero holomorphic map. It remains to identify the conditions on f for which there will exist a matrix a with $\det P_D(qf(0)q^{-1}) \neq 0$.

Let x be a matrix in $M_n(\mathbb{C})$. We will write $\operatorname{cof}_{ij}(x)$ for the cofactor associated to the ij entry of x, that is, $\operatorname{cof}_{ij}(x) := (-1)^{i+j} \det \widetilde{x}$ where $\widetilde{x}$ is the matrix obtained from x by deleting row i and column j. The matrix of cofactors of x and the adjoint of x will be noted $\operatorname{cof} x := [\operatorname{cof}_{ij}(x)]$ and $\operatorname{Adj} x := (\operatorname{cof} x)^t$ respectively.

PROPOSITION 6. *For each matrix* $x \neq 0$, *there exists an invertible matrix* $q \in M^{-1}$ *such that* $\det P_D(qxq^{-1}) \neq 0$.

Proof. Let x be an arbitrary matrix. Define $\psi(q) := qxq^{-1}$ and suppose that $\det P_D(\psi(q))$ is identically zero on M^{-1}. Our goal is to show that this forces $x = 0$. One of the diagonal entries of $\psi(q)$, say the $1,1$ entry, must be identically zero on M^{-1} since these entries are holomorphic functions on M^{-1}. With the help of the formula $q^{-1} = (1/\det q) \operatorname{Adj} q$, one shows with a direct computation that

$$(\star) \qquad 0 = \psi(q)_{11} = \frac{1}{\det q} \sum_{j=1}^{n} \operatorname{cof}_{1j}(q) \sum_{k=1}^{n} q_{1k} x_{kj}.$$

For any vectors $\alpha, \beta \in \mathbb{C}^n$ such that $\beta_1 = 1$, we can construct a matrix $y \in M_n(\mathbb{C})$ such that $y_{1j} = \alpha_j$ and $\operatorname{cof}_{1j}(y) = \beta_j$. Indeed, it is enough to choose

$$y(\alpha, \beta) := \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n \\ -\beta_2 & 1 & 0 & \dots & 0 \\ -\beta_3 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_n & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Moreover, if $\sum_{j=1}^n \alpha_j \beta_j > 0$, then $y(\alpha, \beta)$ is invertible since this sum is exactly the determinant of y. By applying $(\star)$ to the matrix $y(\alpha, \beta)$, we show that, for each pair of vectors $\alpha, \beta \in \mathbb{C}^n$ such that $\beta_1 = 1$ and $\sum_{j=1}^n \alpha_j \beta_j > 0$, we have

$$(\star\star) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j x_{ij} = 0.$$

This is sufficient to deduce that $x = 0$.

For let $v_k \in \mathbb{C}^n$ be the vector

$$v_k := (\underbrace{1, \dots, 1}_{k \text{ times}}, \underbrace{0, \dots, 0}_{n-k \text{ times}}).$$

We first choose $\alpha = \beta = v_1$. Equation $(\star\star)$ shows that $x_{11} = 0$. Then we consider the choices $\alpha = v_1$ and $\beta = v_j$ for j running from 2 to n successively. These give $x_{1j} = 0$ for $j = 2, \dots, n$. Now, we set $\alpha = v_2$ and $\beta = v_j$ for $j = 1, \dots, n$. We find that $x_{2j} = 0$ for each $j \in \{1, \dots, n\}$. Continuing this way up to $\alpha = v_n$, we show that each entry of x is necessarily 0, which ends the proof. ∎

THEOREM 7. *Let V be a neighborhood of* 0 *and let $F : V \to M$ be a holomorphic map such that $F'(0) = I$ and $F(x) \sim x$ for each $x \in V$. Then, for each function $f : V \to M$ such that $f(0) \neq 0$, $\widehat{w}_f$ is not identically zero on V.*

Proof. Let $f : V \to M$ be such that $f(0) \neq 0$. Then by the preceding lemma, there exists an invertible matrix q such that $\det P_D(qf(0)q^{-1}) \neq 0$. Let d be the matrix $\operatorname{diag}(1, 2, \dots, n)$. Define $a := \delta q^{-1} dq$, where $\delta \in \mathbb{C}$ is small enough for a to be in V. Since a is in Γ, Proposition 2 gives

$$\lim_{\varepsilon \to 0} \widehat{w}_f(\varepsilon a) = \det P_D(qf(0)q^{-1}) \neq 0.$$

Therefore, for ε small enough, $\widehat{w}_f(\varepsilon a) \neq 0$ and so $\widehat{w}_f$ is not identically zero on $V \cap \Gamma$. ∎

We now have every tool we need to prove Theorem 5.

Proof of Theorem 5. For each function $f : V \to M$ such that $f(0) \neq 0$, the function $u(x) := w_f(x)$ satisfies the conclusions of the theorem. Indeed, set $Z = \{z \in V \cap \Gamma : \widehat{w}_f(z) = 0\}$. The preceding theorem shows that $Z \neq V \cap \Gamma$. Also, by Proposition 5, $\widehat{w}_f$ is a holomorphic map such that $\widehat{w}_f(x) = 0$ if and only if $w_f(x)$ is invertible. Thus, $w_f(x)$ is invertible for each $x \in V \cap \Gamma \setminus Z$. Finally, the same theorem shows that $w_f(x)F(x) = w_f(x)x$. ∎

5. Examples. Theorem 5 gives rise to a question: can we make a choice of f that will give a map u extendible throughout V and such that $u(x) \in M^{-1}$ for each $x \in V$? Unfortunately, we do not know the answer to this question. An affirmative answer would be a big step toward the complete classification of $\operatorname{Aut} \Omega$. It would only remain to look at the problem of invertibility of $u(x)^{-1} x u(x)$ as a function on the spectral unit ball. Would we have to require that u satisfies the condition $u(q^{-1}xq) = u(x)$ for each $x \in \Omega$ and each invertible q? As we have seen, this condition is sufficient for F to be invertible on Ω.

We are now going to take a look at some examples of choices of f. First of all, in the case where F is already a conjugation, we prove that there is always a good choice of f.

Example 1. Suppose F is a conjugation, that is to say, F is of the form $F(x) = G(x)^{-1}xG(x)$, where G is an M^{-1}-valued holomorphic map defined in a neighborhood of 0 with $G(0) = \lambda e$ $(0 \neq \lambda \in \mathbb{C})$. This last condition is necessary and sufficient to have $F'(0) = I$. Indeed, since $G(x)F(x) = xG(x)$ in a neighborhood of 0, the derivative of each side at 0 applied to the matrix h gives

$$\begin{gathered}G'(0)hF(0) + G(0)F'(0)h = hG(0) + 0G'(0)h,\\ G(0)F'(0)h = hG(0).\end{gathered}$$

If $G(0) = \lambda e$ then clearly $F'(0) = I$. Conversely, if $F'(0) = I$, then $G(0)h = hG(0)$ for each matrix h and so $G(0)$ is a multiple of the identity.

If we make the choice $f := G$ in the proof of Theorem 5, then we find $u(x) = w_G(x) = G(x)$, that is, we get back the original map defining F. This statement is easily proved as follows. Let q, r and d be such that $x = q^{-1}dq$, $F(x) = r^{-1}dr$ and $\det q = \det r$. Then

$$\begin{aligned}F(x) &= G(x)^{-1}xG(x),\\ r^{-1}dr &= G(x)^{-1}q^{-1}dqG(x),\\ qG(x)r^{-1}d &= dqG(x)r^{-1}.\end{aligned}$$

Hence, $qG(x)r^{-1}$ is a diagonal matrix since it commutes with a diagonal matrix in Γ. The definition of w_f now gives the result:

$$w_G(x) = q^{-1}P_D(qG(x)r^{-1})r = q^{-1}(qG(x)r^{-1})r = G(x). \ \blacksquare$$

The next example illustrates the fact that not every choice of f gives rise to nice functions u. Some choices may introduce singularities.

Example 2. Consider the following map:

$$F(x) := \begin{pmatrix} 1 & e^{\operatorname{tr} x} - 1 \\ 0 & e^{\operatorname{tr} x} \end{pmatrix}^{-1} x \begin{pmatrix} 1 & e^{\operatorname{tr} x} - 1 \\ 0 & e^{\operatorname{tr} x} \end{pmatrix}.$$

Here, $\operatorname{tr} x$ is the trace of x. We easily see that F is an automorphism of Ω such that $F(0) = 0$ and $F'(0) = I$. Indeed, it is a conjugation of the form $F(x) := G(x)^{-1}xG(x)$, where $G(0) = e$ and $G(q^{-1}xq) = G(x)$ for every invertible matrix q.

In $M_2(\mathbb{C})$, consider the following curve γ:

$$x = x(\varepsilon) = \begin{pmatrix} \varepsilon & \varepsilon \\ 0 & \varepsilon + \varepsilon^3 \end{pmatrix}.$$

In a neighborhood of 0, this curve is in Γ. For each holomorphic map f with $f(0) \neq 0$, Theorem 5 gives us a solution $u = w_f$. For certain choices of f, we will look at the behavior of these solutions on γ in a neighborhood of 0. Note that on the lines joining 0 to a point of Γ, we know (Proposition 2) that w_f behaves well in a neighborhood of 0. Plainly, γ is not a line here.

Firstly, a direct computation shows that

$$F(x) := G(x)^{-1}xG(x) = \begin{pmatrix} \varepsilon & \varepsilon(1-\varepsilon^2)e^{\varepsilon(2+\varepsilon^2)} + \varepsilon^3 \\ 0 & \varepsilon + \varepsilon^3 \end{pmatrix}.$$

The matrices x and $F(x)$ are diagonalizable and so they can be represented as $x = q^{-1}dq$ and $F(x) = r^{-1}dr$. More explicitly, we define q, r and d to be the matrices exhibited below:

$$x = \begin{pmatrix} 0 & 1 \\ -\varepsilon^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon+\varepsilon^3 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\varepsilon^2 & 1 \end{pmatrix},$$

$$F(x) = \begin{pmatrix} 0 & 1 \\ \frac{-\varepsilon^2}{(1-\varepsilon^2)e^{\varepsilon(2+\varepsilon^2)}+\varepsilon^2} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon+\varepsilon^3 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{-\varepsilon^2}{(1-\varepsilon^2)e^{\varepsilon(2+\varepsilon^2)}+\varepsilon^2} & 1 \end{pmatrix}.$$

Remembering that $w_f(x) := q^{-1}P_D(qf(x)r^{-1})r$, it is now possible to compute $w_f(x)$ for any given f.

(a) For $f(x) := e$, we find

$$w_f(x) = \begin{pmatrix} 1 & -2/\varepsilon - 2 + \mathrm{O}(\varepsilon) \\ 0 & 1 \end{pmatrix},$$

where $\mathrm{O}(\varepsilon)$ is a function of ε for which there exists a constant M such that $\mathrm{O}(\varepsilon) \le M\varepsilon$ in a neighborhood of $\varepsilon = 0$. We realize that with this choice of f, the solution $u = w_f$ has a singularity at 0. Therefore, it is not possible to extend u to the definition domain of F.

(b) Another choice of f shows that the situation may be even worse. Define

$$f(x) := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$w_f(x) = \begin{pmatrix} 0 & 1/\varepsilon^2 \\ 0 & 1 \end{pmatrix}.$$

Here, we not only have a singularity at 0, but also $w_f(x)$ is non-invertible for every point of γ.

(c) Nevertheless, Example 1 shows that if we choose

$$f(x) := \begin{pmatrix} 1 & e^{\operatorname{tr} x} - 1 \\ 0 & e^{\operatorname{tr} x} \end{pmatrix},$$

then we have

$$w_f(x) = \begin{pmatrix} 1 & e^{\operatorname{tr} x} - 1 \\ 0 & e^{\operatorname{tr} x} \end{pmatrix} = G(x),$$

which is clearly a global solution. ∎

6. Criteria for $w_f(x)$ to be invertible. We have seen earlier that the value of w_f is invertible at a point $x \in V \cap \Gamma$ if and only if $\widehat{w}_f$ is non-zero at x. Concretely, this left us with verifying that every diagonal entry of $qf(x)r^{-1}$ is non-zero at a given point x, where r and q are invertible matrices and d is a diagonal matrix such that $x = q^{-1}dq$, $F(x) = r^{-1}dr$ and $\det q = \det r$. Since $\widehat{w}_f(x)$ is independent of the choice of q, r and d, one can ask whether it is possible to write $\widehat{w}_f(x)$ in terms of x, $f(x)$ and $F(x)$ only. We would then have a more tractable condition.

We show in the next theorem that it is possible to realize this idea in the case $n = 2$, that is, when $0 \in V \subset M_2(\mathbb{C})$. Our goal is achieved by rather long and brutal computations. Unfortunately, the generalization to the cases $n > 2$ does not seem to be straightforward.

Lemma 3. *Let x, q and d be matrices such that $x \in \Gamma \cap M_2(\mathbb{C})$, $q \in M_2^{-1}(\mathbb{C})$, $\det q = 1$, $d = \operatorname{diag}(d_1, d_2)$ and $x = q^{-1}dq$. Define $\widehat{q}_{ij} := q_{1i}q_{2j}$. Then, if $\operatorname{tr} x \neq 0$, we have*

$$\widehat{q} = \frac{1}{d_1 - d_2}\begin{pmatrix} -x_{21} & \dfrac{x_{11}d_1 - x_{22}d_2}{\operatorname{tr} x} \\ \dfrac{x_{11}d_2 - x_{22}d_1}{\operatorname{tr} x} & x_{12} \end{pmatrix}$$

and if $\operatorname{tr} x = 0$, then

$$\widehat{q} = \frac{1}{2d_1}\begin{pmatrix} -x_{21} & x_{11} + d_1 \\ -(x_{22} + d_1) & x_{12} \end{pmatrix}.$$

Proof. The following computations lead to the result. Since $\det q = 1$, we have

$$\begin{aligned}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} &= \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}^{-1}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \\ &= \begin{pmatrix} q_{22} & -q_{12} \\ -q_{21} & q_{11} \end{pmatrix}\begin{pmatrix} d_1q_{11} & d_1q_{12} \\ d_2q_{21} & d_2q_{22} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{q}_{12}d_1 - \widehat{q}_{21}d_2 & \widehat{q}_{22}(d_1 - d_2) \\ -\widehat{q}_{11}(d_1 - d_2) & -\widehat{q}_{21}d_1 + \widehat{q}_{12}d_2 \end{pmatrix}.\end{aligned}$$

Since $x \in \Gamma$, we always have $d_1 - d_2 \neq 0$. When $\operatorname{tr} x = d_1 + d_2 \neq 0$, this linear system in $\widehat{q}$ has the solution

$$\widehat{q} = \frac{1}{d_1 - d_2}\begin{pmatrix} -x_{21} & \dfrac{x_{11}d_1 - x_{22}d_2}{\operatorname{tr} x} \\ \dfrac{x_{11}d_2 - x_{22}d_1}{\operatorname{tr} x} & x_{12} \end{pmatrix}.$$

One can verify this by substitution. When $\operatorname{tr} x = 0$, the equation $x = q^{-1}dq$

can be written as

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} (\widehat{q}_{12}+\widehat{q}_{21})d_1 & 2d_1\widehat{q}_{22} \\ -2d_1\widehat{q}_{11} & -(\widehat{q}_{21}+\widehat{q}_{12})d_1 \end{pmatrix},$$

since then $d_2 = -d_1$. Therefore, $\widehat{q}_{12}+\widehat{q}_{21} = x_{11}/d_1 = -x_{22}/d_1$. On the other hand, $1 = \det q = \widehat{q}_{12} - \widehat{q}_{21}$. We deduce from these equalities that $2d_1\widehat{q}_{12} = x_{11}+d_1$ and $2d_1\widehat{q}_{21} = -(x_{22}+d_1)$. ∎

THEOREM 8. *Let V be an open subset of $M_2(\mathbb{C})$ and let $F : V \to M_2(\mathbb{C})$ be a holomorphic map such that $F(x) \sim x$ for each $x \in V$. Let $f : V \to M_2(\mathbb{C})$ be another holomorphic map. Then, for each $x \in V \cap \Gamma_2(\mathbb{C})$, we have*

$$\widehat{w}_f(x) = \frac{\operatorname{tr}(xf(x)F(x)\operatorname{Adj} f(x)) - 2\det x \det f(x)}{(\operatorname{tr} x)^2 - 4\det x}.$$

Moreover, if $f(x)$ is an invertible matrix, then

$$\widehat{w}_f(x) = \frac{\det f(x)\,(\operatorname{tr}(xf(x)F(x)f(x)^{-1}) - 2\det x)}{(\operatorname{tr} x)^2 - 4\det x}.$$

Proof. Let $x \in V \cap \Gamma_2(\mathbb{C})$. We will denote by f_{ij} the components of $f(x)$. Let q and r be invertible matrices and d be a diagonal matrix such that $x = q^{-1}dq$, $F(x) = r^{-1}dr$ and $\det q = \det r = 1$. Then by definition of $\widehat{w}_f$, we have

$$\begin{aligned}\widehat{w}_f(x) &= \det P_D(qf(x)r^{-1}) \\ &= (r_{22}q_{11}f_{11} + r_{22}q_{12}f_{21} - r_{21}q_{11}f_{12} - r_{21}q_{12}f_{22}) \\ &\quad \times (-r_{12}q_{21}f_{11} - r_{12}q_{22}f_{21} + r_{11}q_{21}f_{12} + r_{11}q_{22}f_{22}).\end{aligned}$$

Now, we set $\widehat{q}_{ij} := q_{1i}q_{2j}$ and $\widehat{r}_{ij} := r_{1i}r_{2j}$. By expanding the preceding product, we find

$$\begin{aligned}\widehat{w}_f(x) = {}& f_{21}f_{22}\widehat{q}_{22}\widehat{r}_{12} - f_{11}^2\widehat{q}_{11}\widehat{r}_{22} - f_{12}f_{22}\widehat{q}_{12}\widehat{r}_{11} + f_{21}f_{22}\widehat{q}_{22}\widehat{r}_{21} \\ & - f_{11}f_{21}\widehat{q}_{12}\widehat{r}_{22} - f_{22}^2\widehat{q}_{22}\widehat{r}_{11} + f_{21}f_{12}\widehat{q}_{21}\widehat{r}_{12} - f_{21}^2\widehat{q}_{22}\widehat{r}_{22} \\ & + f_{11}f_{12}\widehat{q}_{11}\widehat{r}_{12} - f_{12}f_{22}\widehat{q}_{21}\widehat{r}_{11} - f_{11}f_{21}\widehat{q}_{21}\widehat{r}_{22} + f_{22}f_{11}\widehat{q}_{21}\widehat{r}_{21} \\ & + f_{11}f_{22}\widehat{q}_{12}\widehat{r}_{12} + f_{12}f_{21}\widehat{q}_{12}\widehat{r}_{21} - f_{12}^2\widehat{q}_{11}\widehat{r}_{11} + f_{11}f_{12}\widehat{q}_{11}\widehat{r}_{21}.\end{aligned}$$

Suppose for the moment that $\operatorname{tr} x = \operatorname{tr} F(x) \neq 0$ and write F_{ij} for the entries of $F(x)$. The preceding lemma applied to x, q and d, and then to $F(x)$, r and d, gives

$$\begin{aligned}&\widehat{w}_f(x)\operatorname{tr} x \operatorname{tr} F(x)(d_1-d_2)^2 \\ &\ = (d_1^2+d_2^2)(-f_{21}f_{12}x_{22}F_{11} + f_{21}f_{22}x_{12}F_{11} + f_{11}^2x_{21}F_{12} + f_{22}^2x_{12}F_{21} \\ &\qquad - f_{21}^2x_{12}F_{12} - f_{11}f_{12}x_{21}F_{11} - f_{11}f_{21}x_{11}F_{12} + f_{12}f_{22}x_{11}F_{21} \\ &\qquad - f_{21}f_{22}x_{12}F_{22} + f_{11}f_{12}x_{21}F_{22} - f_{21}f_{12}x_{11}F_{22} + f_{11}f_{22}x_{22}F_{22} \\ &\qquad + f_{11}f_{22}x_{11}F_{11} + f_{11}f_{21}x_{22}F_{12} - f_{12}^2x_{21}F_{21} - f_{12}f_{22}x_{22}F_{21})\end{aligned}$$

$$+ 2d_1d_2(-f_{12}^2x_{21}F_{21} - f_{11}f_{22}x_{22}F_{11} + f_{22}^2x_{12}F_{21} - f_{11}f_{12}x_{21}F_{11}$$
$$- f_{11}f_{21}x_{11}F_{12} + f_{12}f_{22}x_{11}F_{21} - f_{21}^2x_{12}F_{12} + f_{11}^2x_{21}F_{12}$$
$$+ f_{21}f_{12}x_{11}F_{11} + f_{21}f_{12}x_{22}F_{22} - f_{11}f_{22}x_{11}F_{22} + f_{11}f_{12}x_{21}F_{22}$$
$$+ f_{11}f_{21}x_{22}F_{12} - f_{12}f_{22}F_{21}x_{22} - f_{21}f_{22}x_{12}F_{22} + f_{21}f_{22}x_{12}F_{11}).$$

We now use the equations $d_1d_2 = \det x$ and $d_1 + d_2 = \operatorname{tr} x = \operatorname{tr} F(x)$. We have

$$d_1^2 + d_2^2 = (d_1 + d_2)^2 - 2d_1d_2 = (\operatorname{tr} x)^2 - 2\det x,$$
$$(d_1 - d_2)^2 = d_1^2 + d_2^2 - 2d_1d_2 = (\operatorname{tr} x)^2 - 4\det x.$$

The relation between $\widehat{w}_f(x)$, x, $F(x)$ and $f(x)$ can be simplified to

$$\widehat{w}_f(x)(\operatorname{tr} x)^2((\operatorname{tr} x)^2 - 4\det x)$$
$$- 2\det x(x_{22} + x_{11})(F_{11} + F_{22})(-f_{21}f_{12} + f_{22}f_{11})$$
$$+ (\operatorname{tr} x)^2[f_{11}f_{12}x_{21}F_{22} + f_{22}f_{11}x_{11}F_{11} - f_{11}f_{12}x_{21}F_{11} + f_{22}f_{12}x_{11}F_{21}$$
$$- f_{21}f_{12}x_{11}F_{22} + f_{11}^2x_{21}F_{12} - f_{22}f_{12}x_{22}F_{21} + f_{22}^2x_{12}F_{21}$$
$$- f_{21}^2x_{12}F_{12} - f_{21}f_{11}x_{11}F_{12} - f_{12}^2x_{21}F_{21} - f_{21}f_{22}x_{12}F_{22}$$
$$+ f_{21}f_{22}x_{12}F_{11} + f_{21}f_{11}x_{22}F_{12} - f_{21}f_{12}x_{22}F_{11} + f_{22}f_{11}x_{22}F_{22}].$$

In the first term, one can easily recognize the trace of x, the trace of $F(x)$ and also the determinant of $f(x)$. However, one must have some experience to realize that the expression in the square brackets is nothing else than $\operatorname{tr}(xf(x)F(x)\operatorname{Adj} f(x))$! This can be verified by simply computing the latter expression. Combining all these remarks gives

$$\widehat{w}_f(x)(\operatorname{tr} x)^2((\operatorname{tr} x)^2 - 4\det x)$$
$$= (\operatorname{tr} x)^2(-2\det x\det f(x) + \operatorname{tr}(xf(x)F(x)\operatorname{Adj} f(x))).$$

By hypothesis, $\operatorname{tr} x \neq 0$, which leaves us with

$$\widehat{w}_f(x) = \frac{\operatorname{tr}(xf(x)F(x)\operatorname{Adj} f(x)) - 2\det x\det f(x)}{(\operatorname{tr} x)^2 - 4\det x}.$$

We now go back to the case $\operatorname{tr} x = 0$. The preceding lemma gives

$$\widehat{w}_f(x)\,4d_1^2$$
$$= 2d_1^2(f_{11}f_{22} - f_{12}f_{21}) + d_1(x_{11} + x_{22} + F_{11} + F_{22})(f_{11}f_{22} - f_{12}f_{21})$$
$$+ [f_{11}f_{12}x_{21}F_{22} + f_{22}f_{11}x_{11}F_{11} - f_{11}f_{12}x_{21}F_{11} + f_{22}f_{12}x_{11}F_{21}$$
$$- f_{21}f_{12}x_{11}F_{22} + f_{11}^2x_{21}F_{12} - f_{22}f_{12}x_{22}F_{21} + f_{22}^2x_{12}F_{21}$$
$$- f_{21}^2x_{12}F_{12} - f_{21}f_{11}x_{11}F_{12} - f_{12}^2x_{21}F_{21} - f_{21}f_{22}x_{12}F_{22}$$
$$+ f_{21}f_{22}x_{12}F_{11} + f_{21}f_{11}x_{22}F_{12} - f_{21}f_{12}x_{22}F_{11} + f_{22}f_{11}x_{22}F_{22}].$$

The expression in the square brackets is the same as above. Also, since $\operatorname{tr} x = \operatorname{tr} F(x) = 0$, the term in d_1 is zero. Finally, substituting d_1^2 for $-\det x$ gives

$$\widehat{w}_f(x) = \frac{\operatorname{tr}(xf(x)F(x)\operatorname{Adj} f(x)) - 2\det x \det f(x)}{-4\det x}.$$

We have thus shown that for all $x \in V \cap \Gamma$,

$$\widehat{w}_f(x) = \frac{\operatorname{tr}(xf(x)F(x)\operatorname{Adj} f(x)) - 2\det x \det f(x)}{(\operatorname{tr} x)^2 - 4\det x}.$$

If we know that $f(x) \in M^{-1}$, then $\operatorname{Adj} f(x) = f(x)^{-1} \det f(x)$ and consequently

$$\widehat{w}_f(x) = \frac{\det f(x)(\operatorname{tr}(xf(x)F(x)f(x)^{-1}) - 2\det x)}{(\operatorname{tr} x)^2 - 4\det x}. \blacksquare$$

As mentioned earlier this theorem gives a criterion for $w_f(x)$ to be in M^{-1} and so gives a criterion for the existence of a solution of $F(x) = u(x)^{-1}xu(x)$.

COROLLARY 3. *Let V be an open subset of $M_2(\mathbb{C})$ and let $F : V \to M_2(\mathbb{C})$ be a holomorphic map such that $F(x) \sim x$ for each $x \in V$. For each holomorphic map $f : V \to M_2(\mathbb{C})$, $u(x) := w_f(x)$ is a holomorphic solution of $F(x) = u(x)^{-1}xu(x)$ on $V \cap \Gamma_2(\mathbb{C}) \setminus Z$, where*

$$Z := \{x \in V \cap \Gamma_2(\mathbb{C}) : \operatorname{tr}(xf(x)F(x)\operatorname{Adj} f(x)) = 2\det x \det f(x)\}.$$

Moreover, if $f(x)$ is invertible at each point of $V \cap \Gamma_2(\mathbb{C})$, then

$$Z = \{x \in V \cap \Gamma_2(\mathbb{C}) : \operatorname{tr}(xf(x)F(x)f(x)^{-1}) = 2\det x\}.$$

Proof. Proposition 5 and the preceding theorem give the result. $\blacksquare$

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Received May 9, 2000
Revised version July 22, 2002 (4526)