Fréchet quotients of spaces of real-analytic functions

by

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Dedicated to Aleksander Pełczyński on the occasion of his 70th birthday

Abstract. We characterize all Fréchet quotients of the space $A(\Omega)$ of (complex-valued) real-analytic functions on an arbitrary open set $\Omega \subseteq \mathbb{R}^d$. We also characterize those Fréchet spaces $E$ such that every short exact sequence of the form $0 \to E \to X \to A(\Omega) \to 0$ splits.

Let $A(\Omega)$ denote the space of complex-valued real-analytic functions on the open set $\Omega \subseteq \mathbb{R}^d$, equipped with its natural locally convex topology (see [15] or [1]). The Fréchet structure of the spaces $A(\Omega)$ was closely investigated in recent years and this was a basis of various interesting results. The Fréchet subspaces are characterized in Domani–Langenbruch [5] as being isomorphic to subspaces of $H(\mathbb{D}^d)$ if $\Omega$ has finitely many connected components, and of $H(\mathbb{D}^d)^\mathbb{N}$ if $\Omega$ has infinitely many connected components. In Domani–Vogt [7] it was shown that for $\Omega$ connected the space $A(\Omega)$ admits only finite-dimensional complemented subspaces, and that led to the proof in [7] (cf. [8]) that no space $A(\Omega)$ admits a (Schauder) basis.

About the quotients of $A(\Omega)$ it was only known that they have the very restrictive property $\overline{(\Omega)}$. This was a basic ingredient in the proof that all complemented Fréchet subspaces are finite-dimensional. However, it was not known whether $A(\Omega)$ admits any infinite-dimensional Fréchet quotient besides the space $\omega$ of all sequences. Only recently was it shown in [9] that $A(\Omega)$ admits nontrivial Fréchet quotients, but that is far from being an exact picture.

In the present paper we show that a Fréchet space $E$ is isomorphic to a quotient of $A(\Omega)$ for some, equivalently any, open set $\Omega \subseteq \mathbb{R}^d$ if and only if

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$E$ has property $(\overline{\Omega})$ and is $n^{1/d}$-nuclear (see below for the definition in terms of Kolmogorov diameters), or equivalently (see [25]), if and only if $E$ has property $(\overline{\Omega})$ and is isomorphic to a quotient of $H(\mathbb{D}^d)$. An essential step is to show that a Fréchet space has $(\overline{\Omega})$ if and only if $\text{Ext}^1(A(\Omega), E) = 0$, which means that every topologically exact sequence

$$0 \to E \to X \to A(\Omega) \to 0$$

splits. This result is of independent interest.

1. Preliminaries. We use common notation for locally convex spaces, in particular Fréchet spaces. For the notation and general results we refer to [18] or [13]. For homological notions in locally convex spaces see [26].

For any open $\Omega \subset \mathbb{R}^d$ the space $A(\Omega)$ is equipped with its natural topology given by

$$A(\Omega) = \limproj_n H(K_n)$$

where $K_1 \subset K_2 \subset K_2 \subset \ldots$ is a compact exhaustion of $\Omega$ and $H(K_n)$ denotes the (LB)-space of germs of holomorphic functions on $K_n$. By Martineau [15, Th. 1.9] this topology coincides with the one given by

$$A(\Omega) = \limind_{\omega} H(\omega).$$

Here $\omega$ runs through all complex neighborhoods of $\Omega$, and $H(\omega)$ denotes the Fréchet space of holomorphic functions on $\omega$ with the compact-open topology.

A Fréchet space with a fundamental system of seminorms $\| \cdot \|_1 \leq \| \cdot \|_2 \leq \ldots$ is said to have property $(\overline{\Omega})$ if

$$\forall k \exists l \forall n, 0 < \vartheta < 1 \exists C : \|y\|^*_k \leq C\|y\|^*_{\vartheta}\|y\|^1_{\vartheta}$$

or equivalently

$$\forall k \exists l \forall n, \varepsilon > 0 \exists C \forall r > 0 : \quad U_l \subset C(r^\varepsilon U_n + r^{-1}U_k).$$

Here we set $\|y\|^*_k = \sup\{|y(x)| \mid x \in U_k\}$ and $U_k = \{x \mid \|x\|_k \leq 1\}$. For the role of property $(\overline{\Omega})$ see [1], [2], [7], [22], [24]; for the equivalence of both conditions see [18, Lemma 29.13]. For examples of spaces with $(\overline{\Omega})$ see [16, Ex. 4.12(5)].

Clearly $(\overline{\Omega})$ is a topological linear invariant which is inherited by quotient spaces. We will need another topological invariant.

Let $X$ be a linear space and $V \subset U$ absolutely convex subsets. We define the $n$th Kolmogorov diameter of $V$ with respect to $U$ to be

$$\delta_n(V, U) = \inf\{\delta > 0 \mid V \subset \delta U + F, \dim F \leq n\}.$$ 

Here $F$ denotes a linear subspace of $E$.
Let \( \alpha = (\alpha_0, \alpha_1, \ldots) \) be a nonnegative increasing sequence so that \( \lim_n (\log n)/\alpha_n = 0 \) and \( \sup_n \alpha_{2n}/\alpha_n < \infty \). We call this a \textit{stable exponent sequence}. Using \( V \) and \( U \) for absolutely convex neighborhoods of zero, we define a locally convex space \( X \) to be:

1. \( \textit{weakly } \alpha\text{-nuclear} \text{ if } \forall U, t > 0 : \lim_n e^{t\alpha_n} \delta_n(V, U) = 0,
2. \( \alpha\text{-nuclear} \text{ if } \forall U, t > 0 : \lim_n e^{t\alpha_n} \delta_n(V, U) = 0,
3. \( \textit{strongly } \alpha\text{-nuclear} \text{ if } \forall U, \forall t > 0 : \lim_n e^{t\alpha_n} \delta_n(V, U) = 0.

The assumptions on \( (\alpha_n) \) imply that every weakly \( \alpha\)-nuclear space is nuclear (see [20, p. 296]). These are topological linear invariants, inherited by subspaces, quotients, countable products and direct sums, hence by countable projective and inductive limits (see [25, Lemma 1.3, 1.4]).

It is well known that for any open \( U \subset \mathbb{C}^d \) the space \( H(U) \) of holomorphic functions on \( U \) with the compact-open topology is weakly \( n^{1/d}\)-nuclear. Since for open \( \Omega \subset \mathbb{R}^d \) we can write

\[
\mathcal{A}(\Omega) = \lim \text{proj} \lim \text{ind} H(K_n + k^{-1} \mathbb{D}^d)
\]

we see that \( \mathcal{A}(\Omega) \) is weakly \( n^{1/d}\)-nuclear. Here \( \mathbb{D}^d = \{ z \in \mathbb{C}^d | \sup \nu \| z_\nu \| < 1 \} \).

A locally convex space is called a \((\text{PLB})\)-space ((\text{PLS})-space, resp.) if it is a countable projective limit of \((\text{LB})\)-spaces ((\text{LS})-spaces, resp.). Main examples of \((\text{PLB})\)-spaces are \( \mathcal{A}(\Omega) \), the space \( \mathcal{D}_0 \), of distributions, and all Fréchet and \((\text{LB})\)-spaces; the first two examples are also \((\text{PLS})\)-spaces.

In this paper \( \text{Ext}^1 \) will always be taken in the category of locally convex spaces, i.e. \( \text{Ext}^1(E, F) = 0 \) will mean that every topologically exact sequence

\[
0 \to F \xrightarrow{j} X \xrightarrow{q} E \to 0
\]

of locally convex spaces splits, i.e., \( q \) has a continuous linear right inverse. Recall that the sequence above is \textit{topologically exact} whenever \( j \) is a topological isomorphism onto the kernel of \( q \) and \( q \) is surjective, continuous and open onto its image.

We denote by \( L(X, Y) \) the space of all continuous linear maps from a locally convex space \( X \) into another such space \( Y \).

Consider an arbitrary projective spectrum of linear spaces:

\[
\ldots \to X_{n+1} \xrightarrow{i_{n+1}} X_n \to \ldots \to X_1 \xrightarrow{i_0} X_0.
\]

Then \( \text{Proj}^1_{n \in \mathbb{N}} X_n := \prod_{n \in \mathbb{N}} X_n/\text{im } \sigma \), where

\[
\sigma : \prod_{n \in \mathbb{N}} X_n \to \prod_{n \in \mathbb{N}} X_n, \quad \sigma((x_n)_{n \in \mathbb{N}}) := (i_{n+1}^n x_{n+1} - x_n)_{n \in \mathbb{N}}.
\]

Clearly, \( \text{ker } \sigma = \lim \text{proj}_{n \in \mathbb{N}} X_n \).

Let \( E \) be a Fréchet space, \( (E_n)_{n \in \mathbb{N}} \) its sequence of local Banach spaces, and \( i_m : E_m \to E_n \) the linking maps. There exists the following short
(topologically) exact sequence:

\[ 0 \to E \to \prod_{n \in \mathbb{N}} E_n \xrightarrow{\sigma} \prod_{n \in \mathbb{N}} E_n \to 0, \]

where \( \sigma((x_n)_{n \in \mathbb{N}}) := (i_n^{n+1}x_{n+1} - x_n)_{n \in \mathbb{N}} \), which we call the canonical resolution ([18, definition after 26.14]). Consider the spectrum \((L(X, E_n))_{n \in \mathbb{N}}\), where the linking maps are defined as follows:

\[ I_{n+1}^n : L(X, E_{n+1}) \to L(X, E_n), \quad I_{n+1}^n(T) = i_{n+1}^n T. \]

Then \( \text{Proj}_{n \in \mathbb{N}}^1 L(X, E_n) = 0 \) means exactly that every \( T \in L(X, \prod_{n \in \mathbb{N}} E_n) \) lifts with respect to \( \sigma \), i.e., there is a map \( S \in L(X, \prod_{n \in \mathbb{N}} E_n) \) such that \( \sigma \circ S = T \).

We will need some general homological facts.

**Lemma 1.1.** Let \( E \) be a Fréchet space with local Banach spaces \((E_n)_{n \in \mathbb{N}}\). Let \( F \) be a locally convex space. If either \( E \) or \( F \) is nuclear then \( \text{Ext}^1(F, E) \cong \text{Proj}_{n \in \mathbb{N}}^1 L(F, E_n) \).

**Proof.** Observe that \( L(E, F) \) can be identified with \( \lim \text{proj}_{n \in \mathbb{N}} L(F, E_n) \). We apply the functor \( L(F, \cdot) \) to the canonical resolution of \( E \). By the standard homological argument we obtain the following long exact sequence (see, for instance, [19, Th. 3.4(b)]):

\[ 0 \to L(F, E) \to L \left( F, \prod_{n \in \mathbb{N}} E_n \right) \to L \left( F, \prod_{n \in \mathbb{N}} E_n \right) \to \text{Ext}^1(F, E) \to \text{Ext}^1 \left( F, \prod_{n \in \mathbb{N}} E_n \right) \to \ldots \]

For nuclear \( F \) and any Banach space \( X \) we have \( \text{Ext}^1(F, X) = 0 \) ([23]) and for nuclear \( E \) we may assume \( E_n \) to be injective for every \( n \). In both cases we get \( \text{Ext}^1(F, \prod_{n \in \mathbb{N}} E_n) = 0 \). We obtain the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
\prod_{n \in \mathbb{N}} L(F, E_n) & \xrightarrow{\sigma} & \prod_{n \in \mathbb{N}} L(F, E_n) & \xrightarrow{T} & \text{Proj}_{n \in \mathbb{N}}^1 L(F, E_n) & \to 0 \\
L(F, \prod_{n \in \mathbb{N}} E_n) & \xrightarrow{T} & L(F, \prod_{n \in \mathbb{N}} E_n) & \xrightarrow{T} & \text{Ext}^1(F, E) & \to 0 \\
\end{array}
\]

where the upper row is obtained from the sequence defining \( \text{Proj}_{n \in \mathbb{N}}^1 \) for the spectrum \((L(F, E_n))_{n \in \mathbb{N}}\) and we have omitted the beginning of both rows. Here \( T \) denotes the canonical identification. It is clear that the last vertical arrow is an isomorphism we are looking for.

**Corollary 1.2.** Let \( E \) be a Fréchet space, \( F \) a locally convex space and \( F_0 \subset F \) a subspace. If either \( E \) or \( F \) is nuclear and \( \text{Ext}^1(F, E) = 0 \) then \( \text{Ext}^1(F_0, E) = 0 \).
Proof. Let \((E_n)_{n \in \mathbb{N}}\) be local Banach spaces for \(E\). By our previous comments, we only have to show that \(\text{Proj}_{n \in \mathbb{N}}^1 L(F_0, E_n) = 0\). But this follows easily. Indeed, arguing as in the proof of Lemma 1.1, we show that every \(t_n \in L(F_0, E_n)\) can be extended to \(T_n \in L(F, E_n)\). We construct the following commutative diagram with exact rows:

\[
\begin{array}{c}
\prod_{n \in \mathbb{N}} L(F, E_n) \to \prod_{n \in \mathbb{N}} L(F, E_n) \\
\downarrow S_1 \quad \downarrow S_2
\end{array}
\begin{array}{c}
\to \text{Proj}_{n \in \mathbb{N}}^1 L(F, E_n) \to 0 \\
\to \text{Proj}_{n \in \mathbb{N}}^1 L(F_0, E_n) \to 0
\end{array}
\]

where \(S_1, S_2\) are surjective restriction maps. Thus also the last vertical arrow is surjective and this completes the proof.

We are now in a position to formulate our main theorems.

2. Main results

**Theorem 2.1.** A Fréchet space \(E\) is isomorphic to a quotient of \(\mathcal{A}(\Omega)\) if and only if it is \(n^{1/d}\)-nuclear and has property \((\overline{\Omega})\).

By [25, Th. 4.1], this has an immediate consequence:

**Corollary 2.2.** A Fréchet space \(E\) is isomorphic to a quotient of \(\mathcal{A}(\Omega)\) if and only if it has property \((\overline{\Omega})\) and it is isomorphic to a quotient of \(H(\mathbb{D}^d)\).

**Theorem 2.3.** If \(E\) is a Fréchet space then the following assertions are equivalent:

(a) \(\text{Ext}_{n \in \mathbb{N}}^1 (\mathcal{A}(\Omega), E) = 0\);
(b) \(\text{Proj}_{n \in \mathbb{N}}^1 L(\mathcal{A}(\Omega), E_n) = 0\);
(c) \(E\) has property \((\overline{\Omega})\).

**Remark.** In fact, analyzing the proof of the above result, it follows that if (c) is satisfied then (a) and (b) also hold for any closed subspace \(Y\) of \(\mathcal{A}(\Omega)\) in place of \(\mathcal{A}(\Omega)\) or, more generally, for every complete nuclear space \(Y\) which has \((\text{DN}_\varphi)\) for any \(\varphi\) with \(\lim_{r \to 0} r^{-\varepsilon} \varphi(r) = 0\) for all \(\varepsilon > 0\) (see Definition 4.3 below).

The proof will consist of a series of lemmas and will take the rest of this paper.

3. Necessity of the conditions. We first quote the following result (Theorem 3.4 of [7]):

**Lemma 3.1.** If \(E\) is a Fréchet space isomorphic to a quotient of \(\mathcal{A}(\Omega)\) then it has property \((\overline{\Omega})\).
Due to the preliminary remarks we know that every quotient of $\mathcal{A}(\Omega)$ is weakly $n^{1/d}$-nuclear. This condition is self-improving by the following.

**Lemma 3.2.** If $W \subset V \subset U$ are absolutely convex sets, $\varepsilon > 0$ and

$$V \subset C(r^{-1}U + r^\varepsilon W) \quad \text{for all } r > 0,$$

then there is $D$ so that

$$\delta_n(V, U)^\varepsilon \leq D \delta_n(W, V) \quad \text{for all } n.$$

**Proof.** Let $F \subset E$ be a subspace. We set for the moment $\delta(V, U; F) = \inf \{ \delta > 0 \mid V \subset \delta U + F \}$ and assume $\delta > \delta(W, U; F)$. Then, by assumption,

$$V \subset C(r^{-1} + r^\varepsilon \delta)U + F \quad \text{for all } r > 0,$$

hence

$$\delta(V, U; F) \leq C \inf_{r > 0} (r^{-1} + r^\varepsilon \delta) = C(1 + \varepsilon)^{\varepsilon/(1+\varepsilon)\delta^{1/(1+\varepsilon)}},$$

and therefore $\delta(V, U; F)^{1+\varepsilon} \leq D\delta(W, U; F)$. Since $\delta(W, U; F) \leq \delta(W, V; F) \times \delta(V, U; F)$ we obtain $\delta(V, U; F)^{\varepsilon} \leq D\delta(W, V; F)$. Taking the infimum over all $F$ with dim $F \leq n$ we obtain $\delta_n(V, U)^\varepsilon \leq D\delta_n(W, V)$. □

**Lemma 3.3.** If the Fréchet space $E$ has property $(\overline{\Omega})$ and is weakly $\alpha$-nuclear then it is strongly $\alpha$-nuclear. In particular it is $\alpha$-nuclear.

**Proof.** For $k$ we choose $l > k$ according to $(\overline{\Omega})$, i.e. for every $n$ and $\varepsilon > 0$ there is $C > 0$ so that

$$U_l \subset C(r^{-1}U_k + r^\varepsilon U_n) \quad \text{for all } r > 0.$$

Since $E$ is weakly $\alpha$-nuclear we can find $t > 0$ so that

$$e^{t\alpha_n} \delta_n(U_n, U_l) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Using Lemma 3.2 we obtain

$$e^{t\alpha_n} \delta_n(U_l, U_k) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for every $t > 0$. □

This completes the proof of the necessity of the conditions in Theorem 2.1. We state it as a separate lemma.

**Lemma 3.4.** If $E$ is a Fréchet space isomorphic to a quotient of $\mathcal{A}(\Omega)$ then $E$ is $n^{1/d}$-nuclear.

For the proof of the sufficiency we need a further improvement.

**Lemma 3.5.** If the Fréchet space $E$ is strongly $\alpha$-nuclear then there is a stable exponent sequence $\beta$ with $\lim_n \alpha_n/\beta_n = 0$ so that $E$ is (strongly) $\beta$-nuclear.
Proof. By a recursive choice of the fundamental system \((U_k)_k\) of neighborhoods of zero we may assume that
\[ e^{\alpha_n} \delta_n(U_{k+1}, U_k) \to 0 \quad \text{as } n \to \infty \]
for all \(m\) and \(k\).

We set \(n_0 = 1\) and determine inductively \(n_{m+1} > n_m\) so that
\[ e^{(m+1)^2 \alpha_n} \delta_n(U_{k+1}, U_k) < \frac{1}{m+1} \]
for \(n \geq n_{m+1}\) and \(k = 1, \ldots, m\). Setting \(\beta_n = m \alpha_n\) for \(n_m \leq n < n_{m+1}\) we obtain the result. ■

To prove the necessity of the conditions in Theorem 2.3 we recall that in Domański–Langenbruch [5] it is shown that the space \(A_0(n^{1/d}) \simeq H(\mathbb{D}^d)\) can be imbedded into \(\mathcal{A}(\Omega)\).

**Lemma 3.6.** If \(E\) is a Frechet space and \(\text{Ext}^1(\mathcal{A}(\Omega), E) = 0\) then \(E\) has property \((\overline{\Omega})\).

**Proof.** We choose \(\beta_n = n^{1/d}\) and imbed \(A_0(\beta)\) into \(\mathcal{A}(\Omega)\). Then by Corollary 1.2, \(\text{Ext}^1(A_0(\beta), E) = 0\) and the result follows from [24, Theorem 4.2]. ■

### 4. Sufficiency of conditions in Theorem 2.3.

First notice that, by Lemma 1.1, conditions (a) and (b) in Theorem 2.3 are equivalent. We write property \((\overline{\Omega})\) in a different form. To do this, throughout this section we let \(\varphi\) and \(\psi\) denote increasing unbounded functions \((0, \infty) \to (0, \infty)\).

**Definition 4.1.** \(E\) has property \((\Omega_\varphi)\) if
\[
\forall k \exists l \forall n \exists C \forall r > 0 : \quad U_l \subset C \psi(r) U_n + r^{-1} U_k.
\]

**Remark.** Equivalently we may write (cf. [18, Lemma 29.13])
\[
(1) \quad \forall k \exists l \forall n \exists C \forall r > 0 \forall y \in E' : \quad \|y\|_{\varphi}^* \leq C \psi(r) \|y\|_n^* + r^{-1} \|y\|_{k}^*.
\]

We obtain:

**Lemma 4.2.** \(E\) has property \((\overline{\Omega})\) if and only if there is a function \(\psi\) with \(\lim_{r \to \infty} r^{-\epsilon} \psi(r) = 0\) for all \(\epsilon > 0\) so that \(E\) has property \((\Omega_\varphi)\).

**Proof.** If \(E\) has property \((\Omega_\varphi)\) with \(\psi\) as described, then clearly \(E\) has property \((\overline{\Omega})\).

To prove the converse we find for given \(k\) an \(l = l(k)\) according to property \((\overline{\Omega})\). For \(r > 0\) and \(n \in \mathbb{N}\) we set
\[
\psi_{k,n}(r) = \sup_{x \in U_l} \inf_{y \in r^{-1} U_k} \|x - y\|_n + 1.
\]

Then, clearly,
\[
U_l \subset \psi_{k,n}(r) U_n + r^{-1} U_k
\]
for every \( r > 0 \) and, due to property (\( \overline{\Omega} \)), for every \( \varepsilon > 0 \) we obtain a constant \( C > 0 \) so that \( \psi_{k,n}(r) \leq C r^\varepsilon + 1 \). This implies that \( \lim_{r \to \infty} r^{-\varepsilon} \psi_{k,n}(r) = 0 \) for all \( k, n \) and \( \varepsilon > 0 \). It is easily seen that we can find \( \psi \) so that \( \lim_{r \to \infty} r^{-\varepsilon} \psi(r) = 0 \) for all \( \varepsilon > 0 \) and that for all \( k, n \) there are \( C > 0 \) and \( r_0 \) with \( \psi_{k,n}(r) \leq C \psi(r) \) for all \( r > r_0 \).

Let now \( X \) be a locally convex space, let \( p_0, p, q \) denote continuous seminorms on \( X \), and let \( \varphi \) be a nondecreasing positive unbounded function.

**Definition 4.3.** \( X \) has property (\( \text{DN}_\varphi \)) if

\[
\exists p_0 \ \forall p \ \exists q, C > 0 \ \forall r > 0 : \ p \leq C \left( r p_0 + \frac{1}{\varphi(r)} q \right).
\]

In this case, \( p_0 \) is a norm and it is called a \( \varphi \)-dominating norm.

In the next lemma we assume that \( X \) is an (LB)-space, i.e. there is a sequence \( X_1 \subset X_2 \subset \ldots \) of Banach spaces so that \( X = \bigcup_{n=1}^\infty X_n \) and \( X \) carries the strongest topology so that all the imbeddings are continuous. We denote the norm in \( X_n \) by \( \| \cdot \|_n \); we may assume that \( \| \cdot \|_n \geq \| \cdot \|_{n+1} \) for all \( n \).

**Lemma 4.4.** If there exists a continuous norm \( \| \cdot \| \) on \( X \) so that for every \( n \) there is \( 0 < \tau_n < 1 \) with

\[
\| x \|_{n+1} \leq \| x \|_{\tau_n} \| x \|^{1-\tau_n}
\]

for all \( x \in X_n \), then \( X \) has property (\( \text{DN}_\varphi \)) for every \( \varphi \) with \( \lim_{r \to \infty} r^{-\varepsilon} \varphi(r) = 0 \) for all \( \varepsilon > 0 \). Moreover \( \| \cdot \| \) is a \( \varphi \)-dominating norm.

**Proof.** We set \( A = \{ x \in X \mid \| x \| \leq 1 \} \). We choose a neighborhood \( B \) of zero in \( X \), which may be assumed to be of the form

\[
B := \sum_{\nu=1}^\infty \beta_\nu B_\nu := \bigcup_{n \in \mathbb{N}} \sum_{\nu=1}^n \beta_\nu B_\nu,
\]

where \( (\beta_\nu)_{\nu \in \mathbb{N}} \in (0, 1)^\mathbb{N} \) is a decreasing null sequence, and \( B_\nu \) is the closed unit ball of \( X_\nu \). It is enough to show the existence of \( C, r_0 \geq 1, (\gamma_\nu)_{\nu \in \mathbb{N}} \in (0, 1)^\mathbb{N} \) and \( l : \mathbb{N} \to \mathbb{N} \) with \( \lim_{n \to \infty} l(n) = \infty \) such that

\[
\varphi(r) \| D \cap \frac{1}{2r} A \| \subset 2C B \quad \text{for all } r \geq r_0,
\]

where

\[
D := \sum_{\nu=1}^\infty \gamma_\nu B_{l(\nu)}.
\]

To start we choose an increasing function \( \theta : [1, \infty) \to (0, 1) \) with \( \lim_{r \to \infty} \theta(r) = 1 \) and

\[
\lim_{r \to \infty} \varphi(r)(1/r)^{1-\theta(r)} = 0.
\]
Then we define an increasing function \( n : [1, \infty) \to \mathbb{N} \) with \( n(r) \leq r \), \( \lim_{r \to \infty} n(r) = \infty \), and \( C \geq 1 \) such that
\[
\varphi(r)(1/r)^{1-\theta(r)} \leq C\beta_{n(r)+1} \quad \text{for all } r \geq 1.
\]

We construct an increasing function \( l : \mathbb{N} \to \mathbb{N} \) with \( l(n) \leq n \) and \( \lim_{n \to \infty} l(n) = \infty \) such that
\[
d(n) := \tau_{l(n)} \leq \theta(n) \quad \text{for all } n.
\]
This implies that there is \( r_0 \geq 1 \) such that \( \varphi(r_0) \geq 1 \) and
\[
(2) \quad \varphi(r)(1/r)^{1-d(n(r))} \leq C\beta_{n(r)+1} \quad \text{for all } r \geq r_0
\]
and, by assumption, we have
\[
(3) \quad \|x\|^{l(n)+1} \leq \|x\|^{d(n)} \|x\|^{1-d(n)} \quad \text{for all } n.
\]
For every \( n \) we have \( c_n \geq 1 \) so that \( \|\| \leq c_n \| \|_n \). We set
\[
\gamma_n := \frac{\beta_n}{c_l(n)2^{n+1}} \inf \left\{ \frac{1}{r\varphi(r)} : n(r) \leq n \right\}, \quad n \in \mathbb{N}.
\]
Then
\[
\varphi(r) \sum_{\nu=n(r)+1}^{\infty} \gamma_\nu B_{l(\nu)} \subset \left( \sum_{\nu=n(r)+1}^{\infty} \beta_\nu B_\nu \right) \cap \frac{1}{2r} A \quad \text{for every } r \geq 1.
\]
Let now
\[
D := \sum_{\nu=1}^{\infty} \gamma_\nu B_{l(\nu)}.
\]
If \( r \geq r_0 \) and \( f \in \varphi(r)D \cap \frac{1}{2r} A \) we may write \( f = f_1 + f_2 \), where \( f_1 \in \varphi(r) \sum_{\nu=1}^{n(r)} \gamma_\nu B_{l(\nu)} \) and \( f_2 \in \varphi(r) \sum_{\nu=n(r)+1}^{\infty} \gamma_\nu B_{l(\nu)} \). We obtain
\[
f_2 \in \sum_{\nu=n(r)+1}^{\infty} \beta_\nu B_\nu, \quad f_1 \in \varphi(r)B_{l(n(r))} \cap r^{-1} A.
\]
To prove that \( f \in 2CB \) it is enough to show that \( f_1 \in C\beta_{n(r)+1}B_{n(r)+1} \). We apply (2) and (3) to obtain, for \( r \geq r_0 \),
\[
\|f_1\|_{n(r)+1} \leq \|f_1\|_{l(n(r))} \leq \|f_1\|^{d(n(r))}_{l(n(r))} \|f_1\|^{1-d(n(r))}_{l(n(r))} \leq \varphi(r)^{d(n(r))}(1/r)^{1-d(n(r))} \leq C\beta_{n(r)+1}.
\]
This completes the proof of the lemma.

The following lemma is probably well known. We give a proof for the sake of completeness. We set \( \|f\|_M = \sup_{x \in M} |f(x)| \) for any function \( f \) on the set \( M \). An open bounded subset \( \Omega \subseteq \mathbb{C}^d \) is called hyperconvex whenever it is connected and there is a continuous plurisubharmonic negative function
\( \varrho \) on \( \Omega \) such that the sets \( \{ z \in \Omega \mid \varrho(z) < c \} \) are relatively compact in \( \Omega \) for every negative \( c \) (see [14, p. 80]).

**Lemma 4.5.** Let \( U \subset \mathbb{C}^d \) be open, hyperconvex and connected, and \( \omega \subset U \cap \mathbb{R}^d \) open in \( \mathbb{R}^d \) and nonempty. Then for every open connected set \( V \) with \( \omega \subset V \subset U \) there is \( 0 < \tau < 1 \) so that

\[
\|f\|_V \leq \|f\|_U^\frac{1}{\tau} \|f\|_\omega^{1-\tau}
\]

for all bounded holomorphic functions \( f \) on \( U \).

**Proof.** We choose a small ball \( E \subset \omega \subset \mathbb{R}^d \). We denote by \( v_{E,U}(z) = \sup\{v(z) \mid v \text{ plurisubharmonic on } U, v|_{E} \leq -1, v \leq 0\} \). As \( U \) is hyperconvex we have \( \lim_{z \to w} v_{E,U}(z) = 0 \) for all \( w \in \partial U \) (see [14, Proposition 4.5.2]). We wish to show that \( v_{E,U} \) is continuous.

Let \( V_E \) be the pluricomplex Green function of \( E \) (see [14, pp. 184 ff.]), which is continuous on \( \mathbb{C}^d \) ([14, Theorem 5.4.6]). Therefore \( 2 \varepsilon := \inf \{ V_E(z) \mid z \in \partial U \} > 0 \). This implies that \( u = \max(\varepsilon(v_{E,U} + 1), V_E) \in \mathcal{L}(\mathbb{C}^d) \) (the class of plurisubharmonic functions of minimal growth, see [14, p. 184]) and \( u \leq 0 \) on \( E \). Therefore \( u \leq V_E \), which implies \( v_{E,U} \leq \varepsilon^{-1}V_E - 1 \) on \( U \). Because of the continuity of \( V_E \), the upper semicontinuous regularization satisfies \( v_{E,U} = -1 \) on \( E \). Therefore \( v_{E,U} \) is continuous (see [14, Proposition 4.5.3]).

We set \( \tau = \sup \{ v_{E,U}(z) + 1 \mid z \in V \} \). Then \( 0 < \tau < 1 \).

Let \( f \) be holomorphic, bounded and nonconstant on \( U \). We put

\[
v(z) = \frac{\log |f(z)| - \log \|f\|_U}{\log \|f\|_U - \log \|f\|_E}.
\]

Then \( v \leq v_{E,U} \) on \( U \), hence \( v(z) \leq \tau - 1 \) for \( z \in V \), which means

\[
\frac{\log \|f\|_V - \log \|f\|_U}{\log \|f\|_U - \log \|f\|_E} \leq \tau - 1
\]

and therefore

\[
\|f\|_V \leq \|f\|_U^\frac{1}{\tau} \|f\|_\omega^{1-\tau} \leq \|f\|_U^\frac{1}{\tau} \|f\|_\omega^{1-\tau}.
\]

**Proposition 4.6.** If \( \Omega \subset \mathbb{R}^d \) is open and connected and \( \lim_{r \to \infty} r^{-\varepsilon} \varphi(r) = 0 \), then \( \mathcal{A}(\Omega) \) has property \( (DN_\varphi) \).

**Proof.** We choose \( \omega \subset \subset \Omega \) open. If \( p \) is a continuous seminorm on \( \mathcal{A}(\Omega) \), then there is a compact \( K \subset \Omega \) so that \( p \) extends to a continuous seminorm on \( H(K) \). We may assume that \( \omega \subset K \). We choose a basis \( U_1 \supset U_2 \supset \ldots \) of open connected neighborhoods of \( K \). Then Lemma 4.5 and the fact that every open connected subset in \( \mathbb{R}^d \) has a basis of hyperconvex neighborhoods in \( \mathbb{C}^d \) (this is definitely well known, see [4, proof of Prop. 1] or [12, proof of Props. 6 and 7]; explicitly it follows from [10, Lemma 1.1]) provide the assumption of Lemma 4.4. So \( H(K) \) has property \( (DN_\varphi) \) and \( \| \|_\omega \) is a
\( \varphi \)-dominating norm. But then we find a continuous seminorm \( q \) on \( H(K) \) according to (DN\( _\varphi \)). The restriction of \( q \) to \( \mathcal{A}(\Omega) \) gives the result. \( \blacksquare \)

**Proposition 4.7.** If \( E \) is a Fréchet space with property \( (\overline{\Omega}) \) then there is a nondecreasing positive function \( \varphi \) such that \( \lim_{r \to \infty} r^{-\varepsilon}\varphi(r) = 0 \) for all \( \varepsilon > 0 \) and every nuclear Fréchet space \( F \) having (DN\( _\varphi \)) satisfies \( \text{Ext}^1(F, E) = 0 \).

**Proof.** We proceed in a similar way to [17, Lemma 3]. First we choose, by use of Lemma 4.2, a function \( \psi \) so that \( E \) has property \( (\Omega_\psi) \) and \( \lim_{r \to \infty} r^{-\varepsilon}\psi(r) = 0 \) for all \( \varepsilon > 0 \). We set \( \varphi(r) = \psi(r^2) \). Then \( \varphi \) dominates \( \psi \), i.e. for every \( R \) there is \( D_R \) with \( D_R r^{-\varepsilon}\varphi(r) = 0 \) for all \( \varepsilon > 0 \).

Let \( p_0 \) be a \( \varphi \)-dominating norm in \( F \). If a seminorm \( p \leq p_0 \) is given, then we choose \( q \) according to (DN\( _\varphi \)), i.e. we have

\[
(4) \quad p \leq D \left( r p_0 + \frac{1}{\varphi(r)} q \right).
\]

For \( x \neq 0 \) and \( R > 0 \) we put \( r = R \frac{p(x)}{p_0(x)} \) in (1) to obtain, for any \( y \in E' \),

\[
\|y\|_l^* \leq C \psi \left( R \frac{p(x)}{p_0(x)} \right) \|y\|_n^* + \frac{1}{R} \frac{p_0(x)}{p(x)} \|y\|_k^*,
\]

hence

\[
\|y\|_l^* p(x) \leq C \psi \left( R \frac{p(x)}{p_0(x)} \right) p(x) \|y\|_n^* + \frac{1}{R} \|y\|_k^* p_0(x).
\]

Now we put \( r = \frac{1}{2D} \frac{p(x)}{p_0(x)} \) in (4) to obtain

\[
\varphi \left( \frac{1}{2D} \frac{p(x)}{p_0(x)} \right) \leq 2D \frac{q(x)}{p(x)}
\]

and therefore

\[
\psi \left( R \frac{p(x)}{p_0(x)} \right) \leq D_{2D} \varphi \left( \frac{1}{2D} \frac{p(x)}{p_0(x)} \right) \leq 2DD_{2D} \frac{q(x)}{p(x)}.
\]

We have shown that

\[
\exists p_0 \forall k \exists l \forall n, p, R \exists q, S \forall x, y : \quad \|y\|_l^* p(x) \leq S \|y\|_n^* q(x) + \frac{1}{R} \|y\|_k^* p_0(x).
\]

This is condition \((S_0^*)\) in [24]. As \( F \) is nuclear, [24, Theorem 3.8] yields the result (cf. also [26, Th. 5.2.6], [11, Th. 3.1]). \( \blacksquare \)

The proof of Theorem 2.3 is now completed by:

**Proposition 4.8.** If \( E \) is a Fréchet space with property \( (\overline{\Omega}) \) and \( F \) is a nuclear locally convex space which has (DN\( _\varphi \)) for every \( \varphi \) satisfying \( \lim_{r \to \infty} r^{-\varepsilon}\varphi(r) = 0 \) for all \( \varepsilon > 0 \), then \( \text{Ext}^1(F, E) = 0 \). In particular, this holds for any closed subspace \( F \) of \( \mathcal{A}(\Omega)^d \).
Proof. Consider an arbitrary short topologically exact sequence

\[ 0 \to E \to X \to F \to 0. \]

Weakening the topology of \( X \) and \( F \) in a suitable way, we easily obtain the commutative diagram

\[
\begin{array}{ccc}
0 & \to & E \\
\uparrow \text{id} & & \uparrow \text{id} \\
0 & \to & X
\end{array}
\begin{array}{ccc}
& & \\
& \downarrow j & \\
& & \\
0 & \to & F \\
\downarrow q & & \downarrow q \\
0 & \to & 0
\end{array}
\]

where, by assumption, one can assume that \( F \) has \((\text{DN}_\varphi)\) for \( \varphi \) chosen according to Proposition 4.7. Thus \( J \) lifts with respect to \( q_1 \) and the lower row splits [6, Prop. 1.7(c)]. ■

5. Sufficiency of conditions in Theorem 2.1. We will use a result from [9]. For this we need some notation. Let \( \omega : [0,\infty[ \to [0,\infty[ \) be a continuous increasing function. We call \( \omega \) a quasi-analytic weight function if it has the following properties:

\[
\begin{align*}
(\alpha) & \quad \omega(2t) = O(\omega(t)) \text{ as } t \to \infty. \\
(\beta) & \quad \int_0^\infty \frac{\omega(t)}{1+t^2} \, dt = \infty. \\
(\gamma) & \quad \log t = o(\omega(t)) \text{ as } t \to \infty. \\
(\delta) & \quad \varphi_\omega : t \mapsto \omega(e^t) \text{ is convex.} \\
(\varepsilon) & \quad \omega(t) = o(t) \text{ as } t \to \infty.
\end{align*}
\]

Let \( \omega \) be a weight function and \( \Omega \subset \mathbb{R}^d \) an open set. We define (cf. [3])

\[
E_{(\omega)}(\Omega) = \left\{ f \in C^\infty(\Omega) \bigg| \text{ for every } K \subset \subset \Omega \text{ and every } m \in \mathbb{N} \text{ we have} \right. \\
q_{K,m}(f) = \sup_{j \in \mathbb{N}^d} \sup_{x \in K} |f^{(j)}(x)| \exp \left( -m \sum_{\nu=1}^{d} \varphi_\omega^*(\frac{j_\nu}{m}) \right) < \infty, \\
\left. \right\}
\]

where \( \varphi_\omega^* : [0,\infty[ \to [0,\infty[ \) is the Young conjugate of \( \varphi_\omega \), i.e.,

\[
\varphi_\omega^*(y) := \sup\{xy - \varphi_\omega(x) : x \geq 0\}.
\]

It is easily seen that \( \varphi_\omega^* \) is a convex increasing function. Then \( E_{(\omega)}(\Omega) \) is a nuclear Fréchet space which contains \( \mathcal{A}(\Omega) \) continuously.

We denote by \( A_s(\alpha) \) the finite type (for \( s < \infty \)) and infinite type (for \( s = \infty \)) power series space generated by the exponent sequence \( \alpha = (\alpha_n) \) (cf. [18, §29]). If \( \alpha \) is a stable exponent sequence (sup \( n \alpha_{2n}/\alpha_n < \infty \)) then \( A_s(\alpha) \simeq A_s(\alpha)^2 \), in particular, we then have \( A_s(\alpha,Z) \simeq A_s(\alpha) \), where

\[
A_s(\alpha,Z) = \left\{ \xi \in \mathbb{C}^Z \bigg| |x|_t := \sum_{k \in \mathbb{Z}} |\xi_k| e^{t|k|} < \infty \text{ for all } t < s \right\}.
\]

The following result is Lemma 3 of [9].
Lemma 5.1. If \( \alpha = (\alpha_n)_n \) is a stable exponent sequence such that
\[
\lim_{n \to \infty} n/\alpha_n = 0
\]
then there is a quasi-analytic weight function \( \omega \) and an exact sequence
\[
0 \to A_s(\alpha, \mathbb{Z}) \xrightarrow{J} \mathcal{E}_\omega(I) \xrightarrow{L} \mathcal{E}_\omega(I) \to 0,
\]
where \( s = 1 \) for \( I = (-1, 1) \), \( s = \infty \) for \( I = \mathbb{R} \) and \( \text{im} \, J \subset \mathcal{A}(I) \). The operator \( J \) is given by
\[
J(\xi) = \sum_{k \in \mathbb{Z}} \xi_k e^{\text{sgn}(k)\alpha_{|k|}z},
\]
and \( L \) is an infinite order differential operator which maps \( \mathcal{A}(I) \) into \( \mathcal{A}(I) \).

To extend this lemma to open cubes \( Q = I_1 \times \ldots \times I_d \subset \mathbb{R}^d \), where the \( I_\nu \) are open intervals, we need some preparation. Since the exponentials \( e^{r_kz} = \prod_{\nu=1}^d e^{r_{\nu}z_{\nu}} \) are total in \( \mathcal{E}_\omega(Q) \) it is easily seen that \( \mathcal{E}_\omega(Q) \cong \mathcal{E}_\omega(I_1) \otimes \ldots \otimes \mathcal{E}_\omega(I_d) \). We will need to decompose a power series space into a tensor product of such spaces.

Lemma 5.2. Let \( \alpha \) be a stable exponent sequence with \( \lim_{n \to \infty} n^{1/d}/\alpha_n = 0 \) and \( s \in \{0, \infty\} \). Then there exists a stable exponent sequence \( \beta \) with \( \lim_{n \to \infty} n/\beta_n = 0 \) so that \( A_s(\alpha) \cong A_s(\beta)^{\otimes d} \).

Proof. We set \( \beta_n = \alpha_n^d \). This sequence satisfies the assertions on \( \beta \). To establish the isomorphism we fix an enumeration \( \mathbb{N}^d \ni j \mapsto n = n(j) \in \mathbb{N} \) of \( \mathbb{N}^d \) so that \( m = m(j) := \max_{\nu} j_\nu \) is increasing and set \( \gamma_n = \beta_{j_1} + \ldots + \beta_{j_d} \) for \( n = n(j) \). Then clearly \( (m - 1)^d < n \leq m^d \) and \( \beta_m \leq \gamma_n \leq d\beta_m \). So using the stability of \( \beta \) we obtain, with suitable \( D \),
\[
\frac{1}{D} \alpha_n \leq \beta_{[n^{1/d}]} \leq \beta_m \leq \gamma_n \leq d\beta_m \leq d\beta_{[n^{1/d}]+1} \leq dD\alpha_n.
\]
Since
\[
A_s(\beta)^{\otimes d} \cong \left\{ \xi = (\xi_j)_{j \in \mathbb{N}^d} \left| \left| x \right|_t = \sum_j \left| \xi_j \right| e^{t(\beta_{j_1} + \ldots + \beta_{j_d})} < \infty \right. \text{ for all } t < s \right\},
\]
the map \( \xi = (\xi_j)_{j \in \mathbb{N}^d} \mapsto x = (\xi(n,j))_{n \in \mathbb{N}} \) establishes an isomorphism onto \( A_s(\alpha) \).

We will use the following lemma (we omit the proof).

Lemma 5.3. Let
\[
0 \to X_k \xrightarrow{j_k} Y_k \xrightarrow{l_k} Z_k \to 0, \quad k = 1, 2,
\]
be exact sequences of nuclear Fréchet spaces. Then
\[
0 \to X_1 \otimes X_2 \xrightarrow{J} Y_1 \otimes Y_2 \xrightarrow{L} (Z_1 \otimes Z_2) \oplus (Y_1 \otimes Z_2) \xrightarrow{K} Z_1 \otimes Z_2 \to 0
\]
is an exact sequence, where $J = j_1 \otimes j_2$, $L = (l_1 \otimes \text{id}) \oplus (\text{id} \otimes l_2)$ and $K(u_1 \oplus u_2) = (\text{id} \otimes l_2)u_1 - (l_1 \otimes \text{id})u_2$.

Now we are in a position to give a $d$-dimensional analogue to Lemma 5.1.

**Lemma 5.4.** If $\alpha$ is a stable exponent sequence with $\lim_{n \to \infty} n/\alpha_n = 0$, then there is a quasi-analytic weight function $\omega$ and an exact sequence

$$0 \to \Lambda_s(\alpha, \mathbb{Z})^\otimes \hat{d} \xrightarrow{J} \mathcal{E}(\omega)(Q) \xrightarrow{L_f} X \to 0,$$

where $s = 1$ for $Q = (-1, 1)^d$, $s = \infty$ for $Q = \mathbb{R}^d$, $\text{im} \ J \subset \mathcal{A}(Q)$ and $X$ is a closed topological subspace of $\mathcal{E}(\omega)(Q)^d$. The operator $J$ is given by

$$J(\xi) = \sum_{j \in \mathbb{Z}^d} \xi_j \exp \left( \sum_{\nu=1}^{d} \text{sgn}(j_{\nu})\alpha_{|j_{\nu}|}z_{\nu} \right)$$

and $Lf = (L_1 f, \ldots, L_d f)$, $L_\nu$ being the operator from Lemma 5.1 acting on the $\nu$th variable. Moreover

$$L(\mathcal{E}(\omega)(Q)) = X := \{(f_1, \ldots, f_d) \in \mathcal{E}(\omega)(Q)^d \mid L_\nu f_\mu = L_\mu f_\nu \text{ for all } \nu, \mu\},$$

and

$$L(\mathcal{A}(Q)) \subseteq Y := \{(f_1, \ldots, f_d) \in \mathcal{A}(Q)^d \mid L_\nu f_\mu = L_\mu f_\nu \text{ for all } \nu, \mu\},$$

where $Y$ is a closed topological subspace of $\mathcal{A}(Q)^d$.

**Proof.** This follows by induction on dimension by applying Lemma 5.3 to the $(d-1)$-dimensional exact sequence

$$0 \to \Lambda_s(\alpha, \mathbb{Z})^\otimes (d-1) \xrightarrow{J} \mathcal{E}(\omega)(Q) \xrightarrow{L_f} \text{im} L \to 0,$$

and the 1-dimensional exact sequence of Lemma 5.1. \hfill \blacksquare

We are now ready to prove Theorem 2.1 for cubes, hence for all Cartesian product sets in $\mathbb{R}^d$. To do it for all open sets we need some geometrical preparation.

**Lemma 5.5.** Every open set $\Omega \subset \mathbb{R}^d$ is real-analytically diffeomorphic to an open set $\Omega' \subset \mathbb{R}^d$ so that $(-1, 1)^d \subset \Omega' \subset (-\infty, 1)^d$.

**Proof.** For $\Omega = \mathbb{R}^d$ this is clear, so assume $\Omega \neq \mathbb{R}^d$. We choose $y \in \Omega$ and then a point $w \in \partial B \cap \partial \Omega$ where $B$ is the largest open ball with center $y$ which is contained in $\Omega$. By an affine transformation we may assume that $w = 0$ and $\{x \mid x_\nu > 0 \text{ for all } \nu, |x| < \varepsilon \} \subset \Omega$ for some $\varepsilon > 0$.

The reflection $x \mapsto |x|^{-2}x$ maps $\Omega$ onto $\Omega_1$ with $\{x \mid x_\nu > 0 \text{ for all } \nu, |x| > r \} \subset \Omega_1$ for some $r > 0$.

Finally $x \mapsto (\frac{2s}{\pi} \arctan x_1 + 1 - s, \ldots, \frac{2s}{\pi} \arctan x_d + 1 - s)$ for $s > 0$ large enough maps $\Omega_1$ onto a set as claimed. \hfill \blacksquare

The proof of 2.1 is now completed by:
Proposition 5.6. If $E$ is an $n^{1/d}$-nuclear Fréchet space with property $(\overline{\Omega})$ then $E$ is isomorphic to a quotient space of $\mathcal{A}(\Omega)$.

Proof. We may assume that $\Omega$ is of the form described in Lemma 5.5. We set $Q = (-1, 1)^d$. According to Lemmas 3.3 and 3.5 there is a stable exponent sequence $\alpha$ with $\lim_n n^{1/d}/\alpha_n = 0$ so that $E$ is $\alpha$-nuclear. By use of Lemma 5.2 we find a stable exponent sequence $\beta$ with $\lim_n n/\beta_n = 0$ so that $A_0(\alpha) \cong A_0(\beta)_{\hat{\otimes}d}$. Since $E$ has property $(\overline{\Omega})$, hence $(\Omega)$ (see [18]), it is isomorphic to a quotient space of $A_0(\alpha) \cong A_0(\beta)_{\hat{\otimes}d} \cong A_1(\beta)_{\hat{\otimes}d}$ by [21]. Let $q : A_1(\beta)_{\hat{\otimes}d} \to E$ be a quotient map.

For $\beta$ we find $\omega$ according to Lemma 5.4 and obtain from that lemma the middle row of the following diagram:

\[
0 \to E \xrightarrow{Q} X \xrightarrow{id} 0 \\
0 \to A_1(\beta, Z)_{\hat{\otimes}d} \xrightarrow{J} \mathcal{E}(\omega)(Q) \xrightarrow{L} X \xrightarrow{0} 0 \\
0 \to A_1(\beta)_{\hat{\otimes}d} \xrightarrow{J} \mathcal{A}(\Omega) \xrightarrow{L} Y
\]

Here $\iota$ is the natural imbedding, i.e. for $\xi = (\xi_j)_{j \in \mathbb{N}^d}$ we set $(\iota \xi)_j = \xi_j$ for $j \in \mathbb{N}^d$ and $(\iota \xi)_j = 0$ otherwise. This is an imbedding onto a complemented subspace so the quotient map $q$ yields the surjective map $S$. Moreover, $X$ (resp. $Y$) is the set of elements in $\mathcal{E}(\omega)(Q)^d$ (resp. $\mathcal{A}(\Omega)^d$) satisfying the compatibility conditions. We denote by $\varphi_1$ the restriction map $\mathcal{A}(\Omega) \to \mathcal{A}(Q) \hookrightarrow \mathcal{E}(\omega)(Q)$ and analogously $\varphi_2$ in the last column. The upper row is obtained via the standard procedure as in [6, Prop. 1.7(a)]. The diagram is commutative, the upper and middle rows are topologically exact.

We have to show that the map $J$ in the lower row, which is the restriction to $A_0(\beta)_{\hat{\otimes}d}$ of the map $J$ described in Lemma 5.4, has values in $\mathcal{A}(\Omega)$. The reason is that for $\xi \in A_1(\beta)_{\hat{\otimes}d}$, we have

\[
J(\xi) = \sum_{j \in \mathbb{N}^d} \xi_j \exp \left( \sum_{\nu=1}^d j_\nu \beta_{j_\nu} z_\nu \right).
\]

This series converges uniformly on compact subset of $\{ z \in \mathbb{C}^d \mid \text{Re} \ z_\nu < 1 \ \text{for all} \ \nu \}$, hence defines a holomorphic function on this set. Therefore it defines a real-analytic function on the set $\{ x \in \mathbb{R}^d \mid x_\nu < 1 \ \text{for all} \ \nu \}$ which contains $\Omega$.

By Proposition 4.8 we have $\text{Ext}^1(Y, E) = 0$ and therefore $\varphi_2$ lifts to $Z_1$ with respect to $Q$. A standard proof (cf. [6, Prop. 1.7(c)]) shows that this
implies existence of a map $B : \mathcal{A}(\Omega) \to E$ such that $B \circ J = S \circ \iota$. Since $B \circ J = S \circ \iota = q$ is surjective and open, so is $B$. ■

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