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Smooth operators in the commutant of a contraction

by

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Abstract. For a completely non-unitary contraction T, some necessary (and, in certain cases, sufficient) conditions are found for the range of the H^∞ calculus, $H^\infty(T)$, and the commutant, $\{T\}'$, to contain non-zero compact operators, and for the finite rank operators of $\{T\}'$ to be dense in the set of compact operators of $\{T\}'$. A sufficient condition is given for $\{T\}'$ to contain non-zero operators from the Schatten–von Neumann classes S_p.

1. Introduction. For a given Hilbert space contraction T, we study how "smooth" (compact, etc.) operators in the commutant $\{T\}' = \{A : AT = TA\}$ can be. The problem arises in several applications in control theory, vector-valued Hankel operators or the theory of model operators. Here it is treated in the framework of the Sz.-Nagy–Foiaş functional model and some answers are proposed in the language of the characteristic function Θ_T of the contraction T.

Let $\mathcal{H}$ be a separable Hilbert space, and $L(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$. Let $\mathcal{F}$ and S_∞ denote the subspaces of $L(\mathcal{H})$ consisting respectively of the finite-rank and compact operators. Let S_p $(0 < p < \infty)$ denote the Schatten–von Neumann class consisting of the compact operators on $\mathcal{H}$ for which the sequence of singular numbers belongs to l^p. Only completely non-unitary (c.n.u.) contractions are considered. The questions studied in this paper are the following. Let T be a c.n.u. contraction on $\mathcal{H}$. When does $\{T\}'$ contain non-zero finite-rank operators, non-zero compact operators, or non-zero operators from the class S_p? When is $\{T\}' \cap \mathcal{F}$ dense in $\{T\}' \cap S_\infty$? Some of these questions are also considered for the range space of the functional calculus $H^\infty(T) = \{\varphi(T) : \varphi \in H^\infty\}$ instead of $\{T\}'$. Clearly, $H^\infty(T) \subset \{T\}'$.

For contractions of some specific classes, several facts are known about the above problems. For instance, for operators from the class C_0 (that is, for c.n.u. contractions T for which the H^∞ calculus $\varphi \mapsto \varphi(T)$ has a non-zero kernel) the following is proved:

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(i) (Sz.-Nagy, [SN74]) Always, $\{T\}' \cap S_\infty \neq \{0\}$, but it may happen that $H^\infty(T) \cap S_\infty = \{0\}$.

(ii) (Nordgren, [Nor75]) If $I - T^*T \in S_\infty$ then $H^\infty(T) \cap S_\infty \neq \{0\}$; and moreover, there exists a sequence $(\varphi_n)_{n\geq 1} \subset H^\infty$ such that $\|\varphi_n\|_\infty \leq 1$, $\varphi_n(T) \in S_\infty$ for $n \geq 1$, and (WOT)-$\lim \varphi_n(T) = I$. Here I denotes the identity operator and WOT stands for Weak Operator Topology.

Before answering the above questions, we recall some elements of the Sz.-Nagy–Foiaş model. First, it is worth mentioning that every Hilbert space contraction is an orthogonal sum of a unitary operator and a c.n.u. contraction, and that for the unitary part the questions related to the commutant and functional calculus can be easily answered via the von Neumann spectral theorem. In what follows, $\mathbb{T}$ is the unit circle of the complex plane, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\mathbb{D}$ is the unit disc, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

For a given c.n.u. contraction T, the main object of the functional model approach is the *characteristic function* $\Theta = \Theta_T$ defined by

$$\Theta(z) = [-T + zD_{T^*}(I - zT^*)^{-1}D_T]|_{\mathcal{D}_T}, \qquad z \in \mathbb{D}.$$

Here $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$ are the *defect operators* of T, $\mathcal{D}_T = \operatorname{clos} D_T\mathcal{H}$ and $\mathcal{D}_{T^*} = \operatorname{clos} D_{T^*}\mathcal{H}$ are the *defect spaces* of T. In fact, Θ is an analytic contractive-valued ($\|\Theta(z)\| \leq 1$, $z \in \mathbb{D}$) function from $\mathcal{D}_T$ to $\mathcal{D}_{T^*}$; in particular, Θ belongs to $H^\infty(L(\mathcal{D}_T, \mathcal{D}_{T^*}))$. The main theorem of the model theory says that T is unitarily equivalent to the *model operator* M_Θ defined on the *model space* K_Θ by the following formulas:

$$K_\Theta = \begin{pmatrix} H^2(\mathcal{D}_{T^*}) \\ \operatorname{clos} \Delta L^2(\mathcal{D}_T) \end{pmatrix} \ominus \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(\mathcal{D}_T),$$

$$M_\Theta \in L(K_\Theta), \qquad M_\Theta f = P_\Theta z f, \quad f \in K_\Theta.$$

Here, for a Hilbert space E, $L^2(E)$ denotes the Bochner–Lebesgue space of square integrable E-valued strongly measurable functions on $\mathbb{T}$; $H^2(E)$ denotes the Hardy space of E-valued analytic functions, $H^2(E) \subset L^2(E)$; Δ is the defect operator of Θ defined by $\Delta(\xi) = (I - \Theta(\xi)^*\Theta(\xi))^{1/2}$, which for almost every $\xi \in \mathbb{T}$ is a bounded $L(\mathcal{D}_T)$-valued function on $\mathbb{T}$, that is, $\Delta \in L^\infty(L(\mathcal{D}_T))$; and, finally, P_Θ denotes the orthogonal projection from $L^2(\mathcal{D}_{T^*} \oplus \mathcal{D}_T)$ onto K_Θ. More details on the model operators are given in Section 1 below and a complete exposition can be found in the book [SNF67]. In principle, the notation here follows that of this book.

Some of our answers are valid for the general situation of an arbitrary c.n.u. contraction T, others for particular classes of contractions, mainly in the case when the characteristic function has a scalar multiple. The following theorems are the main results of this paper. T is always a c.n.u. contraction. Inner and outer functions and the different classes of contractions are defined in Subsection 2.2 below.

THEOREM. *Assume that* $H^\infty(T) \cap S_\infty \neq \{0\}$. *Then* Θ_T *is a two-sided inner function. In the case when* T *is an* (SM)*-contraction and* $I - T^*T \in S_\infty$ *the converse is also true.*

This theorem reflects a feeling that the outer factor of Θ_T corresponds to a part of the operator similar in some sense to a unitary one. It is well known that, when a unitary operator has absolutely continuous spectrum, the only compact operator in the commutant is zero.

The following result means that only the trace-class smoothness of D_T^2 can guarantee the existence of non-zero compact H^∞ functions of T. Here $\sigma(T)$ denotes the spectrum of T, and $\sigma_{\mathrm{p}}(T)$ the point spectrum of T, that is, the set of eigenvalues of T.

THEOREM. *Let* $\mathfrak{S} \subset S_\infty$ *be a symmetrically normed ideal of* $L(\mathcal{H})$. *The following are equivalent*:

(i) *For every c.n.u. contraction* $T \in C_{00}$ *such that* $\mathbb{D} \backslash \sigma(T)$ *is non-empty and* $I - T^*T \in \mathfrak{S}$, *we have*

$$H^\infty(T) \cap S_\infty \neq \{0\}.$$

(ii) $\mathfrak{S} = S_1$.

Passing to the commutant $\{T\}'$ we first prove that if $I - T^*T \in S_\infty$ and $\mathbb{D} \setminus \sigma(T) \neq \emptyset$, then $\{T\}' \cap \mathcal{F} \neq \{0\}$ if and only if $\mathbb{D} \cap \sigma(T)$ is non-empty. Next, we obtain the following criterion.

THEOREM. *Let* $T \in L(\mathcal{H})$ *be a c.n.u. contraction. If* $T \in C_{1.} \cup C_{.1}$ (*equivalently*, Θ_T *is either outer or* $*$*-outer*) *then* $\{T\}' \cap S_\infty = \{0\}$. *If*, *moreover*, $T \in$ (SM) *and* $I - T^*T \in S_\infty$, *then the converse is also true*, *and in fact*, $\{T\}' \cap S_\infty = \{0\}$ *implies that* $T \in C_{11}$ (*equivalently*, Θ_T *is two-sided outer*).

In particular, if $\Theta \in H^\infty$ is a non-zero contractive-valued (scalar) function, then $\{M_\Theta\}' \cap S_\infty = \{0\}$ if and only if Θ is outer, and $H^\infty(M_\Theta) \cap S_\infty = \{0\}$ if and only if Θ is not inner.

Next, we pass to the question of the density of $\{T\}' \cap \mathcal{F}$ in $\{T\}' \cap S_\infty$ for $T \in$ (SM) such that $I - T^*T \in S_\infty$. It can be formulated in terms of the restriction T_0 of T to the invariant subspace $\mathcal{H}_0(T) = \{x \in \mathcal{H} : \|T^n x\| \to 0\}$. Then T_0 is a C_0-contraction and density holds if and only if T_0 is complete (equivalently m_{T_0} is a Blaschke product). In this case we always have a linear approximation process, and $\mathcal{H}_0(T)$ coincides with $E_T = \operatorname{clos}(\bigcup\{X\mathcal{H} : X \in \{T\}' \cap S_\infty\})$.

It is likely that $\mathcal{H}_0(T) = E_T$ for any (SM)-contraction with $I - T^*T \in S_\infty$. But at the moment we can only prove that $m_{T_0} = m_{T_E}$, where $T_E = T|_{E_T}$.

The last result deals with operators in the Schatten–von Neumann ideals S_p.

THEOREM. *Let T be an* (SM)*-contraction such that $I - T^*T \in S_p$ with $1 \le p < \infty$. Then $\{T\}' \cap S_p \neq \{0\}$ as soon as one of the following properties is satisfied*:

(i) $\sigma_{\mathrm{p}}(T) \neq \emptyset$, *or equivalently* $\ker \Theta_T(\lambda) \neq \{0\}$ *for some $\lambda \in \mathbb{D}$.*

(ii) *There exists a Beurling–Carleson set $\sigma \subset \mathbb{T}$ such that $H(\sigma) \neq \{0\}$, where $H(\sigma)$ stands for the maximal spectral subspace over σ* (*see Section* 5 *for definitions*)*.*

Moreover if $T \in C_0$ then $H^\infty(T) \cap S_p \neq \{0\}$.

The techniques used for the proofs of the above results are mostly based on the Commutant Lifting Theorem (CLT for short), the cornerstone of the theory of model operators. Via the CLT, the problems are reduced to certain questions about vector-valued Hankel operators. In the case of two-sided inner characteristic functions this reduction was known long ago (see [Nik86]). For a more general case, a new formula is established below to link compact operators in the commutant and Hankel operators (see Lemma 4.4). Then Muhly's and Peller's theories of smooth Hankel operators are used.

The paper is organized as follows. Section 2 contains necessary prerequisites on the Sz.-Nagy–Foiaş functional model. Section 3 is devoted to smooth operators in $H^\infty(T)$. Section 4 deals with compact and finite rank operators in $\{T\}'$, and Section 5 is devoted to the Schatten–von Neumann classes S_p.

2. Some facts about the canonical model. Let $\Theta \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ be any contractive-valued function, where $\mathcal{H}_1$, $\mathcal{H}_2$ are two separable Hilbert spaces. Then Θ is called *pure* if $\|\Theta(0)x\| < \|x\|$ for all $x \in \mathcal{H}_1$, $x \neq 0$. For every contractive-valued $\Theta \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$, there exists a unique pure contractive-valued function Θ^0 and a constant unitary operator U acting between certain subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively such that $\Theta(z) = \Theta^0(z) \oplus U$. This Θ^0 is called the *pure part* of Θ. Let $\Theta \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ be a contractive-valued function, and M_Θ be defined as in Section 1. Then M_Θ is a c.n.u. contraction and the characteristic function of M_Θ coincides with the pure part of Θ.

2.1. *The commutant lifting theorem (CLT).* Here T is identified with M_Θ. It is clear that for every $\varphi \in H^\infty$ the lifting formula $\varphi(M_\Theta) = P_\Theta \varphi|_{K_\Theta}$ holds. As already mentioned, $H^\infty(M_\Theta) \subset \{M_\Theta\}'$. It is known that $\{M_\Theta\}' = H^\infty(M_\Theta)$ when Θ is a scalar inner function [Sar67]. In general, this is not the case. The characterization of the c.n.u. contractions T such that $H^\infty(T) = \{T\}'$ seems to be unknown and is a delicate problem. However the above lifting formula extends to operators from $\{M_\Theta\}'$ thanks to the CLT due to Sz.-Nagy and Foiaş [SNF67]. Namely, $X \in \{M_\Theta\}'$ if and only if there exists an operator Y acting on $\binom{H^2(\mathcal{D}_{T^*})}{\operatorname{clos} \Delta L^2(\mathcal{D}_T)}$ such that $X = P_\Theta Y|_{K_\Theta}$, $Yz = zY$, and

$$Y\begin{pmatrix}\Theta\\ \Delta\end{pmatrix}H^2(\mathcal{D}_T)\subset\begin{pmatrix}\Theta\\ \Delta\end{pmatrix}H^2(\mathcal{D}_T).$$

Then Y is called a *lifting* of X. The space $\left(\begin{smallmatrix}H^2(\mathcal{D}_{T^*})\\ \operatorname{clos}\Delta L^2(\mathcal{D}_T)\end{smallmatrix}\right)$ is actually the space of the minimal isometric dilation of M_Θ.

Notice that in the case when Θ is a two-sided inner function (see below) the two lifting conditions are equivalent to $Y\in H^\infty(L(\mathcal{D}_{T^*}))$ and $\Theta^*Y\Theta\in H^\infty(L(\mathcal{D}_T))$ and the CLT admits an alternative proof due to N. K. Nikolski [Nik86], making use of Hankel operators. Moreover, if Y is a lifting of X, then $X=\Theta H_{\Theta^*Y}|_{K_\Theta}$, as in this case $P_\Theta=\Theta P_-\Theta^*$.

It follows that a lifting Y is an operator of multiplication by a block matrix function of the type

$$Y=\begin{pmatrix}A_1 & 0\\ B_1 & C_1\end{pmatrix},\tag{1}$$

where

$$A_1\in H^\infty(L(\mathcal{D}_{T^*})),\quad B_1\in L^\infty(L(\mathcal{D}_{T^*},\operatorname{clos}\Delta\mathcal{D}_T)),\quad C_1\in L^\infty(L(\operatorname{clos}\Delta\mathcal{D}_T))$$

are operator-valued functions satisfying the following relations:

$$A_1(\xi)\Theta(\xi)=\Theta(\xi)A_0(\xi),\qquad B_1(\xi)\Theta(\xi)+C_1(\xi)\Delta(\xi)=\Delta(\xi)A_0(\xi)$$

a.e. on $\mathbb{T}$, for some $A_0\in H^\infty(L(\mathcal{D}_T))$.

Any lifting of the zero operator is of the form $Y=\left(\begin{smallmatrix}\Theta G & 0\\ \Delta G & 0\end{smallmatrix}\right)$ for some $G\in H^\infty(L(\mathcal{D}_{T^*},\mathcal{D}_T))$. Moreover,

$$\begin{aligned}\|X\|&=\inf\{\|Y\|_\infty : Y \text{ a lifting of } X\}\\ &=\inf\left\{\left\|Y_0+\begin{pmatrix}\Theta G & 0\\ \Delta G & 0\end{pmatrix}\right\| : G\in H^\infty(L(\mathcal{D}_{T^*},\mathcal{D}_T))\right\},\end{aligned}$$

where Y_0 is any lifting of Y; the infimum is always attained. Notice that a lifting of a function of M_Θ, $X=\varphi(M_\Theta)\in\{M_\Theta\}'$, $\varphi\in H^\infty$, corresponds to $B_1=0$, $A_1=\varphi I$, $C_1=\varphi I$.

Another parametrization of the liftings of the operators in $\{M_\Theta\}'$ is

$$Y=\begin{pmatrix}A_* & 0\\ \Delta A\Theta^*+B\Delta_* & \Delta A\Delta-B\Theta\end{pmatrix},\tag{2}$$

where $A\in H^\infty(L(\mathcal{D}_T))$, $A_*\in H^\infty(L(\mathcal{D}_{T^*}))$ satisfy $\Theta A=A_*\Theta$, and $B\in L^\infty(L(\Delta_*\mathcal{D}_{T^*},\Delta\mathcal{D}_T))$; see [NV98].

2.2. *Classes of contractions*

$C_{\alpha\beta}$ classes. Let T be a c.n.u. contraction. Then T is of class $C_{0.}$ if T^n tends SOT (Strong Operator Topology) to zero (i.e. $\lim_{n\to\infty}\|T^nx\|=0$ for every $x\in\mathcal{H}$), and T is $C_{1.}$ if $\|T^nx\|$ does not tend to 0 for every $x\neq 0$. For

$\alpha, \beta = 0, 1$, the contraction T is $C_{.\alpha}$ if T^* is $C_{\alpha.}$, and T is $C_{\alpha\beta}$ if it is both $C_{\alpha.}$ and $C_{.\beta}$.

Let Θ be the characteristic function of T. Then T is $C_{0.}$ (resp. $C_{.0}$, $C_{1.}$, $C_{.1}$) if Θ is $*$-inner (resp. inner, $*$-outer, outer). Recall that a function $F \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ is *inner* if the non-tangential limits on $\mathbb{T}$ are isometric almost everywhere. It is *outer* if $FH^2(\mathcal{H}_1)$ is dense in $H^2(\mathcal{H}_2)$, where F is identified with the operator of multiplication by F. The function F is *$*$-inner* (resp. *$*$-outer*) if F^{t} is inner (resp. outer), where $F^{\mathrm{t}}(z) = F(\bar{z})^*$. It is *two-sided inner* (resp. *two-sided outer*) if F is both inner and $*$-inner (resp. outer and $*$-outer). Every function $F \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ admits a canonical inner-outer factorization $F = F_{\mathrm{inn}} F_{\mathrm{out}}$ through an intermediate Hilbert space (and consequently, also a canonical $*$-outer-$*$-inner factorization).

Let $T \in L(\mathcal{H})$ be a contraction. Then $\mathcal{H}_0 = \{x \in \mathcal{H} : \|T^n x\| \to 0\}$ is an invariant subspace of T and the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ induces a triangular decomposition of T, which is called the $C_{0.}$-$C_{1.}$ *decomposition* of T:

$$\begin{pmatrix} T_0 & * \\ 0 & T_1' \end{pmatrix}, \qquad T_0 \in C_{0.},\ T_1' \in C_{1.}. \tag{3}$$

It is the only decomposition of T satisfying (3). Applying the result to T^*, we find that T admits a unique triangular decomposition of the form

$$\begin{pmatrix} T_1 & * \\ 0 & T_0' \end{pmatrix}, \qquad T_1 \in C_{.1},\ T_0' \in C_{.0}, \tag{4}$$

where $T_1 = T|_{\mathcal{H}_1}$ and $\mathcal{H}_1 = \{x \in \mathcal{H} : T^{*n} x \to 0\}^\perp$. It is called the $C_{.1}$-$C_{.0}$ *decomposition* of T.

C_0-contractions. Let $T \in L(\mathcal{H})$ be a contraction. By definition $T \in C_0$ if T is c.n.u. and there exists a function $u \in H^\infty$, $u \neq 0$, such that $u(T) = 0$. For every $T \in C_0$, there exists a *minimal* (*annihilating*) *function* m_T (unique within a unimodular constant); that is, m_T is an inner function such that $m_T(T) = 0$, and if $u(T) = 0$ for some $u \in H^\infty \setminus \{0\}$, then $u/m_T \in H^\infty$ (see [SNF67, Proposition III.4.4]).

Recall that the *spectrum* $\sigma(\varphi)$ of a contractive-valued function $\varphi \in H^\infty$ is defined by

$$\sigma(\varphi) = (\operatorname{clos}\{\lambda \in \mathbb{D} : \varphi(\lambda) = 0\}) \cup \operatorname{supp}(\Delta_\varphi|_{\mathbb{T}}) \cup \operatorname{supp}(\mu_\varphi^{\mathrm{s}}),$$

where $\Delta_\varphi = (1 - |\varphi|^2)^{1/2}$ and μ_φ^{s} is the singular measure on $\mathbb{T}$ associated to φ through its Nevanlinna–Riesz–Smirnov canonical factorization. Let $T \in L(\mathcal{H})$ be a C_0-contraction with minimal function m_T. Then $\sigma(T) = \sigma(m_T)$ (see [SNF67, Theorem III.5.1]). In particular, $\sigma_{\mathrm{p}}(T) = \sigma(m_T) \cap \mathbb{D} = \sigma(T) \cap \mathbb{D}$ consists of the zeros of m_T, thus $\sigma(T) \cap \mathbb{D}$ is a (possibly empty) Blaschke sequence. Recall also that a C_0-contraction T is *complete* (i.e. $\mathcal{H}$ is spanned by the generalized eigenvectors of T, see Subsection 4.3) if and only if m_T is a Blaschke product.

Scalar multiples and (SM)-*contractions.* A function $\Theta\in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ has a *scalar multiple* $\delta \in H^\infty$, $\delta \neq 0$, if there exists $\Omega \in H^\infty(L(\mathcal{H}_2, \mathcal{H}_1))$ such that $\Omega\Theta = \delta I$ and $\Theta\Omega = \delta I$. Obviously, we then have $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$ and Θ is invertible at every point $z \in \mathbb{D}$ such that $\delta(z) \neq 0$, and then

$$\Theta(z)^{-1} = \frac{1}{\delta(z)}\,\Omega(z).$$

In particular, Θ^{-1} is a meromorphic function in $\mathbb{D}$.

Let $\Theta \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ be a contractive-valued function having a scalar multiple δ. If $\Theta = \Theta_{\mathrm{inn}}\Theta_{\mathrm{out}}$ is the canonical inner-outer factorization of Θ, then Θ_{inn} and Θ_{out} admit respectively δ_{inn} and δ_{out} as scalar multiples, where δ_{inn} and δ_{out} are the inner and outer parts of δ. In particular, Θ_{inn} is two-sided inner and Θ_{out} is two-sided outer. Similarly Θ admits an outer-inner factorization, $\Theta = \Theta'_{\mathrm{out}}\Theta'_{\mathrm{inn}}$, where δ_{out} and δ_{inn} are scalar multiples of Θ'_{out} and Θ'_{inn} (see [SNF67, Theorem V.6.2]).

Let $T \in L(\mathcal{H})$ be a c.n.u. contraction. We write $T \in$ (SM) if Θ_T has a scalar multiple. If T has δ as a scalar multiple, then $\sigma_{\mathrm{p}}(T) = \sigma(T) \cap \mathbb{D} \subset \{\lambda \in \mathbb{D} : \delta(\lambda) = 0\}$, and thus $\sigma(T) \cap \mathbb{D}$ is a (possibly empty) Blaschke sequence; in particular $\mathbb{D} \setminus \sigma(T) \neq \emptyset$.

For $T \in$ (SM), the components T_0, T_1' of the $C_{0\cdot}$-$C_{1\cdot}$ decomposition (3) are (SM)-contractions and the $C_{0\cdot}$-$C_{1\cdot}$ decomposition is in fact a C_0-C_{11} decomposition. If δ is a scalar multiple of Θ_T, then δ_{inn} and δ_{out} are scalar multiples of T_0 and T_1'. Then m_{T_0} is the minimal scalar multiple of T_0. For this reason a scalar multiple δ of Θ_T is called *minimal* if its inner part δ_{inn} coincides with m_{T_0}. Similarly, the $C_{\cdot 1}$-$C_{\cdot 0}$ decomposition (4) of T is in fact a C_{11}-C_0 decomposition and $T_1, T_0' \in$ (SM). Moreover, the invariant subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ defined by (3) and (4) satisfy $\mathcal{H}_0 \cap \mathcal{H}_1 = \{0\}$ and $\operatorname{span}(\mathcal{H}_0, \mathcal{H}_1) = \mathcal{H}$; and we have $\sigma(T) = \sigma(T_0) \cup \sigma(T_1') = \sigma(T_1) \cup \sigma(T_0')$ (see [SNF67, Section VIII.2.1]). Moreover, the spectrum of a C_{11}-contraction in (SM) is contained in $\mathbb{T}$ (see [SNF67, Proposition VI.4.3]).

Weak contractions. $T \in L(\mathcal{H})$ is called a *weak contraction* if $\mathbb{D}\setminus\sigma(T) \neq \emptyset$ and $I - T^*T \in S_1$. A weak contraction $T \in L(\mathcal{H})$ belongs to the class (SM) ([SNF67, Theorem VIII.1.1]). The contractions T_0, T_1', T_1, T_0' appearing in the C_0-C_{11} and C_{11}-C_0 decompositions of T are all weak contractions (see [SNF67, Theorem VIII.2.1]).

3. Smooth operators in $H^\infty(T)$. In this section we first find some necessary conditions for the spaces $H^\infty(T) \cap S_\infty$ not to be reduced to $\{0\}$. Then we explore in which cases these conditions are sufficient. All contractions will be supposed to be c.n.u. Therefore we may work equivalently with T or its canonical model M_Θ, $\Theta = \Theta_T$, as T and M_Θ are unitarily equivalent.

LEMMA 3.1. *Let $\Theta \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ be a contractive-valued function, and P_Θ defined as in Section 1. For $g \in L^2(\mathcal{H}_1)$, we have*

$$\left\| P_\Theta \begin{pmatrix} 0 \\ g \end{pmatrix} \right\|^2 = \|g\|^2 - \|P_+ \Delta g\|^2 = \|\Theta g\|^2 + \|P_- \Delta g\|^2,$$

where P_+ denotes the Riesz projection (i.e. onto the analytic part), and $P_- = I - P_+$.

Proof. The well known block decomposition of P_Θ,

$$P_\Theta = \begin{pmatrix} P_+ - \Theta P_+ \Theta^* & -\Theta P_+ \Delta \\ -\Delta P_+ \Theta^* & I - \Delta P_+ \Delta \end{pmatrix}, \tag{5}$$

and the fact that Δ is a positive selfadjoint operator satisfying $\Delta^2 = I - \Theta^*\Theta$ lead to

$$\begin{aligned} \left\| P_\Theta \begin{pmatrix} 0 \\ g \end{pmatrix} \right\|^2 &= \|\Theta P_+ \Delta g\|^2 + \|(I - \Delta P_+ \Delta) g\|^2 \\ &= \|P_+ \Delta g\|^2 - \|\Delta P_+ \Delta g\|^2 + \|(I - \Delta P_+ \Delta) g\|^2 \\ &= \|P_+ \Delta g\|^2 - \|\Delta P_+ \Delta g\|^2 + \|g\|^2 - 2\|P_+ \Delta g\|^2 + \|\Delta P_+ \Delta g\|^2 \\ &= \|g\|^2 - \|P_+ \Delta g\|^2, \end{aligned}$$

and $\|g\|^2 - \|P_+ \Delta g\|^2 = \|\Theta g\|^2 + \|\Delta g\|^2 - \|\Delta g\|^2 + \|P_- \Delta g\|^2 = \|\Theta g\|^2 + \|P_- \Delta g\|^2$. ∎

LEMMA 3.2. *Let $\Theta \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ be a contractive-valued function, and $M_\Theta \in L(K_\Theta)$ the model operator associated with Θ. Assume that there exists $\varphi \in H^\infty$, $\varphi \neq 0$, such that $\varphi(M_\Theta)^* M_\Theta^{*n}$ tends SOT to 0. Then Θ is an inner function, or equivalently, M_Θ^{*n} tends SOT to 0.*

Notice that both M_Θ^n and M_Θ^{*n} always tend WOT to 0. Indeed, if $u, v \in K_\Theta \subset L^2(\mathcal{D}_T \oplus \mathcal{D}_{T^*})$, then

$$\langle M_\Theta^n u, v \rangle = \langle P_\Theta z^n u, v \rangle = \langle z^n u, v \rangle = \int_{\mathbb{T}} \xi^n \langle u(\xi), v(\xi) \rangle \, d\mu(\xi) = \widehat{w}(-n),$$

where $w = \langle u(\cdot), v(\cdot) \rangle \in L^1$; and therefore $\widehat{w}(-n) \to 0$ from the Riemann–Lebesgue lemma. Thus, for every c.n.u. contraction $T \in L(\mathcal{H})$, T^n and T^{*n} tend WOT to 0.

Proof of Lemma 3.2. Let $\varphi \in H^\infty$. First we compute the SOT limit of $\varphi(M_\Theta)^* M_\Theta^{*n}$ in the general case. Let $\binom{f}{g} \in K_\Theta$. Note that the function $z^n M_\Theta^{*n} \binom{f}{g}$ converges in $L^2(\mathcal{H}_2 \oplus \mathcal{H}_1)$ to the function $\binom{0}{g}$. This follows from

the facts that
$$M_\Theta^{*n}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} P_+\overline{z}^n f \\ \overline{z}^n g \end{pmatrix},$$
and that $\|P_+\overline{z}^n f\| = \sum_{k\geq n}\|\widehat{f}(k)\|^2$ tends to 0 in $L^2(\mathcal{H}_2)$. Then, for any $\varphi \in H^\infty$, $\big\|\varphi(M_\Theta)^* M_\Theta^{*n}\binom{f}{g}\big\|^2$ converges to the same limit as
$$\left\|P_\Theta \overline{\varphi}\begin{pmatrix} 0 \\ \overline{z}^n g \end{pmatrix}\right\|^2 = \|\overline{\varphi} g\|^2 - \|P_+ \Delta \overline{z}^n \overline{\varphi} g\|^2,$$
where the last equality comes from Lemma 3.1. As $\Delta\overline{\varphi} g \in L^2(\mathcal{H}_1)$, the term $\|P_+\Delta\overline{z}^n\overline{\varphi}g\|^2 = \|P_+\overline{z}^n\Delta\overline{\varphi}g\|^2$ tends to 0. Therefore $\big\|\varphi(M_\Theta)^* M_\Theta^{*n}\binom{f}{g}\big\|$ tends to $\|\overline{\varphi}g\|$.

Now suppose that φ is such that $\varphi(M_\Theta)^* M_\Theta^{*n}$ tends SOT to 0. Then, for all $\binom{f}{g} \in K_\Theta$, we have $\|\overline{\varphi}g\| = 0$, which implies that $g = 0$. It remains to show that this result forces Δ to be identically 0. Indeed, we then have
$$K_\Theta \subset \begin{pmatrix} H^2(\mathcal{H}_2) \\ 0 \end{pmatrix},$$
and therefore
$$K_\Theta^\perp \supset \begin{pmatrix} 0 \\ \operatorname{clos} \Delta L^2(\mathcal{H}_1) \end{pmatrix}.$$
As $K_\Theta^\perp = \binom{\Theta}{\Delta} H^2(\mathcal{H}_1)$ contains no non-zero $\overline{z}$-invariant subspace, we necessarily have $\Delta = 0$. ∎

THEOREM 3.3. *Let $T \in L(\mathcal{H})$ be a c.n.u. contraction. If $H^\infty(T) \cap S_\infty \neq \{0\}$, then $\Theta_T \in H^\infty(L(\mathcal{D}_T, \mathcal{D}_{T^*}))$ is a two-sided inner function, in particular $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ are of the same dimension.*

Proof. It suffices to show that $H^\infty(T) \cap S_\infty \neq \{0\}$ implies that Θ_T is inner. Then we can apply the result to T^* and use the fact that $\Theta_{T^*} = \Theta_T^{\mathrm{t}}$. Let $M_\Theta \in L(K_\Theta)$ be the model of T. As M_Θ is unitarily equivalent to T, $H^\infty(T)\cap S_\infty \neq \{0\}$ if and only if $H^\infty(M_\Theta)\cap S_\infty \neq \{0\}$. Now, let $\varphi(M_\Theta) \in S_\infty$, $\varphi \neq 0$. As M_Θ^{*n} tends WOT to 0, $\varphi(M_\Theta)^* M_\Theta^{*n}$ tends SOT to 0. From Lemma 3.2 we deduce that Θ is an inner function. ∎

We are now interested in a converse to Theorem 3.3. Some additional assumptions are necessary to prevent the spectrum from behaving badly. The following lemmas concerning C_0-contractions will be useful.

LEMMA 3.4. *Let $T \in L(\mathcal{H})$ be a C_0-contraction such that $\sigma(T)\cap\mathbb{D}$ is non-empty. If $\lambda \in \sigma(T)\cap\mathbb{D}$, then the Riesz projection $\mathcal{P}_\lambda$ belongs to $H^\infty(T)$.*

Proof. For an outline of proof, see [SNF67, Section III.7.1]. ∎

Now we consider C_0-contractions T with compact defect D_T (equivalently $I - T^*T \in S_\infty$). Such contractions are said to be *essentially uni-*

tary with respect to Fredholm theory (because $T \in C_0$, or more generally $\mathbb{D} \setminus \sigma(T) \neq \emptyset$, and $I - T^*T \in S_\infty$ imply that $T = U + K$, where U is unitary and K is compact).

Notice that for any c.n.u. contraction T such that $\mathbb{D} \setminus \sigma(T) \neq \emptyset$ and $I - T^*T \in S_\infty$, $\sigma(T) \cap \mathbb{D}$ coincides with $\sigma_{\mathrm{p}}(T)$ and consists of an at most countable sequence of normal (finite multiplicity) eigenvalues tending to $\mathbb{T}$. In particular, the set of so-called normal eigenvalues, $\sigma_{\mathrm{np}}(T)$, coincides with $\sigma_{\mathrm{p}}(T)$ and $\sigma(T) \cap \mathbb{D}$.

REMARK 3.5. If $T \in L(\mathcal{H})$ is a c.n.u. contraction such that $H^\infty(T) \cap \mathcal{F} \neq \{0\}$ then $T \in C_0$. Indeed, if $\varphi(T) \in \mathcal{F}$ with $\varphi \in H^\infty$, $\varphi \neq 0$, then the restriction of T to $\varphi(T)\mathcal{H}$ has a minimal annihilating function, say m, and $m\varphi$ is a non-zero H^∞ function annihilating T.

Now, we will see why, for an essentially unitary C_0-contraction T, we have $H^\infty(T) \cap S_\infty \neq \{0\}$. We start with a lemma which will have larger consequences. The method employed for the proof, which consists in using an outer function in $\mathcal{C}_A$ (H^∞ functions continuous on $\overline{\mathbb{D}}$) equal to zero, on a given set of Lebesgue measure zero, was introduced by B. Moore and E. Nordgren to study the existence of compact operators in the weakly closed algebra generated by an essentially unitary C_0-operator [MN75]. Another ingredient of this lemma consists in the characterization of compact Hankel operators in terms of their symbol; namely, for $\Phi \in L^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$, the Hankel operator H_Φ is compact if and only if $\Phi \in H^\infty(L(\mathcal{H}_1, \mathcal{H}_2)) + \mathcal{C}(\mathbb{T}, S_\infty(\mathcal{H}_1, \mathcal{H}_2))$, where $\mathcal{C}(\mathbb{T}, X)$ stands for the space of X-valued continuous functions on $\mathbb{T}$ (see [Muh69], [Muh71], [BP75]). Recall that H_Φ is defined by

$$H_\Phi : H^2(\mathcal{H}_1) \to H^2_-(\mathcal{H}_2) = L^2(\mathcal{H}_2) \ominus H^2(\mathcal{H}_2), \qquad h \mapsto P_- \Phi h.$$

LEMMA 3.6. *Let $T \in L(\mathcal{H})$ be a C_0-contraction such that $I - T^*T \in S_\infty$, and m_T the minimal function of T. Then there exists $\varphi \in H^\infty$ such that $\varphi(T) \neq 0$ and the Hankel operator $H_{\varphi\Theta_T^*}$ is compact. More precisely, we can take $\varphi = fm_T/m_1$, where m_1 is any non-trivial inner factor of m_T such that $\sigma(m_1) \cap \mathbb{T}$ is of Lebesgue measure zero, and f is any outer $\mathcal{C}_A$ function equal to zero on $\sigma(m_1) \cap \mathbb{T}$.*

Proof. Take a non-trivial factorization $m_T = m_1 m_2$ such that m_1, m_2 are inner and $\sigma(m_1) \cap \mathbb{T}$ is contained in a closed subset γ of $\mathbb{T}$ of Lebesgue measure $\mu(\gamma) = 0$. From a theorem of Fatou we obtain the existence of an outer function $f \in \mathcal{C}_A$ such that $f|_\gamma = 0$. Set $\varphi = fm_2$. Then $\varphi/m_T = f/m_1 \notin H^\infty$, and the minimality of m_T implies that $\varphi(T) \neq 0$. Now, there exists an invariant subspace $E \subset \mathcal{H}$ of T such that m_1 and m_2 annihilate $T_1 = T|_E \in L(E)$ and $T_2 = P_{E^\perp} T|_{E^\perp} \in L(E^\perp)$ respectively, where $P_{E^\perp}$ denotes the orthogonal projection from $\mathcal{H}$ onto $E^\perp$ (see [SNF67, Theorem

III.6.3]). Moreover, there exist $\Theta_1 \in H^\infty(L(\mathcal{D}_T,\mathcal{K}))$, $\Theta_2 \in H^\infty(L(\mathcal{K},\mathcal{D}_{T^*}))$, where $\mathcal{K}$ is an auxiliary separable Hilbert space, such that $\Theta = \Theta_2\Theta_1$ and Θ_{T_1} and Θ_{T_2} coincide with the pure part of Θ_1 and Θ_2 respectively (see [SNF67, Theorem VII.1.1 and Proposition VII.2.1]). Then, for $i = 1,2$, the function m_i is a scalar multiple of Θ_{T_i} (see [Ber88, Proposition V.3.2 and Corollary V.3.3]) and Θ_{T_i} is a two-sided inner function. The same properties hold for Θ_i as $\Theta_i = \Theta_{T_i} \oplus U_i$, where U_i is a unitary operator. Let $\Omega_1 \in H^\infty(L(\mathcal{K},\mathcal{D}_T))$ and $\Omega_2 \in H^\infty(L(\mathcal{D}_{T^*},\mathcal{K}))$ be such that $\Theta_1\Omega_1 = m_1 I$ and $\Theta_2\Omega_2 = m_2 I$. Now,

$$\varphi\Theta^* = f m_2 \Theta_1^*\Theta_2^* = f\Theta_1^*(m_2\Theta_2^*) = f\Theta_1^*\Omega_2.$$

The proof will be completed by showing that $f\Theta_1^* \in H^\infty(L(\mathcal{K},\mathcal{D}_T)) + \mathcal{C}(\mathbb{T}, S_\infty(\mathcal{K},\mathcal{D}_T))$, as this implies that $H_{f\Theta_1^*}$ is compact, and then so is $H_{f\Theta_1^*\Omega_2} = H_{\varphi\Theta^*}$. Compactness of $I - T^*T$ implies that $P_E(I - T^*T)|_E = I - T_1^*T_1$ is compact; and thus $\Theta_1(z) = A + K(z)$, $z \in \mathbb{D}$, where $A \in L(\mathcal{D}_T,\mathcal{K})$ and $K \in H^\infty(S_\infty(\mathcal{D}_T,\mathcal{K}))$. Therefore it suffices to show that $f\Theta_1^*$ considered as a function on $\mathbb{T}$ is continuous at every point of $\mathbb{T}$, as this fact implies that $K \in \mathcal{C}(S_\infty(\mathcal{D}_T,\mathcal{K}))$. As $\sigma(\Theta_{T_1}) = \sigma(T_1) = \sigma(m_1) \subset \gamma$, the function Θ_{T_1} has a holomorphic continuation at every $\xi \in \mathbb{T}\setminus\gamma$, and so does Θ_1. It follows that $f\Theta_1^*$ is continuous at every $\xi \in \mathbb{T}\setminus\gamma$. Eventually, if $\xi \in \gamma$, then $f(\xi) = 0$. As Θ_1^* is bounded on $\mathbb{T}$, the continuity of f implies that of $f\Theta_1^*$ at ξ. ∎

Before giving a direct consequence of Lemma 3.6 we deduce the following fact which will be useful later on (see Subsection 4.2).

LEMMA 3.7. *Let $T \in (\mathrm{SM})$, $T \notin C_{11}$, be such that $I - T^*T \in S_\infty$. Then there exists $\psi \in H^\infty$ such that $H_{\psi\Theta_T^{-1}}$ is compact but*

$$\psi\Theta_T^{-1} \notin H^\infty(L(\mathcal{D}_{T^*},\mathcal{D}_T)).$$

More precisely, we can take $\psi = f\delta/m_1$, where δ is a minimal scalar multiple of T, m_1 is any non-trivial inner factor of δ such that $\sigma(m_1)\cap\mathbb{T}$ is of Lebesgue measure zero, and f is any outer function in $\mathcal{C}_A$ equal to zero on $\sigma(m_1)\cap\mathbb{T}$.

Proof. Denote by T_0 the C_0-part of T appearing in the C_0-C_{11} decomposition of T. This T_0 is the restriction of T to $\mathcal{H}_0 \neq \{0\}$ as $T \notin C_{11}$. Therefore $\sigma(T_0) \subset \sigma(T)$ and $I - T_0^*T_0 = P_{\mathcal{H}_0}(I - T^*T)|_{\mathcal{H}_0}$ is compact. On the other hand, the characteristic function of T_0 is the pure part of Θ_{inn} (see [SNF67, Theorem VII.1.1 and Proposition VII.2.1]). If there exists $h \in H^\infty \setminus \{0\}$ such that $h(T) = 0$, then T is in fact C_0 (i.e. T coincides with T_0), and the result is given by Lemma 3.6. Thus we can suppose that $h(T) \neq 0$ for every $h \in H^\infty \setminus \{0\}$. Lemma 3.6 applied to T_0 gives $\varphi = f m_{T_0}/m_1$ such that $H_{\varphi\Theta_{T_0}^*}$ is compact. Let $\Theta_T = \Theta_{\mathrm{out}}\Theta_{\mathrm{inn}}$ be the outer-inner factorization of Θ_T. In fact Θ_{T_0} coincides with the pure part of Θ_{inn} (see [SNF67, Theorem VII.1.1 and Proposition VII.2.1]); thus $H_{\varphi\Theta^*_{\mathrm{inn}}}$ is also compact. Let δ be a minimal scalar multiple of Θ_T, that is, $\delta = m_{T_0}\delta_{\mathrm{out}}$ with δ_{out} outer. Set

$$\psi = \delta_{\mathrm{out}}\varphi = f\,\frac{m_{T_0}}{m_1}\,\delta_{\mathrm{out}} = f\,\frac{\delta}{m_1}.$$

Then $\psi\Theta_T^{-1} = \varphi\Theta^*_{\mathrm{inn}}\delta_{\mathrm{out}}\Theta^{-1}_{\mathrm{out}}$; thus $H_{\psi\Theta_T^{-1}}$ is compact. Further, if we suppose that $\psi\Theta_T^{-1} = F \in H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T))$, then $\psi I = \Theta_T T = F\Theta_T$. Therefore, $\psi = \delta_{\mathrm{out}} m_{T_0}/m_1$ is a scalar multiple of Θ_T, which is impossible since $\delta_{\mathrm{out}} m_{T_0}$ is a minimal scalar multiple. ■

The first part of the following theorem is mostly known [Nor75] but we give a proof for the sake of completeness.

THEOREM 3.8. *Let $T \in L(\mathcal{H})$ be a C_0-contraction such that $I - T^*T \in S_\infty$. Then $H^\infty(T) \cap S_\infty \neq \{0\}$. Moreover, $H^\infty(T) \cap \mathcal{F} \neq \{0\}$ if and only if $\sigma(T) \cap \mathbb{D} \neq \emptyset$.*

Proof. To prove the first assertion we use Lemma 3.6 and get a function $\varphi \in H^\infty$ such that $\varphi(T) \neq 0$ and $H_{\varphi\Theta_T^*}$ is compact. As T is C_0, its characteristic function Θ is two-sided inner. Then $P_\Theta = \Theta P_-\Theta^*$ and, identifying T with its model operator M_Θ, we deduce that $\varphi(T) = \Theta H_{\varphi\Theta_T^*}|_{K_\Theta}$ is compact. Now prove the second assertion. If $H^\infty(T) \cap \mathcal{F} \neq \{0\}$, then T has a non-trivial finite-dimensional invariant subspace, and therefore an eigenvalue. The converse is an immediate consequence of Lemma 3.4. Indeed, if $\lambda \in \sigma(T) \cap \mathbb{D}$, then $\mathcal{P}_\lambda \in H^\infty(T) \cap \mathcal{F}$.

Yet another proof is to observe directly that

$$m_T(T) = b_\lambda(T)\,\frac{m_T}{b_\lambda}(T) = 0,$$

whence

$$\frac{m_T}{b_\lambda}(T) \subset \ker b_\lambda(T),$$

and $\ker b_\lambda(T)$ is finite-dimensional as λ is necessarily of finite type. Here b_λ denotes the elementary Blaschke factor,

$$b_\lambda(z) = \frac{|\lambda|}{\lambda}\,\frac{\lambda - z}{1 - \overline{\lambda}z}. \ ■$$

COROLLARY 3.9. *Let $T \in L(\mathcal{H})$ be an* (SM)*-contraction such that $I - T^*T \in S_\infty$. Then $H^\infty(T) \cap S_\infty \neq \{0\}$ if and only if Θ_T is a two-sided inner function.*

Proof. If Θ_T is inner, then $T \in C_0$ and the result follows from Theorem 3.8. The converse is a consequence of Theorem 3.3. ■

Now, we consider C_{00}-contractions T (that is, contractions having a two-sided inner characteristic function) subject to restrictions of "smoothness" of the defect operator $D_T = (I - T^*T)^{1/2}$. The following result means that only the trace class smoothness of D_T^2 can guarantee the existence of compact H^∞ functions of T.

THEOREM 3.10. *Let $\mathfrak{S} \subset S_\infty$ be a symmetrically normed ideal of $L(\mathcal{H})$. The following are equivalent*:

(i) *For every c.n.u. contraction $T \in C_{00}$ such that $\mathbb{D} \setminus \sigma(T) \neq \emptyset$ and $I - T^*T \in \mathfrak{S}$, we have*

$$H^\infty(T) \cap S_\infty \neq \{0\}.$$

(ii) $\mathfrak{S} = S_1$.

We refer to [GK69] for properties of symmetrically normed ideals of $L(\mathcal{H})$. For the proof of Theorem 3.10 we need the following notion. Let $\sigma \subset \mathbb{D}$. Then σ is called a *determining subset* for (the H^∞ norm on) $\mathbb{D}$ if $\|f\|_\infty = \sup_{z\in\sigma} |f(z)|$ for all $f \in H^\infty = H^\infty(\mathbb{D})$. The following lemma is a result by N. K. Nikolski and S. A. Vinogradov (see [Nik71]).

LEMMA 3.11. *Suppose $0 < r_n < 1$ $(n \geq 1)$ satisfy $\lim r_n = 1$. There exists a determining sequence for $\mathbb{D}$, say $(\lambda_n)_{n\geq 1}$, such that $|\lambda_n| = r_n$ $(n \geq 1)$ if and only if $\sum_{n\geq 1}(1 - r_n) = \infty$.*

Proof of Theorem 3.10. The fact that property (i) is true when $\mathfrak{S} = S_1$ can be deduced from Theorem 3.8. Indeed, in this case T is a weak contraction in C_{00}, and therefore $T \in C_0$. To prove that (i) implies (ii) suppose that $\mathfrak{S} \neq S_1$. Take $A \in \mathfrak{S} \setminus S_1$ and let $A = \sum_{n\geq 0} s_n \langle \cdot, x_n\rangle y_n$ be a Schmidt decomposition of A, that is, $(s_n)_{n\geq 0}$ is a sequence of positive numbers decreasing to 0, and $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are orthonormal families in $\mathcal{H}$. As $A \notin S_1$, we have $\sum_{n\geq 0} s_n = \infty$, and hence $s_n \neq 0$ for all $n \geq 0$. With $\mathfrak{S}$ being an ideal, we can suppose that $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ coincide with the same orthonormal basis $(e_n)_{n\geq 0}$ of $\mathcal{H}$. Define $T = \operatorname{diag}(\lambda_n)_{n\geq 0}$ with respect to this basis, where we choose $(\lambda_n)_{n\geq 0} \subset \mathbb{D}$ to be a determining subset for $\mathbb{D}$ satisfying $|\lambda_n|^2 = 1 - s_n$, which is possible due to Lemma 3.11, where we take $r_n = \sqrt{1 - s_n}$. Under these conditions, T is a c.n.u. C_{00}-contraction and $I - T^*T = \sum_{n\geq 0} s_n \langle \cdot, e_n\rangle e_n \in \mathfrak{S}$. But if $\varphi \in H^\infty$, then $\varphi(T) = \operatorname{diag}(\varphi(\lambda_n))_{n\geq 0}$, thus $\varphi(T) \in S_\infty$ if and only if $\lim_{n\to\infty} \varphi(\lambda_n) = 0$, which implies that $\varphi = 0$. Indeed, if $(\lambda_n)_{n\geq 0} \subset \mathbb{D}$ is a determining sequence for $\mathbb{D}$, then so is $(\lambda_n)_{n\geq N} \subset \mathbb{D}$ for every $N \geq 0$. Therefore $\|\varphi\|_\infty = \sup_{n\geq N} |\varphi(\lambda_n)|$, which tends to 0 as $N \to \infty$. ∎

4. Smooth operators in the commutant

4.1. *Finite-rank operators in the commutant*

THEOREM 4.1. *Let $T \in L(\mathcal{H})$ be a c.n.u. contraction such that $\mathbb{D} \setminus \sigma(T) \neq \emptyset$. Suppose $I - T^*T \in S_\infty$. Then $\{T\}' \cap \mathcal{F} \neq \{0\}$ if and only if $\mathbb{D} \cap \sigma(T)$ is non-empty. The condition $\mathbb{D} \setminus \sigma(T) \neq \emptyset$ cannot be omitted.*

Proof. First suppose that $\{T\}' \cap \mathcal{F} \neq \{0\}$. Let $A \in \{T\}' \cap \mathcal{F}$, $A \neq 0$. Set $E = A\mathcal{H} \subset \mathcal{H}$. Then $E \neq \{0\}$ is a finite-dimensional invariant subspace of T. Therefore, $T|_E \in L(E)$ admits an eigenvalue λ. Then λ is also an eigenvalue of T and has to be in $\mathbb{D}$.

Now suppose $\mathbb{D} \cap \sigma(T) \neq \emptyset$. Then $\sigma_{\mathrm{np}}(T) = \sigma_{\mathrm{p}}(T) = \mathbb{D} \cap \sigma(T) \neq \emptyset$, therefore there exists $\lambda \in \mathbb{D}$ which is a normal eigenvalue of T. Consequently, the corresponding Riesz projection $\mathcal{P}_\lambda \neq 0$ is of finite rank. As $\mathcal{P}_\lambda$ belongs to $\{T\}'$, we deduce that $\{T\}' \cap \mathcal{F}$ cannot be reduced to $\{0\}$.

Finally, consider the case when $T = S$, the shift operator on H^2. Then $I - S^*S = 0 \in S_\infty$ but $\sigma(T) \supset \mathbb{D}$. Moreover $\{S\}' = H^\infty(S)$ obviously contains no non-zero finite-rank operator, and even no non-zero compact operator. ■

4.2. *Compact operators in the commutant.* First, we note that the condition $\{T\}' \cap S_\infty \neq \{0\}$ is not in general sufficient for Θ_T to be a two-sided inner function (in contrast to the case of $H^\infty(T)$, see Theorem 3.3). For example, it is easy to see that if Θ is a scalar H^∞ function and $\Theta = 0$, then M_Θ can be identified with $S \oplus S^*$, where S is the shift operator on H^2, and

$$\begin{pmatrix} 0 & 0 \\ \Gamma_\varphi & 0 \end{pmatrix} \in \{M_\Theta\}' \cap S_\infty \quad \text{for every } \varphi \in H^\infty + \mathcal{C}(\mathbb{T})$$

where Γ_φ denotes the Hankel operator on H^2 with symbol φ. Thus we then have $\{T\}' \cap S_\infty \neq \{0\}$.

Now, we give a characterization of operators Y on $H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta L^2(\mathcal{D}_T)}$ given by formula (1) which are liftings of compact operators in $\{M_\Theta\}'$.

Lemma 4.2. *Let $T \in L(\mathcal{H})$ be a c.n.u. contraction and $\Theta = \Theta_T$. If $X \in \{M_\Theta\}'$ is represented via the CLT by $X = P_\Theta Y|_{K_\Theta}$, then X is compact if and only if the operator $P_\Theta Y$ acting on*

$$\begin{pmatrix} H^2(\mathcal{D}_{T^*}) \\ \operatorname{clos} \Delta L^2(\mathcal{D}_T) \end{pmatrix}$$

is compact.

Proof. This is an immediate consequence of the condition

$$Y \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(\mathcal{D}_T) \subset \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(\mathcal{D}_T). \ ■$$

Lemma 4.3. *Let $T \in L(\mathcal{H})$ be an* (SM)*-contraction and $\Theta = \Theta_T$. Let $X \in \{M_\Theta\}'$ be represented via the CLT by $X = P_\Theta Y|_{K_\Theta}$, where Y is the lifting of X with parameters A_1, B_1, C_1 given by formula* (1)*. If X is compact, then $C_1 = 0$.*

Proof. If X is compact, it follows from Lemma 4.2 that $P_\Theta Y$ is compact. Then $P_\Theta Y|_{0\oplus\overline{\Delta L^2(\mathcal{D}_T)}}$ is also compact. If $g \in \overline{\Delta L^2(\mathcal{D}_T)}$, then

$$Y\begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ C_1 g \end{pmatrix}.$$

According to Lemma 3.1 we have

$$\left\| P_\Theta Y \begin{pmatrix} 0 \\ g \end{pmatrix} \right\|^2 = \|\Theta C_1 g\|^2 + \|P_- \Delta C_1 g\|^2 \geq \|\Theta C_1 g\|^2.$$

Therefore the map $g \in \overline{\Delta L^2(\mathcal{D}_T)} \mapsto \Theta C_1 g$ is compact. Let $\delta \in H^\infty$ be a scalar multiple of Θ, and $\Omega \in H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T))$ be such that $\Omega\Theta = \delta$. Then $g \mapsto \Omega\Theta C_1 g = \delta C_1 g$ is compact, which implies that $C_1(\xi)\Delta(\xi) = 0$ for a.e. $\xi \in \mathbb{T}$. Indeed, if there exists g_0 such that $\|\delta(\xi)C_1(\xi)g_0(\xi)\| \geq \varepsilon > 0$ on a set of positive measure then the operator $h \mapsto \delta C_1 g_0 h$ cannot be compact. Therefore $C_1 = 0$ as $C_1 \in L^\infty(L(\overline{\Delta L^2(\mathcal{D}_T)}))$. ∎

LEMMA 4.4. *Let $T \in L(\mathcal{H})$ be an* (SM)*-contraction and $\Theta = \Theta_T$. Let $X \in \{M_\Theta\}'$ be represented via the CLT by $X = P_\Theta Y|_{K_\Theta}$, where Y is the lifting of X with parameters A_1, B_1, C_1 given by* (1)*. If $C_1 = 0$ then*

$$P_\Theta Y = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} P_- \Theta^{-1} A_* P_*, \tag{6}$$

where P_* stands for the orthogonal projection from $H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta L^2(\mathcal{D}_T)}$ onto $H^2(\mathcal{D}_{T^*})$ and $A_* \in H^\infty(L(\mathcal{D}_{T^*}))$ is such that $\Theta^{-1}A_*\Theta \in H^\infty(L(\mathcal{D}_T))$. Conversely, if $P_\Theta Y$ satisfies (6)*, with $A_* \in H^\infty(L(\mathcal{D}_{T^*}))$ such that $\Theta^{-1}A_*\Theta \in H^\infty(L(\mathcal{D}_T))$, then $X = P_\Theta Y|_{K_\Theta} \in \{M_\Theta\}'$.*

Proof. Suppose $X \in \{M_\Theta\}' \cap S_\infty$ and Y is a lifting of X with parameters A, A_* and B given by (2). The fact that $C_1 = 0$ in (1) means that $\Delta A \Delta = B\Theta$. As $T \in$ (SM), Θ is invertible a.e. on $\mathbb{T}$ and we can write $B = \Delta A \Delta \Theta^{-1}$. Then we use the following two intertwining relations: $\Theta A = A_*\Theta$ and $\Theta\Delta = \Delta_*\Theta$. We deduce that $B = \Delta A \Theta^{-1}\Delta_* = \Delta\Theta^{-1}A_*\Delta_*$ and $B\Delta_* = \Delta A\Theta^{-1}(1-\Theta\Theta^*) = \Delta A(\Theta^{-1} - \Theta^*)$. Therefore $\Delta A\Theta^* + B\Delta_* = \Delta A\Theta^{-1} = \Delta\Theta^{-1}A_*$ and Y has the following form:

$$Y = \begin{pmatrix} A_* & 0 \\ \Delta\Theta^{-1}A_* & 0 \end{pmatrix}. \tag{7}$$

We compute $P_\Theta Y$ using the block matrix decomposition (5) and the following identities:

$$\begin{aligned} 1 - \Theta P_+ \Theta^* - \Theta P_+ \Delta^2 \Theta^{-1} &= \Theta(1 - P_+\Theta^*\Theta - P_+\Delta^2)\Theta^{-1} \\ &= \Theta(1 - P_+)\Theta^{-1} = \Theta P_- \Theta^{-1}, \\ -\Delta P_+ \Theta^* + (1 - \Delta P_+ \Delta)\Delta\Theta^{-1} &= \Delta(-P_+\Theta^*\Theta + 1 - P_+\Delta^2)\Theta^{-1} \\ &= \Delta(1 - P_+)\Theta^{-1}. \end{aligned}$$

The result follows. The converse is clear because Y, as defined by (7), with $A_* \in H^\infty(L(\mathcal{D}_{T^*}))$ such that $\Theta^{-1}A_*\Theta \in H^\infty(L(\mathcal{D}_{T^*}))$, satisfies the required conditions for Y to be a lifting. ∎

Corollary 4.5. *Let $T \in L(\mathcal{H})$ be an* (SM)*-contraction and $\Theta = \Theta_T$. Let $X \in \{M_\Theta\}'$. Then X is compact if and only if*

$$X = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} P_- \Theta^{-1} A_* P_*|_{K_\Theta} \tag{8}$$

for some $A_* \in H^\infty(L(\mathcal{D}_{T^*}))$ satisfying $\Theta^{-1}A_*\Theta \in H^\infty(L(\mathcal{D}_{T^*}))$ and $\Theta^{-1}A_* \in H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T)) + \mathcal{C}(\mathbb{T}, S_\infty(\mathcal{D}_{T^*}, \mathcal{D}_T))$, where P_* stands for the orthogonal projection from $H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta L^2(\mathcal{D}_T)}$ onto $H^2(\mathcal{D}_{T^*})$. In this case, $X \neq 0$ if and only if $\Theta^{-1}A_* \notin H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T))$.

Proof. If X is compact we get the expression of X by Lemmas 4.2 and 4.4. As the operator $\binom{\Theta}{\Delta}$ is an isometry the compactness of such an X is equivalent to the compactness of $H_{\Theta^{-1}A_*}$ and also to $\Theta^{-1}A_*$ belonging to $H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T)) + \mathcal{C}(\mathbb{T}, S_\infty(\mathcal{D}_{T^*}, \mathcal{D}_T))$. Moreover, $X = 0$ is equivalent to $P_-\Theta^{-1}A_* = 0$. Conversely, if X is defined by (8) the last assertion of Lemma 4.4 tells us that X is in $\{M_\Theta\}'$. ∎

Remark 4.6. The conclusions of Lemmas 4.3 and 4.4 and of Corollary 4.5 remain valid if we replace the condition $T \in$ (SM) by the weaker condition that $\Theta_T(\xi)$ be invertible a.e. on $\mathbb{T}$.

For $T \in$ (SM), the following theorem gives a necessary and sufficient condition for $\{T\}' \cap S_\infty \neq \{0\}$ in terms of Θ_T, namely Θ_T must be neither outer nor $*$-outer.

Theorem 4.7. *Let $T \in L(\mathcal{H})$ be a c.n.u. contraction. If $T \in C_{1.} \cup C_{.1}$* (*equivalently Θ_T is either outer or $*$-outer*) *then $\{T\}' \cap S_\infty = \{0\}$. If moreover $T \in$* (SM) *and $I - T^*T \in S_\infty$, then the converse is also true, and in fact, $\{T\}' \cap S_\infty = \{0\}$ implies that $T \in C_{11}$.*

Proof. To prove the first assertion suppose that there exists a non-zero $K \in \{T\}' \cap S_\infty$. As T^n tends WOT to 0, we deduce that KT^n tends SOT to 0. But $KT^n = T^nK$ for all $n \geq 0$. Therefore $T^n|_{\operatorname{Im} K}$ tends SOT to 0. As $\operatorname{Im} K \neq \{0\}$, $T \notin C_{1.}$. The same reasoning applies to T^*, thus $T^* \notin C_{1.}$, or equivalently $T \in C_{.1}$.

Now suppose that $T \in$ (SM), $I - T^*T \in S_\infty$ and $T \notin C_{11}$. As $T \in$ (SM), that means $T \notin C_{1.} \cup C_{.1}$. Therefore $\Theta = \Theta_T$ is not outer. By Corollary 4.5, it remains to find $A_* \in H^\infty(L(\mathcal{D}_{T^*}))$ such that $\Theta^{-1}A_*\Theta \in H^\infty(L(\mathcal{D}_T))$ and $\Theta^{-1}A_* \in H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T)) + \mathcal{C}(\mathbb{T}, S_\infty(\mathcal{D}_{T^*}, \mathcal{D}_T))$, but $P_-\Theta^{-1}A_* \neq 0$. But we know from Lemma 3.7 that there exists $u \in H^\infty$ such that $u\Theta^{-1} \in H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T)) + \mathcal{C}(S_\infty(\mathcal{D}_{T^*}, \mathcal{D}_T))$ with $u\Theta^{-1} \notin H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T))$. The choice of $A_* = uI$ gives the result. ∎

4.3. *Finite-rank approximation of compact operators in the commutant.* In this subsection we are interested in the density of finite-rank operators in the commutant of a given c.n.u. contraction $T \in L(\mathcal{H})$ in the space $\{T\}' \cap S_\infty$ of compact operators from the commutant.

If $B \in L(\mathcal{H})$ and $\lambda \in \sigma_{\mathrm{p}}(B)$, we denote by $C_\lambda(B) = \bigcup_{N\geq 1} \ker(B-\lambda)^N$ the root manifold corresponding to the eigenvalue λ of B. We denote by

$$C(B) = \bigcup_{\lambda\in\sigma_{\mathrm{p}}(B)} C_\lambda(B) \tag{9}$$

the set consisting of all *generalized eigenvectors* (root vectors) of B. Now B is said to be *complete* if the family of generalized eigenvectors of B is total in X, so that $X = \operatorname{clos} C(B)$.

Let $T \in L(\mathcal{H})$ be a c.n.u. contraction, and define E_T to be the total image of $\{T\}' \cap S_\infty$, that is,

$$E_T = \operatorname{clos}\Big(\bigcup\{X\mathcal{H} : X \in \{T\}' \cap S_\infty\}\Big). \tag{10}$$

Then E_T is hyperinvariant for T, that is, invariant for every $X \in \{T\}'$. We set $T_E = T|_{E_T}$.

LEMMA 4.8. *In the notation above, if $\{T\}' \cap \mathcal{F}$ is SOT dense in $\{T\}' \cap S_\infty$ then T_E is complete.*

Proof. Let $X \in \{T\}' \cap S_\infty$ and $(X_n)_{n\geq 1} \subset \{T\}' \cap \mathcal{F}$ be such that $X_n x \to Xx$ for all $x \in \mathcal{H}$. For every $n \geq 1$, $X_n\mathcal{H}$ is a finite-dimensional subspace of E_T, invariant for T and T_E. Therefore, $X_n\mathcal{H}$ is generated by some generalized eigenvectors of T_E. Thus $X_n\mathcal{H} \subset \operatorname{span} C(T_E)$ and $E_T \subset \operatorname{span} C(T_E)$. ∎

LEMMA 4.9. *Let $T \in L(\mathcal{H})$ be a c.n.u. contraction such that $\mathbb{D}\setminus\sigma(T) \neq \emptyset$ and $I - T^*T \in S_\infty$, $T_E = T|_{E_T}$. Then $\lambda \in \sigma_{\mathrm{p}}(T)$ if and only if $\lambda \in \sigma_{\mathrm{p}}(T_E)$ and for all $\lambda \in \sigma_{\mathrm{p}}(T)$, $C_\lambda(T) = C_\lambda(T_E)$. Therefore* $\operatorname{span} C(T_E) = \operatorname{span} C(T)$.

Proof. If $\mathbb{D} \setminus \sigma(T) \neq \emptyset$ and $I - T^*T \in S_\infty$, then $\sigma(T) \cap \mathbb{D} = \sigma_{\mathrm{p}}(T) = \sigma_{\mathrm{np}}(T)$ and for every $\lambda \in \sigma_{\mathrm{p}}(T)$, the Riesz projection $\mathcal{P}_\lambda$ is of finite rank and $C_\lambda(T) = \mathcal{P}_\lambda\mathcal{H}$. As $\mathcal{P}_\lambda \in \{T\}'$, $C_\lambda(T) \subset E_T$ and $C_\lambda(T) = C_\lambda(T_E)$. ∎

The following corollary is an obvious consequence of Lemmas 4.8 and 4.9.

COROLLARY 4.10. *Let $T \in L(\mathcal{H})$ be a c.n.u. contraction such that $\mathbb{D} \setminus \sigma(T) \neq \emptyset$ and $I - T^*T \in S_\infty$, $T_E = T|_{E_T}$. If $\{T\}' \cap \mathcal{F}$ is SOT dense in $\{T\}' \cap S_\infty$ then $E_T = \operatorname{span} C(T)$, that is, E_T is generated by the generalized eigenvectors of T.* ∎

Recall that for a c.n.u. contraction $T \in L(\mathcal{H})$,

$$\mathcal{H}_0(T) = \{x \in \mathcal{H} : \|T^n x\| \to 0\}. \tag{11}$$

Recall also that if $T \in (\mathrm{SM})$ then the $C_{0.}$ part $T_0 = T|_{\mathcal{H}_0(T)}$ of T is in fact a C_0-contraction. Moreover Θ_{T_0} is the pure part of the inner factor Θ_{inn} of Θ_T and the minimal function m_{T_0} of T_0 is the minimal scalar multiple of Θ_{inn}.

LEMMA 4.11. *Let $T \in L(\mathcal{H})$ be an* (SM)*-contraction. Let E_T and $\mathcal{H}_0(T)$ be defined by* (10) *and* (11)*. Then $E_T \subset \mathcal{H}_0$. In particular $T_E = T|_{E_T}$ is a C_0-contraction and m_{T_E} divides m_{T_0}.*

Proof. Let $X \in \{T\}' \cap S_\infty$. We know that T^n tends WOT to 0. But $T^n X = XT^n$ and X is compact, therefore $T^n X = XT^n$ tends SOT to 0. Thus $X\mathcal{H} \subset \mathcal{H}_0(T)$. Consequently, $E_T \subset \mathcal{H}_0(T)$. As E_T is invariant for T, it is invariant for T_0, thus $T_E = T_0|_{E_T}$ is a C_0-contraction and $m(T_E)$ divides m_T (see [SNF67, Proposition III.6.1]). ∎

The following theorem is a completed version of the theorem given in Section 1.

THEOREM 4.12. *Let $T \in L(\mathcal{H})$ be an* (SM)*-contraction such that $I - T^*T \in S_\infty$. Set $\Theta = \Theta_T$, and let $\Theta = \Theta_{\mathrm{out}}\Theta_{\mathrm{inn}}$ be the outer-inner factorization of Θ. Let E_T and $\mathcal{H}_0(T)$ be defined by* (10) *and* (11), *and $T_E = T|_{E_T}$. The following are equivalent*:

(i) *$\{T\}' \cap \mathcal{F}$ is dense in $\{T\}' \cap S_\infty$.*
(ii) *$\{T\}' \cap \mathcal{F}$ is SOT dense in $\{T\}' \cap S_\infty$.*
(iii) *$\{M_{\Theta_{\mathrm{inn}}}\}' \cap \mathcal{F}$ is (SOT) dense in $\{M_{\Theta_{\mathrm{inn}}}\}' \cap S_\infty$.*
(iv) *Θ_{inn} has a Blaschke scalar multiple.*
(v) *$T_0 = T|_{\mathcal{H}_0}$ is complete.*
(vi) *The minimal function m_{T_E} of T_E is a Blaschke product.*

Moreover, *if* (i)–(vi) *are satisfied*, *then we have the following linear approximation*:

$$\lim_{n\to\infty} \|B_n(T)A - A\| = 0, \qquad \forall A \in \{T\}' \cap S_\infty,$$

where $B = \prod_{k\ge1} b_{\lambda_k}$ is the minimal scalar multiple of Θ_{inn}, $B_n = \prod_{k>n} b_{\lambda_k}$, and $B_n(T)A \in \{T\}' \cap \mathcal{F}$ for all $n \ge 1$.

REMARK 4.13. (1) Let $T \in L(\mathcal{H})$ be a c.n.u. contraction such that $\mathbb{D} \setminus \sigma(T) \neq \emptyset$ and $I - T^*T \in S_\infty$, and let T_0 be the $C_{0.}$ part of T. Then $C(T) = C(T_0) = C(T_E)$, in other words the generalized eigenvectors of T, T_0 and T_E are all the same. Indeed, if $\lambda \in \sigma(T)$, then the associated Riesz projection $\mathcal{P}_\lambda$ is in $\{T\}' \cap \mathcal{F}$. Thus $C_\lambda(T) = \operatorname{Im} \mathcal{P}_\lambda \subset E$ and $C(\lambda) \subset E$. The fact that E is invariant gives the conclusion.

(2) In the situation of Theorem 4.12 this means that m_{T_0} and m_{T_E} can differ within a singular inner factor only. If moreover m_{T_0} is a Blaschke product, then so is m_{T_E} (because m_{T_E} divides m_{T_0}), therefore $m_{T_E} = m_{T_0}$ and $E_T = \operatorname{span} C(T_E) = \operatorname{span} C(T_0) = \mathcal{H}_0(T)$ (as a C_0-contraction having a Blaschke minimal function is complete), in other words $T_0 = T_E$.

(3) It is likely that the latter equality remains true even if m_{T_0} is not a Blaschke product. Nevertheless we can only prove that $m_{T_0} = m_{T_E}$ for every (SM)-contraction with compact defect.

Proof of Theorem 4.12. We prove that (iv)⇔(v)⇒(vi)⇒(i)⇒(ii)⇒(iv). The equivalence (iii)⇔(iv) then follows from the equivalence (i)⇔(ii)⇔(iv) applied to $M_{\Theta_{\mathrm{inn}}}$.

The equivalence (v)⇔(iv) is clear because a C_0-contraction is complete if and only if its minimal function is a Blaschke product, and the minimal function of T_0 is the minimal scalar multiple of Θ_{inn}.

We shall see now that (vi)⇒(i). Let $m_{T_E} = B = \prod_{k\ge1} b_{\lambda_k}$ and $B_n = \prod_{k>n} b_{\lambda_k}$, $n \ge 1$. As $|B_n(\xi)| = 1$ a.e. on $\mathbb{T}$ we have

$$\int_{\mathbb{T}} |B_n(\xi) - 1|^2\, d\mu(\xi) = \int_{\mathbb{T}} |B_n(\xi)|^2\, d\mu(\xi) + 1 - 2\Re\Big(\int_{\mathbb{T}} B_n(\xi)\, d\mu(\xi)\Big)$$
$$= 2(1 - B_n(0)) \to 0.$$

This implies that $\|B_n(T)x - x\| \to 0$ for every $x \in E_T$, so $\|B_n(T)X - X\| = \|XB_n(T) - X\| \to 0$ for every $X \in \{T\}' \cap S_\infty$. But $B_n(T)X \in \{T\}' \cap \mathcal{F}$ as

$$B_n(T)X\mathcal{H} \subset \ker \frac{B}{B_n}(T) \subset \operatorname{span}\{C_{\lambda_k}(T) : 1 \le k < n\}$$

and each $C_{\lambda_k}(T_E)$ is finite-dimensional (all eigenvalues are of finite type). This also proves the last assertion of the theorem.

(i)⇒(ii) is obvious. Now, we show that (ii)⇒(iv). Suppose (ii) and suppose that the minimal scalar multiple of Θ has a non-trivial singular part. We shall work with the model M_Θ of T to show that this leads to a contradiction. We know that if $A \in \{T\}' \cap \mathcal{F}$, then $A\mathcal{H}$ is a finite-dimensional invariant subspace of T, and hence $A\mathcal{H} \subset C(T)$ and $B(T)A = 0$, where $C(T)$ is defined according to (9). Condition (ii) implies that $B(M_\Theta)A = 0$ for every $A \in \{M_\Theta\}' \cap S_\infty$.

Let δ be a minimal scalar multiple of Θ. Assuming that δ contains a non-trivial singular inner factor we obtain a contradiction to the above property of $B(M_\Theta)$. Let m_1 be a non-trivial singular factor of δ such that $\sigma(m_1)$ has Lebesgue measure zero, and $f \in \mathcal{C}_A$ an outer function equal to zero on $\sigma(m_1)$. According to Lemma 3.7, if we set $\psi = f\delta/m_1$ then $H_{\psi\Theta^{-1}}$ is compact. Then, Corollary 4.5 says that the operator $X = \binom{\Theta}{\Delta} P_- \Theta^{-1}\psi P_*|_{K_\Theta}$ is in $\{M_\Theta\}' \cap S_\infty$. Suppose $\psi\Theta^{-1} = G \in H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T))$, then

$$\psi I = f\,\frac{\delta}{m_1}\, I = \Theta G,$$

and hence $f\delta/m_1$ is a scalar multiple of Θ_{inn}, which is impossible since δ is a minimal scalar multiple. Therefore, $\psi\Theta^{-1} \notin H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T))$.

Now, consider $B(M_\Theta)X$. As

$$Y = \begin{pmatrix} \psi I & 0 \\ \psi \Delta \Theta^{-1} & 0 \end{pmatrix}$$

is a lifting of X (see the proof of Lemma 4.4), a lifting of $B(M_\Theta)X$ is

$$\begin{pmatrix} B\psi & 0 \\ B\psi \Delta \Theta^{-1} & 0 \end{pmatrix}.$$

Thus

$$B(M_\Theta)X = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} P_- \Theta^{-1} B\psi P_*|_{K_\Theta}.$$

Clearly, $B\psi\Theta^{-1}$ is still in $H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T)) + \mathcal{C}(\mathbb{T}, S_\infty(\mathcal{D}_{T^*}, \mathcal{D}_T))$ but is not in $H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T))$ by the same minimality reason as above. Thus $B(M_\Theta)X$ is a non-zero operator. This gives a contradiction to the above property of $B(M_\Theta)$. ■

5. Schatten–von Neumann operators in the commutant. The following theorem contains some sufficient conditions for $\{T\}' \cap S_p \neq \{0\}$. The meaning of these conditions is that there exist some part of the operator T which has "thin" spectrum. To express this "thinness" we make use of the notions of maximal spectral subspace and of Beurling–Carleson subsets of the unit circle. Recall that, for a compact set $\sigma \subset \mathbb{C}$, the maximal *spectral subspace* $H(\sigma)$ over σ is defined by the following requirements:

(i) $H(\sigma) \subset \mathcal{H}$ is an invariant subspace for T such that $\sigma(T|_{H(\sigma)}) \subset \sigma$;

(ii) if E is another T-invariant subspace with $\sigma(T|_E) \subset \sigma$ then $E \subset H(\sigma)$.

It is known that for c.n.u. contractions of the class (SM), the maximal spectral subspaces $H(\sigma)$ exist for all closed sets $\sigma \subset \overline{\mathbb{D}}$ (see [SNF67]). Moreover, if δ is a minimal scalar multiple of Θ_T and $\sigma \cap \mathbb{T}$ is a subset of Lebesgue measure 0, then $H(\sigma) \neq \{0\}$ if and only if $\delta_\sigma \neq 1$, where

$$\delta_\sigma(z) = \Big(\prod_{\lambda_n \in \sigma\cap\mathbb{D}} b_{\lambda_n}(z)\Big) \exp\Big(-\int_{\sigma\cap\mathbb{T}} \frac{\xi+z}{\xi-z}\, d\mu_\delta(\xi)\Big),$$

and

$$\delta = \Big(\prod_{n\ge 1} b_{\lambda_n}(z)\Big) \exp\Big(-\int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\, d\mu_\delta(\xi)\Big)$$

is the canonical factorization of δ (the measure $\mu_\delta = \log(1/|\delta|)d\mu + \mu_\delta^{\mathrm{s}}$ contains both absolutely continuous and singular parts); here μ denotes the Lebesgue measure on $\mathbb{T}$. Therefore, for a subset $\sigma \subset \mathbb{T}$ of Lebesgue measure zero, $H(\sigma) \neq \{0\}$ if and only if $\mu_\delta^{\mathrm{s}}(\sigma) > 0$.

Recall also that a closed subset $\sigma \subset \mathbb{T}$ is called a *Beurling–Carleson set* if

$$\int_{\mathbb{T}} \log(\operatorname{dist}(\xi,\sigma))\, d\mu(\xi) > -\infty,$$

or equivalently, if $\mu(\sigma) = 0$ and $\sum_{k\ge 1} |I_k| \log |I_k|^{-1} < \infty$, where $(I_k)_{k\ge 1}$ is the sequence of complementary arcs of σ. For more information on Beurling–Carleson sets see, for example, V. Havin and B. Jöricke [HJ94, Section II.3.1].

THEOREM 5.1. *Let T be an* (SM)*-contraction such that $I - T^*T \in S_p$ with $1 \le p < \infty$. Then $\{T\}' \cap S_p \neq \{0\}$ as soon as one of the following properties is satisfied*:

(i) $\sigma_{\mathrm{p}}(T) \neq \emptyset$, *or equivalently* $\ker \Theta(\lambda) \neq \{0\}$ *for some* $\lambda \in \mathbb{D}$.

(ii) *There exists a Beurling–Carleson set $\sigma \subset \mathbb{T}$ such that $H(\sigma) \neq \{0\}$.*

Moreover if $T \in C_0$ then $H^\infty(T) \cap S_p \neq \{0\}$.

Before starting the proof, we give modified versions of Lemmas 3.6 and 3.7 for the S_p classes.

LEMMA 5.2. *Let $T \in L(\mathcal{H})$ be a C_0-contraction such that $I - T^*T \in S_p$ $(1 \le p < \infty)$ and m_T the minimal function of T. Assume that there exists a Beurling–Carleson set $\sigma \subset \mathbb{T}$ such that $(m_T)_\sigma$ is non-constant. Then there exists $\varphi \in H^\infty$ such that $\varphi(T) \neq 0$ and the Hankel operator $H_{\varphi\Theta_T^*}$ belongs to S_p. More precisely, we can take $\varphi = f m_T/m_1$, where $m_1 = (m_T)_\sigma$ and f is any outer $\mathcal{C}_A^\infty$ function equal to zero on σ.*

For the proof the following characterization of Hankel operators from the class S_p in terms of their symbols is used. Let $\Phi \in L^\infty(L(\mathcal{H}_1, \mathcal{H}_2))$ and $1 \le p < \infty$. Then $H_\Phi \in S_p$ if and only if $P_-\Phi \in B_p^{1/p}(S_p)$, where $B_p^{1/p}(S_p)$ denotes the Besov class with values in S_p (see [Pel82]).

Proof of Lemma 5.2. The proof is similar to that of Lemma 3.6. The only difference is that f can be chosen to be an outer function in $\mathcal{C}_A^\infty = \mathcal{C}_A \cap \mathcal{C}^\infty$ such that $|f(\xi)| = o((\operatorname{dist}(\xi,\sigma))^N)$ as ξ tends to σ, for every $N \ge 1$ (see [HJ94, Section II.3.1] for the existence of such a function). Here $\mathcal{C}^\infty$ stands for the space of infinitely differentiable functions on $\mathbb{T}$. We denote by $\mathcal{C}^\infty(X)$ the space of X-valued $\mathcal{C}^\infty$ functions. Then, in the notation of Lemma 3.6, $f\Theta_1(z)^* = fA^* + fK(z)^*$, where $f\Theta_1^* \in \mathcal{C}^\infty(L(\mathcal{K}, \mathcal{D}_T))$ and consequently $K \in \mathcal{C}^\infty(S_p(\mathcal{D}_T, \mathcal{K}))$. On the other hand, it is easy to see that $\mathcal{C}^\infty(X) \subset B_p^{1/p}(X)$ for any Banach space X and any $0 < p < \infty$. Therefore, by Peller's theorem the Hankel operator $H_{f\Theta_1^*}$ is in S_p, and the same holds for $H_{f\Theta^*}$. ∎

LEMMA 5.3. *Let $T \in$* (SM), *$T \notin C_{11}$, be such that $I - T^*T \in S_p$, $1 \le p < \infty$, and let δ be a minimal scalar multiple of T. Assume that there exists* *a Beurling–Carleson set $\sigma \subset \mathbb{T}$ such that δ_σ is non-constant. Then there exists $\psi \in H^\infty$ such that $H_{\psi\Theta_T^{-1}}$ is in S_p but $\psi\Theta_T^{-1} \notin H^\infty(L(\mathcal{D}_{T^*}, \mathcal{D}_T))$. More precisely, we can take $\psi = f\delta/m_1$, where $m_1 = (\delta_{\rm inn})_\sigma$, $\delta_{\rm inn}$ being the inner part of δ, and f is any outer $\mathcal{C}_A^\infty$ function equal to zero on σ.*

Proof. The proof follows that of Lemma 3.7 with the obvious modifications, in particular Lemma 3.6 is replaced by Lemma 5.2. ∎

Proof of Theorem 5.1. If (i) is satisfied, then the Riesz projection $\mathcal{P}_\lambda$ belongs to $\{T\}' \cap \mathcal{F}$. If (ii) is satisfied, identify T with its model operator M_Θ and define X by

$$X = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} P_- \Theta^{-1} \psi P_*|_{K_\Theta},$$

where ψ is defined as in Lemma 5.3. Then, as in the proof of Theorem 4.7, X is a compact operator in $\{T\}'$ which is also non-zero and in S_p due to Lemma 5.3. ∎

In the case when Θ is a scalar H^∞ function, Theorem 5.1 extends to all p, $0 < p < \infty$, as the characterization of Hankel operators of class S_p is valid for $0 < p < \infty$ in the case when the symbols are scalar-valued (see [Pel83]).

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