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The existence of solutions for elliptic systems with nonuniform growth

by

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Abstract. We study the Dirichlet problems for elliptic partial differential systems with nonuniform growth. By means of the Musielak–Orlicz space theory, we obtain the existence of weak solutions, which generalizes the result of Acerbi and Fusco [1].

1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. It is our purpose to study the following systems:

$$\frac{\partial A_\alpha^i}{\partial x^\alpha}(x, u(x), Du(x)) = B^i(x, u(x), Du(x)), \quad x \in \Omega,\ i = 1, \ldots, N, \tag{1.1}$$

$$u^i(x) = 0, \quad x \in \partial\Omega,\ i = 1, \ldots, N, \tag{1.2}$$

where $u : \Omega \to \mathbb{R}^N$ is a vector-valued function. We use the summation convention throughout with i, j running from 1 to N and α, β running from 1 to n.

Because problems with nonuniform growth have important applications in mechanics, in recent years numerous papers have been devoted to the study of elliptic equations with nonuniform growth (see [2], [3], [7]–[10], [13], [14], [16] and the references therein). The results of these papers show that problems with nonuniform growth conditions are much more complicated than those with standard growth conditions. These works motivate our study of the Dirichlet problem (1.1)–(1.2) in the setting of Musielak–Orlicz spaces.

In this paper, we suppose that the coefficients of (1.1) satisfy:

(H1) $A_\alpha^i : \Omega \times \mathbb{R}^N \times M^{N\times n} \to \mathbb{R}$, $B^i : \Omega \times \mathbb{R}^N \times M^{N\times n} \to \mathbb{R}$ are Carathéodory functions, $i = 1, \ldots, N$, $\alpha = 1, \ldots, n$.

(H2) $|A(x, s, \xi)| \le C_1|\xi|^{p(x)-1} + C_2|s|^{p(x)-1} + G(x)$, where $G \in L^{p'(\cdot)}(\Omega)$, $C_1, C_2 \ge 0$ and C_2 small.

(H3) $|B(x, s, \xi)| \le C_1'|\xi|^{p(x)-1} + C_2'|s|^{p(x)-1} + \overline{G}(x)$, where $\overline{G} \in L^{p'(\cdot)}(\Omega)$, $C_1', C_2' \ge 0$ and small.

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(H4) $A^i_\alpha(x,s,\xi)\xi^i_\alpha \ge \lambda_0|\xi|^{p(x)} - C|s|^{p(x)} + h(x)$, where $\lambda_0 > 0$, $C \ge 0$ small and $h \in L^1(\Omega)$.

(H5) For almost every $x_0 \in \Omega$, $s_0 \in \mathbb{R}^N$, the mapping $\xi \mapsto A(x_0, s_0, \xi)$ satisfies

$$\int_G A^i_\alpha(x_0, s_0, \xi_0 + Dz(x)) z^i_{,\alpha}(x)\, dx \ge \nu \int_G |Dz(x)|^{p(x)}\, dx$$

for each $\xi_0 \in M^{N\times n}$, $G \subset \mathbb{R}^n$, $z \in C^1_0(G, \mathbb{R}^N)$ where $\nu > 0$ and $(Du(x))^i_\alpha = \partial u^i(x)/\partial x^\alpha = u^i_{,\alpha}(x)$.

Here $p : \Omega \to [1,\infty]$ is a measurable function and p' is its conjugate function (see Section 2).

For a simple case of (1.1), the Euler–Lagrange systems:

$$\sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha} F^i_{u^i_{,\alpha}}(x, u(x), Du(x)) - F^i_u(x, u(x), Du(x)) = 0, \quad x \in \Omega,\ i = 1, \ldots, N,$$

which can be reduced to finding the stationary points of the functional

$$\int_\Omega F^i(x, u(x), Du(x))\, dx, \qquad i = 1, \ldots, N,$$

it is immediate to obtain the existence of weak solutions in Sobolev spaces by applying Acerbi and Fusco [1]. From this point of view, the existence of weak solutions for (1.1) in a Musielak–Orlicz space (Theorem 3.1) is a generalization of their result.

2. Preliminaries

Definition 2.1. Let $M^{N\times n}$ be the set of real $N\times n$ matrices. A function $f : \mathbb{R}^n \times \mathbb{R}^N \times M^{N\times n} \to \mathbb{R}$ is called a *Carathéodory function* if it satisfies: for all $(s,\xi) \in \mathbb{R}^N \times M^{N\times n}$, $x \mapsto f(x,s,\xi)$ is measurable; for almost every $x \in \mathbb{R}^n$, $(s,\xi) \mapsto f(x,s,\xi)$ is continuous.

Lemma 2.1 (see [6]). *$f : \mathbb{R}^n \times \mathbb{R}^N \times M^{N\times n} \to \mathbb{R}$ is a Carathéodory function if and only if for each compact set $K \subset \mathbb{R}^n$ and every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset K$ satisfying $\operatorname{meas}(K \setminus K_\varepsilon) < \varepsilon$ such that f is continuous on $K_\varepsilon \times \mathbb{R}^N \times M^{N\times n}$.*

Lemma 2.2 (see [5]). *Let $G \subset \mathbb{R}^n$ be measurable and $\operatorname{meas}(G) < \infty$. Suppose that $\{M_k\}$ is a sequence of subsets of G such that for some $\varepsilon > 0$,*

$$\operatorname{meas}(M_k) \ge \varepsilon \quad \text{for each } k \in \mathbb{N}.$$

Then there exists a subsequence $\{M_{k_h}\}$ such that $\bigcap_{h\in\mathbb{N}} M_{k_h} \ne \emptyset$.

Lemma 2.3 (see [1]). *Let $\{f_k\}$ be a sequence of bounded functions in $L^1(\mathbb{R}^n)$. For each $\varepsilon > 0$ there exists $(A_\varepsilon, \delta, S)$ (where A_ε is measurable and* meas$(A_\varepsilon) < \varepsilon$, $\delta > 0$, *S is an infinite subset of* $\mathbb{N}$) *such that for each* $k \in S$,

$$\int_B |f_k(x)|\,dx < \varepsilon$$

where B and A_ε are disjoint and meas$(B) < \delta$.

DEFINITION 2.2. For $u \in C_0^1(\mathbb{R}^n)$, define

$$(M^*u)(x) = (Mu)(x) + \sum_{\alpha=1}^{n} (MD_\alpha u)(x)$$

where

$$(Mu)(x) = \sup_{r>0} \frac{1}{\operatorname{meas}(B_r(x))} \int_{B_r(x)} f(x)\,dx,$$

$B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ and $D_\alpha u = \partial u/\partial x^\alpha$.

LEMMA 2.4 (see [12]). *If $u \in C_0^\infty(\mathbb{R}^n)$, then $M^*u \in C^0(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$,*

$$|u(x)| + \sum_{\alpha=1}^{n} |D_\alpha u(x)| \le (M^*u)(x).$$

Furthermore, if $p > 1$, then

$$\|M^*u\|_{L^p(\mathbb{R}^n)} \le C(n,p)\|u\|_{W_0^{1,p}(\mathbb{R}^n)}$$

and if $p = 1$, then

$$\operatorname{meas}(\{x \in \mathbb{R}^n : (M^*u)(x) \le \lambda\}) \le \frac{C(n)}{\lambda}\,\|u\|_{W^{1,1}(\mathbb{R}^n)}$$

for all $\lambda > 0$.

LEMMA 2.5 (see [12]). *Let $u \in C_0^\infty(\mathbb{R}^n)$. Define*

$$U(x,y) = \frac{|u(y) - u(x) - \sum_{\alpha=1}^{n} D_\alpha u(x)(y^\alpha - x^\alpha)|}{|y-x|}.$$

For all $x \in \mathbb{R}^n$, $r > 0$, we have

$$\int_{B_r(x)} U(x,y)\,dy \le 2\operatorname{meas}(B_r(x))(M^*u)(x).$$

LEMMA 2.6 (see [1]). *Let $u \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$. Set*

$$H^\lambda = \{x \in \mathbb{R}^n : (M^*u)(x) < \lambda\}.$$

Then for all $x, y \in H^\lambda$, we have

$$|u(y) - u(x)| \le C(n)\lambda|y-x|.$$

LEMMA 2.7 (see [15]). *Let X be a metric space, E a subspace of X, and k a positive number. Then any k-Lipschitz mapping from E into $\mathbb{R}$ can be extended to a k-Lipschitz mapping from X into $\mathbb{R}$.*

Let $\mathcal{P}(\Omega)$ be the family of all Lebesgue measurable functions $p(\cdot) : \Omega \to [1,\infty]$. For $p(\cdot) \in \mathcal{P}(\Omega)$, we put $\Omega_1^{p(\cdot)} = \{x \in \Omega : p(x) = 1\}$, $\Omega_\infty^{p(\cdot)} = \{x \in \Omega : p(x) = \infty\}$, $\Omega_0^{p(\cdot)} = \Omega \setminus (\Omega_1^{p(\cdot)} \cup \Omega_\infty^{p(\cdot)})$, $p_* = \operatorname{ess\,inf}_{x\in\Omega_0^{p(\cdot)}} p(x)$ and $p^* = \operatorname{ess\,sup}_{\Omega_0^{p(\cdot)}} p(x)$ if $\operatorname{meas}(\Omega_0^{p(\cdot)}) > 0$, $p_* = p^* = 1$ if $\operatorname{meas}(\Omega_0^{p(\cdot)}) = 0$. We use the convention $1/\infty = 0$.

Let $p(\cdot) \in \mathcal{P}(\Omega)$. On the set of all functions on Ω, we define $\varrho_{p(\cdot)}$ and $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ by

$$\varrho_{p(\cdot)}(f) = \int_{\Omega\setminus\Omega_\infty^{p(\cdot)}} |f(x)|^{p(x)}\,dx + \operatorname*{ess\,sup}_{x\in\Omega_\infty^{p(\cdot)}} |f(x)|,$$

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \le 1\}.$$

The space $L^{p(\cdot)}(\Omega)$ is the class of all functions f such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$. Thus $L^{p(\cdot)}(\Omega)$ is a Musielak–Orlicz space.

Given $p(\cdot) \in \mathcal{P}(\Omega)$, we define the conjugate function $p'(\cdot) \in \mathcal{P}(\Omega)$ by

$$p'(x) = \begin{cases} \infty & \text{if } x \in \Omega_1^{p(\cdot)}, \\ 1 & \text{if } x \in \Omega_\infty^{p(\cdot)}, \\ \dfrac{p(x)}{p(x)-1} & \text{if } x \in \Omega_0^{p(\cdot)}. \end{cases}$$

LEMMA 2.8. *Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then*

$$\int_\Omega |f(x)g(x)|\,dx \le C(p(\cdot))\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)}$$

for every $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$.

We shall say that $\{f_n\} \subseteq L^{p(\cdot)}(\Omega)$ *converges modularly* to a function $f \in L^{p(\cdot)}(\Omega)$ if $\lim_{n\to\infty} \varrho_{p(\cdot)}(f - f_n) = 0$.

LEMMA 2.9. (1) *The topology of $L^{p(\cdot)}(\Omega)$ given by the norm coincides with the topology of modular convergence if and only if $p^* < \infty$.*

(2) *$L^{p(\cdot)}(\Omega)$ is complete.*

(3) *The dual space to $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ if and only if $p(\cdot) \in L^\infty(\Omega)$.*

(4) *The space $L^{p(\cdot)}(\Omega)$ is reflexive if and only if $1 < p_* \le p^* < \infty$.*

LEMMA 2.10. *Let $p(\cdot) \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$.*

(1) *$C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$,*

(2) *$L^{p(\cdot)}(\Omega)$ is separable.*

Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, we set $|\alpha| = \alpha_1 + \dots + \alpha_N$ and $D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N}$, where $D_i = \partial/\partial x_i$ is the generalized derivative operator.

The space $W^{k,p(\cdot)}(\Omega)$ is the class of all functions f on Ω such that $D^\alpha f \in L^{p(\cdot)}(\Omega)$ for every multiindex α with $|\alpha| \le k$, endowed with the norm

$$\|f\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha f\|_{W^{k,p(\cdot)}(\Omega)}.$$

We denote by $W_0^{k,p(\cdot)}(\Omega)$ the subspace of $W^{k,p(\cdot)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm of $W^{k,p(\cdot)}(\Omega)$.

LEMMA 2.11. *$W^{k,p(\cdot)}(\Omega)$ and $W_0^{k,p(\cdot)}(\Omega)$ are Banach spaces, which are separable if $p(\cdot) \in L^\infty(\Omega)$ and reflexive if $p(\cdot)$ satisfies*

$$1 < p_* \le p^* < \infty.$$

We shall say that a function $p(\cdot) \in \mathcal{P}(\Omega)$ is $*$-*continuous* on Ω if

$$\lim_{y\to x,\, y\in\Omega} p(y) = p(x) \qquad \text{for every } x \in \Omega$$

(i.e. even if $p(x) = \infty$).

Throughout this paper, we suppose that $p(\cdot)$ is $*$-continuous on $\overline{\Omega}$ and $p(\cdot) \in L^\infty(\Omega)$.

LEMMA 2.12. (1) *Let $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$. If $q(x) \le p(x)$ for a.e. $x \in \Omega$, then the embedding $W^{k,p(\cdot)}(\Omega) \subseteq W^{k,q(\cdot)}(\Omega)$ is continuous.*

(2) *Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $p(\cdot)$ is $*$-continuous on $\overline{\Omega}$, then the embedding $W_0^{k,p(\cdot)}(\Omega) \subseteq L^{p(\cdot)}(\Omega)$ is compact.*

LEMMA 2.13. *Let $p(\cdot) \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$. Then for every G in the dual space $(W_0^{k,p(\cdot)}(\Omega))^*$, there exists a unique system $\{g_\alpha \in L^{p'(\cdot)}(\Omega) : |\alpha| \le k\}$ of functions such that*

$$\langle G, f\rangle = \sum_{|\alpha|\le k} \int_\Omega D^\alpha f(x) g_\alpha(x)\, dx, \qquad f \in W_0^{k,p(\cdot)}(\Omega).$$

In view of Lemma 2.13, we denote $(W_0^{k,p(\cdot)}(\Omega))^*$ by $W^{-k,p'(\cdot)}(\Omega)$ and endow it with the norm

$$\|v\|_{W^{-k,p'(\cdot)}(\Omega)} = \sup_{u\in W_0^{k,p(\cdot)}(\Omega)} |\langle u, v\rangle|.$$

We refer to O. Kováčik and J. Rákosnik [11] for the notions and lemmas mentioned above.

Lemma 2.14. *If* $p(\cdot) \in L^\infty(\Omega)$ *and* $u \in W_0^{1,p(\cdot)}(\Omega)$, *then*

$$\int_\Omega |u|^{p(x)}\,dx \le C \int_\Omega |Du|^{p(x)}\,dx$$

where C *is a constant depending on* Ω.

Proof. Set $R = \operatorname{diam}\Omega$. By translation, we may assume that $0 < x^n < R$ in Ω. Then we can extend u to be zero outside Ω, so

$$u(x) = \int_0^{x^n} D_n u(x', t)\,dt \qquad \text{a.e. } x = (x', x^n) \in \Omega.$$

Integrating with respect to x^n, we have

$$\begin{aligned}
&\int_0^R \left(\frac{|u(x', x^n)|}{R}\right)^{p(x)} dx^n \\
&\quad \le \int_0^R \left(\int_0^{x^n} \frac{|D_n u(x', t)|}{R} dt\right)^{p(x)} dx^n \\
&\quad \le \int_0^R \int_0^{x^n} \left(\frac{x^n |D_n u(x', t)|}{R}\right)^{p(x)} dt\,|x^n|^{p(x)-1}\,dx^n \\
&\quad \le C \int_0^R \int_0^{x^n} |D_n u(x', t)|^{p(x)}\,dt\,dx^n = C \int_0^R \int_t^R |D_n u(x', t)|^{p(x)}\,dx^n\,dt \\
&\quad = C \int_0^R (R - t)|D_n u(x', t)|^{p(x)}\,dt \le C \int_0^R |D_n u(x', t)|^{p(x)}\,dt \\
&\quad = C \int_0^R |D_n u(x)|^{p(x)}\,dx^n.
\end{aligned}$$

Finally we integrate with respect to x' over $\mathbb{R}^{n-1}$ and the conclusion follows. ■

Lemma 2.15. *If* $p(\cdot) \in L^\infty(\Omega)$, *then*

$$\lim_{\operatorname{meas}(E)\to 0} \|u\chi_E\|_{L^{p(\cdot)}(\Omega)} = 0$$

for all $u \in L^{p(\cdot)}(\Omega)$.

Proof. By Lemma 2.10, for each $\varepsilon > 0$ and each $u \in L^{p(\cdot)}(\Omega)$ there exists $w \in C_0^\infty(\Omega)$ such that $\|u - w\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$. Suppose that $|w(x)| \le C$ for all $x \in \Omega$. Let $\operatorname{meas}(E) < 1$. Then $\|\chi_E\|_{L^{p(\cdot)}(\Omega)} \le (\operatorname{meas}(E))^{1/p^*} \to 0$ as $\operatorname{meas}(E) \to 0$. So there exists $\delta > 0$ such that if $\operatorname{meas}(E) < \delta$, then

$\|\chi_E\|_{L^{p(\cdot)}(\Omega)} < \varepsilon/(2C)$. Now we get

$$\begin{aligned}\|u\chi_E\|_{L^{p(\cdot)}(\Omega)} &\le \|(u-w)\chi_E\|_{L^{p(\cdot)}(\Omega)} + \|w\chi_E\|_{L^{p(\cdot)}(\Omega)}\\ &\le \|u-w\|_{L^{p(\cdot)}(\Omega)} + C\|\chi_E\|_{L^{p(\cdot)}(\Omega)} < \varepsilon,\end{aligned}$$

that is to say, $\lim_{\text{meas}(E)\to 0}\|u\chi_E\|_{L^{p(\cdot)}(\Omega)} = 0$. ∎

LEMMA 2.16. *Suppose that* $p(\cdot)\in L^\infty(\Omega)$. *Let* $\{u_k\}_{k=1}^\infty$ *be bounded in* $L^{p(\cdot)}(\Omega)$. *If* $u_k\to u$ *a.e. on* Ω, *then* $u_k \rightharpoonup u$ *weakly in* $L^{p(\cdot)}(\Omega)$.

Proof. Suppose that $\|u_k\|_{L^{p(\cdot)}(\Omega)} \le C$ for each integer k. By Fatou's Lemma,

$$\int_\Omega \left(\frac{u}{C}\right)^{p(x)} dx = \int_\Omega \lim_{k\to\infty}\left(\frac{u_k}{C}\right)^{p(x)} dx \le \liminf_{k\to\infty}\int_\Omega \left(\frac{u_k}{C}\right)^{p(x)} dx \le 1,$$

hence $\|u\|_{L^{p(\cdot)}(\Omega)} \le C$. Let $\varepsilon>0$ and $g\in L^{p'(\cdot)}(\Omega)$. By Lemma 2.15, $\lim_{\text{meas}(E)\to 0}\|g\chi_E\|_{L^{p'(\cdot)}(\Omega)} = 0$ and so there exists $\delta>0$ such that for all E satisfying $\text{meas}(E)<\delta$, we have

$$\|g\chi_E\|_{L^{p'(\cdot)}(\Omega)} < \frac{\varepsilon}{4C}.$$

By Egorov's Theorem, there exists a set B such that $u_k\to u$ uniformly on B and $\text{meas}(\Omega\setminus B)<\delta$. Finally choose K such that $k>K$ implies

$$\max_{x\in B}|u-u_k|\cdot\|g\|_{L^{p'(\cdot)}(\Omega)}\|\chi_\Omega\|_{L^{p(\cdot)}(\Omega)} < \frac{\varepsilon}{2}$$

for all $x\in B$. Thus taking $E=\Omega\setminus B$, we have

$$\Big|\int_\Omega ug\,dx - \int_\Omega u_k g\,dx\Big| \le \int_B |u_k-u|\cdot|g|\,dx + \int_{\Omega\setminus B}|u_k-u|\cdot|g|dx$$
$$\le \|g\|_{L^{p'(\cdot)}(\Omega)}\|\chi_\Omega\|_{L^{p(\cdot)}(\Omega)}\max_{x\in B}|u_k-u| + \|u_k-u\|_{L^{p(\cdot)}(\Omega)}\|g\chi_{\Omega\setminus B}\|_{L^{p'(\cdot)}(\Omega)} < \varepsilon$$

for all $k>K$, that is to say, $u_k\rightharpoonup u$ weakly in $L^{p(\cdot)}(\Omega)$. ∎

3. Main theorem

THEOREM 3.1. *Under the conditions* (H1)–(H5), *the Dirichlet problem* (1.1)–(1.2) *has at least one weak solution in* $W_0^{1,p(\cdot)}(\Omega,\mathbb{R}^N)$, *that is to say, there exists at least one* $u\in W_0^{1,p(\cdot)}(\Omega,\mathbb{R}^N)$ *satisfying*

$$\int_\Omega [A^i_\alpha(x,u,Du)z^i_{,\alpha}(x) + B^i(x,u,Du)z^i(x)]\,dx = 0 \tag{3.1}$$

for all $z\in W_0^{1,p(\cdot)}(\Omega,\mathbb{R}^N)$.

Proof. Set $V = W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$. For $u \in V$, define $T : V \to V^*$ in the following way: for each $w \in V$,

$$(Tu, w) = \int_\Omega [A_\alpha^i(x, u, Du) w_{,\alpha}^i(x) + B^i(x, u, Du) w^i(x)]\, dx = 0. \tag{3.2}$$

Now we only need to show that there exists $u \in V$ such that $(Tu, w) = 0$ for all $w \in V$. We will prove this in several steps.

1) *T is strong-weakly continuous.* Suppose that $u_k \to u$ strongly in $W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$. Then $\|u\|_V \le C$ for some constant C independent of k. By (H2)–(H3), $A_\alpha^i(x, u_k, Du_k)$ and $B^i(x, u_k, Du_k)$ are bounded in $L^{p'(\cdot)}(\Omega)$. Then by (H1) and Lemma 2.16, we know

$$\lim_{k\to\infty} (Tu_k, w) = (T(\lim_{k\to\infty} u_k), w) = (Tu, w). \tag{3.3}$$

That is to say, T is strong-weakly continuous.

2) *T is coercive, i.e.*

$$\lim_{\|u\|_V\to\infty} \frac{(Tu, u)}{\|u\|_V} = +\infty. \tag{3.4}$$

By (H1)–(H2) and Lemma 2.14,

$$\begin{aligned}
(Tu, u) \ge & \int_\Omega [\lambda_0 |Du|^{p(x)} - C|u|^{p(x)} + h(x) - C_1'|Du|^{p(x)-1}|u| \\
& - C_2'|u|^{p(x)} - \overline{G}(x)|u|]\, dx \\
\ge & \int_\Omega [\lambda_0 |Du|^{p(x)} - C|u|^{p(x)} + h(x) - C_1'|Du|^{p(x)} - C_1'|u|^{p(x)} \\
& - C_2'|u|^{p(x)} - \mu|u|^{p(x)} - C(\mu)(\overline{G}(x))^{p'(x)}]\, dx \\
\ge & \int_\Omega [(\lambda_0 - C_1' - C^*(C + C_1' + C_2' + \mu))|Du|^{p(x)} \\
& + h(x) - C(\mu)(\overline{G}(x))^{p'(x)}]\, dx
\end{aligned}$$

where C^* is the constant in Lemma 2.14.

When C, C_1', C_2', μ are small, we can get

$$\lambda_0 - C_1' - C^*(C + C_1' + C_2' + \mu) > 0.$$

By Lemma 2.12, we have

$$(1 + C_1^*)\| \, |Du| \, \|_{L^{p(\cdot)}(\Omega)} \ge \|u\|_{L^{p(\cdot)}(\Omega, \mathbb{R}^N)} + \| \, |Du| \, \|_{L^{p(\cdot)}(\Omega)} \ge \|u\|_V \tag{3.5}$$

where C_1^* is the imbedding constant. In view of (3.5), it is easy to see that $\| \, |Du| \, \|_{L^{p(\cdot)}(\Omega)} \to \infty$ as $\|u\|_V \to \infty$. Taking ε sufficiently small, for example

$$\varepsilon = \frac{1}{2}(\| \, |Du| \, \|_{L^{p(\cdot)}(\Omega)} - e^{\frac{2}{p_*+1} \ln \| \, |Du| \, \|_{L^{p(\cdot)}(\Omega)}}),$$

we have

$$\begin{aligned}\frac{\int_\Omega |Du|^{p(x)}\,dx}{\|u\|_V} &= \int_\Omega \biggl(\frac{|Du|}{\|\,|Du|\,\|_{L^{p(\cdot)}(\Omega)}-\varepsilon}(\|\,|Du|\,\|_{L^{p(\cdot)}(\Omega)}-\varepsilon)\biggr)^{p(x)}\frac{1}{\|u\|_V}\,dx\\ &\ge \int_\Omega \biggl(\frac{|Du|}{\|\,|Du|\,\|_{L^{p(\cdot)}(\Omega)}-\varepsilon}\biggr)^{p(x)}dx\,\frac{(\|\,|Du|\,\|_{L^{p(\cdot)}(\Omega)}-\varepsilon)^{p_*}}{\|u\|_V}\\ &\ge \frac{(\|\,|Du|\,\|_{L^{p(\cdot)}(\Omega)}-\varepsilon)^{(p_*+1)/2}}{(1+C_1^*)\|\,|Du|\,\|_{L^{p(\cdot)}(\Omega)}}(\|\,|Du|\,\|_{L^{p(\cdot)}(\Omega)}-\varepsilon)^{(p_*-1)/2}\\ &\ge \frac{1}{1+C_1^*}\,(\|\,|Du|\,\|_{L^{p(\cdot)}(\Omega)}-\varepsilon)^{(p_*-1)/2}\to\infty\end{aligned}$$

as $\|u\|_V\to\infty$. As $\int_\Omega[h(x)-C(\mu)(\overline{G}(x))^{p'(x)}]\,dx$ is bounded, we conclude that (3.4) holds.

3) Now we construct an approximating sequence. By Lemma 2.10, we can choose a basis $\{w_k\}$ of V such that the union of subspaces finitely generated from $\{w_k\}$ is dense in V. Let B_s be the subspace of V generated by $w_1,\dots,w_s$. By the coerciveness of T and Morrey [15], there exists $u_s\in B_s$ such that

$$(Tu_s,w)=0$$

for all $w\in B_s$. By the coerciveness of T again, we know that $\|u_s\|_V\le C$ where C is independent of s. As V is reflexive, we can extract a subsequence $\{u_k\}$ such that

$$u_k\rightharpoonup u_0 \text{ weakly in } V,\qquad Tu_k\rightharpoonup\xi \text{ weakly in } V^*,\qquad (\xi,w)=0$$

where w is in a dense subset of V. For fixed ξ, by the continuity of $(\xi,\cdot)$, we get $(\xi,w)=0$ for all $w\in V$. Considering (Tu_k,u_k-u_0), we have

$$(Tu_k,u_k-u_0)=(Tu_k,u_k)-(Tu_k,u_0)=-(Tu_k,u_0)\to 0$$

as $k\to\infty$. Set $z_k=u_k-u_0$. Then

$$z_k\rightharpoonup 0\quad \text{weakly in } V \text{ as } k\to\infty.$$

Consider (Tu_k,u_k-u_0) once more:

$$\begin{aligned}&(Tu_k,u_k-u_0)\\ &=\int_\Omega[A^i_\alpha(x,u_0+z_k,Du_0+Dz_k)z^i_{k,\alpha}+B^i(x,u_0+z_k,Du_0+Dz_k)z^i_k]\,dx\to 0\end{aligned}$$

as $k\to\infty$. By applying Lemma 2.12, we get

$$z_k\to 0\quad\text{strongly in } L^{p(\cdot)}(\Omega,\mathbb{R}^N). \tag{3.6}$$

In view of (H3) and (3.6), it is immediate that

$$\int_\Omega B^i(x,u_0+z_k,Du_0+Dz_k)z^i_k\,dx\to 0$$

as $k \to \infty$, that is to say,

$$\int_{\Omega} A^i_\alpha(x, u_0 + z_k, Du_0 + Dz_k) z^i_{k,\alpha}\, dx \to 0 \tag{3.7}$$

as $k \to \infty$.

Now if we can prove that there exists a subsequence of $\{z_k\}$ which is strongly convergent in V, then from the strong-weak continuity of T, we get $Tu_k \rightharpoonup Tu_0 = \xi$ weakly in V as $k \to \infty$ and u_0 will be a weak solution of (1.1)–(1.2).

4) We will find a subsequence of $\{z_k\}$ which is strongly convergent in V. For each measurable set $S \subset \Omega$, define

$$F(v, S) = \int_S A^i_\alpha(x, u_0 + v, Du_0 + Dv) v^i_{,\alpha}\, dx$$

where $v \in W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$. Similarly to the remark in step 1, we can show $F(v, S)$ is strongly continuous in $W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$. Since $C_0^\infty(\Omega, \mathbb{R}^N)$ is dense in $W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$, there exists $\{f_k\} \subset C_0^\infty(\Omega, \mathbb{R}^N)$ such that

$$\|f_k - z_k\|_V < 1/k, \qquad |F(f_k, \Omega) - F(z_k, \Omega)| < 1/k.$$

So we can suppose $\{z_k\}$ is in $C_0^\infty(\Omega, \mathbb{R}^N)$ and bounded in $W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$.

Next we define

$$z_k(x) = 0 \qquad \text{when } x \in \mathbb{R}^n \setminus \Omega.$$

In this way, we extend the domain of z_k to $\mathbb{R}^n$ and $\{z_k\} \subset W_0^{1,p(x)}(\mathbb{R}^n, \mathbb{R}^N)$ and $\{z_k\}$ is bounded and $\operatorname{supp} z_k \subset \Omega$.

Let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function satisfying $\eta(0) = 0$ and for each measurable set $B \subset \Omega$,

$$\sup_k \int_B [(g(x))^{p'(x)} + h(x) + 1 + C(|u_0|^{p(x)} + |Du_0|^{p(x)} + |z_k|^{p(x)})]\, dx \\ \le \eta(\operatorname{meas}(B))$$

where $C = C_1 + C_2$ and C_1, C_2 are the two constants in (H2).

Let $\{\varepsilon_j\}$ be a positive decreasing sequence with $\varepsilon_j \to 0$ as $j \to \infty$. For ε_1, applying Lemma 2.3 to each of the N sequences $\{(M^* z^i_k)^{p(x)}\}$, $1 \le i \le N$, we get a subsequence $\{z_{k_1}\}$, a set $A_{\varepsilon_1} \subset \Omega$ satisfying $\operatorname{meas}(A_{\varepsilon_1}) < \varepsilon_1$, and a real number $\delta_1 > 0$ such that

$$\int_B (M^* z^i_{k_1})^{p(x)}\, dx < \varepsilon_1$$

for all k_1, $1 \le i \le N$ and $B \subset \Omega \setminus A_{\varepsilon_1}$ satisfying $\operatorname{meas}(B) < \delta_1$. By Lemma 2.4, we can choose $\lambda > 1$ so large that for all i and k_1,

$$\operatorname{meas}(\{x \in \mathbb{R}^n : (M^* z^i_{k_1})(x) \ge \lambda\}) \le \min\{\varepsilon_1, \delta_1\}.$$

For all i and k_1, define

$$H^\lambda_{i,k_1} = \{x \in \mathbb{R}^n : (M^* z^i_{k_1})(x) < \lambda\}, \qquad H^\lambda_{k_1} = \bigcap_{i=1}^{N} H^\lambda_{i,k_1}.$$

By Lemma 2.6, we have

$$\frac{|z^i_{k_1}(y) - z^i_{k_1}(x)|}{|y-x|} \le C(n)\lambda$$

for all $x, y \in H^\lambda_{k_1}$ and $1 \le i \le N$. From Lemma 2.7, there exists a Lipschitz function $g^i_{k_1}$ which extends $z^i_{k_1}$ outside $H^\lambda_{k_1}$ and the Lipschitz constant of $g^i_{k_1}$ is no more than $C(n)\lambda$. As $H^\lambda_{k_1}$ is an open set, we have $g^i_{k_1}(x) = z^i_{k_1}(x)$ and $Dg^i_{k_1}(x) = Dz^i_{k_1}(x)$ for all $x \in H^\lambda_{k_1}$, and

$$\| \, |Dg^i_{k_1}| \, \|_{L^\infty(\mathbb{R}^n)} \le C(n)\lambda.$$

In view of Lemma 2.4, we can further suppose that

$$\|g^i_{k_1}\|_{L^\infty(\mathbb{R}^n)} \le \|z^i_{k_1}\|_{L^\infty(H^\lambda_{k_1})} \le \lambda, \qquad \|g_{k_1}\|_{W^{1,p(\cdot)}(\Omega,\mathbb{R}^N)} \le C.$$

By the boundedness of $\|g_{k_1}\|_{W^{1,\infty}(\Omega,\mathbb{R}^N)}$, there exists a subsequence of $\{g^i_{k_1}\}$ (still denoted by $\{g^i_{k_1}\}$) such that

$$g^i_{k_1} \to v^i \qquad *\text{-weakly in } W^{1,\infty}(\Omega) \text{ as } k_1 \to \infty \tag{3.8}$$

for $1 \le i \le N$. Set $(g^1_{k_1}, \ldots, g^N_{k_1}) = g_{k_1}$ and $(v^1, \ldots, v^N) = v$. We have

$$\begin{aligned} F(z_{k_1}, \Omega) &= F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap H^\lambda_{k_1}) + F(z_{k_1}, A_{\varepsilon_1} \cup (\Omega \setminus H^\lambda_{k_1})) \\ &= F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) + F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus H^\lambda_{k_1}) \\ &\quad + F(z_{k_1}, A_{\varepsilon_1} \cup (\Omega \setminus H^\lambda_{k_1})). \end{aligned} \tag{3.9}$$

Since

$$\operatorname{meas}((\Omega \setminus A_{\varepsilon_1}) \cap H^\lambda_{k_1}) \le \sum_{i=1}^{N} \operatorname{meas}((\Omega \setminus A_{\varepsilon_1}) \cap H^\lambda_{i,k_1}) \le N \min(\varepsilon_1, \delta_1)$$

from (H2), (H4) and the choice of A_{ε_1}, we get

$$\begin{aligned} &|F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus H^\lambda_{k_1})| \\ &\quad \le \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H^\lambda_{k_1}} |A^i_\alpha(x, u_0 + g_{k_1}, Du_0 + Dg_{k_1}) g^i_{k_1,\alpha}| \, dx \\ &\quad \le \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H^\lambda_{k_1}} [C_1 |Du_0 + Dg_{k_1}|^{p(x)-1} |Dg_{k_1}| \\ &\qquad + C_2 |u_0 + g_{k_1}|^{p(x)-1} |Dg_{k_1}| + G(x) |Dg_{k_1}|] \, dx \end{aligned} \tag{3.10}$$

$$\leq \int_{(\Omega\setminus A_{\varepsilon_1})\setminus H^\lambda_{k_1}} [C_1|Du_0+Dg_{k_1}|^{p(x)}+C_1|Dg_{k_1}|^{p(x)}+C_2|u_0+g_{k_1}|^{p(x)}$$

$$+C_2|Dg_{k_1}|^{p(x)}+(G(x))^{p'(x)}+|Dg_{k_1}|^{p(x)}]\,dx$$

$$\leq \int_{(\Omega\setminus A_{\varepsilon_1})\setminus H^\lambda_{k_1}} [C_1 2^{p^*-1}|Du_0|^{p(x)}+C_1 2^{p^*-1}|Dg_{k_1}|^{p(x)}$$

$$+C_1|Dg_{k_1}|^{p(x)}+C_2 2^{p^*-1}|u_0|^{p(x)}+C_2 2^{p^*-1}|g_{k_1}|^{p(x)}$$

$$+C_2|Dg_{k_1}|^{p(x)}+(G(x))^{p'(x)}+|Dg_{k_1}|^{p(x)}]\,dx$$

$$\leq 2^{p^*-1}\eta(\mathrm{meas}((\Omega\setminus A_{\varepsilon_1})\setminus H^\lambda_{k_1}))$$

$$+2^{p^*-1}(C_1+C_2+1)\int_{(\Omega\setminus A_{\varepsilon_1})\setminus H^\lambda_{k_1}} (|g_{k_1}|^{p(x)}+|Dg_{k_1}|^{p(x)})\,dx$$

$$\leq 2^{p^*-1}C(n,\Omega,C_1+C_2)\int_{(\Omega\setminus A_{\varepsilon_1})\setminus H^\lambda_{k_1}} \lambda^{p(x)}\,dx+2^{p^*-1}\eta(N\varepsilon_1)$$

$$\leq 2^{p^*-1}C(n,\Omega,C_1+C_2)\sum_{i=1}^{N}\int_{(\Omega\setminus A_{\varepsilon_1})\setminus H^\lambda_{i,k_1}} (M^* z^i_{k_1})^{p(x)}\,dx+2^{p^*-1}\eta(N\varepsilon_1)$$

$$\leq 2^{p^*-1}\eta(N\varepsilon_1)+2^{p^*-1}C(n,\Omega,C_1+C_2)N\varepsilon_1=V_1(\varepsilon_1),$$

while

$$F(z_{k_1},A_{\varepsilon_1}\cup(\Omega\setminus H^\lambda_{k_1})) \tag{3.11}$$

$$=\int_{A_{\varepsilon_1}\cup(\Omega\setminus H^\lambda_{k_1})} A^i_\alpha(x,u_0+z_{k_1},Du_0+Dz_{k_1})z^i_{k_1,\alpha}\,dx$$

$$=\int_{A_{\varepsilon_1}\cup(\Omega\setminus H^\lambda_{k_1})} A^i_\alpha(x,u_0+z_{k_1},Du_0+Dz_{k_1})(u^i_{0,\alpha}+z^i_{k_1,\alpha})\,dx$$

$$-\int_{A_{\varepsilon_1}\cup(\Omega\setminus H^\lambda_{k_1})} A^i_\alpha(x,u_0+z_{k_1},Du_0+Dz_{k_1})u^i_{0,\alpha}\,dx$$

$$\geq \int_{A_{\varepsilon_1}\cup(\Omega\setminus H^\lambda_{k_1})} [\lambda_0|Du_0+Dz_{k_1}|^{p(x)}-C|u_0+z_{k_1}|^{p(x)}+h(x)]\,dx$$

$$-\int_{A_{\varepsilon_1}\cup(\Omega\setminus H^\lambda_{k_1})} [C_1|Du_0+Dz_{k_1}|^{p(x)-1}|Du_0|$$

$$+C_2|u_0+z_{k_1}|^{p(x)-1}|Du_0|+G(x)|Du_0|]\,dx$$

$$\geq \left(\frac{\lambda_0}{2^{p^*-1}} - \mu\right) \int\limits_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda)} |Dz_{k_1}|^{p(x)}\, dx$$

$$- C(\mu, \lambda_0, p(\cdot), C_1, C_2, C)\eta(\mathrm{meas}(A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda)))$$

where $\mu > 0$ is arbitrary. Taking

$$0 < \mu < \frac{\lambda_0}{2^{(p^*-1)/2}},$$

we have

$$F(z_{k_1}, A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda)) \geq \frac{\lambda_0}{2^{(p^*-1)/2}} \int\limits_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda)} |Dz_{k_1}|^{p(x)}\, dx - V_2(\varepsilon_1)$$

where $V_1(\varepsilon), V_2(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Set $A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda) = U_{\varepsilon_1,k_1}^1$, $\alpha_0 = \lambda_0/2^{(p^*-1)/2}$, $V_3(\varepsilon) = V_1(\varepsilon) + V_2(\varepsilon)$. From (3.9)–(3.11), we get

$$F(z_{k_1}, \Omega) \geq F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) + \alpha_0 \int\limits_{U_{\varepsilon_1,k_1}^1} |Dz_{k_1}|^{p(x)}\, dx - V_3(\varepsilon_1). \tag{3.12}$$

Next, set

$$h_{k_1} = g_{k_1} - v$$

where v is defined by (3.8). Then

$$h_{k_1} \rightharpoonup 0 \quad *\text{-weakly in } W^{1,\infty}(\Omega, \mathbb{R}^N) \text{ as } k_1 \to \infty$$

and

$$\|h_{k_1}\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq 2\lambda, \quad \|\,|Dh_{k_1}|\,\|_{L^\infty(\Omega)} \leq 2C(n)\lambda.$$

Set $G = \{x \in \Omega : v(x) \neq 0\}$. According to Acerbi and Fusco [1], we have

$$\mathrm{meas}(G) \leq (N+1)\varepsilon_1$$

and

$$\begin{aligned} F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) &= F(h_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus G) + F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^\lambda \cap G) \\ &\quad + F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap (G \setminus H_{k_1}^\lambda)) \\ &= F(h_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus G) + F(z_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^\lambda \cap G) \\ &\quad + F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap (G \setminus H_{k_1}^\lambda)). \end{aligned} \tag{3.13}$$

Define

$$\begin{aligned} U_{\varepsilon_1}^2 &= (\Omega \setminus A_{\varepsilon_1}) \setminus G, \\ U_{\varepsilon_1,k_1}^3 &= (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^\lambda \cap G, \\ U_{\varepsilon_1,k_1}^4 &= (\Omega \setminus A_{\varepsilon_1}) \cap (G \setminus H_{k_1}^\lambda). \end{aligned}$$

Similarly to the proof of (3.12), we get

$$F(z_{k_1}, U^3_{\varepsilon_1,k_1}) \geq \alpha_0 \int\limits_{U^3_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx - V_4(\varepsilon_1). \tag{3.14}$$

On $U^4_{\varepsilon_1,k_1}$, we have

$$\int\limits_{U^4_{\varepsilon_1,k_1}} (|g_{k_1}|^{p(x)} + |Dg_{k_1}|^{p(x)})\,dx \leq NC(n,\Omega)\varepsilon_1.$$

Then similarly to the proof of (3.10), we have

$$\begin{aligned} |F(g_{k_1}, U^4_{\varepsilon_1,k_1})| &\leq C(C_1, C_2, p(\cdot))NC(n,\Omega)\varepsilon_1 + \eta((N+1)\varepsilon_1) \\ &= V_3(\varepsilon_1). \end{aligned} \tag{3.15}$$

From (3.13)–(3.15), we get

$$F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) \geq F(h_{k_1}, U^2_{\varepsilon_1}) + \alpha_0 \int\limits_{U^3_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx - V_4(\varepsilon_1) - V_5(\varepsilon_1).$$

Define

$$U^5_{\varepsilon_1,k_1} = U^3_{\varepsilon_1,k_1} \cup U^1_{\varepsilon_1,k_1}.$$

From (3.12),

$$F(z_{k_1}, \Omega) \geq F(h_{k_1}, U^2_{\varepsilon_1}) + \alpha_0 \int\limits_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx - V_6(\varepsilon_1) \tag{3.16}$$

where $V_6(\varepsilon) = V_3(\varepsilon) + V_4(\varepsilon) + V_5(\varepsilon)$.

Choose an open set $\Omega' \subset \Omega$ which contains $U^2_{\varepsilon_1}$ such that

$$|F(h_{k_1}, \Omega') - F(h_{k_1}, U^2_{\varepsilon_1})| < \varepsilon_1.$$

In view of (3.16), we get

$$F(z_{k_1}, \Omega) \geq F(h_{k_1}, \Omega') + \alpha_0 \int\limits_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx - V_7(\varepsilon_1)$$

where $V_7(\varepsilon) = V_6(\varepsilon) + \varepsilon$.

Next approximate Ω' by hypercubes with edges parallel to the coordinate axes, i.e. construct

$$\begin{cases} H_j = \bigcup_{s=1}^{I_j} D_{j,s}, \\ \operatorname{meas}(\Omega' \setminus H_j) \to 0 \quad \text{as } j \to \infty, \\ \operatorname{meas}(D_{j,s}) = 1/2^{nj}, \quad 1 \leq s \leq I_j, \\ H_j \subset \Omega'. \end{cases}$$

Let $j > 0$ be so large that for all $k_1 > 0$,

$$(3.17)\qquad |F(h_{k_1}, \Omega') - F(h_{k_1}, H_j)| < \varepsilon_1, \qquad \int\limits_{\Omega' \setminus H_j} |Dh_{k_1}|^{p(x)} dx < \varepsilon_1$$

and

$$\operatorname{meas}(\Omega' \setminus H_j) < \min(\varepsilon_1, \delta_1).$$

Then

$$(3.18)\qquad F(z_{k_1}, \Omega) \geq F(h_{k_1}, H_j) + \alpha_0 \int\limits_{U^5_{\varepsilon_1, k_1}} |Dz_{k_1}|^{p(x)}\, dx - V_8(\varepsilon_1)$$

where $V_8(\varepsilon) = V_7(\varepsilon) + \varepsilon$.

Let

$$M = 2C(n)\lambda \geq \| \, |Dh_{k_1}| \, \|_{L^\infty(\Omega)}$$

and $\alpha > 0$ be so large that for $E = \{x \in \Omega' : a(x) \leq \alpha\}$, we have

$$\operatorname{meas}(\Omega' \setminus E) \leq \varepsilon_1 / M, \qquad \int\limits_{\Omega' \setminus E} a(x)\, dx \leq \varepsilon_1$$

where

$$a(x) = 2^{p^* - 1}[(1 + C_1 + C_2)|Du_0(x)|^{p(x)} + C_2|u_0(x)|^{p(x)} + (G(x))^{p'(x)}].$$

For $x \in \Omega$, $s \in \mathbb{R}^N$, $\xi \in M^{N \times n}$, define

$$f(x, s, \xi) := A^i_\alpha(x, u_0(x) + s, Du_0(x) + \xi)\xi^i_\alpha.$$

By Lemma 2.1, there exists a compact subset $K \subset H_j$ such that $f(x, s, \xi)$ is continuous on $K \times \mathbb{R}^N \times M^{N \times n}$ and

$$\operatorname{meas}(H_j \setminus K) < \frac{\varepsilon_1}{\alpha + M}.$$

Divide each $D_{j,s}$ into 2^{nm} hypercubes $Q^m_{h,s,j}$ with edge length 2^{-jm}, $1 \leq h \leq 2^{nm}$. For all j, s, m, h, take $x^m_{h,s,j} \in Q^m_{h,s,j} \cap K \cap E$ (if this set is empty, take $x^m_{h,s,j} \in Q^m_{h,s,j}$) such that

$$a(x^m_{h,s,j}) \operatorname{meas}(Q^m_{h,s,j}) \leq \int\limits_{Q^m_{h,s,j}} a(x)\, dx.$$

Then

$$(3.19)\qquad F(h_{k_1}, H_j)$$
$$= F(h_{k_1}, H_j \cap K \cap E) + F(h_{k_1}, H_j \setminus E) + F(h_{k_1}, (H_j \cap E) \setminus K)$$

$$\begin{aligned}
&\geq F(h_{k_1}, H_j \cap K \cap E) - \int_{H_j \setminus E} a(x)\,dx - \int_{(H_j\cap E)\setminus K} a(x)\,dx \\
&\quad - 2^{p^*-1}(1+C_1+C_2)\Big(\int_{H_j\setminus E} [|Dh_{k_1}|^{p(x)} + |h_{k_1}|^{p(x)}]\,dx \\
&\quad + \int_{(H_j\cap E)\setminus K} [|Dh_{k_1}|^{p(x)} + |h_{k_1}|^{p(x)}]\,dx \Big) \\
&= F(h_{k_1}, H_j \cap K \cap E) - V_9(\varepsilon_1) \\
&= a^j_{k_1} + b^{m,j}_{k_1} + c^{m,j}_{k_1} + d^{m,j}_{k_1} - V_9(\varepsilon_1)
\end{aligned}$$

where

$$\begin{aligned}
a^j_{k_1} &= \int_{H_j\cap K\cap E} [f(x, h_{k_1}(x), Dh_{k_1}(x)) - f(x, 0, Dh_{k_1}(x))]\,dx \\
b^{m,j}_{k_1} &= \sum_{h,s} \int_{Q^m_{h,s,j}\cap K\cap E} [f(x, 0, Dh_{k_1}(x)) - f(x^m_{h,s,j}, 0, Dh_{k_1}(x))]\,dx \\
c^{m,j}_{k_1} &= \sum_{h,s} \int_{Q^m_{h,s,j}} f(x^m_{h,s,j}, 0, Dh_{k_1}(x))\,dx \\
d^{m,j}_{k_1} &= -\sum_{h,s} \int_{Q^m_{h,s,j}\setminus(K\cap E)} f(x^m_{h,s,j}, 0, Dh_{k_1}(x))\,dx.
\end{aligned}$$

By the uniform continuity of f on bounded sets of $K \times \mathbb{R}^N \times M^{N\times n}$ and (3.7), we have

$$\lim_{k_1\to\infty} a^j_{k_1} = 0, \qquad \lim_{k_1\to\infty} F(z_{k_1}, \Omega) = 0$$

and the pointwise convergence of $u_0(x^m_{h,s,j})$, $Du_0(x^m_{h,s,j})$ implies

$$\lim_{m\to\infty} h^{m,j}_{k_1} = 0$$

uniformly with respect to k_1, for fixed j, and

$$\begin{aligned}
|d^{m,j}_{k_1}| &\leq \sum_{h,s} \int_{Q^m_{h,s,j}\setminus(K\cap E)} [a(x^m_{h,s,j}) + 2^{p^*}(1+C_1+C_2)M]\,dx \\
&\leq C(\alpha + M)\operatorname{meas}((H_j\cap E)\setminus K) + C\int_{H_j\setminus E} [a(x) + M]\,dx \\
&\leq C(C_1, C_2, p(\cdot))\varepsilon_1.
\end{aligned}$$

Now we suppose that m is so large that $|b^{m,j}_{k_1}| < \varepsilon_1$ for each $k_1 > 0$ and there exists $\overline{k}_1 > 0$ such that $F(z_{k_1}, \Omega) < \varepsilon_1$, $|a^j_{k_1}| < \varepsilon_1$ for $k_1 > \overline{k}_1$. Therefore from (3.7), (3.18) and (3.19), we have

$$\begin{aligned}(3.20)\qquad \varepsilon_1 &\ge F(z_{k_1},\Omega)\\ &\ge c_{k_1}^{m,j} + \alpha_0 \int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx - V_8(\varepsilon_1) - V_9(\varepsilon_1)\\ &\quad - 2\varepsilon_1 - C(C_1,C_2,p(\cdot))\varepsilon_1\\ &= c_{k_1}^{m,j} + \alpha_0 \int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx - V_{10}(\varepsilon_1).\end{aligned}$$

As $h_{k_1} \rightharpoonup 0$ weakly in $W^{1,\infty}(\Omega,\mathbb{R}^N)$ as $k_1\to\infty$, we obtain

$$R_{h,s,j}^{k_1,m} = \|\,|h_{k_1}|\,\|_{L^\infty(Q^m_{h,s,j})} \to 0 \quad \text{as } k_1\to\infty$$

for fixed m.

Define a hypercube $E_{h,s,j}^{k_1,m}$ contained in $Q^m_{h,s,j}$ with edge length $1/2^{jm} - 2R_{h,s,j}^{k_1,m}$ such that

$$\operatorname{dist}(\partial Q^m_{h,s,j}, E^{k_1,m}_{h,s,j}) = R^{k_1,m}_{h,s,j}.$$

Next define

$$f_{k_1}(x) = \begin{cases} 0, & x\in \partial Q^m_{h,s,j},\\ h_{k_1}(x), & x\in E^{k_1,m}_{h,s,j}.\end{cases}$$

Since f_{k_1} is a Lipschitz mapping on the set where it is defined and its Lipschitz constant is no more than $2C(n)\lambda$, by Lemma 2.7, f_{k_1} can be extended to the whole $Q^m_{h,s,j}$, where it is also a Lipschitz mapping with the same Lipschitz constant. We still denote the extension by f_{k_1} and suppose that it is defined on the whole H_j. Then by [4],

$$Df_{k_1}(x) - Dh_{k_1}(x) \to 0 \quad \text{a.e. on } H_j.$$

So there exists a $\overline{\overline{k}}_1 > \overline{k}_1$ such that for all $k_1 > \overline{\overline{k}}_1$, we have

$$\int_{H_j} |Df_{k_1} - Dh_{k_1}|^{p(x)}\,dx \le \frac{\varepsilon_1}{2},$$

$$\Big|\sum_{h,s}\int_{Q^m_{h,s,j}} [f(x^m_{h,s,j},0,Dh_{k_1}) - f(x^m_{h,s,j},0,Df_{k_1})]\,dx\Big| \le \frac{\varepsilon_1}{2}.$$

In view of (H5),

$$\begin{aligned}c_{k_1}^{m,j} &= \sum_{h,s}\int_{Q^m_{h,s,j}} f(x^m_{h,s,j},0,Dh_{k_1})\,dx \ge \sum_{h,s}\int_{Q^m_{h,s,j}} f(x^m_{h,s,j},0,Df_{k_1})\,dx - \frac{\varepsilon_1}{2}\\ &\ge \sum_{h,s}\nu\int_{Q^m_{h,s,j}} |Df_{k_1}|^{p(x)}\,dx - \frac{\varepsilon_1}{2} \ge \frac{\nu}{2^{p^*-1}}\int_{H_j} |Dh_{k_1}|^{p(x)}\,dx - \frac{(\nu+1)\varepsilon_1}{2}.\end{aligned}$$

Thus in (3.20) for $k_1 \geq \overline{\overline{k}}_1$, we have

$$\varepsilon_1 \geq \alpha_0 \int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx + \frac{\nu}{2^{p^*-1}} \int_{H_j} |Dh_{k_1}|^{p(x)}\,dx - \frac{(\nu+1)\varepsilon_1}{2} - V_{10}(\varepsilon_1).$$

Set

$$K(\varepsilon) = \frac{V_{10}(\varepsilon) + (\nu+3)\varepsilon/2}{\min(\alpha_0, \nu/2^{p^*-1})}.$$

Then

$$\int_{H_j} |Dh_{k_1}|^{p(x)}\,dx + \int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx \leq K(\varepsilon_1) \tag{3.21}$$

for $k_1 > \overline{\overline{k}}_1$. From (3.17) and (3.21), we deduce that

$$\int_{\Omega'} |Dh_{k_1}|^{p(x)}\,dx \leq K(\varepsilon_1) + \varepsilon_1, \qquad \int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,dx \leq K(\varepsilon_1).$$

According to the definition of Ω', we have

$$\int_{U^2_{\varepsilon_1}} |Dg_{k_1}|^{p(x)}\,dx \leq K(\varepsilon_1) + \varepsilon_1.$$

Since $Dg_{k_1}(x) = Dz_{k_1}(x)$ for each $x \in H^\lambda_{k_1}$, we get

$$\int_{U^2_{\varepsilon_1} \cap H^\lambda_{k_1}} |Dz_{k_1}|^{p(x)}\,dx \leq K(\varepsilon_1) + \varepsilon_1.$$

By the definition of $U^2_{\varepsilon_1}$ and $U^5_{\varepsilon_1,k_1}$, it is immediate that

$$(U^2_{\varepsilon_1} \cap H_{k_1}) \cup U^5_{\varepsilon_1,k_1} = \Omega,$$

which implies that

$$\int_{\Omega} |Dz_{k_1}|^{p(x)}\,dx \leq 2K(\varepsilon_1) + \varepsilon_1 = W(\varepsilon_1)$$

where $W(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

For $\varepsilon_2 > 0$ and the sequence $\{z_{k_1}\}$, repeating the above arguments we can extract a subsequence of $\{z_{k_1}\}$, denoted by $\{z_{k_2}\}$, such that

$$\int_{\Omega} |Dz_{k_2}|^{p(x)}\,dx \leq W(\varepsilon_2)$$

whenever $k_2 > \overline{\overline{k}}_2$ for some $\overline{\overline{k}}_2$. If $\{z_{k_n}\}$ has been obtained, repeating the above process, we can extract a subsequence of $\{z_{k_n}\}$, denoted by $\{z_{k_{n+1}}\}$, which satisfies

$$\int_{\Omega} |Dz_{k_{n+1}}|^{p(x)}\,dx \leq W(\varepsilon_{n+1})$$

whenever $k_{n+1} > \overline{\overline{k}}_{n+1}$ for some $\overline{\overline{k}}_{n+1}$. Finally, by a diagonal argument we get a subsequence $\{z_{k_i}\}_{i=1}^{\infty}$ of $\{z_k\}$ which satisfies

$$\int_{\Omega} |Dz_{k_i}|^{p(x)}\,dx \to 0 \quad \text{as } i \to \infty.$$

By Lemma 2.9, we have

$$\|\,|Dz_{k_i}|\,\|_{L^{p(\cdot)}(\Omega)} \to 0 \quad \text{as } i \to \infty$$

and furthermore $\{z_{k_i}\}_{i=1}^{\infty}$ converges to zero strongly in $W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$ as $i \to \infty$. This completes the proof of Theorem 3.1. ■

If we choose $p(x) \equiv p$, $1 < p < \infty$, then we get

Corollary 3.1. *Assume the following conditions*:

(G1) *The same as* (H1).

(G2) $|A(x,s,\xi)| \le C_1|\xi|^{p-1} + C_2|s|^{p-1} + G(x)$, *where* $G \in L^{p'}(\Omega)$, $C_1, C_2 \ge 0$ *and* C_2 *is small*, $1/p + 1/p' = 1$.

(G3) $|B(x,s,\xi)| \le C_1'|\xi|^{p-1} + C_2'|s|^{p-1} + \overline{G}(x)$, *where* $\overline{G} \in L^{p'}(\Omega)$, $C_1', C_2' \ge 0$ *and are small.*

(G4) $A_\alpha^i(x,s,\xi)\xi_\alpha^i \ge \lambda_0|\xi|^p - C|s|^p + h(x)$, *where* $\lambda_0 > 0$, $C \ge 0$ *is small and* $h \in L^1(\Omega)$.

(G5) *For almost every* $x_0 \in \Omega$, $s_0 \in \mathbb{R}^N$, *the mapping* $\xi \mapsto A(x_0, s_0, \xi)$ *satisfies*

$$\int_G A_\alpha^i(x_0, s_0, \xi_0 + Dz(x))z_{,\alpha}^i(x)\,dx \ge \nu \int_G |Dz(x)|^p\,dx$$

for each $\xi_0 \in M^{N\times n}$, $G \subset \mathbb{R}^n$, $z \in C_0^1(G, \mathbb{R}^N)$ *where* $\nu > 0$.

Then the system (1.1)–(1.2) *has at least one weak solution which satisfies*

$$u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$$

and

$$\int_\Omega [A_\alpha^i(x,u,Du)z_{,\alpha}^i(x) + B^i(x,u,Du)z^i(x)]\,dx = 0$$

for all $z \in W_0^{1,p}(\Omega, \mathbb{R}^N)$.

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