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Fréchet quotients of spaces of real-analytic functions

by

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Dedicated to Aleksander Pełczyński on the occasion of his 70th birthday

Abstract. We characterize all Fréchet quotients of the space $\mathscr{A}(\Omega)$ of (complex-valued) real-analytic functions on an arbitrary open set $\Omega \subseteq \mathbb{R}^d$. We also characterize those Fréchet spaces E such that every short exact sequence of the form $0 \to E \to X \to \mathscr{A}(\Omega) \to 0$ splits.

Let $\mathscr{A}(\Omega)$ denote the space of complex-valued real-analytic functions on the open set $\Omega \subset \mathbb{R}^d$, equipped with its natural locally convex topology (see [15] or [1]). The Fréchet structure of the spaces $\mathscr{A}(\Omega)$ was closely investigated in recent years and this was a basis of various interesting results. The Fréchet subspaces are characterized in Domański–Langenbruch [5] as being isomorphic to subspaces of $H(\mathbb{D}^d)$ if Ω has finitely many connected components, and of $H(\mathbb{D}^d)^{\mathbb{N}}$ if Ω has infinitely many connected components. In Domański–Vogt [7] it was shown that for Ω connected the space $\mathscr{A}(\Omega)$ admits only finite-dimensional complemented subspaces, and that led to the proof in [7] (cf. [8]) that no space $\mathscr{A}(\Omega)$ admits a (Schauder) basis.

About the quotients of $\mathscr{A}(\Omega)$ it was only known that they have the very restrictive property $(\overline{\overline{\Omega}})$. This was a basic ingredient in the proof that all complemented Fréchet subspaces are finite-dimensional. However, it was not known whether $\mathscr{A}(\Omega)$ admits any infinite-dimensional Fréchet quotient besides the space ω of all sequences. Only recently was it shown in [9] that $\mathscr{A}(\Omega)$ admits nontrivial Fréchet quotients, but that is far from being an exact picture.

In the present paper we show that a Fréchet space E is isomorphic to a quotient of $\mathscr{A}(\Omega)$ for some, equivalently any, open set $\Omega \subseteq \mathbb{R}^d$ if and only if

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E has property $(\overline{\overline{\Omega}})$ and is $n^{1/d}$-nuclear (see below for the definition in terms of Kolmogorov diameters), or equivalently (see [25]), if and only if E has property $(\overline{\overline{\Omega}})$ and is isomorphic to a quotient of $H(\mathbb{D}^d)$. An essential step is to show that a Fréchet space has $(\overline{\overline{\Omega}})$ if and only if $\mathrm{Ext}^1(\mathscr{A}(\Omega), E) = 0$, which means that every topologically exact sequence

$$0 \to E \to X \to \mathscr{A}(\Omega) \to 0$$

splits. This result is of independent interest.

1. Preliminaries. We use common notation for locally convex spaces, in particular Fréchet spaces. For the notation and general results we refer to [18] or [13]. For homological notions in locally convex spaces see [26].

For any open $\Omega \subset \mathbb{R}^d$ the space $\mathscr{A}(\Omega)$ is equipped with its natural topology given by

$$\mathscr{A}(\Omega) = \lim_{n} \mathrm{proj}\, H(K_n)$$

where $K_1 \subset \mathring{K}_2 \subset K_2 \subset \ldots$ is a compact exhaustion of Ω and $H(K_n)$ denotes the (LB)-space of germs of holomorphic functions on K_n. By Martineau [15, Th. 1.9] this topology coincides with the one given by

$$\mathscr{A}(\Omega) = \lim_{\omega} \mathrm{ind}\, H(\omega).$$

Here ω runs through all complex neighborhoods of Ω, and $H(\omega)$ denotes the Fréchet space of holomorphic functions on ω with the compact-open topology.

A Fréchet space with a fundamental system of seminorms $\| \ \|_1 \leq \| \ \|_2 \leq \ldots$ is said to have *property* $(\overline{\overline{\Omega}})$ if

$$\forall k\ \exists l\ \forall n, 0 < \vartheta < 1\ \exists C: \quad \| \ \|_l^* \leq C \| \ \|_k^{*\,\vartheta} \| \ \|_n^{*\,1-\vartheta}$$

or equivalently

$$\forall k\ \exists l\ \forall n, \varepsilon > 0\ \exists C\ \forall r > 0: \quad U_l \subset C(r^\varepsilon U_n + r^{-1} U_k).$$

Here we set $\|y\|_k^* = \sup\{|y(x)| \mid x \in U_k\}$ and $U_k = \{x \mid \|x\|_k \leq 1\}$. For the role of property $(\overline{\overline{\Omega}})$ see [1], [2], [7], [22], [24]; for the equivalence of both conditions see [18, Lemma 29.13]. For examples of spaces with $(\overline{\overline{\Omega}})$ see [16, Ex. 4.12(5)].

Clearly $(\overline{\overline{\Omega}})$ is a topological linear invariant which is inherited by quotient spaces. We will need another topological invariant.

Let X be a linear space and $V \subset U$ absolutely convex subsets. We define the nth *Kolmogorov diameter of* V *with respect to* U to be

$$\delta_n(V, U) = \inf\{\delta > 0 \mid V \subset \delta U + F,\ \dim F \leq n\}.$$

Here F denotes a linear subspace of E.

Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ be a nonnegative increasing sequence so that $\lim_n (\log n)/\alpha_n = 0$ and $\sup_n \alpha_{2n}/\alpha_n < \infty$. We call this a *stable exponent sequence.* Using V and U for absolutely convex neighborhoods of zero, we define a locally convex space X to be:

(1) *weakly α-nuclear* if $\forall U\ \exists V, t > 0 : \lim_n e^{t\alpha_n}\delta_n(V, U) = 0$,
(2) *α-nuclear* if $\forall U, t > 0\ \exists V : \lim_n e^{t\alpha_n}\delta_n(V, U) = 0$,
(3) *strongly α-nuclear* if $\forall U\ \exists V\ \forall t > 0 : \lim_n e^{t\alpha_n}\delta_n(V, U) = 0$.

The assumptions on (α_n) imply that every weakly α-nuclear space is nuclear (see [20, p. 296]). These are topological linear invariants, inherited by subspaces, quotients, countable products and direct sums, hence by countable projective and inductive limits (see [25, Lemma 1.3, 1.4]).

It is well known that for any open $U \subset \mathbb{C}^d$ the space $H(U)$ of holomorphic functions on U with the compact-open topology is weakly $n^{1/d}$-nuclear. Since for open $\Omega \subset \mathbb{R}^d$ we can write

$$\mathscr{A}(\Omega) = \lim_n \operatorname{proj} \lim_k \operatorname{ind} H(K_n + k^{-1}\mathbb{D}^d)$$

we see that $\mathscr{A}(\Omega)$ is weakly $n^{1/d}$-nuclear. Here $\mathbb{D}^d = \{z \in \mathbb{C}^d \mid \sup_\nu |z_\nu| < 1\}$.

A locally convex space is called a (PLB)-*space* ((PLS)-*space*, resp.) if it is a countable projective limit of (LB)-spaces ((LS)-spaces, resp.). Main examples of (PLB)-spaces are $\mathscr{A}(\Omega)$, the space $\mathscr{D}'(\Omega)$ of distributions, and all Fréchet and (LB)-spaces; the first two examples are also (PLS)-spaces.

In this paper Ext^1 will always be taken in the category of locally convex spaces, i.e. $\mathrm{Ext}^1(E, F) = 0$ will mean that every topologically exact sequence

$$0 \to F \xrightarrow{j} X \xrightarrow{q} E \to 0$$

of locally convex spaces splits, i.e., q has a continuous linear right inverse. Recall that the sequence above is *topologically exact* whenever j is a topological isomorphism onto the kernel of q and q is surjective, continuous and open onto its image.

We denote by $L(X, Y)$ the space of all continuous linear maps from a locally convex space X into another such space Y.

Consider an arbitrary projective spectrum of linear spaces:

$$\ldots \to X_{n+1} \xrightarrow{i_n^{n+1}} X_n \to \ldots \to X_1 \xrightarrow{i_0^1} X_0.$$

Then $\mathrm{Proj}^1_{n\in\mathbb{N}} X_n := \prod_{n\in\mathbb{N}} X_n / \operatorname{im} \sigma$, where

$$\sigma : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} X_n, \qquad \sigma((x_n)_{n\in\mathbb{N}}) := (i_n^{n+1} x_{n+1} - x_n)_{n\in\mathbb{N}}.$$

Clearly, $\ker \sigma = \lim \operatorname{proj}_{n\in\mathbb{N}} X_n$.

Let E be a Fréchet space, $(E_n)_{n\in\mathbb{N}}$ its sequence of local Banach spaces, and $i_n^m : E_m \to E_n$ the linking maps. There exists the following short (topologically) exact sequence:

$$0 \to E \to \prod_{n\in\mathbb{N}} E_n \xrightarrow{\sigma} \prod_{n\in\mathbb{N}} E_n \to 0,$$

where $\sigma((x_n)_{n\in\mathbb{N}}) := (i_n^{n+1}x_{n+1} - x_n)_{n\in\mathbb{N}}$, which we call the *canonical resolution* ([18, definition after 26.14]). Consider the spectrum $(L(X,E_n))_{n\in\mathbb{N}}$, where the linking maps are defined as follows:

$$I_n^{n+1} : L(X,E_{n+1}) \to L(X,E_n), \qquad I_n^{n+1}(T) = i_n^{n+1} \circ T.$$

Then $\mathrm{Proj}^1_{n\in\mathbb{N}} L(X,E_n) = 0$ means exactly that every $T \in L(X, \prod_{n\in\mathbb{N}} E_n)$ lifts with respect to σ, i.e., there is a map $S \in L(X, \prod_{n\in\mathbb{N}} E_n)$ such that $\sigma \circ S = T$.

We will need some general homological facts.

LEMMA 1.1. *Let E be a Fréchet space with local Banach spaces $(E_n)_{n\in\mathbb{N}}$. Let F be a locally convex space. If either E or F is nuclear then* $\mathrm{Ext}^1(F,E) \cong \mathrm{Proj}^1_{n\in\mathbb{N}} L(F,E_n)$.

Proof. Observe that $L(E,F)$ can be identified with $\lim\mathrm{proj}_{n\in\mathbb{N}} L(F,E_n)$. We apply the functor $L(F,\cdot)$ to the canonical resolution of E. By the standard homological argument we obtain the following long exact sequence (see, for instance, [19, Th. 3.4(b)]):

$$0 \to L(F,E) \to L\Big(F, \prod_{n\in\mathbb{N}} E_n\Big) \to L\Big(F, \prod_{n\in\mathbb{N}} E_n\Big) \to \mathrm{Ext}^1(F,E) \to \mathrm{Ext}^1\Big(F, \prod_{n\in\mathbb{N}} E_n\Big) \to \ldots$$

For nuclear F and any Banach space X we have $\mathrm{Ext}^1(F,X) = 0$ ([23]) and for nuclear E we may assume E_n to be injective for every n. In both cases we get $\mathrm{Ext}^1(F, \prod_{n\in\mathbb{N}} E_n) = 0$. We obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\prod_{n\in\mathbb{N}} L(F,E_n) & \xrightarrow{\sigma} & \prod_{n\in\mathbb{N}} L(F,E_n) & \longrightarrow & \mathrm{Proj}^1_{n\in\mathbb{N}} L(F,E_n) & \longrightarrow & 0 \\
{\scriptstyle T}\uparrow & & {\scriptstyle T}\uparrow & & \uparrow & & \\
L(F, \prod_{n\in\mathbb{N}} E_n) & \longrightarrow & L(F, \prod_{n\in\mathbb{N}} E_n) & \longrightarrow & \mathrm{Ext}^1(F,E) & \longrightarrow & 0
\end{array}$$

where the upper row is obtained from the sequence defining Proj^1 for the spectrum $(L(F,E_n))_{n\in\mathbb{N}}$ and we have omitted the beginning of both rows. Here T denotes the canonical identification. It is clear that the last vertical arrow is an isomorphism we are looking for. ∎

COROLLARY 1.2. *Let E be a Fréchet space, F a locally convex space and $F_0 \subset F$ a subspace. If either E or F is nuclear and* $\mathrm{Ext}^1(F,E) = 0$ *then* $\mathrm{Ext}^1(F_0,E) = 0$.

Proof. Let $(E_n)_{n\in\mathbb{N}}$ be local Banach spaces for E. By our previous comments, we only have to show that $\mathrm{Proj}^1_{n\in\mathbb{N}} L(F_0, E_n) = 0$. But this follows easily. Indeed, arguing as in the proof of Lemma 1.1, we show that every $t_n \in L(F_0, E_n)$ can be extended to $T_n \in L(F, E_n)$. We construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\prod_{n\in\mathbb{N}} L(F, E_n) & \xrightarrow{\sigma} & \prod_{n\in\mathbb{N}} L(F, E_n) & \longrightarrow & \mathrm{Proj}^1_{n\in\mathbb{N}} L(F, E_n) & \longrightarrow & 0 \\
{\scriptstyle S_1}\downarrow & & {\scriptstyle S_2}\downarrow & & \downarrow & & \\
\prod_{n\in\mathbb{N}} L(F_0, E_n) & \xrightarrow{\sigma} & \prod_{n\in\mathbb{N}} L(F_0, E_n) & \longrightarrow & \mathrm{Proj}^1_{n\in\mathbb{N}} L(F_0, E_n) & \longrightarrow & 0
\end{array}$$

where S_1, S_2 are surjective restriction maps. Thus also the last vertical arrow is surjective and this completes the proof. ∎

We are now in a position to formulate our main theorems.

2. Main results

THEOREM 2.1. *A Fréchet space E is isomorphic to a quotient of $\mathscr{A}(\Omega)$ if and only if it is $n^{1/d}$-nuclear and has property $(\overline{\overline{\Omega}})$.*

By [25, Th. 4.1], this has an immediate consequence:

COROLLARY 2.2. *A Fréchet space E is isomorphic to a quotient of $\mathscr{A}(\Omega)$ if and only if it has property $(\overline{\overline{\Omega}})$ and it is isomorphic to a quotient of $H(\mathbb{D}^d)$.*

THEOREM 2.3. *If E is a Fréchet space then the following assertions are equivalent*:

(a) $\mathrm{Ext}^1(\mathscr{A}(\Omega), E) = 0$;
(b) $\mathrm{Proj}^1_{n\in\mathbb{N}} L(\mathscr{A}(\Omega), E_n) = 0$;
(c) *E has property $(\overline{\overline{\Omega}})$.*

REMARK. In fact, analyzing the proof of the above result, it follows that if (c) is satisfied then (a) and (b) also hold for any closed subspace Y of $\mathscr{A}(\Omega)$ in place of $\mathscr{A}(\Omega)$ or, more generally, for every complete nuclear space Y which has (DN_φ) for any φ with $\lim_{r\to\infty} r^{-\varepsilon}\varphi(r) = 0$ for all $\varepsilon > 0$ (see Definition 4.3 below).

The proof will consist of a series of lemmas and will take the rest of this paper.

3. Necessity of the conditions. We first quote the following result (Theorem 3.4 of [7]):

LEMMA 3.1. *If E is a Fréchet space isomorphic to a quotient of $\mathscr{A}(\Omega)$ then it has property $(\overline{\overline{\Omega}})$.*

Due to the preliminary remarks we know that every quotient of $\mathscr{A}(\Omega)$ is weakly $n^{1/d}$-nuclear. This condition is self-improving by the following.

LEMMA 3.2. *If $W \subset V \subset U$ are absolutely convex sets, $\varepsilon > 0$ and*

$$V \subset C(r^{-1}U + r^{\varepsilon}W) \quad \text{for all } r > 0,$$

then there is D so that

$$\delta_n(V,U)^{\varepsilon} \le D\delta_n(W,V) \quad \text{for all } n.$$

Proof. Let $F \subset E$ be a subspace. We set for the moment $\delta(V,U;F) = \inf\{\delta > 0 \mid V \subset \delta U + F\}$ and assume $\delta > \delta(W,U;F)$. Then, by assumption,

$$V \subset C(r^{-1} + r^{\varepsilon}\delta)U + F \quad \text{for all } r > 0,$$

hence

$$\delta(V,U;F) \le C \inf_{r>0}(r^{-1} + r^{\varepsilon}\delta) = C(1+\varepsilon)\varepsilon^{-\varepsilon/(1+\varepsilon)}\delta^{1/(1+\varepsilon)}$$

and therefore $\delta(V,U;F)^{1+\varepsilon} \le D\delta(W,U;F)$. Since $\delta(W,U;F) \le \delta(W,V;F) \times \delta(V,U;F)$ we obtain $\delta(V,U;F)^{\varepsilon} \le D\delta(W,V;F)$. Taking the infimum over all F with $\dim F \le n$ we obtain $\delta_n(V,U)^{\varepsilon} \le D\delta_n(W,V)$. ∎

LEMMA 3.3. *If the Fréchet space E has property $(\overline{\overline{\Omega}})$ and is weakly α-nuclear then it is strongly α-nuclear. In particular it is α-nuclear.*

Proof. For k we choose $l > k$ according to $(\overline{\overline{\Omega}})$, i.e. for every n and $\varepsilon > 0$ there is $C > 0$ so that

$$U_l \subset C(r^{-1}U_k + r^{\varepsilon}U_n) \quad \text{for all } r > 0.$$

Since E is weakly α-nuclear we can find $t > 0$ so that

$$e^{t\alpha_n}\delta_n(U_n,U_l) \to 0 \quad \text{as } n \to \infty.$$

Using Lemma 3.2 we obtain

$$e^{t\alpha_n}\delta_n(U_l,U_k) \to 0 \quad \text{as } n \to \infty$$

for every $t > 0$. ∎

This completes the proof of the necessity of the conditions in Theorem 2.1. We state it as a separate lemma.

LEMMA 3.4. *If E is a Fréchet space isomorphic to a quotient of $\mathscr{A}(\Omega)$ then E is $n^{1/d}$-nuclear.*

For the proof of the sufficiency we need a further improvement.

LEMMA 3.5. *If the Fréchet space E is strongly α-nuclear then there is a stable exponent sequence β with $\lim_n \alpha_n/\beta_n = 0$ so that E is (strongly) β-nuclear.*

Proof. By a recursive choice of the fundamental system $(U_k)_k$ of neighborhoods of zero we may assume that

$$e^{m\alpha_n}\delta_n(U_{k+1},U_k)\to 0 \quad \text{as } n\to\infty$$

for all m and k.

We set $n_0=1$ and determine inductively $n_{m+1}>n_m$ so that

$$e^{(m+1)^2\alpha_n}\delta_n(U_{k+1},U_k)<\frac{1}{m+1}$$

for $n\ge n_{m+1}$ and $k=1,\dots,m$. Setting $\beta_n=m\alpha_n$ for $n_m\le n<n_{m+1}$ we obtain the result. ■

To prove the necessity of the conditions in Theorem 2.3 we recall that in Domański–Langenbruch [5] it is shown that the space $\Lambda_0(n^{1/d})\simeq H(\mathbb{D}^d)$ can be imbedded into $\mathscr{A}(\Omega)$.

Lemma 3.6. *If E is a Fréchet space and* $\mathrm{Ext}^1(\mathscr{A}(\Omega),E)=0$ *then E has property* $(\overline{\overline{\Omega}})$.

Proof. We choose $\beta_n=n^{1/d}$ and imbed $\Lambda_0(\beta)$ into $\mathscr{A}(\Omega)$. Then by Corollary 1.2, $\mathrm{Ext}^1(\Lambda_0(\beta),E)=0$ and the result follows from [24, Theorem 4.2]. ■

4. Sufficiency of conditions in Theorem 2.3. First notice that, by Lemma 1.1, conditions (a) and (b) in Theorem 2.3 are equivalent. We write property $(\overline{\overline{\Omega}})$ in a different form. To do this, throughout this section we let φ and ψ denote increasing unbounded functions $(0,\infty)\to(0,\infty)$.

Definition 4.1. E has *property* (Ω_ψ) if

$$\forall k\ \exists l\ \forall n\ \exists C\ \forall r>0: \quad U_l\subset C\psi(r)U_n+r^{-1}U_k.$$

Remark. Equivalently we may write (cf. [18, Lemma 29.13])

(1) $$\forall k\ \exists l\ \forall n\ \exists C\ \forall r>0\ \forall y\in E': \quad \|y\|_l^*\le C\psi(r)\|y\|_n^*+r^{-1}\|y\|_k^*.$$

We obtain:

Lemma 4.2. *E has property $(\overline{\overline{\Omega}})$ if and only if there is a function ψ with* $\lim_{r\to\infty} r^{-\varepsilon}\psi(r)=0$ *for all $\varepsilon>0$ so that E has property (Ω_ψ).*

Proof. If E has property (Ω_ψ) with ψ as described, then clearly E has property $(\overline{\overline{\Omega}})$.

To prove the converse we find for given k an $l=l(k)$ according to property $(\overline{\overline{\Omega}})$. For $r>0$ and $n\in\mathbb{N}$ we set

$$\psi_{k,n}(r)=\sup_{x\in U_l}\inf_{y\in r^{-1}U_k}\|x-y\|_n+1.$$

Then, clearly,

$$U_l\subset\psi_{k,n}(r)U_n+r^{-1}U_k$$

for every $r>0$ and, due to property $(\overline{\overline{\Omega}})$, for every $\varepsilon>0$ we obtain a constant $C>0$ so that $\psi_{k,n}(r)\le Cr^{\varepsilon}+1$. This implies that $\lim_{r\to\infty} r^{-\varepsilon}\psi_{k,n}(r)=0$ for all k,n and $\varepsilon>0$. It is easily seen that we can find ψ so that $\lim_{r\to\infty} r^{-\varepsilon}\psi(r)=0$ for all $\varepsilon>0$ and that for all k,n there are $C>0$ and r_0 with $\psi_{k,n}(r)\le C\psi(r)$ for all $r>r_0$. ■

Let now X be a locally convex space, let p_0, p, q denote continuous seminorms on X, and let φ be a nondecreasing positive unbounded function.

DEFINITION 4.3. X has *property* (DN_φ) if

$$\exists p_0\ \forall p\ \exists q, C>0\ \forall r>0: \quad p\le C\Big(rp_0+\frac{1}{\varphi(r)}\,q\Big).$$

In this case, p_0 is a norm and it is called a *φ-dominating norm.*

In the next lemma we assume that X is an (LB)-*space*, i.e. there is a sequence $X_1\subset X_2\subset\ldots$ of Banach spaces so that $X=\bigcup_{n=1}^\infty X_n$ and X carries the strongest topology so that all the imbeddings are continuous. We denote the norm in X_n by $\|\ \|_n$; we may assume that $\|\ \|_n\ge\|\ \|_{n+1}$ for all n.

LEMMA 4.4. *If there exists a continuous norm $\|\cdot\|$ on X so that for every n there is $0<\tau_n<1$ with*

$$\|x\|_{n+1}\le\|x\|_n^{\tau_n}\|x\|^{1-\tau_n}$$

for all $x\in X_n$, then X has property (DN_φ) for every φ with $\lim_{r\to\infty} r^{-\varepsilon}\varphi(r)=0$ for all $\varepsilon>0$. Moreover $\|\ \|$ is a φ-dominating norm.

Proof. We set $A=\{x\in X\mid \|x\|\le1\}$. We choose a neighborhood B of zero in X, which may be assumed to be of the form

$$B:=\sum_{\nu=1}^\infty\beta_\nu B_\nu:=\bigcup_{n\in\mathbb{N}}\sum_{\nu=1}^n\beta_\nu B_\nu,$$

where $(\beta_\nu)_{\nu\in\mathbb{N}}\in(0,1)^{\mathbb{N}}$ is a decreasing null sequence, and B_ν is the closed unit ball of X_ν. It is enough to show the existence of $C,r_0\ge1$, $(\gamma_\nu)_{\nu\in\mathbb{N}}\in(0,1)^{\mathbb{N}}$ and $l:\mathbb{N}\to\mathbb{N}$ with $\lim_{n\to\infty} l(n)=\infty$ such that

$$\varphi(r)D\cap\frac{1}{2r}\,A\subset 2CB\qquad\text{for all } r\ge r_0,$$

where

$$D:=\sum_{\nu=1}^\infty\gamma_\nu B_{l(\nu)}.$$

To start we choose an increasing function $\theta:[1,\infty)\to(0,1)$ with $\lim_{r\to\infty}\theta(r)=1$ and

$$\lim_{r\to\infty}\varphi(r)(1/r)^{1-\theta(r)}=0.$$

Then we define an increasing function $n : [1, \infty) \to \mathbb{N}$ with $n(r) \le r$, $\lim_{r\to\infty} n(r) = \infty$, and $C \ge 1$ such that

$$\varphi(r)(1/r)^{1-\theta(r)} \le C\beta_{n(r)+1} \quad \text{for all } r \ge 1.$$

We construct an increasing function $l : \mathbb{N} \to \mathbb{N}$ with $l(n) \le n$ and $\lim_{n\to\infty} l(n) = \infty$ such that

$$d(n) := \tau_{l(n)} \le \theta(n) \quad \text{for all } n.$$

This implies that there is $r_0 \ge 1$ such that $\varphi(r_0) \ge 1$ and

$$\varphi(r)(1/r)^{1-d(n(r))} \le C\beta_{n(r)+1} \quad \text{for all } r \ge r_0 \tag{2}$$

and, by assumption, we have

$$\|x\|_{l(n)+1} \le \|x\|_{l(n)}^{d(n)} \|x\|^{1-d(n)} \quad \text{for all } n. \tag{3}$$

For every n we have $c_n \ge 1$ so that $\|\ \| \le c_n \|\ \|_n$. We set

$$\gamma_n := \frac{\beta_n}{c_{l(n)}2^{n+1}} \inf\left\{\frac{1}{r\varphi(r)} : n(r) \le n\right\}, \quad n \in \mathbb{N}.$$

Then

$$\varphi(r) \sum_{\nu=n(r)+1}^{\infty} \gamma_\nu B_{l(\nu)} \subset \Big(\sum_{\nu=n(r)+1}^{\infty} \beta_\nu B_\nu\Big) \cap \frac{1}{2r} A \quad \text{for every } r \ge 1.$$

Let now

$$D := \sum_{\nu=1}^{\infty} \gamma_\nu B_{l(\nu)}.$$

If $r \ge r_0$ and $f \in \varphi(r)D \cap \frac{1}{2r}A$ we may write $f = f_1 + f_2$, where $f_1 \in \varphi(r)\sum_{\nu=1}^{n(r)} \gamma_\nu B_{l(\nu)}$ and $f_2 \in \varphi(r)\sum_{\nu=n(r)+1}^{\infty} \gamma_\nu B_{l(\nu)}$. We obtain

$$f_2 \in \sum_{\nu=n(r)+1}^{\infty} \beta_\nu B_\nu, \quad f_1 \in \varphi(r) B_{l(n(r))} \cap r^{-1} A.$$

To prove that $f \in 2CB$ it is enough to show that $f_1 \in C\beta_{n(r)+1}B_{n(r)+1}$. We apply (2) and (3) to obtain, for $r \ge r_0$,

$$\begin{aligned}\|f_1\|_{n(r)+1} \le \|f_1\|_{l(n(r))+1} &\le \|f_1\|_{l(n(r))}^{d(n(r))} \|f_1\|^{1-d(n(r))} \\ &\le \varphi(r)^{d(n(r))}(1/r)^{1-d(n(r))} \le C\beta_{n(r)+1}.\end{aligned}$$

This completes the proof of the lemma. ∎

The following lemma is probably well known. We give a proof for the sake of completeness. We set $\|f\|_M = \sup_{x\in M} |f(x)|$ for any function f on the set M. An open bounded subset $\Omega \subseteq \mathbb{C}^d$ is called *hyperconvex* whenever it is connected and there is a continuous plurisubharmonic negative function

ϱ on Ω such that the sets $\{z \in \Omega \mid \varrho(z) < c\}$ are relatively compact in Ω for every negative c (see [14, p. 80]).

LEMMA 4.5. *Let $U \subset \mathbb{C}^d$ be open, hyperconvex and connected, and $\omega \subset\subset U \cap \mathbb{R}^d$ open in $\mathbb{R}^d$ and nonempty. Then for every open connected set V with $\omega \subset\subset V \subset\subset U$ there is $0 < \tau < 1$ so that*

$$\|f\|_V \leq \|f\|_U^{\tau} \|f\|_{\omega}^{1-\tau}$$

for all bounded holomorphic functions f on U.

Proof. We choose a small ball $E \subset \omega \subset \mathbb{R}^d$. We denote by $v_{E,U}$ the relative extremal function, i.e.

$$v_{E,U}(z) = \sup\{v(z) \mid v \text{ plurisubharmonic on } U, v|_E \leq -1, v \leq 0\}.$$

As U is hyperconvex we have $\lim_{z\to w} v_{E,U}(z) = 0$ for all $w \in \partial U$ (see [14, Proposition 4.5.2]). We wish to show that $v_{E,U}$ is continuous.

Let V_E be the pluricomplex Green function of E (see [14, pp. 184 ff.]), which is continuous on $\mathbb{C}^d$ ([14, Theorem 5.4.6]). Therefore $2\varepsilon := \inf\{V_E(z) \mid z \in \partial U\} > 0$. This implies that $u = \max(\varepsilon(v_{E,U}+1), V_E) \in \mathcal{L}(\mathbb{C}^d)$ (the class of plurisubharmonic functions of minimal growth, see [14, p. 184]) and $u \leq 0$ on E. Therefore $u \leq V_E$, which implies $v_{E,U} \leq \varepsilon^{-1} V_E - 1$ on U. Because of the continuity of V_E, the upper semicontinuous regularization satisfies $v_{E,U}^* = -1$ on E. Therefore $v_{E,U}$ is continuous (see [14, Proposition 4.5.3]).

We set $\tau = \sup\{v_{E,U}(z) + 1 \mid z \in V\}$. Then $0 < \tau < 1$.

Let f be holomorphic, bounded and nonconstant on U. We put

$$v(z) = \frac{\log|f(z)| - \log\|f\|_U}{\log\|f\|_U - \log\|f\|_E}.$$

Then $v \leq v_{E,U}$ on U, hence $v(z) \leq \tau - 1$ for $z \in V$, which means

$$\frac{\log\|f\|_V - \log\|f\|_U}{\log\|f\|_U - \log\|f\|_E} \leq \tau - 1$$

and therefore

$$\|f\|_V \leq \|f\|_U^{\tau}\|f\|_E^{1-\tau} \leq \|f\|_U^{\tau}\|f\|_{\omega}^{1-\tau}. \ \blacksquare$$

PROPOSITION 4.6. *If $\Omega \subset \mathbb{R}^d$ is open and connected and $\lim_{r\to\infty} r^{-\varepsilon}\varphi(r) = 0$, then $\mathscr{A}(\Omega)$ has property* (DN_φ).

Proof. We choose $\omega \subset\subset \Omega$ open. If p is a continuous seminorm on $\mathscr{A}(\Omega)$, then there is a compact $K \subset \Omega$ so that p extends to a continuous seminorm on $H(K)$. We may assume that $\omega \subset K$. We choose a basis $U_1 \supset\supset U_2 \supset\supset \ldots$ of open connected neighborhoods of K. Then Lemma 4.5 and the fact that every open connected subset in $\mathbb{R}^d$ has a basis of hyperconvex neighborhoods in $\mathbb{C}^d$ (this is definitely well known, see [4, proof of Prop. 1] or [12, proof of Props. 6 and 7]; explicitly it follows from [10, Lemma 1.1]) provide the assumption of Lemma 4.4. So $H(K)$ has property (DN_φ) and $\| \ \|_\omega$ is a

φ-dominating norm. But then we find a continuous seminorm q on $H(K)$ according to (DN_φ). The restriction of q to $\mathscr{A}(\Omega)$ gives the result. ■

PROPOSITION 4.7. *If E is a Fréchet space with property $(\overline{\overline{\Omega}})$ then there is a nondecreasing positive function φ such that $\lim_{r\to\infty} r^{-\varepsilon}\varphi(r) = 0$ for all $\varepsilon > 0$ and every nuclear Fréchet space F having (DN_φ) satisfies $\mathrm{Ext}^1(F,E) = 0$.*

Proof. We proceed in a similar way to [17, Lemma 3]. First we choose, by use of Lemma 4.2, a function ψ so that E has property (Ω_ψ) and $\lim_{r\to\infty} r^{-\varepsilon}\psi(r) = 0$ for all $\varepsilon > 0$. We set $\varphi(r) = \psi(r^2)$. Then φ dominates ψ, i.e. for every R there is D_R with $\psi(Rr) \le D_R\,\varphi(r)$, and $\lim_{r\to\infty} r^{-\varepsilon}\varphi(r) = 0$ for all $\varepsilon > 0$.

Let p_0 be a φ-dominating norm in F. If a seminorm $p \ge p_0$ is given, then we choose $q \ge p$ according to (DN_φ), i.e. we have

$$p \le D\biggl(rp_0 + \frac{1}{\varphi(r)}\,q\biggr). \tag{4}$$

For $x \ne 0$ and $R > 0$ we put $r = R\frac{p(x)}{p_0(x)}$ in (1) to obtain, for any $y \in E'$,

$$\|y\|_l^* \le C\psi\biggl(R\,\frac{p(x)}{p_0(x)}\biggr)\|y\|_n^* + \frac{1}{R}\,\frac{p_0(x)}{p(x)}\,\|y\|_k^*,$$

hence

$$\|y\|_l^* p(x) \le C\psi\biggl(R\,\frac{p(x)}{p_0(x)}\biggr)p(x)\|y\|_n^* + \frac{1}{R}\,\|y\|_k^* p_0(x).$$

Now we put $r = \frac{1}{2D}\frac{p(x)}{p_0(x)}$ in (4) to obtain

$$\varphi\biggl(\frac{1}{2D}\,\frac{p(x)}{p_0(x)}\biggr) \le 2D\,\frac{q(x)}{p(x)}$$

and therefore

$$\psi\biggl(R\,\frac{p(x)}{p_0(x)}\biggr) \le D_{2DR}\varphi\biggl(\frac{1}{2D}\,\frac{p(x)}{p_0(x)}\biggr) \le 2DD_{2DR}\,\frac{q(x)}{p(x)}.$$

We have shown that

$$\exists p_0\ \forall k\ \exists l\ \forall n,p,R\ \exists q,S\ \forall x,y: \quad \|y\|_l^*\,p(x) \le S\|y\|_n^* q(x) + \frac{1}{R}\,\|y\|_k^*\,p_0(x).$$

This is condition $(\mathrm{S}_1^*)_0$ in [24]. As F is nuclear, [24, Theorem 3.8] yields the result (cf. also [26, Th. 5.2.6], [11, Th. 3.1]). ■

The proof of Theorem 2.3 is now completed by:

PROPOSITION 4.8. *If E is a Fréchet space with property $(\overline{\overline{\Omega}})$ and F is a nuclear locally convex space which has (DN_φ) for every φ satisfying $\lim_{r\to\infty} r^{-\varepsilon}\varphi(r) = 0$ for all $\varepsilon > 0$, then $\mathrm{Ext}^1(F,E) = 0$. In particular, this holds for any closed subspace F of $\mathscr{A}(\Omega)^d$.*

Proof. Consider an arbitrary short topologically exact sequence

$$0 \to E \to X \to F \to 0.$$

Weakening the topology of X and F in a suitable way, we easily obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & X_1 & \xrightarrow{q_1} & F_1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle J} & & \\
0 & \longrightarrow & E & \longrightarrow & X & \xrightarrow{q} & F & \longrightarrow & 0
\end{array}$$

where, by assumption, one can assume that F_1 has (DN_φ) for φ chosen according to Proposition 4.7. Thus J lifts with respect to q_1 and the lower row splits [6, Prop. 1.7(c)]. ∎

5. Sufficiency of conditions in Theorem 2.1. We will use a result from [9]. For this we need some notation. Let $\omega : [0,\infty[\to [0,\infty[$ be a continuous increasing function. We call ω a *quasi-analytic weight function* if it has the following properties:

(α) $\omega(2t) = O(\omega(t))$ as $t \to \infty$.
(β) $\int_0^\infty \frac{\omega(t)}{1+t^2}\,dt = \infty$.
(γ) $\log t = o(\omega(t))$ as $t \to \infty$.
(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.
(ε) $\omega(t) = o(t)$ as $t \to \infty$.

Let ω be a weight function and $\Omega \subset \mathbb{R}^d$ an open set. We define (cf. [3])

$$\mathscr{E}_{(\omega)}(\Omega) = \Big\{ f \in C^\infty(\Omega) \;\Big|\; \text{for every } K \subset\subset \Omega \text{ and every } m \in \mathbb{N} \text{ we have}$$
$$q_{K,m}(f) = \sup_{j\in\mathbb{N}^d} \sup_{x\in K} |f^{(j)}(x)| \exp\Big(-m \sum_{\nu=1}^{d} \varphi_\omega^*\Big(\frac{j_\nu}{m}\Big)\Big) < \infty \Big\},$$

where $\varphi_\omega^* : [0,\infty[\to [0,\infty[$ is the Young conjugate of φ_ω, i.e.,

$$\varphi_\omega^*(y) := \sup\{xy - \varphi_\omega(x) : x \ge 0\}.$$

It is easily seen that φ_ω^* is a convex increasing function. Then $\mathscr{E}_{(\omega)}(\Omega)$ is a nuclear Fréchet space which contains $\mathscr{A}(\Omega)$ continuously.

We denote by $\Lambda_s(\alpha)$ the finite type (for $s < \infty$) and infinite type (for $s = \infty$) power series space generated by the exponent sequence $\alpha = (\alpha_n)$ (cf. [18, §29]). If α is a stable exponent sequence ($\sup_n \alpha_{2n}/\alpha_n < \infty$) then $\Lambda_s(\alpha) \simeq \Lambda_s(\alpha)^2$, in particular, we then have $\Lambda_s(\alpha,\mathbb{Z}) \cong \Lambda_s(\alpha)$, where

$$\Lambda_s(\alpha,\mathbb{Z}) = \Big\{ \xi \in \mathbb{C}^{\mathbb{Z}} \;\Big|\; |x|_t := \sum_{k\in\mathbb{Z}} |\xi_k| e^{t\alpha_{|k|}} < \infty \text{ for all } t < s \Big\}.$$

The following result is Lemma 3 of [9].

Lemma 5.1. *If $\alpha = (\alpha_n)_n$ is a stable exponent sequence such that*

$$\lim_{n\to\infty} n/\alpha_n = 0$$

then there is a quasi-analytic weight function ω and an exact sequence

$$0 \to \Lambda_s(\alpha, \mathbb{Z}) \xrightarrow{J} \mathscr{E}_{(\omega)}(I) \xrightarrow{L} \mathscr{E}_{(\omega)}(I) \to 0,$$

where $s = 1$ for $I = (-1, 1)$, $s = \infty$ for $I = \mathbb{R}$ and $\operatorname{im} J \subset \mathscr{A}(I)$. *The operator J is given by*

$$J(\xi) = \sum_{k\in\mathbb{Z}} \xi_k \, e^{\operatorname{sgn}(k)\alpha_{|k|} z},$$

and L is an infinite order differential operator which maps $\mathscr{A}(I)$ into $\mathscr{A}(I)$.

To extend this lemma to open cubes $Q = I_1 \times \ldots \times I_d \subset \mathbb{R}^d$, where the I_ν are open intervals, we need some preparation. Since the exponentials $e^{\xi z} = \prod_{\nu=1}^d e^{\xi_\nu z_\nu}$ are total in $\mathscr{E}_{(\omega)}(Q)$ it is easily seen that $\mathscr{E}_{(\omega)}(Q) \cong \mathscr{E}_{(\omega)}(I_1) \hat{\otimes} \ldots \hat{\otimes} \mathscr{E}_{(\omega)}(I_d)$. We will need to decompose a power series space into a tensor product of such spaces.

Lemma 5.2. *Let α be a stable exponent sequence with* $\lim_{n\to\infty} n^{1/d}/\alpha_n = 0$ *and $s \in \{0, \infty\}$. Then there exists a stable exponent sequence β with* $\lim_{n\to\infty} n/\beta_n = 0$ *so that $\Lambda_s(\alpha) \cong \Lambda_s(\beta)^{\hat{\otimes} d}$.*

Proof. We set $\beta_n = \alpha_{n^d}$. This sequence satisfies the assertions on β. To establish the isomorphism we fix an enumeration $\mathbb{N}^d \ni j \mapsto n = n(j) \in \mathbb{N}$ of $\mathbb{N}^d$ so that $m = m(j) := \max_\nu j_\nu$ is increasing and set $\gamma_n = \beta_{j_1} + \ldots + \beta_{j_d}$ for $n = n(j)$. Then clearly $(m-1)^d < n \le m^d$ and $\beta_m \le \gamma_n \le d\beta_m$. So using the stability of β we obtain, with suitable D,

$$\frac{1}{D}\,\alpha_n \le \beta_{[n^{1/d}]} \le \beta_m \le \gamma_n \le d\beta_m \le d\beta_{[n^{1/d}]+1} \le dD\alpha_n.$$

Since

$$\Lambda_s(\beta)^{\hat{\otimes} d} \cong \Big\{\xi = (\xi_j)_{j\in\mathbb{N}^d} \,\Big|\, |x|_t = \sum_j |\xi_j| e^{t(\beta_{j_1} + \ldots + \beta_{j_d})} < \infty \text{ for all } t < s\Big\},$$

the map $\xi = (\xi_j)_{j\in\mathbb{N}^d} \mapsto x = (\xi_{j(n)})_{n\in\mathbb{N}}$ establishes an isomorphism onto $\Lambda_s(\alpha)$. ■

We will use the following lemma (we omit the proof).

Lemma 5.3. *Let*

$$0 \to X_k \xrightarrow{j_k} Y_k \xrightarrow{l_k} Z_k \to 0, \qquad k = 1, 2,$$

be exact sequences of nuclear Fréchet spaces. Then

$$0 \to X_1 \hat{\otimes} X_2 \xrightarrow{J} Y_1 \hat{\otimes} Y_2 \xrightarrow{L} (Z_1 \hat{\otimes} Y_2) \oplus (Y_1 \hat{\otimes} Z_2) \xrightarrow{K} Z_1 \hat{\otimes} Z_2 \to 0$$

is an exact sequence, where $J = j_1 \otimes j_2$, $L = (l_1 \otimes \mathrm{id}) \oplus (\mathrm{id} \otimes l_2)$ *and* $K(u_1 \oplus u_2) = (\mathrm{id} \otimes l_2)u_1 - (l_1 \otimes \mathrm{id})u_2$.

Now we are in a position to give a d-dimensional analogue to Lemma 5.1.

LEMMA 5.4. *If* α *is a stable exponent sequence with* $\lim_{n\to\infty} n/\alpha_n = 0$, *then there is a quasi-analytic weight function* ω *and an exact sequence*

$$0 \to \Lambda_s(\alpha, \mathbb{Z})^{\hat{\otimes} d} \xrightarrow{J} \mathscr{E}_{(\omega)}(Q) \xrightarrow{L} X \to 0,$$

where $s = 1$ *for* $Q = (-1,1)^d$, $s = \infty$ *for* $Q = \mathbb{R}^d$, $\mathrm{im}\, J \subset \mathscr{A}(Q)$ *and* X *is a closed topological subspace of* $\mathscr{E}_{(\omega)}(Q)^d$. *The operator* J *is given by*

$$J(\xi) = \sum_{j\in\mathbb{Z}^d} \xi_j \exp\Big(\sum_{\nu=1}^{d} \mathrm{sgn}(j_\nu)\alpha_{|j_\nu|} z_\nu\Big)$$

and $Lf = (L_1 f, \dots, L_d f)$, L_ν *being the operator from Lemma* 5.1 *acting on the* ν*th variable. Moreover*

$$L(\mathscr{E}_{(\omega)}(Q)) = X := \{(f_1, \dots, f_d) \in \mathscr{E}_{(\omega)}(Q)^d \mid L_\nu f_\mu = L_\mu f_\nu \text{ for all } \nu, \mu\},$$

and

$$L(\mathscr{A}(Q)) \subseteq Y := \{(f_1, \dots, f_d) \in \mathscr{A}(Q)^d \mid L_\nu f_\mu = L_\mu f_\nu \text{ for all } \nu, \mu\},$$

where Y *is a closed topological subspace of* $\mathscr{A}(Q)^d$.

Proof. This follows by induction on dimension by applying Lemma 5.3 to the $(d-1)$-dimensional exact sequence

$$0 \to \Lambda_s(\alpha, \mathbb{Z})^{\hat{\otimes}(d-1)} \xrightarrow{J} \mathscr{E}_{(\omega)}(Q) \xrightarrow{L} \mathrm{im}\, L \to 0,$$

and the 1-dimensional exact sequence of Lemma 5.1. ∎

We are now ready to prove Theorem 2.1 for cubes, hence for all Cartesian product sets in $\mathbb{R}^d$. To do it for all open sets we need some geometrical preparation.

LEMMA 5.5. *Every open set* $\Omega \subset \mathbb{R}^d$ *is real-analytically diffeomorphic to an open set* $\Omega' \subset \mathbb{R}^d$ *so that* $(-1,1)^d \subset \Omega' \subset (-\infty, 1)^d$.

Proof. For $\Omega = \mathbb{R}^d$ this is clear, so assume $\Omega \neq \mathbb{R}^d$. We choose $y \in \Omega$ and then a point $w \in \partial B \cap \partial\Omega$ where B is the largest open ball with center y which is contained in Ω. By an affine transformation we may assume that $w = 0$ and $\{x \mid x_\nu > 0 \text{ for all } \nu,\ |x| < \varepsilon\} \subset \Omega$ for some $\varepsilon > 0$.

The reflection $x \mapsto |x|^{-2}x$ maps Ω onto Ω_1 with $\{x \mid x_\nu > 0 \text{ for all } \nu,\ |x| > r\} \subset \Omega_1$ for some $r > 0$.

Finally $x \mapsto \big(\frac{2s}{\pi}\arctan x_1 + 1 - s, \dots, \frac{2s}{\pi}\arctan x_d + 1 - s\big)$ for $s > 0$ large enough maps Ω_1 onto a set as claimed. ∎

The proof of 2.1 is now completed by:

Proposition 5.6. *If E is an $n^{1/d}$-nuclear Fréchet space with property $(\overline{\overline{\Omega}})$ then E is isomorphic to a quotient space of $\mathscr{A}(\Omega)$.*

Proof. We may assume that Ω is of the form described in Lemma 5.5. We set $Q = (-1,1)^d$. According to Lemmas 3.3 and 3.5 there is a stable exponent sequence α with $\lim_n n^{1/d}/\alpha_n = 0$ so that E is α-nuclear. By use of Lemma 5.2 we find a stable exponent sequence β with $\lim_n n/\beta_n = 0$ so that $\Lambda_0(\alpha) \cong \Lambda_0(\beta)^{\hat{\otimes} d}$. Since E has property $(\overline{\overline{\Omega}})$, hence $(\overline{\Omega})$ (see [18]), it is isomorphic to a quotient space of $\Lambda_0(\alpha) \cong \Lambda_0(\beta)^{\hat{\otimes} d} \cong \Lambda_1(\beta)^{\hat{\otimes} d}$ by [21]. Let $q : \Lambda_1(\beta)^{\hat{\otimes} d} \to E$ be a quotient map.

For β we find ω according to Lemma 5.4 and obtain from that lemma the middle row of the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & Z_1 & \xrightarrow{Q} & X & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle S} & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \Lambda_1(\beta,\mathbb{Z})^{\hat{\otimes} d} & \xrightarrow{J} & \mathscr{E}_{(\omega)}(Q) & \xrightarrow{L} & X & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \varrho_1} & & \uparrow{\scriptstyle \varrho_2} & & \\
0 & \longrightarrow & \Lambda_1(\beta)^{\hat{\otimes} d} & \xrightarrow{J} & \mathscr{A}(\Omega) & \xrightarrow{L} & Y & &
\end{array}$$

Here ι is the natural imbedding, i.e. for $\xi = (\xi_j)_{j\in\mathbb{N}^d}$ we set $(\iota\xi)_j = \xi_j$ for $j \in \mathbb{N}^d$ and $(\iota\xi)_j = 0$ otherwise. This is an imbedding onto a complemented subspace so the quotient map q yields the surjective map S. Moreover, X (resp. Y) is the set of elements in $\mathscr{E}_{(\omega)}(Q)^d$ (resp. $\mathscr{A}(\Omega)^d$) satisfying the compatibility conditions. We denote by ϱ_1 the restriction map $\mathscr{A}(\Omega) \to \mathscr{A}(Q) \hookrightarrow \mathscr{E}_{(\omega)}(Q)$ and analogously ϱ_2 in the last column. The upper row is obtained via the standard procedure as in [6, Prop. 1.7(a)]. The diagram is commutative, the upper and middle rows are topologically exact.

We have to show that the map J in the lower row, which is the restriction to $\Lambda_0(\beta)^{\hat{\otimes} d}$ of the map J described in Lemma 5.4, has values in $\mathscr{A}(\Omega)$. The reason is that for $\xi \in \Lambda_1(\beta)^{\hat{\otimes} d}$, we have

$$J(\xi) = \sum_{j\in\mathbb{N}^d} \xi_j \exp\Big(\sum_{\nu=1}^{d} j_\nu \beta_{j_\nu} z_\nu\Big).$$

This series converges uniformly on compact subset of $\{z \in \mathbb{C}^d \mid \operatorname{Re} z_\nu < 1$ for all $\nu\}$, hence defines a holomorphic function on this set. Therefore it defines a real-analytic function on the set $\{x \in \mathbb{R}^d \mid x_\nu < 1$ for all $\nu\}$ which contains Ω.

By Proposition 4.8 we have $\mathrm{Ext}^1(Y,E) = 0$ and therefore ϱ_2 lifts to Z_1 with respect to Q. A standard proof (cf. [6, Prop. 1.7(c)]) shows that this implies existence of a map $B : \mathscr{A}(\Omega) \to E$ such that $B \circ J = S \circ \iota$. Since $B \circ J = S \circ \iota = q$ is surjective and open, so is B. ∎

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