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Extremal sections of complex l_p-balls, $0 < p \leq 2$

by

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Abstract. We study the extremal volume of central hyperplane sections of complex n-dimensional l_p-balls with $0 < p \leq 2$. We show that the minimum corresponds to hyperplanes orthogonal to vectors $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{C}^n$ with $|\xi^1| = \ldots = |\xi^n|$, and the maximum corresponds to hyperplanes orthogonal to vectors with only one non-zero coordinate.

1. Introduction. This article continues the study of extremal sections of l_p-balls. We denote by $B_p(\mathbb{R}^n)$ and $B_p(\mathbb{C}^n)$ the unit balls of the real and complex n-dimensional l_p-spaces, $l_p(\mathbb{R}^n)$ and $l_p(\mathbb{C}^n)$, respectively.

The extremal hyperplane sections of the cube are known in both real and complex cases. Hadwiger [Ha] proved that the minimal volume of hyperplane sections of the real unit cube is equal to 1 and corresponds to the sections parallel to the faces. Different proofs of this fact were later given by Vaaler [V], who generalized the result to sections of arbitrary dimensions, Hensley [He] and Ball [B]. It was shown in [BK] that this property of the cube is in some sense stable, i.e. for every $0 < t < 3/4$ the slab parallel to the face has minimal volume among all central slabs of the cube with fixed width t. The exact upper bound $\sqrt{2}$ for the volume of hyperplane sections of the real unit cube was found by Ball [B] and corresponds to the hyperplane orthogonal to the vector $(1, 1, 0, \ldots, 0)$. The case of the complex cube was studied by Oleszkiewicz and Pełczyński [OP], who proved that the minimal sections are the ones orthogonal to vectors with only one non-zero coordinate, and the maximal sections are orthogonal to vectors of the form $e_j + \sigma e_k$, where $j \neq k$, e_j and e_k are standard basic vectors, and $\sigma \in \mathbb{C}$, $|\sigma| = 1$. Note that the "minimal" part also follows from an earlier result of Meyer and Pajor [MP, Corollary 2.5].

The critical sections of l_p-balls, $0 < p < \infty$, are different for $p > 2$ and $p < 2$. Meyer and Pajor [MP] proved that the section orthogonal to the vector $(1, 0, \ldots, 0)$ is minimal for $p > 2$ and maximal for $1 \leq p < 2$. The

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latter result also holds for $0 < p < 1$, as proved by Caetano [Ca]. In the same paper, Meyer and Pajor proved that the minimal hyperplane section of $B_1(\mathbb{R}^n)$ is the one perpendicular to the vector $(1,\dots,1)$ and conjectured that this is also true for every $p \in [1,2]$. This conjecture was proved in [K1] for $0 < p \le 2$. It is still an open question what are the maximal sections of $B_p(\mathbb{R}^n)$ when $2 < p < \infty$. Oleszkiewicz [O] showed that the answer must depend on p and the dimension.

In this article we characterize the extremal sections of complex l_p-balls $B_p(\mathbb{C}^n)$ for $0 < p \le 2$.

THEOREM 1. *Let $0 < p \le 2$. For $\xi = (\xi^1,\dots,\xi^n) \in \mathbb{C}^n$, $\xi \ne 0$ denote by $H_\xi = \{x \in \mathbb{C}^n : (x,\xi) = 0\}$ the complex hyperplane in $\mathbb{C}^n$ orthogonal to ξ. The $(n-1)$-dimensional complex volume of $B_p(\mathbb{C}^n) \cap H_\xi$ is minimal if $|\xi^1| = \dots = |\xi^n|$, and it is maximal if ξ has only one non-zero coordinate.*

The part of this theorem related to the maximal sections was established earlier by Meyer and Pajor [MP, Corollary 2.5] for $1 \le p \le 2$, and by Barthe [Ba] for $0 < p < 1$. In fact, these papers cover a more general case of the unit balls of the real spaces $l_p^n(l_2^m)$ and show that, for every integer k, the "standard" sections of these balls of dimension km are minimal for $p \ge 2$ and maximal for $0 < p \le 2$.

We prove Theorem 1 by generalizing the method of [K1] to the complex case. As in the real case in [K1], the minimal and maximal sections are identified simultaneously.

2. Preliminaries and notation. We identify $l_p(\mathbb{C}^n)$ with the real $2n$-dimensional space equipped with the norm

$$\|x\|_p = [(x_1^2 + x_2^2)^{p/2} + \dots + (x_{2n-1}^2 + x_{2n}^2)^{p/2}]^{1/p}, \tag{1}$$

where

$$\mathbb{C}^n \ni x = (x_1 + ix_2, \dots, x_{2n-1} + ix_{2n}) \mapsto (x_1, x_2, \dots, x_{2n-1}, x_{2n}) \in \mathbb{R}^{2n}.$$

It is easily seen that this mapping identifies the complex hyperplane H_ξ with a $(2n-2)$-dimensional subspace E_ξ of $\mathbb{R}^{2n}$, orthogonal to the vectors

$$\eta = (\xi_1, \xi_2, \dots, \xi_{2n-1}, \xi_{2n}), \qquad \vartheta = (-\xi_2, \xi_1, \dots, -\xi_{2n}, \xi_{2n-1}).$$

Let $H_\xi^\perp$ be the set of vectors in $\mathbb{C}^n$ perpendicular to H_ξ. This set corresponds to a 2-dimensional subspace in $\mathbb{R}^{2n}$, which is denoted by $E_\xi^\perp$ and is orthogonal to E_ξ. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^{2n}$. If $|\xi| = 1$ the vectors η, ϑ form an orthonormal basis in $E_\xi^\perp$.

The $(n-1)$-dimensional *complex volume* of the section of $B_p(\mathbb{C}^n)$ by H_ξ is defined as the $(2n-2)$-dimensional volume of the section of the unit ball of the norm $\|\cdot\|_p$ by the subspace E_ξ. We write

$$\mathrm{volc}_{n-1}(B_p(\mathbb{C}^n) \cap H_\xi) = \mathrm{vol}_{2n-2}(B_p(\mathbb{C}^n) \cap E_\xi),$$

using the same notation $B_p(\mathbb{C}^n)$ for the unit ball of the complex l_p-space and for the unit ball of the norm $\|\cdot\|_p$ in $\mathbb{R}^{2n}$:

$$B_p(\mathbb{C}^n) = \Big\{(x_1 + ix_2, \ldots, x_{2n-1} + ix_{2n}) \in \mathbb{C}^n : \sum_{j=1}^{n} (x_{2j-1}^2 + x_{2j}^2)^{p/2} \le 1\Big\}$$
$$= \Big\{(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \in \mathbb{R}^{2n} : \sum_{j=1}^{n} (x_{2j-1}^2 + x_{2j}^2)^{p/2} \le 1\Big\}.$$

3. The Fourier transform formula for sections of $B_p(\mathbb{C}^n)$. As in many of the papers cited in the introduction, our result is based on a certain Fourier transform formula for the volume of sections. We use the following general result proved in [K2, Th. 2]: for any infinitely smooth symmetric star body K in $\mathbb{R}^n$, the volume of its section by an $(n-k)$-dimensional subspace H of $\mathbb{R}^n$, $1 \le k < n$, is equal to

$$\text{(2)} \qquad \mathrm{vol}_{n-k}(K \cap H) = \frac{1}{n-k} \int\limits_{S^{n-1}\cap H} \|x\|_K^{-n+k}\, dx = \frac{1}{(2\pi)^k}\,\frac{1}{n-k} \int\limits_{S^{n-1}\cap H^{\perp}} (\|x\|_K^{-n+k})^{\wedge}(\theta)\, d\theta,$$

where $\|\cdot\|_K$ is the Minkowski functional of K. Although the bodies $B_p(\mathbb{C}^n)$ are not always smooth, we assume that formula (2) holds for the norm $\|\cdot\|_p$ introduced in (1). In Section 5 we present a simple approximation argument proving this assumption.

Throughout this paper we use the Fourier transform of distributions. We denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^{2n}$ with values in $\mathbb{C}$. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. Every locally integrable real-valued function f on $\mathbb{R}^{2n}$ with power growth at infinity represents a distribution acting by integration: for every $\phi \in \mathcal{S}$,

$$\langle f, \phi\rangle = \int\limits_{\mathbb{R}^{2n}} f(x)\phi(x)\, dx.$$

The Fourier transform of a distribution f is defined by $\langle \widehat{f}, \phi\rangle = \langle f, \widehat{\phi}\rangle$ for every test function ϕ, where

$$\widehat{\phi}(x) = \int\limits_{\mathbb{R}^{2n}} \phi(\xi)\exp(-i(x,\xi))\, d\xi$$

is the Fourier transform of ϕ.

Lemma 1. *Let $0 < p < \infty$, $y = (y_1, \ldots, y_{2n}) \in \mathbb{R}^{2n}$. Then the Fourier transform of $\|\cdot\|_p^{-2n+2}$ (in the sense of distributions in $\mathbb{R}^{2n}$) is equal to a* *locally integrable function on* $\mathbb{R}^{2n}$:

$$(\|\cdot\|_p^{-2n+2})^\wedge(y)$$
$$= \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t\Big(\prod_{j=1}^n \int_{\mathbb{R}^2} e^{-it(y_{2j-1}x_{2j-1}+y_{2j}x_{2j})} e^{-(x_{2j-1}^2+x_{2j}^2)^{p/2}}\, dx_{2j-1}\, dx_{2j}\Big) dt.$$

Proof. From the definition of the Gamma function, we have

$$\|x\|_p^{-2n+2} = \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t^{2n-3} e^{-t^p\|x\|_p^p}\, dt. \tag{3}$$

We first fix $t > 0$ and compute the Fourier transform of the function $x \mapsto e^{-t^p\|x\|_p^p}$: for any $y \in \mathbb{R}^{2n}$, making a change of variables $tx = z$ we get

$$\begin{aligned}(e^{-t^p\|x\|_p^p})^\wedge(y) &= \int_{\mathbb{R}^{2n}} e^{-i(y,x)} e^{-t^p\|x\|_p^p}\, dx = \int_{\mathbb{R}^{2n}} e^{-i(y,z/t)} e^{-\|z\|_p^p} t^{-2n}\, dz \\ &= t^{-2n} \prod_{j=1}^n \int_{\mathbb{R}^2} e^{-i(y_{2j-1}z_{2j-1}/t+y_{2j}z_{2j}/t)} e^{-(z_{2j-1}^2+z_{2j}^2)^{p/2}}\, dz_{2j-1}\, dz_{2j}.\end{aligned} \tag{4}$$

The function $\|x\|_p^{-2n+2}$ is locally integrable on $\mathbb{R}^{2n}$. Using (3), (4), Fubini and the change of variables $1/t = s$, we get, for any even test function ϕ,

$$\langle (\|\cdot\|_p^{-2n+2})^\wedge, \phi\rangle = \langle \|x\|_p^{-2n+2}, \widehat{\phi}\rangle$$
$$= \int_{\mathbb{R}^{2n}} \|x\|_p^{-2n+2}\widehat{\phi}(x)\, dx = \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_{\mathbb{R}^{2n}} \Big(\int_0^\infty t^{2n-3} e^{-t^p\|x\|_p^p}\, dt\Big)\widehat{\phi}(x)\, dx$$
$$= \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t^{2n-3} \int_{\mathbb{R}^{2n}} e^{-t^p\|x\|_p^p}\widehat{\phi}(x)\, dx\, dt$$
$$= \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t^{2n-3} \int_{\mathbb{R}^{2n}} (e^{-t^p\|x\|_p^p})^\wedge(y)\phi(y)\, dy\, dt$$
$$= \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t^{2n-3} t^{-2n}$$
$$\times \int_{\mathbb{R}^{2n}} \Big(\prod_{j=1}^n \int_{\mathbb{R}^2} e^{-i(y_{2j-1}z_{2j-1}/t+y_{2j}z_{2j}/t)} e^{-(z_{2j-1}^2+z_{2j}^2)^{p/2}} dz_{2j-1}\, dz_{2j}\, \phi(y)\, dy\Big) dt$$
$$= \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_{\mathbb{R}^{2n}} \phi(y)$$
$$\times \int_0^\infty s\Big(\prod_{j=1}^n \int_{\mathbb{R}^2} e^{-is(y_{2j-1}z_{2j-1}+y_{2j}z_{2j})} e^{-(z_{2j-1}^2+z_{2j}^2)^{p/2}}\, dz_{2j-1}\, dz_{2j}\Big)\, ds\, dy.$$

Since ϕ is an arbitrary even test function, the result follows.

REMARK 1. We define a function g on $\mathbb{R}^2$ by

$$g(y_{2j-1}, y_{2j}) := \int_{\mathbb{R}^2} e^{-i(y_{2j-1}z_{2j-1}+y_{2j}z_{2j})} e^{-(z_{2j-1}^2+z_{2j}^2)^{p/2}}\, dz_{2j-1}\, dz_{2j}.$$

The function $(z_{2j-1}, z_{2j}) \mapsto e^{-(z_{2j-1}^2+z_{2j}^2)^{p/2}}$ is a radial function on $\mathbb{R}^2$, so is its Fourier transform. Therefore,

$$g(ty_{2j-1}, ty_{2j}) = g(t\sqrt{y_{2j-1}^2 + y_{2j}^2}, 0),$$

$$f(t(y_{2j-1}^2 + y_{2j}^2)^{1/2}) = \int_{\mathbb{R}^2} e^{-it(y_{2j-1}^2+y_{2j}^2)^{1/2} z_{2j-1}} e^{-(z_{2j-1}^2+z_{2j}^2)^{p/2}}\, dz_{2j-1}\, dz_{2j}.$$

The latter formula defines a function f on $\mathbb{R}$ that we are going to use throughout the paper:

$$f(u) = \int_{\mathbb{R}^2} e^{-iuz_{2j-1}} e^{-(z_{2j-1}^2+z_{2j}^2)^{p/2}}\, dz_{2j-1}\, dz_{2j}, \qquad u \in \mathbb{R}.$$

THEOREM 2. *Let* $0 < p \le 2$, $\xi \in \mathbb{C}^n$, $|\xi| = 1$, *and* $H_\xi = \{x \in \mathbb{C}^n : (x, \xi) = 0\}$. *Then the complex volume of the section of the unit ball* $B_p(\mathbb{C}^n)$ *by the complex hyperplane* H_ξ *is*

$$(*) \qquad \mathrm{volc}_{n-1}(B_p(\mathbb{C}^n) \cap H_\xi) = \frac{1}{2\pi}\, \frac{1}{2n-2}\, \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t \prod_{j=1}^n f(t(\xi_{2j-1}^2 + \xi_{2j}^2)^{1/2})\, dt.$$

Proof. Using the definition of complex volume and formula (2) with $K = B_p(\mathbb{C}^n)$ and $H = E_\xi$ (via the approximation argument of Section 5), we get

$$\mathrm{volc}_{n-1}(B_p(\mathbb{C}^n) \cap H_\xi) = \mathrm{vol}_{2n-2}(B_p(\mathbb{C}^n) \cap E_\xi) = \frac{1}{(2\pi)^2}\, \frac{1}{2n-2} \int_{S^{2n-1}\cap E_\xi^\perp} (\|x\|_p^{-2n+2})^\wedge(y)\, dy.$$

By Lemma 1, Remark 1 and Fubini's theorem, the latter quantity equals

$$(5) \qquad \frac{1}{(2\pi)^2}\, \frac{1}{2n-2}\, \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_{S^{2n-1}\cap E_\xi^\perp} \int_0^\infty t \prod_{j=1}^n f(t(y_{2j-1}^2 + y_{2j}^2)^{1/2})\, dt\, dy$$

$$= \frac{1}{(2\pi)^2}\, \frac{1}{2n-2}\, \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t \int_{S^{2n-1}\cap E_\xi^\perp} \prod_{j=1}^n f(t(y_{2j-1}^2 + y_{2j}^2)^{1/2})\, dy\, dt.$$

We shall now prove that the function under the inner integral in (5) is constant on $S^{2n} \cap E_\xi^\perp$. As mentioned in Section 2, $E_\xi^\perp = \mathrm{span}\{\eta, \vartheta\}$, where $\eta = (\xi_1, \ldots, \xi_{2n})$, $\vartheta = (-\xi_2, \xi_1, \ldots, -\xi_{2n}, \xi_{2n-1})$. Let $y \in E_\xi^\perp$; then y can be written as a linear combination of η, ϑ, i.e. there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ so that $y = \lambda_1\eta + \lambda_2\vartheta$ and, for all $j = 1,\dots,n$, $y_{2j-1} = \lambda_1\xi_{2j-1} - \lambda_2\xi_{2j}$, $y_{2j} = \lambda_1\xi_{2j} + \lambda_2\xi_{2j-1}$. Therefore,

$$y_{2j-1}^2 + y_{2j}^2 = (\lambda_1^2+\lambda_2^2)(\xi_{2j-1}^2+\xi_{2j}^2), \qquad j=1,\dots,n,$$

and hence

$$\begin{aligned}|y|^2 &= y_1^2+y_2^2+\dots+y_{2n-1}^2+y_{2n}^2 = (\lambda_1^2+\lambda_2^2)(\xi_1^2+\xi_2^2+\dots+\xi_{2n-1}^2+\xi_{2n}^2)\\ &= (\lambda_1^2+\lambda_2^2)|\xi|^2 = \lambda_1^2+\lambda_2^2,\end{aligned}$$

since $|\xi|=1$. Thus, if $y \in S^{2n-1}\cap E_\xi^\perp$ then $\lambda_1^2+\lambda_2^2=1$, and we get

$$\begin{aligned}\prod_{j=1}^n f(t(y_{2j-1}^2+y_{2j}^2)^{1/2}) &= \prod_{j=1}^n f(t(\lambda_1^2+\lambda_2^2)^{1/2}(\xi_{2j-1}^2+\xi_{2j}^2)^{1/2})\\ &= \prod_{j=1}^n f(t(\xi_{2j-1}^2+\xi_{2j}^2)^{1/2}).\end{aligned}$$

Hence, the inner integral in (5) is equal to

$$\begin{aligned}\int\limits_{S^{2n-1}\cap E_\xi^\perp}\prod_{j=1}^n f(t(y_{2j-1}^2+y_{2j}^2)^{1/2})\,dy &= \int\limits_{S^{2n-1}\cap E_\xi^\perp}\prod_{j=1}^n f(t(\xi_{2j-1}^2+\xi_{2j}^2)^{1/2})\,dy\\ &= 2\pi\prod_{j=1}^n f(t(\xi_{2j-1}^2+\xi_{2j}^2)^{1/2}),\end{aligned}$$

since $S^{2n-1}\cap E_\xi^\perp$ is a 2-dimensional unit circle. The latter equality and (5) imply $(*)$.

4. Proof of Theorem 1. The result of Theorem 1 immediately follows from Theorem 2 and the following lemma.

LEMMA 2. *If $0<p\le 2$ then the function $f(\sqrt{\cdot})$ is log-convex on $[0,\infty)$.*

Proof. For every $0<p\le 2$, the function $\exp(-|\cdot|^{p/2})$ is completely monotone, so by Bernstein's Theorem (see [W]), there exists a measure μ_p on $[0,\infty)$ so that, for every $t\in\mathbb{R}$,

$$e^{-|t|^{p/2}} = \int_0^\infty e^{-ut}\,d\mu_p(u),$$

which implies

$$e^{-|t|^p} = \int_0^\infty e^{-ut^2}\,d\mu_p(u),$$

and so

$$e^{-(x_1^2+x_2^2)^{p/2}} = \int_0^\infty e^{-u(x_1^2+x_2^2)}\,d\mu_p(u), \qquad x_1,x_2\in\mathbb{R}.$$

Therefore, by Fubini's theorem,

$$\int_{\mathbb{R}} e^{-isx_1} e^{-(x_1^2+x_2^2)^{p/2}}\, dx_1 = \int_{\mathbb{R}} e^{-isx_1} \int_0^\infty e^{-u(x_1^2+x_2^2)}\, d\mu_p(u)\, dx_1$$
$$= \int_0^\infty e^{-ux_2^2} \int_{\mathbb{R}} e^{-isx_1} e^{-ux_1^2}\, dx_1\, d\mu_p(u)$$
$$= \sqrt{\pi} \int_0^\infty e^{-ux_2^2} \frac{1}{\sqrt{u}}\, e^{-s^2/4u}\, d\mu_p(u).$$

Now integrate the latter in x_2 over $\mathbb{R}$. Using Fubini and the well known identity $\int_{\mathbb{R}} e^{-t^2}\, dt = \sqrt{\pi}$, we get the following expression for the function f:

$$f(s) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-isx_1} e^{-(x_1^2+x_2^2)^{p/2}}\, dx_1\, dx_2 = \sqrt{\pi} \int_{\mathbb{R}}\int_0^\infty e^{-ux_2^2} \frac{1}{\sqrt{u}}\, e^{-s^2/4u}\, d\mu_p(u)\, dx_2$$
$$= \sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{u}}\, e^{-s^2/4u} \Big(\int_{\mathbb{R}} e^{-ux_2^2}\, dx_2 \Big)\, d\mu_p(u)$$

and hence

$$f(s) = \pi \int_0^\infty \frac{1}{u}\, e^{-s^2/4u}\, d\mu_p(u).$$

Now, for any $\alpha_1, \alpha_2 > 0$, using the latter formula and the Cauchy–Schwarz inequality, we get

$$\Big(f\Big(\sqrt{\frac{a_1 + a_2}{2}} \Big) \Big)^2 = \pi^2 \Big[\int_0^\infty \frac{1}{\sqrt{u}}\, e^{-a_1/8u} e^{-a_2/8u}\, \frac{1}{\sqrt{u}}\, d\mu_p(u) \Big]^2$$
$$\le \pi^2 \Big(\int_0^\infty \frac{1}{u}\, e^{-a_1/4u}\, d\mu_p(u) \Big) \Big(\int_0^\infty \frac{1}{u}\, e^{-a_2/4u}\, d\mu_p(u) \Big)$$
$$= f(\sqrt{a_1}) f(\sqrt{a_2}),$$

which implies that $f(\sqrt{\cdot})$ is log-convex.

Now, to prove Theorem 1, note that the log-convexity of f immediately implies that for any $0 < \alpha_1 < \beta_1 < \beta_2 < \alpha_2$ with $\alpha_1^2 + \alpha_2^2 = \beta_1^2 + \beta_2^2 = 1$, we have

$$f(t\beta_1) f(t\beta_2) \le f(t\alpha_1) f(t\alpha_2), \qquad \forall t > 0.$$

Therefore, the integrand in the formula of Theorem 2 decreases pointwise when we change the vector $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{C}^n$ so that it remains a unit vector but the absolute values of any two coordinates become closer to each other. In particular, the integrand is maximal when only one of the coordinates is non-zero, and minimal when the absolute values of the coordinates are equal. The latter property immediately implies the result of Theorem 1.

5. An approximation argument for $B_p(\mathbb{C}^n)$. We prove in this section that formula (2) can be applied to the bodies $B_p(\mathbb{C}^n)$ and subspaces E_ξ in spite of the fact that these bodies are not always smooth. This will give a formally correct proof of the formula of Theorem 2.

For $\varepsilon > 0$, we introduce a star body $B_{p,\varepsilon}(\mathbb{C}^n)$ defined as the unit ball of the norm

$$\|x\|_{p,\varepsilon} = [((x_1^2 + x_2^2) + \varepsilon(x_3^2 + \ldots + x_{2n}^2))^{p/2} + \ldots \\ + \ldots + ((x_{2n-1}^2 + x_{2n}^2) + \varepsilon(x_1^2 + \ldots + x_{2n-2}^2))^{p/2}]^{1/p}.$$

Clearly, $\|x\|_{p,\varepsilon}$ is a continuous function of ε, and $\|\cdot\|_{p,\varepsilon} \in C^\infty(S^{2n-1})$. Moreover, $\|x\|_{p,\varepsilon} \to \|x\|_p$ as $\varepsilon \to 0^+$, uniformly with respect to $x \in S^{2n-1}$.

Applying formula (2) to $K = B_{p,\varepsilon}(\mathbb{C}^n)$ and $H = E_\xi$, we get

$$(6) \qquad \int_{S^{n-1}\cap E_\xi} \|x\|_{p,\varepsilon}^{-2n+2}\, dx = \frac{1}{(2\pi)^2} \int_{S^{n-1}\cap E_\xi^\perp} (\|x\|_{p,\varepsilon}^{-2n+2})^\wedge(\theta)\, d\theta.$$

Obviously, the left-hand side of the latter equality converges to the same integral with $\|x\|_p$ in place of $\|x\|_{p,\varepsilon}$, as $\varepsilon \to 0$. Therefore, it suffices to prove that the same happens on the right-hand side.

Recall that the measure μ_p, $0 < p \le 2$, introduced in Section 4 has the property that, for any $x_1, \ldots, x_{2n} \in \mathbb{R}$ and $\varepsilon > 0$,

$$(7) \qquad e^{-(x_1^2+x_2^2+\varepsilon(x_3^2+\ldots+x_{2n}^2))^{p/2}} = \int_0^\infty e^{-v(x_1^2+x_2^2+\varepsilon(x_3^2+\ldots+x_{2n}^2))}\, d\mu_p(v),$$

where $\varepsilon > 0$. Let $u = (u_1, \ldots, u_n) \in \mathbb{R}_+^n = [0,\infty) \times \ldots \times [0,\infty)$. We shall use the same notation μ_p to denote the product measure on $\mathbb{R}_+^n$, $\mu_p(u) = \mu_p(u_1)\ldots\mu_p(u_n)$.

Following the steps of Lemma 1 and using formula (7), one can easily show that the Fourier transform of $\|\cdot\|_{p,\varepsilon}^{-2n+2}$ (in the sense of distributions) is given by the formula

$$(8) \qquad (\|\cdot\|_{p,\varepsilon}^{-2n+2})^\wedge(y) \\ = \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t^{-3} \int_{\mathbb{R}_+^n} \prod_{j=1}^n \frac{\pi}{U_j(u)}\, e^{-\frac{1}{4U_j(u)t^2}(y_{2j-1}^2+y_{2j}^2)}\, d\mu_p(u)\, dt,$$

where $U_j(u) = u_j + \varepsilon \sum_{i=1,\, i\ne j}^n u_i$. Therefore, the right-hand side of (6) is equal to

$$\text{(9)}\quad \frac{1}{(2\pi)^2}\frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \times \int_0^\infty t^{-3} \int_{\mathbb{R}_+^n} \int_{S^{2n-1}\cap E_\xi^\perp} \prod_{j=1}^n \frac{\pi}{U_j(u)}\, e^{-\frac{1}{4U_j(u)t^2}(y_{2j-1}^2+y_{2j}^2)}\, dy\, d\mu_p(u)\, dt.$$

In the same way as in the proof of Theorem 2, one can show that for every $y \in S^{2n-1} \cap E_\xi^\perp$,

$$\prod_{j=1}^n \frac{\pi}{U_j(u)}\, e^{-\frac{1}{4U_j(u)t^2}(y_{2j-1}^2+y_{2j}^2)} = \prod_{j=1}^n \frac{\pi}{U_j(u)}\, e^{-\frac{1}{4U_j(u)t^2}(\xi_{2j-1}^2+\xi_{2j}^2)},$$

so the inner integral in (9) equals

$$2\pi \prod_{j=1}^n \frac{\pi}{U_j(u)}\, e^{-\frac{1}{4U_j(u)t^2}(\xi_{2j-1}^2+\xi_{2j}^2)},$$

and the expression in (9) equals

$$\text{(10)}\quad \frac{1}{2\pi}\frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t^{-3} \int_{\mathbb{R}_+^n} \prod_{j=1}^n \frac{\pi}{U_j(u)}\, e^{-\frac{1}{4U_j(u)t^2}(\xi_{2j-1}^2+\xi_{2j}^2)}\, d\mu_p(u)\, dt.$$

It remains to prove that the latter quantity converges to

$$\text{(11)}\quad \frac{1}{2\pi}\frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t^{-3} \int_{\mathbb{R}_+^n} \prod_{j=1}^n \frac{\pi}{u_j}\, e^{-\frac{1}{4u_jt^2}(\xi_{2j-1}^2+\xi_{2j}^2)}\, d\mu_p(u)\, dt$$

as $\varepsilon \to 0$, because (11) is equal to

$$\frac{1}{(2\pi)^2} \int_{S^{n-1}\cap E_\xi^\perp} (\|x\|_p^{-2n+2})^\wedge(\theta)\, d\theta,$$

which follows from Lemma 1 and (7), in the same way as it was done for the norm $\|\cdot\|_{p,\varepsilon}$.

The pointwise convergence of functions under the integral in (10) is obvious, so we can apply the dominated convergence theorem to finish our argument. To do that, recall the properties of the measure μ_p on $\mathbb{R}$ (see for example [Z]). The measure μ_p has density that decreases at infinity like $|v|^{-1-p/2}$. Moreover, $\int_0^\infty v^{-1}\, d\mu_p(v) < \infty$. Now, break the integral over dt in (10) into two integrals: from 1 to ∞ and from 0 to 1. To find a dominating function in the integral from 1 to ∞, just estimate the exponential by 1. In the integral from 0 to 1, use the fact that $\exp(-1/x^2) \le kx^{1+p/8}$ for every $x \in [0,\infty)$ and some fixed $k > 0$. The integrability of the dominating function follows from the order of decay of the density of the measure μ_p.

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