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Subspaces of L_p, $p > 2$, determined by partitions and weights

by

DALE E. ALSPACH (Stillwater, OK) and SIMEI TONG (Eau Claire, WI)

Abstract. Many of the known complemented subspaces of L_p have realizations as sequence spaces. In this paper a systematic approach to defining these spaces which uses partitions and weights is introduced. This approach gives a unified description of many well known complemented subspaces of L_p. It is proved that the class of spaces with such norms is stable under $(p, 2)$ sums. By introducing the notion of an envelope norm, we obtain a necessary condition for a Banach sequence space with norm given by partitions and weights to be isomorphic to a subspace of L_p. Using this we define a space Y_n with norm given by partitions and weights with distance to any subspace of L_p growing with n. This allows us to construct an example of a Banach space with norm given by partitions and weights which is not isomorphic to a subspace of L_p.

1. Introduction. Prior to Rosenthal's 1970 paper [R], only a few complemented subspaces of L_p were known: ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$, $(\sum \ell_2)_p$ and L_p itself ([P], [F]). Rosenthal's paper added several new spaces but more importantly it was seminal. In 1975 Schechtman [S] combined Rosenthal's results with a tensor product construction to show that there are infinitely many isomorphically distinct complemented subspaces of L_p. A few years later, it was shown by Bourgain, Rosenthal and Schechtman that up to isomorphism, there are uncountably many complemented subspaces of L_p (see [B-R-S]). Recently the first author proposed a new approach to describing the complemented subspaces of $L_p[0,1]$, $p > 2$. For any partition $P = \{N_i\}$ of $\mathbb{N}$ and weight function $W : \mathbb{N} \to (0, 1]$ define

$$\|(a_i)\|_{P,W} = \Big(\sum_i \Big(\sum_{j\in N_i} a_j^2 w_j^2\Big)^{p/2}\Big)^{1/p}.$$

Now suppose that $\mathcal{P} = (P_k, W_k)_{k\in K}$ is a family of pairs of partitions and functions as above. Define

$$\|(a_i)\|_{\mathcal{P}} = \sup_{k\in K} \|(a_i)\|_{(P_k,W_k)}. \tag{1.1}$$

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There are two fundamental questions which we begin to study in this paper. What conditions on $(P_k, W_k)_{k\in K}$ imply that $\|(a_i)\|_{\mathcal{P}}$ defines a norm on a space of sequences X so that X is isomorphic to a complemented subspace of $L_p[0,1]$? Is every complemented subspace of L_p (other than L_p) isomorphic to a space of this form?

This paper includes four major sections in addition to this introduction. Unless otherwise noted we will assume that $p > 2$ throughout. We will also assume that the scalar field is $\mathbb{R}$.

In Section 2, we present well known examples of complemented subspaces of L_p with norm given by partitions and weights. We discuss some natural conditions on the families and in particular normalization of the basis by inclusion of discrete partitions. We also prove that the natural sums of such Banach spaces are stable under these norms, i.e., have norms which are also given by partitions and weights.

In Section 3, we first observe that if the norm on a space X is given by finitely many partitions and weights, then X is isomorphic to a subspace of L_p. Then we give the definition of an envelope norm and we prove the existence of the envelope norm generated by a family of partitions and weights. We also give a lower bound on a norm which is necessary for a space to be isomorphic to a subspace of L_p. Finally we show that if a space with norm given by partitions and weights is isomorphic to a subspace of L_p, then its norm is equivalent to the associated envelope norm.

In Section 4, we construct examples which demonstrate the difference between a norm given by partitions and weights and the corresponding envelope norm. As a consequence we obtain an estimate of the distance between a certain Banach space Y_n with norm given by partitions and weights and $\bigotimes_{k=1}^n X_p$. Finally we give an example of a Banach space with norm given by partitions and weights which is not isomorphic to a subspace of L_p by applying the results from Section 3. Thus we find that not every sequence space with norm given by partitions and weights is a $\mathcal{L}_p$ space. (See [L-P] and [L-R].)

In the last section we pose some questions for further study. In particular we discuss the Bourgain, Rosenthal and Schechtman construction and define spaces X_p^α with norm given by partitions and weights which are natural candidates for sequence space realizations of the spaces R_p^α.

We will use standard terminology and reults in Banach theory as may be found in the books [L-T-1], [L-T-2] and [J-L]. Many results on subspaces of L_p may be found in the exposition [A-O] and its references.

2. Norms determined by partitions and weights.

In this section, we examine some examples of complemented subspaces of L_p in order to motivate the idea of a norm given by partitions and weights. Then we develop the formal definition of a norm given by an admissible family of partitions and weights. Finally we give some results about sums of spaces with these norms.

In the following we will see that many well known complemented subspaces of L_p have equivalent norms of the form defined in the Introduction. Here it is sometimes convenient to take partitions and weights defined on sets other than $\mathbb{N}$. For each example we will have a family of partitions (P_k) of $\mathbb{N}^m$ for some m and weights (W_k) for k in some index set K.

EXAMPLE 2.1 (examples with one partition and weight; $K = \{1\}$).

(1) If $P = \{\{i\} : i \in \mathbb{N}\}$ and $W = (w_n)$ is any sequence of positive numbers, then $X \sim \ell_p$ since

$$\|(x_n)\| = \|(x_n)\|_{P,W} = \Big(\sum_{n=1}^{\infty}(|x_n|^2 w_n^2)^{p/2}\Big)^{1/p} = \Big(\sum_{n=1}^{\infty}|x_n|^p w_n^p\Big)^{1/p}.$$

(2) If $P = \{\mathbb{N}\}$ and $W = (w_n)$ is any sequence of positive numbers, then $X \sim \ell_2$ since

$$\|(x_n)\| = \Big(\Big(\sum_{n=1}^{\infty}|x_n|^2 w_n^2\Big)^{p/2}\Big)^{1/p} = \Big(\sum_{n=1}^{\infty}|x_n|^2 w_n^2\Big)^{1/2}.$$

(3) If the index set is $\mathbb{N}\times\mathbb{N}$, the partition $P = \{\{n\}\times\mathbb{N} : n \in \mathbb{N}\}$, and $W = (w_{n,m})_{n,m\in\mathbb{N}}$, then $X \sim (\sum \ell_2)_{\ell_p}$ since

$$\|(x_n)\| = \Big(\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty}|x_{n,m}|^2 w_{n,m}^2\Big)^{p/2}\Big)^{1/p}.$$

EXAMPLE 2.2 (examples with two partitions and weights; $K = \{1,2\}$).

(1) If $P_1 = \{\{n\}\}$ with weight $W_1 = (1)$ and $P_2 = \{\mathbb{N}\}$ with weight $W_2 = (w_n)$, then X is the space X_{p,W_2}, defined by Rosenthal, with norm

$$\|(a_i)\| = \max\Big\{\Big(\sum |a_n|^p\Big)^{1/p}, \Big(\sum |w_n a_n|^2\Big)^{1/2}\Big\}.$$

Rosenthal [R] proved the following:

(a) If $\inf_n w_n > 0$, then $X_{p,W_2} \sim \ell_2$.

(b) If $\sum w_n^{2p/(p-2)} < \infty$, then $X_{p,W_2} \sim \ell_p$.

(c) If there is some $\varepsilon > 0$ for which $\{n : w_n \ge \varepsilon\}$ and $\{n : w_n < \varepsilon\}$ are both infinite and for which $\sum_{w_n<\varepsilon} w_n^{2p/(p-2)} < \infty$, then $X_{p,W_2} \sim \ell_2 \oplus \ell_p$.

(d) For each $\varepsilon > 0$,

$$(*) \qquad \sum_{w_n<\varepsilon} w_n^{2p/(p-2)} = \infty.$$

If W_2 satisfies $(*)$, then $X_{p,W_2} \sim X_p$.

(2) If $P_1 = \{\{(i,j)\}\}$ with weight $W_1 = (1)$ and $P_2 = \{\{n\} \times \mathbb{N}\}$ with weight $W_2 = (w_{n,m})$ where $w_{n,m} = (1/n)$ for all n,m, then $X \sim (\sum_n X_{p,(1/n)})_p$. Moreover, $X_n = X_{p,(1/n)}$ is isomorphic to ℓ_2 and we have $\sup_{n\in\mathbb{N}} d(X_n, \ell_2) = \infty$, so $X \sim B_p$, as defined by Rosenthal.

EXAMPLE 2.3 (an example with four partitions and weights). Let $K = \{0,1,2,3\}$. Let i represent the first index and j represent the second index in the set $\mathbb{N}\times\mathbb{N}$. Assume the sequences (w_i) and (w'_j) satisfy $(*)$. Let

$$\begin{aligned} P_0 &= \{\mathbb{N}\times\mathbb{N}\} && \text{with weight } W_0 = (w_i \cdot w'_j),\\ P_1 &= \{\{n\}\times\mathbb{N} : n\in\mathbb{N}\} && \text{with weight } W_1 = (1\cdot w'_j),\\ P_2 &= \{\mathbb{N}\times\{n\} : n\in\mathbb{N}\} && \text{with weight } W_2 = (w_i\cdot 1),\\ P_3 &= \{\{(i,j)\} : i,j\in\mathbb{N}\} && \text{with weight } W_3 = (1\cdot 1). \end{aligned}$$

Then this is Schechtman's [S] basic example $X \sim X_p \otimes X_p$, with norm

$$\max\Big\{\Big(\sum_{i,j}|a_{i,j}|^2 w_i^2 w_j'^2\Big)^{1/2}, \Big(\sum_i\Big(\sum_j |a_{i,j}|^2 w_j'^2\Big)^{p/2}\Big)^{1/p}, \\ \Big(\sum_j\Big(\sum_i |a_{i,j}|^2 w_i^2\Big)^{p/2}\Big)^{1/p}, \Big(\sum_{i,j}|a_{i,j}|^p\Big)^{1/p}\Big\} \approx \Big\|\sum_{i,j} a_{i,j}(x_i\otimes y_j)\Big\|_{L_p(I\times I)}. \tag{2.1}$$

This example can be generalized by using the index set $\mathbb{N}^n$. If $|K| = 2^n$ and the partitions and weights are chosen in a manner similar to the above, then $X \sim \bigotimes_{k=1}^n X_p$. Thus we get isomorphs of Schechtman's examples. Additional detail about these examples is contained in Section 4.

We will now give a general definition of a space with norm given by partitions and weights. Below A is any countable set.

DEFINITION 2.4. Let $P = \{N_i\}$ be a partition of A and $W : A \to (0,1]$ be a function, which we refer to as the *weights*. Let $x_j \in \mathbb{R}$ and $w_j = W(j)$ for all $j \in A$. Define

$$\|(x_j)_{j\in A}\|_{P,W} = \Big(\sum_i\Big(\sum_{j\in N_i} x_j^2 w_j^2\Big)^{p/2}\Big)^{1/p}.$$

Suppose that $(P_k, W_k)_{k\in K}$ is a family of pairs of partitions and functions as above. Define a (possibly infinite) norm on the real-valued functions on A, $(x_i)_{i\in A}$, by $\|(x_i)\| = \sup_{k\in K}\|(x_i)\|_{(P_k,W_k)}$ and let X be the subspace of elements of finite norm. In this case we say that X has a *norm given by partitions and weights*.

REMARK 2.5. Because of the nature of this norm, X will have a natural unconditional basis. Thus this approach to describing the complemented subspaces of L_p is limited to complemented subspaces of L_p with unconditional basis. At this time, no complemented subspace of L_p without unconditional basis is known.

Proposition 2.6. *Suppose that X has a norm given by partitions and weights. Then X is a Banach space.*

We leave the straightforward proof to the reader.

Proposition 2.7. *Suppose X is a Banach space with norm given by one partition and weight. Then $X \sim \ell_p$, $X \sim \ell_2$, $X \sim \ell_2 \oplus \ell_p$, or $X \sim (\sum^{\oplus} \ell_2)_{\ell_p}$.*

Notice that these are the spaces given in Example 2.1 and their direct sums. The proof is a routine computation after normalization of the basis.

Since normalization of the basis is an important first step to understanding the spaces, we now introduce admissible families of partitions and weights to incorporate this and some other properties.

Definition 2.8. The partition $\{\{a\} : a \in A\}$ of A will be called the *discrete* partition. The partition $\{A\}$ of A will be called the *indiscrete* partition.

Definition 2.9. A family of partitions and weights is called *admissible* if it contains the discrete partition with the trivial weight $(w(a))_{a\in A} = (1)$ and the indiscrete partition with some weight.

The discrete partition is included to force the natural coordinate basis to be normalized. This requirement is not really a restriction because every normalized unconditional basic sequence in L_p has a lower ℓ_p estimate. (See the Preliminaries section of [A-O].) The indiscrete partition gives a candidate for a natural ℓ_2 structure on the space X. Because we are concerned with embedding these spaces into L_p, $p > 2$, there always must be some ℓ_2 structure on the space.

Notice that in the previous examples, Rosenthal's space and Schechtman's space have norms given by admissible families of partitions and weights. Each of the other cases can be equivalently renormed using an admissible family of partitions and weights. Unless otherwise noted we will assume from now on that a Banach space X with norm given by partitions and weights is actually given by an admissible family of partitions and weights.

Next we are going to prove some stability results for sums of spaces when the spaces are equipped with these norms.

Definition 2.10. Let (X_n) be a sequence of subspaces of $L_p(\Omega, \mu)$ for some probability measure μ, and let (w_n) be a sequence of real numbers, $0 < w_n \le 1$. For any sequence (x_n) such that $x_n \in X_n$ for all n, let

$$\|(x_n)\|_{p,2,(w_n)} = \max\Big\{ \Big(\sum \|x_n\|_p^p\Big)^{1/p}, \Big(\sum \|x_n\|_2^2 w_n^2\Big)^{1/2}\Big\}$$

and let

$$X = \Big(\sum X_n\Big)_{(p,2,(w_n))} = \{(x_n) : x_n \in X_n \text{ for all } \|(x_n)\|_{p,2,(w_n)} < \infty\}.$$

We will say that X is the $(p,2,(w_n))$ *sum of* $\{X_n\}$.

Let A be a countable set and let $(X_a)_{a\in A}$ be a family of Banach spaces of functions defined on sets $(B_a)_{a\in A}$, respectively. That is, for each $a \in A$, X_a has a norm given by a family of partitions of B_a and weights on B_a. Let I_a denote the index set of the corresponding family for X_a. For each $i(a) \in I_a$, let $P^{a,i(a)}$ be a partition of B_a and $W^{a,i(a)}$ be a weight function, i.e., $W^{a,i(a)} : B_a \to (0,1]$. For each $a \in A$ and $i(a) \in I_a$, the norm on X_a with respect to $P^{a,i(a)}, W^{a,i(a)}$ is given by

$$\|(x_{a,b})_{b\in B_a}\|_{P^{a,i(a)},W^{a,i(a)}} = \Big(\sum_{Q\in P^{a,i(a)}}\Big(\sum_{b\in Q}(x_{a,b})^2(w^{a,i(a)}(b))^2\Big)^{p/2}\Big)^{1/p}.$$

For each a, we assume that there is one distinguished indiscrete partition and weight. We will denote the index of this partition and weight as $(\,)$. Let $P^{a,()} = \{B_a\}$, and $W^{a,()}$ be the associated weight. For each a, define

$$\|(x_{a,b})_{b\in B_a}\|_2 = \Big(\sum_{b\in B_a}(x_{a,b})^2(w^{a,(\,)}(b))^2\Big)^{1/2}.$$

Suppose that for the index set A, we have an associated weight function $W : A \to (0,1]$. Let $(\sum_{a\in A} X_a)_{p,2,W}$ be defined on $B = \coprod_{a\in A} B_a$ as above using the norm $\|(x_{a,b})_{b\in B_a}\|_2$ as the $\|\cdot\|_2$ in the definition. Let $I = \prod_{a\in A} I_a \cup \{(\ \ )\}$. Let $(i(a))_{a\in A} \in I$. Then there is a natural partition of B and weight on B given by $P_{(i(a))} = \{\{a\}\times P : P \in P^{a,i(a)},\ a\in A\}$ and $W_{(i(a))} = (w_b^{a,i(a)})_{b\in B_a, a\in A}$. We define as a special case the partition and weight for $(\ \ )$ as $P_{(\ )} = \{\coprod_{a\in A} B_a\}$ and $W_{(\ )} = (W(a)w^{a,(\ )}(b))_{b\in B_a,a\in A}$.

If we expand the definition of the norm we have

$$\begin{aligned}
(2.2)\quad &\|(x_{a,b})_{a\in A,b\in B_a}\|_{p,2,W}\\
&= \max\Big\{\Big(\sum_{a\in A}\|(x_{a,b})_{b\in B_a}\|_{X_a}^p\Big)^{1/p}, \Big(\sum_{a\in A}\|(x_{a,b})_{b\in B_a}\|_2^2(W(a))^2\Big)^{1/2}\Big\}\\
&= \max\Big\{\Big(\sum_{a\in A}\sup_{i(a)\in I_a}\{\|(x_{a,b})_{b\in B_a}\|^p_{P^{a,i(a)},W^{a,i(a)}}\}\Big)^{1/p},\\
&\qquad\qquad \Big(\sum_{a\in A} W(a)^2\sum_{b\in B_a}(w^{a,()}(b))^2|x_{a,b}|^2\Big)^{1/2}\Big\}.
\end{aligned}$$

Notice that for each $a \in A$, we take a supremum over I_a, then we take summation of those supremums, and finally we take the maximum of two sums. If we consider the index $(i(a))$ which for each a approximates the supremum, it is one element in I. So instead of taking the maximum over each I_a, we can compute the norm for each index in I, and then take the supremum of them only once. Hence the norm becomes

$$\|(x_{a,b})_{a\in A,b\in B_a}\|_{p,2,W} = \sup_{(i(a))\in I} \|(x_{a,b})_{b\in B_a}\|_{P_{(i(a))},W_{(i(a))}}.$$

This gives us the following result:

PROPOSITION 2.11. *Let $(X_a)_{a\in A}$ be a family of Banach spaces each with norm given by partitions and weights. Then the norm of the space $(\sum_a X_a)_{p,2,W}$ can also be expressed as a norm given by partitions and weights. In other words, the class of spaces with norm given by partitions and weights is stable under $(p,2)$ sums.*

COROLLARY 2.12. *Let $(X_a)_{a\in A}$ be a family of Banach spaces with norm given by an admissible family of partitions and weights. Then the norm of the space $(\sum_a X_a)_{\ell_p}$ can also be expressed with partitions and weights.*

Proof. Choose W such that $\sum_{a\in A} W(a)^{2p/(p-2)} < 1$ in the previous result.

3. Embedding into L_p. In this section, we first show that any space X with norm given by finitely many partitions and weights is isomorphic to a subspace of L_p. Then we give the definition of an envelope norm. We prove the existence of the envelope norm generated by a family of partitions and weights. We also determine a necessary condition for a space with unconditional basis to be isomorphic to a subspace of L_p, $p > 2$, by giving a lower bound on the L_p norm of elements of any subspace of L_p with unconditional basis in terms of blocks relative to the basis. Finally we show by using this necessary condition that if a space with norm given by partitions and weights is isomorphic to a subspace of L_p, then its norm is equivalent to the natural envelope norm.

PROPOSITION 3.1. *Any sequence space X with norm given by finitely many partitions and weights is isomorphic to a subspace of L_p.*

Proof. Let X be the sequence space with partitions and weights $(P_n, W_n)_{n=1}^N$. Let X_n be the space of sequences with norm given by one partition and weight (P_n, W_n), $1 \le n \le N$. Then it is easy to see that

$$\Big(\sum_{n=1}^N X_n\Big)_{\ell_\infty} \sim \Big(\sum_{n=1}^N X_n\Big)_{\ell_p}$$

Take the isometric embedding from X into $(\sum_{n=1}^N X_n)_{\ell_\infty}$ defined by $x \mapsto (x)_{n=1}^N$. Since for each $n = 1,\dots,N$, X_n is isomorphic to a complemented subspace of L_p by Proposition 2.7, we see that $(\sum_{n=1}^N X_n)_{\ell_p}$ is isomorphic to a complemented subspace of L_p. Hence X is isomorphic to a subspace of L_p. ∎

This trivial approach fails if there are infinitely many partitions and weights. We do not know any general sufficient condition on the partitions and weights to guarantee that the space is isomorphic to a subspace of L_p. By using the fact that these spaces have unconditional basis, we can give a useful necessary condition.

Definition 3.2. Let $X = \{(a_b)_{b\in B}\}$ be a Banach space defined on a countable set B with norm given by a set of partitions and weights $\mathcal{P} = \{(P^j, W^j) : j \in K\}$. Then $\mathcal{P}$ has the *envelope property* if for any partition Q of B and any function $i : Q \to K$, the partition and weight (P_0, W_0) belongs to $\mathcal{P}$ where

$$P_0 = \{K' : K' = q \cap K_{i(q)} \neq \emptyset \text{ for some } q \in Q,\ \text{some } K_{i(q)} \in P^{i(q)}\}$$

and

$$W_0 = (w_b^{i(q)})_{b\in q, q\in Q} \quad \text{where} \quad W^{i(q)} = (w_b^{i(q)})_{b\in B}.$$

In this case we will say that $\|\!|\cdot|\!\| = \sup_{i\in K} \|\cdot\|_{P^i,W^i}$ is an *envelope norm.*

Note that the function i in this definition induces a map $\phi : Q \to \mathcal{P}$ by $\phi(q) = (P^{i(q)}, W^{i(q)})$. Conversely any map $\phi : Q \to \mathcal{P}$ induces a corresponding map i. In what follows we will often start with ϕ when applying the definition.

Example 3.3. Let X_p be Rosenthal's space with the norm

$$\|(a_i)\| = \max\Big\{\Big(\sum |a_n|^p\Big)^{1/p}, \Big(\sum |w_n a_n|^2\Big)^{1/2}\Big\}$$

where (w_n) satisfies $(*)$. Let $P_1 = \{\{n\}\}$ with weight $W_1 = (1)$ and $P_2 = \{\mathbb{N}\}$ with weight $W_2 = (w_n)$. Then $\mathcal{P} = \{(P_1, W_1), (P_2, W_2)\}$ defines the norm on X_p. It is easy to see that $\mathcal{P}$ does not have the envelope property. To get a family of partitions and weights which has the envelope property we need to add all the possible combinations of the given two. Let $\mathcal{Q}$ be the set of all partitions on $\mathbb{N}$. Let $Q \in \mathcal{Q}$ and $T : Q \to \mathcal{P}$. Define

$$\begin{aligned} P(Q,T) = \{K : K = \{n\} \text{ if } n \in q \text{ and } T(q) = (P_1, W_1) \text{ for some } q \in Q\} \\ \cup \{K : K = q \text{ if } T(q) = (P_2, W_2) \text{ for some } q \in Q\} \end{aligned}$$

and

$$W(Q,T) = (w(n))_{n\in q, q\subset \mathbb{N}} \quad \text{where} \quad w(n) = \begin{cases} 1 & \text{if } n \in q,\ T(q) = (P_1, W_1), \\ w_n & \text{if } n \in q,\ T(q) = (P_2, W_2). \end{cases}$$

Then an equivalent envelope norm is defined by $\sup_{(P,W)\in\widetilde{\mathcal{P}}} \|\cdot\|_{(P,W)}$ where $\widetilde{\mathcal{P}} = \{(P(Q,T), W(Q,T)) : Q \in \mathcal{Q},\ T : Q \to \mathcal{P}\}$.

REMARK 3.4. In the case of the norm of X_p in Example 3.3 the envelope norm can be written in the form

$$\max_{q\subset\mathbb{N}}\Big(\sum_{n\in q}|a_n|^p+\Big(\sum_{n\notin q}|a_n|^2w_n^2\Big)^{p/2}\Big)^{1/p}.$$

Next we show by generalizing the construction above that there is a natural envelope norm associated to each norm given by partitions and weights.

PROPOSITION 3.5. *Suppose X is a Banach space defined on a countable set B with norm given by a family $\mathcal{P}$ of partitions and weights. Then there exists a natural family of partitions and weights $\widetilde{\mathcal{P}}$ (defined below) such that $|||\cdot||| = \sup_{(P,W)\in\widetilde{\mathcal{P}}}\|\cdot\|_{P,W}$ is an envelope norm.*

Proof. Let $\mathcal{Q}$ be the set of all partitions of B. Let $\mathcal{P} = \{(P_i, W_i) : i \in K\}$ be the given family of partitions and weights for X. Let $Q \in \mathcal{Q}$. Letting T be a map from Q into $\mathcal{P}$ denote $T(q)$ by $(P^{i(q)}, W^{i(q)})$ for all $q \in Q$. Define $P(Q,T) = \{K : K = q\cap p \neq \emptyset,\ q \in Q,\ p \in P^{i(q)}\}$ and $W(Q,T) = (w^{i(q)}(b))_{b\in q,\, q\in Q}$ where $W^{i(q)} = (w^{i(q)}(b))_{b\in B}$. Let $\widetilde{\mathcal{P}} = \{(P(Q,T), W(Q,T)) : Q \in \mathcal{Q},\ T : Q \to \mathcal{P}\}$. Define a norm on X as $|||(x_i)||| = \sup_{(P,W)\in\widetilde{\mathcal{P}}}\|(x_i)\|_{P,W}$. We claim that $\widetilde{\mathcal{P}}$ has the envelope property and thus $|||\cdot|||$ is an envelope norm.

Let $\overline{Q}$ be any partition of B. Let S be any map from $\overline{Q}$ into $\widetilde{\mathcal{P}}$, i.e.,

$$S(\bar q) = (P(Q_{\bar q}, T_{\bar q}), W(Q_{\bar q}, T_{\bar q}))$$

for all $\bar q \in \overline{Q}$. For any $\bar q \in \overline{Q}$, let $T_{\bar q}(q_0) = (P^{i(\bar q,q_0)}, W^{i(\bar q,q_0)})$ for all $q_0 \in Q_{\bar q}$ and let $\overline{\overline{Q}} = \{\bar{\bar q} \neq \emptyset : \bar{\bar q} = q_0\cap\bar q,\ \bar q\in\overline{Q},\ q_0\in Q_{\bar q}\}$. Because $\overline{Q}$ and $Q_{\bar q}$ are partitions, $\bar{\bar q}$ uniquely determines $\bar q \in \overline{Q}$ and $q_0 \in Q_{\bar q}$ such that $\bar{\bar q} = q_0 \cap \bar q$. From Definition 3.2 we have $P_0 = \{\overline{K} \neq \emptyset : \overline{K} = \bar q \cap \overline{K}_{\bar q},\ \bar q \in \overline{Q},\ \overline{K}_{\bar q} \in P(Q_{\bar q}, T_{\bar q})\}$, which is exactly what the definition above gives for the partition $P(\overline{Q}, S)$ of B determined by $\overline{Q}$ and S. Thus

$$\begin{aligned}
P_0 &= \overline{P}(\overline{Q}, S)\\
&= \{\overline{K} : \overline{K} = \bar q\cap\overline{K}_{\bar q}\neq\emptyset,\ \bar q\in\overline{Q},\ \overline{K}_{\bar q}\in P(Q_{\bar q},T_{\bar q})\}\\
&= \{\overline{K} : \overline{K} = \bar q\cap(q_0\cap p)\neq\emptyset,\ \bar q\in\overline{Q},\ q_0\in Q_{\bar q},\ p\in P^{i(\bar q,q_0)}\} &\text{(3.1)}\\
&= \{\overline{K} : \overline{K} = (\bar q\cap q_0)\cap p\neq\emptyset,\ \bar q\cap q_0\in\overline{\overline{Q}},\ p\in P^{i(\bar q,q_0)}\} &\text{(3.2)}\\
&= \{\overline{K} : \overline{K} = \bar{\bar q}\cap p\neq\emptyset,\ \bar{\bar q} = \bar q\cap q_0\in\overline{\overline{Q}},\ p\in P^{i(\bar q,q_0)}\} &\text{(3.3)}
\end{aligned}$$

where (3.1) follows from the definition of $P(Q_{\bar q}, T_{\bar q})$, (3.2) by the definition of $\overline{\overline{Q}}$, and (3.3) by the uniqueness of $\bar q$ and q_0. Define $\overline{\overline{T}} : \overline{\overline{Q}} \to \mathcal{P}$ by $\overline{\overline{T}}(\bar{\bar q}) = (P^{i(\bar q,q_0)}, W^{i(\bar q,q_0)})$ where $\bar{\bar q} = \bar q\cap q_0$, $q_0\in Q_{\bar q}$, $\bar q\in\overline{Q}$. Then we have shown that $\overline{P}(\overline{Q}, S) = P(\overline{\overline{Q}}, \overline{\overline{T}})$.

Because $T_{\bar q}(q) = (P^{i(\bar q,q)}, W^{i(\bar q,q)}) = (P^{i(\bar q,q)}, (w^{i(\bar q,q)}(b))_{b\in B})$, we have

$$S(\bar q) = (P(Q_{\bar q}, T_{\bar q}), W(Q_{\bar q}, T_{\bar q})) = (P(Q_{\bar q}, T_{\bar q}), (w^{i(\bar q,q)}(b))_{b\in q,\, q\in Q_{\bar q}}).$$

Suppose $W_0 = (w_b)_{b\in B}$. If $\bar q \in \overline{Q}$ and $b \in \bar q$, then as in Definition 3.2, $w_b = w^{i(\bar q,q_0)}(b)$ where $b \in q_0$ and $q_0 \in Q_{\bar q}$. Hence for $b \in \bar{\bar q} = \bar q \cap q_0$, $w^{i(\bar q,q_0)}(b)$ is also the choice specified by $\overline{\overline{T}}(\bar{\bar q})$. Hence $W_0 = W(\overline{\overline{Q}}, \overline{\overline{T}})$. So $(P_0, W_0) \in \widetilde{\mathcal P}$. ∎

Corollary 3.6. *If X has a norm defined by a finite number of partitions and weights, then there is an equivalent envelope norm on X.*

The next result follows by simply checking that all the partitions and weights in the construction are required for the envelope property.

Proposition 3.7. *Suppose $\mathcal P$ is a family of partitions and weights on B. Then $\widetilde{\mathcal P}$ as in Proposition* 3.5 *is the minimal family of partitions and weights on B containing $\mathcal P$ and having the envelope property.*

Corollary 3.8. *Let $\mathcal P$ be a non-empty family of partitions and weights on B. Let $(\mathcal P_\lambda)_{\lambda\in\Lambda}$ be any chain of families of partitions and weights on B such that each $\mathcal P_\lambda$ contains $\mathcal P$ and has the envelope property. Then $\bigcap_{\lambda\in\Lambda}\mathcal P_\lambda$ has the envelope property.*

Our purpose in introducing envelope norms is to show that the envelope norm is related to a property of subspaces of L_p with unconditional basis that we will now explain.

Let X be a Banach space defined on B with a norm given by partitions and weights. Let ϕ be a one-to-one map from $\mathbb N$ onto B such that $x_n = e_{\phi(n)}$ where $(e_b)_{b\in B}$ is the natural unit vector basis of X. Then (x_n) is an unconditional basis for X. Let $x = \sum_{n=1}^\infty a_n x_n$ for some (a_n). Let Q be any partition of B. Let $\{F_k\}_{k=1}^\infty$ be the corresponding partition of $\mathbb N$, i.e., $\phi(F_k) = q$ for some $q \in Q$. Then

$$x = \sum_{k=1}^\infty \sum_{n\in F_k} a_n x_n = \sum_{k=1}^\infty z_k = \sum_{q\in Q} z'_q$$

where

$$z_k = \sum_{n\in F_k} a_n x_n = \sum_{n\in F_k} a_n e_{\phi(n)} = \sum_{b\in q=\phi(F_k)} a_{\phi^{-1}(b)} e_b = z'_{\phi(F_k)} = z'_q.$$

Since (x_n) is an unconditional basis and (z_k) (hence (z'_q)) is a block of (x_n), we see that (z_k) is an unconditional basic sequence with unconditional constant 1.

Remark 3.9. We are abusing the terminology a little here. It would be more correct to use partitions $\{F_k\}$ containing only finite subsets of $\mathbb N$

and the map ϕ which has the property that for all k, if $n \in \phi^{-1}(F_k)$ and $m \in \phi^{-1}(F_{k+1})$ then $n < m$.

In the lemma below we use the notation introduced in Proposition 3.5.

LEMMA 3.10. *Let $\mathcal{P} = \{(P^i, W^i) : i \in K\}$ be a family of partitions and weights on B. Let X be the corresponding Banach space defined on B. If X is isomorphic to a subspace of L_p, then there exists a constant C, depending only on the Banach–Mazur distance of X to a subspace of L_p, such that for any partition Q of B and any map $T : Q \to \mathcal{P}$, $\|x\| \geq C\|x\|_{(P(Q,T),W(Q,T))}$ where $T(q) = (P^{i(q)}, W^{i(q)})$.*

Proof. Let $\phi : \mathbb{N} \to B$ be as above and $T : Q \to \mathcal{P}$ such that $T(q) = (P^{i(q)}, W^{i(q)})$. If X is isomorphic to a subspace of L_p, with isomorphism R, then (Rz_k) (hence (Rz'_q)) is a block of (Rx_n) which is an unconditional basic sequence in L_p with constant λ. So

$$\begin{aligned}
\|x\|_X &= \Big\|\sum_{k=1}^{\infty} z_k\Big\| \geq \|R\|^{-1}\Big\|\sum_k Rz_k\Big\|_{L_p} \\
&\geq \|R\|^{-1}\lambda^{-1}\Big(\sum_k \|Rz_k\|_p^p\Big)^{1/p} \qquad (3.4)\\
&\geq \|R\|^{-1}\lambda^{-1}\Big(\sum_k \frac{\|z_k\|_X^p}{\|R^{-1}\|^p}\Big)^{1/p} = \frac{\lambda^{-1}}{\|R\|\,\|R^{-1}\|}\Big(\sum_{q\in Q}\|z_q\|_X^p\Big)^{1/p} \\
&\geq C\Big(\sum_{q\in Q}\sum_{r\in P^{i(q)}}\Big(\sum_{\phi(n)\in r\cap q}|a_n|^2(w_{\phi(n)}^{i(q)})^2\Big)^{p/2}\Big)^{1/p} \qquad (3.5)\\
&= C\Big(\sum_{\overline{q}\in P(Q,T)}\Big(\sum_{\phi(n)\in\overline{q}}|a_n|^2(w_{\phi(n)}^{i(q)})^2\Big)^{p/2}\Big)^{1/p} = C\|x\|_{P(Q,T),W(Q,T)}
\end{aligned}$$

where (3.4) follows from the standard lower ℓ_p estimate [A-O], and (3.5) is true since $z_q = \sum_{n\in F_k,\phi(F_k)=q} a_n x_n$, and $\|z'_q\|_X \geq \|z'_q\|_{P^{i(q)},W^{i(q)}}$. In (3.5), q is the unique element of Q such that $\overline{q} \subset q$. ∎

THEOREM 3.11. *Suppose X has a norm given by a family $\mathcal{P}$ of partitions and weights and X is isomorphic to a subspace of L_p. Then there is an envelope norm $|||\cdot|||$ such that $|||\cdot||| \sim \|\cdot\|_X$.*

Proof. If we take a supremum over all the choices of Q and T in Lemma 3.10, we have $\|x\|_X \geq C|||x|||$, where $|||\cdot|||$ is the envelope norm defined by $\widetilde{\mathcal{P}}$ in the proof of Proposition 3.5. On the other hand, since $\mathcal{P} \subset \widetilde{\mathcal{P}}$, we get $\|x\|_X \leq |||x|||$. Hence $\|x\|_X \sim |||x|||$. ∎

Remark 3.12. Because the natural basis of a space with norm given by partitions and weights is 1-unconditional, the unconditional constant of the image of any block basis under an isomorphism R is at most $\|R\|\,\|R^{-1}\|$. Hence the constant λ in the proof of Lemma 3.10 and consequently the equivalence in Theorem 3.11 depend only on the distance to a subspace of L_p.

Proposition 3.13. *Let $(X_a)_{a\in A}$ be a family of Banach spaces each with norm given by partitions and weights which satisfies the envelope property. Then the norm of the space $(\sum_a X_a)_{p,2,W}$ can also be expressed as a norm given by partitions and weights which also satisfies the envelope property.*

Proof. This follows from equation (2.2). Indeed, if Q is any partition of $\coprod_{a\in A} B_a$ and $\phi : Q \to \{(P^{(i(a))}, W^{(i(a))}) : (i(a)) \in I\}$ then for each $q \in Q$ let $\phi(q) = (P^{(i(a),q)}, W^{(i(a),q)})$. Then (Q_a, W_a), where $Q_a = \{q \cap q' \neq \emptyset : q \in Q,\ q' \in P^{i(a),q}\}$ and $W_a = (w_b^{a,i(a,q)})_{b\in B_a}$, must be one of the partitions and weights in $\{(P^{a,i(a)}, W^{a,i(a)}) : i(a) \in I_a\}$ since this family of partitions and weights has the envelope property. ∎

Next we consider the L_p tensor product of spaces with norms given by partitions and weights. Because the tensor product is only defined for subspaces of L_p, we will assume that the two spaces are also isomorphic to subspaces of L_p and thus the defining families of partitions and weights must have the envelope property.

Proposition 3.14. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are families of partitions and weights defined on sets A and B, respectively, and having the envelope property. Let X and Y be the corresponding spaces on A and B and let $(x_a)_{a\in A}$ and $(y_b)_{b\in B}$, respectively, be the natural bases. Suppose that S is an isomorphism from X into $L_p[0,1]$ and T is an isomorphism from Y into $L_p[0,1]$. Then there is a constant $C>0$ such that for any constants $(c_{a,b})_{a\in A,\, b\in B}$ with only finitely many non-zero, any $(P,(w_a)) \in \mathcal{P}$ and any $(Q,(w'_b)) \in \mathcal{Q}$,*

$$\Big\|\sum c_{a,b} Sx_a \otimes Ty_b\Big\|_p \ge C\Big(\sum_{p\in P,\, q\in Q}\Big(\sum_{a\in p,\, b\in q}|c_{a,b}|^2(w_a)^2(w'_b)^2\Big)^{p/2}\Big)^{1/p}.$$

Proof. With the given notation we can directly estimate the norm as follows:

$$\Big\|\sum c_{a,b} Sx_a \otimes Ty_b\Big\|_p^p = \int_0^1\!\!\int_0^1 \Big|\sum c_{a,b} Sx_a(s) Ty_b(t)\Big|^p\,dt\,ds$$
$$= \int_0^1\!\!\int_0^1 \Big|\sum_{b\in B}\Big(\sum_{a\in A} c_{a,b} Sx_a(s)\Big) Ty_b(t)\Big|^p\,dt\,ds$$

$$= \int_0^1 \Big\| T\Big(\sum_{b\in B}\Big(\sum_{a\in A} c_{a,b} Sx_a(s)\Big) y_b\Big)\Big\|_p^p \, ds$$

$$\sim \int_0^1 \max_{(Q_1,(w_b''))\in\mathcal{Q}} \Big(\sum_{q\in Q_1}\Big(\sum_{b\in q}\Big|\sum_{a\in A} c_{a,b} Sx_a(s)\Big|^2 (w_b'')^2\Big)^{p/2}\Big)\, ds$$

(since T is an isomorphism)

$$\geq \int_0^1 \Big(\sum_{q\in Q}\Big(\sum_{b\in q}\Big|\sum_{a\in A} c_{a,b} Sx_a(s)\Big|^2 (w_b')^2\Big)^{p/2}\Big)\, ds$$

$$\sim \sum_{q\in Q} \int_0^1\int_0^1 \Big|\sum_{b\in q}\sum_{a\in A} c_{a,b} Sx_a(s) w_b' r_b(u)\Big|^p \, du\, ds$$

(by Khinchin's inequality, where $(r_b)_{b\in B}$ is a sequence of Rademacher functions)

$$= \sum_{q\in Q} \int_0^1 \Big\|\sum_{a\in A}\Big(\sum_{b\in q} c_{a,b} w_b' r_b(u)\Big) Sx_a(s)\Big\|^p \, du$$

$$\sim \sum_{q\in Q} \int_0^1 \max_{(P_1,(w_a''))\in\mathcal{P}} \Big(\sum_{p\in P_1}\Big(\sum_{a\in p}\Big(\sum_{b\in q} c_{a,b} w_b' r_b(u)\Big)^2 w_a''^2\Big)^{p/2}\Big)\, du$$

(since S is an isomorphism)

$$\geq \sum_{q\in Q} \int_0^1 \Big(\sum_{p\in P}\Big(\sum_{a\in p}\Big(\sum_{b\in q} c_{a,b} w_b' r_b(u)\Big)^2 w_a^2\Big)^{p/2}\Big)\, du$$

$$= \sum_{q\in Q}\sum_{p\in P} \int_0^1 \Big(\sum_{a\in p}\Big(\sum_{b\in q} c_{a,b} w_b' w_a r_b(u)\Big)^2\Big)^{p/2} \, du$$

$$\sim \sum_{q\in Q}\sum_{p\in P} \int_0^1\int_0^1 \Big|\sum_{b\in q}\sum_{a\in p} c_{a,b} w_b' w_a r_b(u) r_a'(w)\Big|^p \, dw\, du$$

where $(r_a')_{a\in A}$ is another sequence of Rademacher functions, independent of $(r_b)_{b\in B}$)

$$\sim \sum_{q\in Q}\sum_{p\in P}\Big(\sum_{b\in q}\sum_{a\in p} c_{a,b}^2 (w_b')^2 w_a^2\Big)^{p/2}. \ \blacksquare$$

In the proof above there are two inequalities which result from taking only one of the partitions and weights defining the norm. If the norm is equivalent to a norm given by finitely many partitions and weights, then we can remove the inequality lines, insert maximums, and complete the argument as above except that at a few places we must interchange the maximum with an integral. This is possible with a constant depending on the number of partitions and weights. Therefore we have

COROLLARY 3.15. *If X and Y have norms given by finitely many partitions and weights, then the L_p tensor product of X and Y also has a norm given by finitely many partitions and weights.*

COROLLARY 3.16. *With the same hypothesis as in Proposition* 3.14, *there is a constant $C' > 0$ such that for any partition R of $A \times B$ and partitions and weights $(P_r, (w_{r,a}))_{r\in R}$ and $(Q_r, (w'_{r,a}))_{r\in R}$ such that for each $r \in R$, there exist $p_r \in P_r$ and $q_r \in Q_r$ with $r \subset p_r \times q_r$, we have*

$$\Big\|\sum c_{a,b} Sx_a \otimes Ty_b\Big\| \ge C'\Big(\sum_{r\in R}\Big(\sum_{a,b\in r} |c_{a,b}|^2 (w_{r,a})^2 (w'_{r,b})^2\Big)^{p/2}\Big)^{1/p}.$$

Proof (sketch). Because $SX \otimes TY$ is a subspace of L_p, this follows by the argument given in the proof of Lemma 3.10. ∎

4. An isomorph of $\bigotimes_{j=1}^n X_p$. In this section, we construct an example which demonstrates the difference between a norm given by partitions and weights and the corresponding envelope norm. We also obtain an estimate of the distance between a certain Banach space Y_n, isomorphic to $\bigotimes_{j=1}^n X_p$, with norm given by partitions and weights, and any subspace of L_p. Finally we give an example of a Banach space with norm given by partitions and weights which is not isomorphic to a subspace of L_p.

We will define Y_n to be a Banach space with norm given by partitions and weights which has essentially the same form as the norm on the sequence space realization of $\bigotimes_{j=1}^n X_p$ introduced by Schechtman [S] in 1975. First we will estimate the distance between Y_n and Y_n with the associated envelope norm for the case $n = 3$. Then for any $n \in \mathbb{N}$ we can easily extend the argument to Y_n with the original norm given by partitions and weights and Y_n with the corresponding envelope norm. Consequently, we prove that not every sequence space with norm given by partitions and weights is isomorphic to a subspace of L_p and the envelope norm on the sequence space realization of $\bigotimes_{j=1}^n X_p$ may be a better choice for some purposes.

We will define Y_3 on $\mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2$. Let $(w_i)_{i=1}^\infty$ be a sequence of weights such that $w_i \to 0$ as $i \to \infty$. Let $\mathbf{i}_1, \mathbf{i}_2$, and $\mathbf{i}_3$ represent indices for the first, second and third pair of coordinates, respectively. Define weights on $\mathbb{N}^2$ by $w_{\mathbf{i}} = w_{(m,n)} = w_m$ where $\mathbf{i} = (m,n)$ for all $m, n \in \mathbb{N}$. Let $(e_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3})^2_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\in\mathbb{N}}$ be the natural unit vector basis of Y_3. The partitions of $\mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2$ and the corresponding weights are given as follows:

$$
\begin{array}{ll}
P_0 = \{\mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2\}, & W_0 = (w_{\mathbf{i}_1} \cdot w_{\mathbf{i}_2} \cdot w_{\mathbf{i}_3}), \\
P_1 = \{\{(m,n)\} \times \mathbb{N}^2 \times \mathbb{N}^2 : m,n \in \mathbb{N}\}, & W_1 = (1 \cdot w_{\mathbf{i}_2} \cdot w_{\mathbf{i}_3}), \\
P_2 = \{\mathbb{N}^2 \times \{(n,m)\} \times \mathbb{N}^2 : m,n \in \mathbb{N}\}, & W_2 = (w_{\mathbf{i}_1} \cdot 1 \cdot w_{\mathbf{i}_3}), \\
P_3 = \{\mathbb{N}^2 \times \mathbb{N}^2 \times \{(m,n)\} : m,n \in \mathbb{N}\}, & W_3 = (w_{\mathbf{i}_1} \cdot w_{\mathbf{i}_2} \cdot 1), \\
P_4 = \{\{(m,n)\} \times \{(s,t)\} \times \mathbb{N}^2 : m,n,s,t \in \mathbb{N}\}, & W_4 = (1 \cdot 1 \cdot w_{\mathbf{i}_3}), \\
P_5 = \{\mathbb{N}^2 \times \{(m,n)\} \times \{(s,t)\} : m,n,s,t \in \mathbb{N}\}, & W_5 = (w_{\mathbf{i}_1} \cdot 1 \cdot 1), \\
P_6 = \{\{(m,n)\} \times \mathbb{N}^2 \times \{(s,t)\} : m,n,s,t \in \mathbb{N}\}, & W_6 = (1 \cdot w_{\mathbf{i}_2} \cdot 1), \\
P_7 = \{\{(l,m,n,s,t,u)\} : l,m,n,s,t,u \in \mathbb{N}\}, & W_7 = (1 \cdot 1 \cdot 1).
\end{array}
$$

Then the norm on Y_3 can be calculated by

$$
\begin{aligned}
&\Big\| \sum_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3} a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3} e_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3} \Big\|_{Y_3} = \max_{I \subset \{1,2,3\}} \Big\{ \Big(\sum_{\mathbf{i}:k\in I} \Big(\sum_{\mathbf{i}_l : l \in I^c} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 \prod_{l \in I^c} (w_{\mathbf{i}_l})^2 \Big)^{p/2} \Big)^{1/p} \Big\} \\
&= \max \Big\{ \Big(\sum_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 (w_{\mathbf{i}_1})^2 (w_{\mathbf{i}_2})^2 (w_{\mathbf{i}_3})^2 \Big)^{1/2}, \\
&\qquad \Big(\sum_{\mathbf{i}_1} \Big(\sum_{\mathbf{i}_2,\mathbf{i}_3} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 (w_{\mathbf{i}_2})^2 (w_{\mathbf{i}_3})^2 \Big)^{p/2} \Big)^{1/p}, \\
&\qquad \Big(\sum_{\mathbf{i}_2} \Big(\sum_{\mathbf{i}_1 \mathbf{i}_3} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 (w_{\mathbf{i}_1})^2 (w_{\mathbf{i}_3})^2 \Big)^{p/2} \Big)^{1/p}, \\
&\qquad \Big(\sum_{\mathbf{i}_3} \Big(\sum_{\mathbf{i}_1,\mathbf{i}_2} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 (w_{\mathbf{i}_1})^2 (w_{\mathbf{i}_2})^2 \Big)^{p/2} \Big)^{1/p}, \\
&\qquad \Big(\sum_{\mathbf{i}_1,\mathbf{i}_2} \Big(\sum_{\mathbf{i}_3} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 (w_{\mathbf{i}_3})^2 \Big)^{p/2} \Big)^{1/p}, \Big(\sum_{\mathbf{i}_2,\mathbf{i}_3} \Big(\sum_{\mathbf{i}_1} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 (w_{\mathbf{i}_1})^2 \Big)^{p/2} \Big)^{1/p}, \\
&\qquad \Big(\sum_{\mathbf{i}_1,\mathbf{i}_3} \Big(\sum_{\mathbf{i}_2} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 (w_{\mathbf{i}_2})^2 \Big)^{p/2} \Big)^{1/p}, \Big(\sum_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^p \Big)^{1/p} \Big\} \\
&= \max\{S_i\}_{i=0}^{7}
\end{aligned}
$$

where S_i for $i = 0, 1, \ldots, 7$ are the sums in the previous expression in the same order.

Let Z_3 denote Y_3 with the envelope norm generated. Next we will show that

$$
\sup_{x \in Y_3} \frac{\|x\|_{Z_3}}{\|x\|_{Y_3}} \geq 3^{1/p}.
$$

Since $w_i \to 0$ as $i \to \infty$, for any given ε, $0 < \varepsilon \leq 3$, there exists an N such that if $n > N$, then $w_n < (\varepsilon/3)^{1/2} \leq (\varepsilon/3)^{1/p}$. Let $n_1, n_2, n_3 > N$.

Choose integers K_1, K_2, K_3 such that

$$w_{n_1} K_1^{1/2-1/p} > (3/\varepsilon)^{1/p} \geq 1, \quad w_{n_2} K_2^{1/2-1/p} > (3/\varepsilon)^{1/p} \geq 1,$$
$$w_{n_3} K_3^{1/2-1/p} > (3/\varepsilon)^{1/p} \geq 1.$$

Now take three blocks with constant coefficients as follows:

- a block x_1 of size K_1 with coefficient $(w_{n_1})^{-1} K_1^{-1/2}$ and support

  $\{(n_1, 1, n_2, 1, n_3, 1), (n_1, 2, n_2, 1, n_3, 1), \ldots, (n_1, K_1, n_2, 1, n_3, 1)\}$,

- a block x_2 of size K_2 with coefficient $(w_{n_2})^{-1} K_2^{-1/2}$ and support

  $\{(n_1, K_1+1, n_2, 2, n_3, 2), (n_1, K_1+1, n_2, 3, n_3, 2), \ldots, (n_1, K_1+1, n_2, K_2+1, n_3, 2)\}$,

- a block x_3 of size K_3 with coefficient $(w_{n_3})^{-1} K_3^{-1/2}$ and support

  $\{(n_1, K_1+2, n_2, K_2+2, n_3, 3), (n_1, K_1+2, n_2, K_2+2, n_3, 4), \ldots, (n_1, K_1+2, n_2, K_2+2, n_3, K_3+2)\}$.

Now we estimate the eight sums to get an estimate of the norm of the element

$$x_1 + x_2 + x_3 = \sum_{\mathbf{i}_1=(n_1,1)}^{(n_1,K_1)} w_{n_1}^{-1} K_1^{-1/2} e_{\mathbf{i}_1, n_2, 1, n_3, 1} + \sum_{\mathbf{i}_2=(n_2,2)}^{(n_2,K_2+1)} w_{n_2}^{-1} K_2^{-1/2} e_{n_1, K_1+1, \mathbf{i}_2, n_3, 2} + \sum_{\mathbf{i}_3=(n_3,3)}^{(n_3,K_3+2)} w_{n_3}^{-1} K_3^{-1/2} e_{n_1, K_1+2, n_2, K_2+2, \mathbf{i}_3}. \tag{4.1}$$

First,

$$\begin{aligned} S_0 &= [(w_{n_1})^{-2} K_1^{-1} (w_{n_1})^2 (w_{n_2})^2 (w_{n_3})^2 K_1 \\ &\quad + (w_{n_2})^{-2} K_2^{-1} (w_{n_1})^2 (w_{n_2})^2 (w_{n_3})^2 K_2 \\ &\quad + (w_{n_3})^{-2} K_3^{-1} (w_{n_1})^2 (w_{n_2})^2 (w_{n_3})^2 K_3]^{1/2} \\ &= [(w_{n_2})^2 (w_{n_3})^2 + (w_{n_1})^2 (w_{n_3})^2 + (w_{n_1})^2 (w_{n_2})^2]^{1/2} < \varepsilon^{1/2}, \\ S_1 &= [((w_{n_1})^{-2} K_1^{-1} (w_{n_2})^2 (w_{n_3})^2)^{p/2} K_1 + ((w_{n_2})^{-2} K_2^{-1} (w_{n_2})^2 (w_{n_3})^2 K_2)^{p/2} \\ &\quad + ((w_{n_3})^{-2} K_3^{-1} (w_{n_2})^2 (w_{n_3})^2 K_3)^{p/2}]^{1/p} \\ &= [(w_{n_1} K_1^{1/2-1/p})^{-p} (w_{n_2})^p (w_{n_3})^p + (w_{n_3})^p + (w_{n_2})^p]^{1/p} < \varepsilon^{1/p}. \end{aligned}$$

Similarly we have $S_2 < \varepsilon^{1/p}$ and $S_3 < \varepsilon^{1/p}$. Next,

$$\begin{aligned} S_4 &= (K_1((w_{n_1})^{-2}K_1^{-1}(w_{n_3})^2)^{p/2} \\ &\quad + K_2((w_{n_2})^{-2}K_2^{-1}(w_{n_3})^2)^{p/2} + ((w_{n_3})^{-2}K_3^{-1}(w_{n_3})^2K_3)^{p/2})^{1/p} \\ &= ((w_{n_1}K_1^{1/2-1/p})^{-p}(w_{n_3})^p + (w_{n_2}K_2^{1/2-1/p})^{-p}(w_{n_3})^p + 1)^{1/p} \\ &< (\varepsilon+1)^{1/p}. \end{aligned}$$

Similarly $S_5 < (\varepsilon+1)^{1/p}$ and $S_6 < (\varepsilon+1)^{1/p}$. Finally,

$$S_7 = ((w_{n_1}K_1^{1/2-1/p})^{-p} + (w_{n_2}K_2^{1/2-1/p})^{-p} + (w_{n_3}K_3^{1/2-1/p})^{-p})^{1/p} < \varepsilon^{1/p}.$$

Since ε can be arbitrarily small, if we take the maximum of these eight sums, the norm will be as close to 1 as we want.

Now let us look at the envelope norm of the element $x_1+x_2+x_3$ in (4.1).

Let Q be a partition of $\mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2$ such that the support of each of the above three blocks is an element of Q. (The other sets in the partition do not matter.) Let $\mathcal{P}$ be the given family of weights and partitions, i.e., $\mathcal{P} = \{(P_i, W_i) : i = 0, 1, \ldots, 7\}$. Let $T : Q \to \mathcal{P}$ be a map such that $T(\operatorname{supp} x_1) = (P_6, W_6)$, $T(\operatorname{supp} x_2) = (P_5, W_5)$, and $T(\operatorname{supp} x_3) = (P_4, W_4)$. Then the envelope norm of $x_1 + x_2 + x_3$ can be estimated from below using $P(Q, T)$:

$$\begin{aligned} &\Big|\Big|\Big| \sum_{\mathbf{i}_1=(n_1,1)}^{(n_1,K_1)} w_{n_1}^{-1}K_1^{-1/2} e_{\mathbf{i}_1,n_2,1,n_3,1} + \sum_{\mathbf{i}_2=(n_2,2)}^{(n_2,K_2+1)} w_{n_2}^{-1}K_2^{-1/2} e_{n_1,K_1+1,\mathbf{i}_2,n_3,2} \\ &\qquad + \sum_{\mathbf{i}_3=(n_3,3)}^{(n_3,K_3+2)} w_{n_3}^{-1}K_3^{-1/2} e_{n_1,K_1+2,n_2,K_2+2,\mathbf{i}_3} \Big|\Big|\Big| \\ &\quad \ge \bigg(\Big(\sum_{\mathbf{i}_1=(n_1,1)}^{(n_1,K_1)} (w_{n_1}^{-1}K_1^{-1/2})^2 w_{\mathbf{i}_1}^2\Big)^{p/2} \\ &\qquad + \Big(\sum_{\mathbf{i}_2=(n_2,2)}^{(n_2,K_2+1)} (w_{n_2}^{-1}K_2^{-1/2})^2 w_{\mathbf{i}_2}^2\Big)^{p/2} + \Big(\sum_{\mathbf{i}_3=(n_3,3)}^{(n_3,K_3+2)} (w_{n_3}^{-1}K_3^{-1/2})^2 w_{\mathbf{i}_3}^2\Big)^{p/2}\bigg)^{1/p} \\ &\quad \ge (((w_{n_1})^{-2}K_1^{-1}(w_{n_1})^2K_1)^{p/2} \\ &\qquad + ((w_{n_2})^{-2}K_2^{-1}(w_{n_2})^2K_2)^{p/2} + ((w_{n_3})^{-2}K_3^{-1}(w_{n_3})^2K_3)^{p/2})^{1/p} = 3^{1/p}. \end{aligned}$$

Hence the envelope norm on Y_3 is at best $3^{1/p}$-equivalent to the given norm.

Next we will describe how this computation can be generalized. We define Y_n for any $n \in \mathbb{N}$ on $\mathbb{N}^2 \times \ldots \times \mathbb{N}^2$ (n times). Let $w_{\mathbf{i}} = w_{s,t} = w_s$ for $\mathbf{i} = (s,t)$, $s, t \in \mathbb{N}$, as above. Let $I \subset \{1, \ldots, n\}$. Define

$$P_I = \left\{ \prod_{k=1}^{n} A_k : A_k = \mathbb{N}^2,\ k \notin I;\ A_k = \{(m_k, l_k)\},\ k \in I,\ m_k, l_k \in \mathbb{N} \right\}$$

and

$$W_I = \Big(\prod_{k \notin I} w_{\mathbf{i}_k} \Big).$$

For a given sequence (w_i) such that $w_i \to 0$ as $i \to \infty$ and any $0 < \varepsilon \le 1$, there exists N such that if $m > N$, then $w_m < (\varepsilon/n)^{1/2} \le (\varepsilon/n)^{1/p}$.

Let $m_1, \ldots, m_n > N$. We choose n blocks with size K_l for $l = 1, \ldots, n$ in $(\mathbb{N}^2)^n$ so that $w_{m_l} {K_l}^{1/2-1/p} > (n/\varepsilon)^{1/p}$.

The block of size K_{l+1} would have coefficient $w_{m_{l+1}}^{-1} K_{l+1}^{-1/2}$ and support

$$\begin{gathered}
(m_1, K_1 + l, m_2, K_2 + l, \ldots, m_{l+1}, l+1, \ldots, m_n, l+1) \\
(m_1, K_1 + l, m_2, K_2 + l, \ldots, m_{l+1}, l+2, \ldots, m_n, l+1) \\
\vdots \\
(m_1, K_1 + l, m_2, K_2 + l, \ldots, m_{l+1}, l+K_{l+1}, \ldots, m_n, l+1)
\end{gathered}$$

where $0 \le l \le n-1$.

By applying similar arguments to that for $n = 3$ we deduce that the value of the envelope norm of the sum of these blocks is at least $n^{1/p}$ while the value of the norm given by partitions and weights remains approximately 1.

THEOREM 4.1. *The distance from Y_n to a subspace of L_p goes to ∞ with n, i.e., there is a sequence $(K(n))$, $K(n) \to \infty$, such that for all isomorphisms $T : Y_n \to Z \subset L_p$, $\|T\|\,\|T^{-1}\| \ge K(n)$.*

Proof. If $T : Y_n \to Z \subset L_p$ is an isomorphism, then by Theorem 3.11, the norm of Y_n given by partitions and weights is equivalent to the envelope norm with a constant depending on $\|T\|\,\|T^{-1}\|$. Since the envelope norm of some element of Y_n of norm 1 has value at least $n^{1/p}$, we have

$$\|T\|\,\|T^{-1}\| \ge \lambda^{-1} n^{1/p} \ge \frac{n^{1/p}}{\|T\|\,\|T^{-1}\|}. \ \blacksquare$$

COROLLARY 4.2. *The distance between Y_n and $\bigotimes_{j=1}^{n} X_p$ goes to ∞ with n.*

COROLLARY 4.3. *$(\sum_n Y_n)_{\ell_p}$ with norm given by partitions and weights is not isomorphic to a subspace of L_p.*

5. Remarks and open problems. We introduced the envelope property to show that there is a space with norm given by partitions and weights which is not isomorphic to a subspace of L_p. It seems unlikely that this property alone determines whether a space with norm given by partitions and weights is isomorphic to a subspace of L_p.

QUESTION 5.1. *Is there a space with norm given by partitions and weights which has the envelope property but is not isomorphic to a subspace of L_p?*

QUESTION 5.2. *What are necessary and sufficient conditions for a space with norm given by partitions and weights to be isomorphic to a subspace of L_p?*

We showed that the tensor product of two spaces with norms given by finitely many partitions and weights is isomorphic to a space with norm given by partitions and weights.

QUESTION 5.3. *Suppose X and Y have norms given by partitions and weights and are each isomorphic to a subspace of L_p. Is $X \otimes Y$ isomorphic to a subspace of L_p? What if X and Y are each isomorphic to a complemented subspace of L_p?*

The construction of uncountably many complemented subspaces of L_p given by Bourgain, Rosenthal and Schechtman is based on two fundamental operations. If X and Y are subspaces of $L_p[0,1]$ then a distributional version of the ℓ_p sum is used, $X \oplus_p Y$, with each space being isometrically mapped to a space supported on half the interval in a canonical way. The second operation is used with sequences of subspaces of L_p. Let (X_n) be such a sequence with each X_n containing the constant functions and let $X_{n,0}$ denote the subspace of mean zero functions in X_n. An infinite sum is created by placing (isometrically) $X_{n,0}$ onto an infinite product of probability spaces as functions depending only on the nth coordinate. The space $(\sum X_n)_I$ is the span of these transported copies of X_n, $n = 1, 2, \ldots$, and the constant functions.

With these two operations an induction on ω_1 is used to construct the spaces. Let R_p^0 be the constant functions on $[0,1]$. If R_p^α has been defined, define $R_p^{\alpha+1} = R_p^\alpha \oplus_p R_p^\alpha$. If α is a limit ordinal, define $R_p^\alpha = (\sum_{\beta<\alpha} R_p^\beta)_I$.

These operations and the construction are investigated in [A]. In particular it is shown there that the constant functions do not play an important role in the construction. If we make a few adjustments we can mimic this construction using $(p, 2, W)$ sums. For the ℓ_p sum of two spaces, we use the $(p, 2, W_1)$ sum with two equal weights, $W_1 = (2^{(2-p)/(2p)}, 2^{(2-p)/(2p)})$. For the independent sum we use the $(p, 2, W_2)$ sum with W_2 equal to the constantly 1 sequence. In each case we will also assume that we take the envelope norm generated. In addition in order to strengthen the correspondence between this construction and the very distributional construction of R_p^α, we will use at each step the fact that the space $R_{p,0}^\alpha$ has an unconditional basis which is orthogonal in L_2. Define $Y_p^0 = [1_{[0,1/2)} - 1_{[1/2,1]}]$. This is just a one-dimensional space. Let X_p^0 be the sequences of length one. If Y_p^α and X_p^α

have been defined, define $Y_p^{\alpha+1} = Y_p^\alpha \oplus_p Y_p^\alpha$ and $X_p^{\alpha+1} = (X_p^\alpha, X_p^\alpha)_{(p,2,W_1)}$. If α is a limit ordinal, let $Y_p^\alpha = (\sum_{\beta<\alpha} Y_p^\beta)_I$ and $X_p^{\alpha+1} = (\sum_{\beta<\alpha} X_p^\beta)_{(p,2,W_2)}$.

It follows from the results in Chapter 2 of [A] that for $\alpha < \omega^2$ the spaces R_p^α, Y_p^α, and X_p^α are isomorphic. The choice of weights W_1 and W_2 is such that the natural mapping from $(Y_p^\alpha, \|\cdot\|_2)$ to $(X_p^\alpha, \|\cdot\|_2)$ will be an isometry.

Question 5.4. *Is X_p^α isomorphic to R_p^α for all $\alpha < \omega_1$?*

We think it unlikely to be true but it would be nice to know the answer to the following simple question.

Question 5.5. *Is L_p isomorphic to a space with norm given by partitions and weights?*

Finally we note that the envelope norm suggests that a useful alternative form of Rosenthal's inequality for a sequence of mean zero independent random variables (f_n) might be

$$c_p \max_{Q\subset\mathbb{N}} \Big\{ \Big(\sum_{n\notin Q} \|f_n\|_p^p + \Big(\sum_{n\in Q} \|f_n\|_2^2 \Big)^{p/2} \Big)^{1/p} \Big\}$$
$$\le \Big\| \sum_n f_n \Big\|_p \le C_p \max_{Q\subset\mathbb{N}} \Big\{ \Big(\sum_{n\notin Q} \|f_n\|_p^p + \Big(\sum_{n\in Q} \|f_n\|_2^2 \Big)^{p/2} \Big)^{1/p} \Big\}.$$

Question 5.6. *What are the best constants in this form of Rosenthal's inequality?*

(See [J-S-Z] for results on the constants in the original form.)

At this time very little is known about the isomorphic properties of the whole class of spaces with norm given by partitions and weights. The development of the isomorphic properties for X_p by Rosenthal [R] suggests that it is very helpful to be able to choose special representations of X_p which allow the use of standard gliding hump arguments. In the case of X_p a weight sequence in which every weight is repeated infinitely often is such a special representation. The proof that any $X_{p,w}$ with $w = (w_n)$ satisfying $(*)$ is isomorphic to X_p and hence to the special representation of X_p relies on the ability to construct projections on special blockings. This type of detailed result for general spaces with norm given by partitions and weights seems unlikely. Among more restrictive classes of these spaces such as the $\mathcal{L}_p$ spaces similar results are far more likely.

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Department of Mathematics
Oklahoma State University
Stillwater, OK 74078, U.S.A.
E-mail: alspach@math.okstate.edu

Department of Mathematics
University of Wisconsin-Eau Claire
Eau Claire, WI 54702, U.S.A.
E-mail: tongs@uwec.edu

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