Inductive extreme non-Arens regularity of the Fourier algebra $A(G)$

by

ZHIGUO HU (Windsor, ON)

Abstract. Let $G$ be a non-discrete locally compact group, $A(G)$ the Fourier algebra of $G$, $VN(G)$ the von Neumann algebra generated by the left regular representation of $G$ which is identified with $A(G)^*$, and $WAP(\hat{G})$ the space of all weakly almost periodic functionals on $A(G)$. We show that there exists a directed family $\mathcal{H}$ of open subgroups of $G$ such that: (1) for each $H \in \mathcal{H}$, $A(H)$ is extremely non-Arens regular; (2) $VN(G) = \bigcup_{H \in \mathcal{H}} VN(H)$ and $VN(G)/WAP(\hat{G}) = \bigcup_{H \in \mathcal{H}} [VN(H)/WAP(\hat{H})]$; (3) $A(G) = \bigcup_{H \in \mathcal{H}} A(H)$ and it is a $WAP$-strong inductive union in the sense that the unions in (2) are strongly compatible with it. Furthermore, we prove that the family $\{A(H) : H \in \mathcal{H}\}$ of Fourier algebras has a kind of inductively compatible extreme non-Arens regularity.

1. Introduction. For a Banach algebra $A$, there exist two Banach algebra multiplications on $A^{**}$ (known as Arens products) which extend the multiplication of $A$ (see Arens [1]). When these two multiplications coincide on $A^{**}$, the algebra $A$ is said to be Arens regular. Every $C^*$-algebra is Arens regular. If $A$ is a commutative Banach algebra, then $A$ is Arens regular if and only if $A^{**}$ is commutative with respect to either (and hence both) of the Arens products. Let $WAP(A)$ be the space of all weakly almost periodic functionals on $A$, i.e., $WAP(A) = \{T \in A^* : \{u \cdot T : u \in A \text{ and } \|u\| \leq 1\} \text{ is relatively weakly compact in } A^*\}$, where $\langle u \cdot T, v \rangle = \langle T, uv \rangle$ for $v \in A$. It is known that $A$ is Arens regular if and only if $WAP(A) = A^*$ (see Pym [15], and also Duncan and Hosseinium [3]). Hence, the quotient Banach space $A^*/WAP(A)$ measures the non-Arens regularity of $A$ in some sense. In particular, Granirer introduced the concept of “extreme non-Arens regularity”. $A$ is called extremely non-Arens regular if $A^*/WAP(A)$ contains a closed linear subspace which has $A^*$ as a continuous linear image (see [7]).
Let $G$ be a locally compact group and $A(G)$ the Fourier algebra of $G$. Lau proved that if $G$ is amenable then $A(G)$ is Arens regular if and only if $G$ is finite (see [13, Proposition 3.3]). Generally, Forrest showed that if $A(G)$ is Arens regular then $G$ must be discrete (he even showed this for the Figà-Talamanca Herz algebra $A_p(G)$; see [6]). It is still open whether Lau’s result is true for non-amenable groups $G$ or for algebras $A_p(G)$ with $p \neq 2$. Recently, Granirer investigated the non-Arens regularity of quotients of $A(G)$. A special case of his Corollary 7 in [7] implies that $A(G)$ is extremely non-Arens regular if $G$ is non-discrete and second countable. Let $b(G)$ be the smallest cardinality of an open basis at the unit $e$ of $G$, and $d(G)$ the smallest cardinality of a covering of $G$ by compact sets. It is proved that Granirer’s result holds for all non-discrete locally compact groups $G$ satisfying $b(G) < d(G)$ (see Hu [10, Corollary 4.2 and Remark 4.7]). In particular, $A(G)$ is extremely non-Arens regular if $G$ is a $\sigma$-compact non-discrete locally compact group.

In this paper we will investigate the non-Arens regularity of $A(G)$ when $b(G) < d(G)$. Let $VN(G)$ be the von Neumann algebra generated by the left regular representation of $G$. It is well known that $A(G)$ can be identified with the predual of $VN(G)$, i.e., $VN(G) = A(G)^*$. Let $WAP(\hat{G})$ denote the space of all weakly almost periodic functionals on $A(G)$ (i.e., $WAP(\hat{G}) = WAP(A(G))$). We show (Theorem 5.3) that, for any non-discrete locally compact group $G$ satisfying $b(G) < d(G)$, there exists a directed family $\mathcal{H}$ of open subgroups of $G$ such that:

(1) For each $H \in \mathcal{H}$, $A(H)$ is extremely non-Arens regular, i.e., for each $H \in \mathcal{H}$, there exists a closed linear subspace $Z_H$ of $VN(H)/WAP(\hat{H})$ and a continuous linear map $\Pi_H : Z_H \to VN(H)$ such that $\Pi_H(Z_H) = VN(H)$.

(2) $VN(G) = \bigcup_{H \in \mathcal{H}} VN(H)$ is an inductive union of von Neumann algebras and $VN(G)/WAP(\hat{G}) = \bigcup_{H \in \mathcal{H}} [VN(H)/WAP(\hat{H})]$ is an inductive union of Banach spaces (see Definition 3.1).

(3) $A(G) = \bigcup_{H \in \mathcal{H}} A(H)$ is an inductive union of Banach algebras and it is a WAP-strong inductive union (see Definition 3.3) in the sense that the two inductive unions in (2) are strongly compatible with the inductive union $A(G) = \bigcup_{H \in \mathcal{H}} A(H)$.

In particular, if $G$ is metrizable, then $H$ is a $\sigma$-compact open subgroup of $G$ for all $H \in \mathcal{H}$, and $A(G)$ is a WAP-strong inductive union of the separable Fourier algebras $\{A(H)\}_{H \in \mathcal{H}}$. Furthermore, we obtain the inductive extreme non-Arens regularity of $A(G)$ by showing that $\{\|\Pi_H\| : H \in \mathcal{H}\}$ is bounded and the pairs $\{Z_H, \Pi_H\}$ $(H \in \mathcal{H})$ are inductively compatible (Theorem 5.10).

The analysis of the relation between open subgroups of $G$ and the support of operators in $VN(G)$ plays a key role in our discussion of the inductive
extreme non-Arens regularity of $A(G)$. We show that if $H$ is an open subgroup of a non-discrete locally compact group $G$, then, for any operator $T$ in $VN(G)$, the support of $T$ can be covered by no more than $b(G)$ cosets of $H$ in $G$ (Proposition 4.1).

Motivated by the inductive limits of $C^*$-algebras, in Section 3 we introduce the concept of “inductive union”, which provides a natural mechanism to relate the Fourier algebra of a locally compact group to the Fourier algebras of its open subgroups.

2. Preliminaries and notations. Let $G$ be a locally compact group with unit $e$ and a fixed left Haar measure. The Fourier–Stieltjes algebra $B(G)$ is the linear span of positive-definite continuous functions on $G$ and is identified with the Banach dual of the group $C^*$-algebra $C^*(G)$ of $G$. With the dual norm and the pointwise multiplication, $B(G)$ is a commutative Banach algebra. Let $C_0(G)$ be the space of all continuous functions on $G$ with compact support. Then the Fourier algebra $A(G)$ is the closed ideal in $B(G)$ generated by elements in $B(G) \cap C_0(G)$. Let $VN(G)$ be the von Neumann algebra generated by the left regular representation of $G$. Then $A(G)$ can be identified with the predual of $VN(G)$ (i.e., $VN(G) = A(G)^*$) and $VN(G)$ becomes a $B(G)$-module under the action $\langle u \cdot T, v \rangle = \langle T, uv \rangle$ for $u \in B(G)$, $v \in A(G)$, and $T \in VN(G)$. Also, $VN(G)$ coincides with the space of all bounded linear operators on $L^2(G)$ which satisfy $T(f \ast g) = T(f) \ast g$ for all $f \in L^2(G)$ and $g \in C_0(G)$. See Eymard [5] for more information on $A(G)$, $B(G)$, and $VN(G)$.

The space $\{T \in VN(G) : u \mapsto u \cdot T \text{ is a weakly compact operator from } A(G) \text{ into } VN(G)\}$ is called the space of weakly almost periodic functionals on $A(G)$ and is denoted by $WAP(\hat{G})$. It turns out that $WAP(\hat{G})$ is a self-adjoint closed $B(G)$-submodule of $VN(G)$. When $G$ is a locally compact abelian group, $WAP(\hat{G})$ is identified with the space of weakly almost periodic functions on the dual group of $G$. See Dunkl and Ramirez [4] for more details on $WAP(\hat{G})$.

The support of a function $f$ in $L^2(G)$ is defined by saying that $x \notin \text{supp } f$ if and only if there exists a neighbourhood $V$ of $e$ such that $\int_G f(x)v(x) \, dx = 0$ for all $v \in C_0(G)$ with $\text{supp } v \subseteq V$. The support of an operator $T$ in $VN(G)$ is defined by saying that $x \notin \text{supp } T$ if and only if there exists a neighbourhood $U$ of $e$ such that $x \notin \text{supp}(Tu)$ for all $u \in C_0(G)$ with $\text{supp } u \subseteq U$. An equivalent description for $\text{supp } T$ is that $x \in \text{supp } T$ if and only if $u(x) = 0$ for all $u \in A(G)$ (see Eymard [5] and Herz [8]).

Let $b(G)$ be the smallest cardinality of an open basis at $e$ and $d(G)$ denote the smallest cardinality of a covering of $G$ by compact sets. It is known that $b(G) = d(\hat{G})$ when $G$ is abelian with dual group $\hat{G}$ (see Hewitt and Ross [12, (24.48)])]. Clearly, $G$ is metrizable if and only if $b(G) \leq \aleph_0$. 
3. Inductive unions. Inspired by the inductive limits of $C^*$-algebras, we introduce the concept of “inductive union”, which is of importance for our investigation on the non-Arens regularity of the Fourier algebra $A(G)$.

**Definition 3.1.** Let $A$ be a Banach space (Banach algebra, $C^*$-algebra, respectively) and let $\{A_i\}_{i \in I}$ be a family of Banach spaces (Banach algebras, $C^*$-algebras, respectively) indexed by a directed set $I$. We say that $A$ is an *inductive union* of $\{A_i\}_{i \in I}$ (denoted by $A = \bigsqcup_{i \in I} A_i$) if there exists a linear isometry (isometric isomorphism, $*$-isomorphism, respectively) $\Lambda_i : A_i \to A$ for each $i \in I$ such that $\Lambda_i(A_i) \subseteq A_j(A_j)$ for all $i, j \in I$ with $i \leq j$ and $A = \bigsqcup_{i \in I} A_i(A_i)$.

Immediately, we can show the existence of maps $\Lambda_{ij}$ ($i \leq j$) compatible with $\{A_i\}_{i \in I}$.

**Corollary 3.2.** Let $A = \bigsqcup_{i \in I} A_i$ be an inductive union of the family $\{A_i\}_{i \in I}$ of Banach spaces (Banach algebras, $C^*$-algebras, respectively) via the linear isometries (isometric isomorphisms, $*$-isomorphisms, respectively) $\{A_i\}_{i \in I}$. Then, for all $i, j \in I$ with $i \leq j$, there exists a unique linear isometry (isometric isomorphism, $*$-isomorphism, respectively) $\Lambda_{ij} : A_i \to A_j$ such that:

(a) $\Lambda_j \Lambda_{ij} = \Lambda_i$ for all $i, j \in I$ with $i \leq j$.
(b) $\Lambda_{jk} \Lambda_{ij} = \Lambda_{ik}$ if $i, j, k \in I$ and $i \leq j \leq k$.

**Proof.** Let $i, j \in I$ and $i \leq j$. Note that $\Lambda_i(A_i) \subseteq A_j(A_j)$ and hence $\Lambda_i(A_i)$ is a closed linear subspace (subalgebra, $C^*$-subalgebra, respectively) of $A_j(A_j)$. Define $\Lambda_{ij} = (A_j)^{-1}|_{\Lambda_i(A_i)} A_i$. Then $\Lambda_{ij} : A_i \to A_j$ is a linear isometry (isometric isomorphism, $*$-isomorphism, respectively). By the definition of $\Lambda_{ij}$, it can be seen that (a) holds and the map $\Lambda_{ij}$ satisfying (a) is unique.

Suppose that $i, j, k \in I$ and $i \leq j \leq k$. By (a), we have $\Lambda_k(\Lambda_{jk} \Lambda_{ij}) = A_j \Lambda_{ij} = A_i = A_k \Lambda_{ik}$, i.e., $\Lambda_{jk} \Lambda_{ij} = \Lambda_{ik}$ since $\Lambda_k$ is one-to-one. Therefore, (b) is true. ■

When $A$ is an inductive union of $\{A_i\}_{i \in I}$, it is interesting to know if $A^*$ is an inductive union of $\{A_i^*\}_{i \in I}$ and if a quotient space of $A^*$ is an inductive union of the corresponding quotient spaces of $A_i^*$ ($i \in I$), etc. For our purpose, we only consider the following “WAP” strongly compatible inductive unions of Banach algebras. Recall that, for a Banach algebra $A$, WAP($A$) denotes the space of all weakly almost periodic functionals on $A$.

**Definition 3.3.** Let $A$ be a Banach algebra and let $A = \bigsqcup_{i \in I} A_i$ be an inductive union of the Banach algebras $\{A_i\}_{i \in I}$ via the isometric isomor-
phisms \( \{A_i\}_{i \in I} \). We say that \( A \) is a WAP-strong inductive union of \( \{A_i\}_{i \in I} \) if the following hold.

(1) \( A^* = \bigcup_{i \in I} A_i^* \) is an inductive union of the Banach spaces \( \{A_i^*\}_{i \in I} \) via some linear isometries \( \{\Phi_i\}_{i \in I} \) such that, for all \( i \in I \), \( A_i^* \Phi_i = \text{Id} \) and \( \Phi_i(u \cdot T) = A_i(u) \cdot \Phi_i(T) \) for \( u \in A_i \) and \( T \in A_i^* \).

(2) For all \( i \in I \), \( \Phi_i(\text{WAP}(A_i)) = \text{WAP}(A) \cap \Phi_i(A_i^*) \) and \( \Phi_i \) lifts a linear isometry \( \Gamma_i : A_i^*/\text{WAP}(A_i) \to A^*/\text{WAP}(A) \).

It is easy to see that (1) and (2) in Definition 3.3 are equivalent to the following two conditions.

**Corollary 3.4.** Let \( A = \bigcup_{i \in I} A_i \) be an inductive union of the Banach algebras \( \{A_i\}_{i \in I} \) via \( \{A_i\}_{i \in I} \). Then \( A \) is a WAP-strong inductive union of \( \{A_i\}_{i \in I} \) if and only if the following conditions are satisfied:

(1) \( A^* = \bigcup_{i \in I} A_i^* \) is an inductive union of \( \{A_i^*\}_{i \in I} \) via \( \{\Phi_i\}_{i \in I} \) such that, for all \( i \in I \), \( \Phi_i \Phi_i^* : A^* \to A^* \) is a \( \Lambda_i(A_i) \)-invariant projection (i.e., \( (\Phi_i \Phi_i^*)^2 = \Phi_i \Phi_i^* \) and \( \Phi_i \Phi_i^*(v \cdot T) = v \cdot [\Phi_i \Phi_i^*(T)] \) for all \( v \in \Lambda_i(A_i) \) and \( T \in A^* \)).

(2) \( \text{WAP}(A) = \bigcup_{i \in I} \text{WAP}(A_i) \) is an inductive union of the Banach spaces \( \{\text{WAP}(A_i)\}_{i \in I} \) via the restrictions \( \{\Phi_i|_{\text{WAP}(A)}\}_{i \in I} \) and \( A^*/\text{WAP}(A) = \bigcup_{i \in I} [A_i^*/\text{WAP}(A_i)] \) is an inductive union of the quotient Banach spaces \( \{A_i^*/\text{WAP}(A_i)\}_{i \in I} \) via \( \{\Gamma_i\}_{i \in I} \) such that \( \Gamma_i \varrho_i = \varrho \Phi_i \) for all \( i \in I \), where \( \varrho_i : A_i^* \to A_i^*/\text{WAP}(A_i) \) and \( \varrho : A^* \to A^*/\text{WAP}(A) \) are the canonical quotient maps.

Analogously to Corollary 3.2, we are able to get maps \( \Phi_{ij} \) and \( \Gamma_{ij} \) (\( i \leq j \)) which are compatible with \( \{\Phi_i\}_{i \in I} \) and \( \{\Gamma_i\}_{i \in I} \), respectively.

**Corollary 3.5.** Let \( A = \bigcup_{i \in I} A_i \) be a WAP-strong inductive union of the Banach algebras \( \{A_i\}_{i \in I} \) via the maps \( \{A_i\}_{i \in I} \) with \( A^* = \bigcup_{i \in I} A_i^* \) via \( \{\Phi_i\}_{i \in I} \) and \( A^*/\text{WAP}(A) = \bigcup_{i \in I} [A_i^*/\text{WAP}(A_i)] \) via \( \{\Gamma_i\}_{i \in I} \). Then, for all \( i,j \in I \) with \( i \leq j \), there exist unique linear isometries \( \Phi_{ij} : A_i^* \to A_j^* \) and \( \Gamma_{ij} : A_i^*/\text{WAP}(A_i) \to A_j^*/\text{WAP}(A_j) \) such that the following hold:

(a) \( \Phi_j \Phi_{ij} = \Phi_i \) and \( \Gamma_j \Gamma_{ij} = \Gamma_i \) for all \( i,j \in I \) with \( i \leq j \).

(b) \( \Phi_{jk} \Phi_{ij} = \Phi_{ik} \) and \( \Gamma_{jk} \Gamma_{ij} = \Gamma_{ik} \) if \( i,j,k \in I \) and \( i \leq j \leq k \).

(c) \( A_i^* \Phi_{ij} \Phi_j = \text{Id} \) and \( \Phi_{ij} \Phi_j(u \cdot T) = A_i(u) \cdot \Phi_{ij}(T) \) for all \( i,j \in I \) with \( i \leq j \), \( u \in A_i \) and \( T \in A_i^* \), where \( A_i : A_i \to A_j \) is the same map as in Corollary 3.2.

(d) \( \Phi_{ij}(\text{WAP}(A_i)) = \text{WAP}(A_j) \cap \Phi_{ij}(A_i^*) \) and \( \Gamma_{ij} \varrho_i = \varrho_j \Phi_{ij} \) if \( i,j \in I \) and \( i \leq j \) (i.e., \( \Gamma_{ij} \) is the map lifted by \( \Phi_{ij} \)).

**Proof.** It can be seen that (a) and (b) hold by the same argument as in the proof of Corollary 3.2. Clearly, the maps \( \Phi_{ij} \) and \( \Gamma_{ij} \) satisfying (a) are unique.
Let $i,j \in I$ and $i \leq j$. Note that $A_i^* \Phi_i = \text{Id}$, $A_i^* = A_i^* A_i^*$ (by Corollary 3.2), and $\Phi_i = \Phi_j \Phi_{ij}$. Therefore, $A_i^* \Phi_{ij} = A_i^* (A_j^* \Phi_j) \Phi_{ij} = (A_i^* A_j^*)(\Phi_j \Phi_{ij}) = A_i^* \Phi_i = \text{Id}$, i.e., $A_i^* \Phi_{ij} = \text{Id}$. Suppose that $u \in A_i$ and $T \in A_i^*$. Then

$$
\Phi_j[\Phi_{ij}(u \cdot T)] = \Phi_i(u \cdot T) = A_i(u) \cdot \Phi_i(T) = A_j[A_{ij}(u)] \cdot \Phi_j[\Phi_{ij}(T)] = \Phi_j[A_{ij}(u) \cdot \Phi_{ij}(T)].
$$

We conclude that $\Phi_{ij}(u \cdot T) = A_{ij}(u) \cdot \Phi_{ij}(T)$ since the map $\Phi_j$ is one-to-one. Therefore, (c) is true.

Note that $\Phi_i(WAP(A_i)) \subseteq \Phi_j(WAP(A_j)) \subseteq WAP(A)$ and hence we have $\Phi_i(WAP(A_i)) = \Phi_j(WAP(A_j)) \cap \Phi_i(A_i^*)$, that is, $\Phi_j[\Phi_{ij}(WAP(A_i))] = \Phi_j[WAP(A_j) \cap \Phi_{ij}(A_i^*)]$. Therefore, $\Phi_{ij}(WAP(A_i)) = WAP(A_j) \cap \Phi_{ij}(A_i^*)$.

Finally, by using the facts that $\Gamma_j \Gamma_{ij} = \Gamma_i$, $\Gamma_i \varrho_i = \varrho_i \Phi_i$, and $\Phi_i = \Phi_j \Phi_{ij}$, we have $\Gamma_j(\Gamma_{ij} \varrho_i) = \Gamma_i \varrho_i = \varrho_i \Phi_i = \varrho_i \Phi_j \Phi_{ij} = \Gamma_j(\varrho_i \Phi_{ij})$. It follows that $\Gamma_{ij} \varrho_i = \varrho_i \Phi_{ij}$ since $\Gamma_j$ is one-to-one. Therefore, (d) holds.

4. Open subgroups, support of $T$ in $VN(G)$, and isometric embeddings. In this section, $G$ is a locally compact group and $H$ is an open subgroup of $G$. Let $VN_H(G)$ denote the von Neumann subalgebra of $VN(G)$ generated by $\{\lambda_G(x) : x \in H\}$, where $\lambda_G$ is the left regular representation of $G$. Then $VN_H(G) = \{T \in VN(G) : \text{supp} T \subseteq H\}$ (see Chou [2, Lemma 4.2]). Let $1_H \in B(G)$ be the characteristic function of $H$. Then $1_H \cdot T \in VN_H(G)$ for all $T \in VN(G)$ and $T = 1_H \cdot T$ if $T \in VN_H(G)$. Therefore, $VN_H(G) = 1_H \cdot VN(G)$.

It is known that if an element $T$ of $VN(G)$ is the left convolution operator by a bounded complex-valued regular Borel measure $\mu$ on $G$, then the support of $T$ is just the support of the measure $\mu$ and hence it is a countable union of compact sets in $G$ by the regularity of $\mu$.

Generally, for an arbitrary operator $T$ in $VN(G)$, we are concerned with the question of how many cosets $gH$ we will need at least to cover the support of $T$. If $G$ is discrete, then every element $T$ of $VN(G)$ is identified with a left convolution operator by a function in $l^2(G)$ and so the support of $T$ is a countable subset of $G$. In the following, we will consider the case when $G$ is non-discrete.

**Proposition 4.1.** Let $G$ be a non-discrete locally compact group and let $H$ be an open subgroup of $G$. Then, for any $T \in VN(G)$, there are at most $b(G)$ cosets $gH$ ($g \in G$) such that $\text{supp} T \cap gH \neq \emptyset$.

**Proof.** Replacing $H$ by a $\sigma$-compact open subgroup of $H$, we may assume that $H$ is a $\sigma$-compact open subgroup of $G$.

Let $\mathcal{U}$ be a compact neighbourhood system at $e$ such that $\text{card}(\mathcal{U}) = b(G)$. Then $\mathcal{U}$ is a directed set under the relation $U \preceq V$ if and only if
For each $T \in \mathcal{U}$, let $h_U = (1/|U|)1_U$ and $T_U = T(h_U) \in L^2(G)$, where $|U|$ is the left Haar measure of $U$ and $1_U$ denotes the characteristic function of $U$. By [12, (20.15)], for all $f \in L^2(G)$, $\lim_U \|h_U * f - f\|_2 = 0$. Therefore, $T$ is completely determined by the net $(T_U)_{U \in \mathcal{U}}$. For each $U \in \mathcal{U}$, since $T_U \in L^2(G)$, there exists a sequence $\{g_n^U\}_n$ in $G$ such that $\supp T_U \subseteq \bigcup_{n=1}^{\infty} g_n^U H$.

Fix a compact neighbourhood $V$ of $e$. Since $H$ is $\sigma$-compact, $HV$ and hence $\bigcup_{n=1}^{\infty} g_n^U HV$ is a countable union of compact sets. Therefore, $\bigcup_{n=1}^{\infty} g_n^UHV$ can be covered by countably many cosets $gH$. Note that $\text{card}(U) = b(G) \geq n_0$. It follows that there exists a subset $B$ of $G$ such that $\text{card}(B) \leq b(G) = \text{card}(U)$ and $\bigcup_{U \in \mathcal{U}} \bigcup_{n=1}^{\infty} g_n^UHV \subseteq \bigcup_{g \in B} gH$.

To complete the proof, we only need to show that $\supp T \subseteq \bigcup_{g \in B} gH$.

Suppose $x \in G \backslash \bigcup_{g \in B} gH$. In the following, we will prove that $x \notin \supp T(f)$ for all $f \in C_{00}(G)$ with $\supp f \subseteq V$ and it follows that $x \notin \supp T$.

Let $f \in C_{00}(G)$ and $\supp f \subseteq V$. Then $T(f) = \lim_U (T_U * f)$ in the $\| \cdot \|_2$-norm. Recall that, for each $U \in \mathcal{U}$, $\supp T_U \subseteq \bigcup_{n=1}^{\infty} g_n^U H$ and hence $\supp(T_U * f) \subseteq \bigcup_{n=1}^{\infty} g_n^U HV \subseteq \bigcup_{g \in B} gH$. Also note that $\bigcup_{g \in B} gH$ is closed in $G$. Therefore, $\supp T(f) \subseteq \bigcup_{g \in B} gH$ and we have $x \notin \supp T(f)$.

**Corollary 4.2.** Let $G$ be a metrizable locally compact group and let $H$ be an open subgroup of $G$. Then, for any $T \in \text{VN}(G)$, there exists a sequence $\{g_n\}_n$ in $G$ such that $\supp T \subseteq \bigcup_{n=1}^{\infty} g_n H$.

**Remark 4.3.** Let $G$ be a locally compact group and let $H$ be an open subgroup of $G$. If $T \in \overline{\text{span}} [\lambda_G(G) \text{VN}_H(G)]$ (the norm closed linear span generated by the translates of elements in $\text{VN}_H(G)$), then the support of any operator in $\text{VN}(G)$ can be covered by countably many cosets $gH$. However, it is possible that the support of any operator in $\text{VN}(G)$ can be covered by countably many cosets $gH$ (e.g., when $G$ is metrizable or $\sigma$-compact) but $\text{VN}(G) \neq \overline{\text{span}} [\lambda_G(G) \text{VN}_H(G)]$. For example, let $G$ be a non-compact metrizable locally compact group containing a compact open subgroup $H$. Then $\text{VN}(H) = \text{UC}(\hat{H})$ (the $C^*$-algebra of uniformly continuous functionals on $A(H)$ introduced by Granirer) and thus $\overline{\text{span}} [\lambda_G(G) \text{VN}_H(G)] = \text{UC}(\hat{G})$ (see Hu [11, Proposition 3.5]). Now $\overline{\text{span}} [\lambda_G(G) \text{VN}_H(G)] = \text{UC}(\hat{G}) \nsubseteq \text{VN}(G)$ because $G$ is non-compact.

**Corollary 4.4.** Let $G$ be a metrizable locally compact group. Then, for any $T \in \text{VN}(G)$, there exists a $\sigma$-compact open subgroup $H$ of $G$ such that $\supp T \subseteq H$.

**Proof.** Let $G_0$ be a $\sigma$-compact open subgroup of $G$. Let $T \in \text{VN}(G)$. By Corollary 4.2, there exists a sequence $\{g_n\}_n$ in $G$ such that $\supp T \subseteq \bigcup_{n=1}^{\infty} g_n H$.
\[ \bigcup_{n=1}^{\infty} g_n G_0. \] Let \( H \) be the open subgroup of \( G \) generated by \( G_0 \cup \bigcup_{n=1}^{\infty} g_n G_0. \) Then \( H \) is a \( \sigma \)-compact open subgroup of \( G \) and \( \text{supp} T \subseteq H. \)

Let \( r : A(G) \to A(H) \) be the restriction map. According to Eymard [5], \( r \) is a linear contractive surjection and its adjoint \( r^* \) is a *-isomorphism of the von Neumann algebra \( VN(H) \) onto the von Neumann subalgebra \( VN_H(G) \) of \( VN(G) \) (see [5, (3.21)]), where \( r^*(T) \) is denoted as \( T^\circ \) for \( T \in VN(H) \). It is known that \( r^*(WAP(H)) = WAP(\hat{G}) \cap VN_H(G) \) (see Chou [2, Lemma 4.2]). Therefore, the *-isomorphism \( r^* \) lifts a linear map from the quotient Banach space \( VN(H)/WAP(\hat{H}) \) into the quotient Banach space \( VN(G)/WAP(\hat{G}) \).

Let \( VN_H(G)/WAP(\hat{G}) \) denote the linear subspace \( \{ T + WAP(\hat{G}) : T \in VN_H(G) \} \) of \( VN(G)/WAP(\hat{G}) \). In the following we will show that in fact \( r^* \) lifts a linear isometry between \( VN(H)/WAP(\hat{H}) \) and \( VN_H(G)/WAP(\hat{G}) \).

**Proposition 4.5.** For \( T \in VN(H) \), define \( \tilde{r}^*(T + WAP(\hat{H})) = r^*(T) + WAP(\hat{G}) \). Then \( \tilde{r}^* : VN(H)/WAP(\hat{H}) \to VN(G)/WAP(\hat{G}) \) is a linear isometry with range \( VN_H(G)/WAP(\hat{G}) \) and the following diagram commutes:

\[
\begin{array}{ccc}
VN(H) & \overset{r^*}{\longrightarrow} & VN(G) \\
\downarrow g_H & & \downarrow g \\
VN(H)/WAP(\hat{H}) & \overset{\tilde{r}^*}{\longrightarrow} & VN(G)/WAP(\hat{G})
\end{array}
\]

where \( g_H \) and \( g \) are the canonical quotient maps.

**Proof.** Since \( r^*(VN(H)) = VN_H(G) \) and \( r^*(WAP(\hat{H})) = WAP(\hat{G}) \cap VN_H(G) \), by the definition, \( \tilde{r}^* : VN(H)/WAP(\hat{H}) \to VN(G)/WAP(\hat{G}) \) is well defined, linear, and onto the linear subspace \( VN_H(G)/WAP(\hat{G}) \) of \( VN(G)/WAP(\hat{G}) \). According to the definition of \( \tilde{r}^* \), it is clear that the diagram is commutative. To complete the proof, we only need to show that \( \tilde{r}^* \) is an isometry.

Let \( T \in VN(H) \). Obviously, \( \| \tilde{r}^*(T + WAP(\hat{H})) \| \leq \| T + WAP(\hat{H}) \| \) since \( \| \tilde{r}^* \| \leq \| r^* \| = 1 \). Conversely, let \( W \in WAP(\hat{G}) \). Then \( W = W_1 + W_2 \), where \( W_1 = 1_H \cdot W \) and hence \( W_1 \in WAP(\hat{G}) \cap VN_H(G) \), and \( W_2 = W - W_1 \in WAP(\hat{G}) \) with \( \text{supp} W_2 \subseteq G \setminus H \). Thus, \( W_1 = r^*(V_1) \) for some \( V_1 \in WAP(\hat{H}) \).

So,

\[
\| r^*(T) + W \| = \| r^*(T) + r^*(V_1) + W_2 \| \\
\geq \| 1_H \cdot (r^*(T) + r^*(V_1) + W_2) \| \\
= \| r^*(T) + r^*(V_1) \| \quad \text{(since} \ 1_H \cdot W_2 = 0) \\
= \| T + V_1 \| \\
\geq \| T + WAP(\hat{H}) \|. 
\]
Since $W \in \text{WAP}(\hat{G})$ is arbitrary, it follows that
\[ \|r^*(T) + \text{WAP}(\hat{G})\| \geq \|T + \text{WAP}(\hat{H})\|, \]
i.e., \( \|\tilde{r}^*(T + \text{WAP}(\hat{H}))\| \geq \|T + \text{WAP}(\hat{H})\| \). Therefore, \( \tilde{r}^* \) is a linear isometry. □

**Remark 4.6.** Let \( V \) be any closed \( B(G) \)-submodule of \( \text{VN}(G) \) and let \( V_H = (r^*)^{-1}[V \cap \text{VN}_H(G)] \). Then \( V_H \) is a closed \( B(H) \)-submodule of \( \text{VN}(H) \) and \( r^*(V_H) = V \cap \text{VN}_H(G) \). From the proof it can be seen that Proposition 4.5 holds if \( \text{WAP}(\hat{G}) \) and \( \text{WAP}(\hat{H}) \) are replaced by \( V \) and \( V_H \), respectively. In particular, if we take \( V = \text{AP}(\hat{G}), \text{UC}(\hat{G}), C^*_r(G), \text{and } C^*_b(G) \) (the space of almost periodic functionals on \( A(G) \), the space of uniformly continuous functionals on \( A(G) \), the reduced group \( C^* \)-algebra of \( G \), and the \( C^* \)-algebra generated by \( \{\lambda_G(x) : x \in G\} \), respectively), then we will get \( V_H = \text{AP}(\hat{H}), \text{UC}(\hat{H}), C^*_r(H), \text{and } C^*_b(H) \), respectively (cf. [11]).

5. **Inductive extreme non-Arens regularity of \( A(G) \).** Throughout this section, we assume that \( G \) is a non-discrete locally compact group and \( G_0 \) is a \( \sigma \)-compact open subgroup of \( G \).

Let \( T \in \text{VN}(G) \). By Proposition 4.1, there exists a subset \( B \) of \( G \) such that \( \text{card}(B) \leq b(G) \) and \( \text{supp} T \cap gG_0 = \emptyset \) for all \( g \in G \setminus B \). Hence, \( \text{supp} T \subseteq \bigcup_{g \in B} gG_0 \). Let \( H_B \) be the open subgroup of \( G \) generated by \( G_0 \cup \bigcup_{g \in B} gG_0 \), i.e.,
\[ H_B = \bigcup_{n=1}^{\infty} \left\{ G_0 \cup \bigcup_{g \in B} gG_0 \right\}^{-1}. \]

Then we have \( T \in VN_{H_B}(G) \) and \( H_B \) can be covered by no more than \( b(G) \) compact sets (since \( G_0 \) is \( \sigma \)-compact and \( b(G) \geq \aleph_0 \)). Therefore, \( d(H_B) \leq b(H_B) (= b(G)) \). According to the result of Hu [10, Corollary 4.2 and Remark 4.7], \( A(H_B) \) is extremely non-Arens regular.

To obtain the inductive extreme non-Arens regularity of \( A(G) \), we need to consider the following maps.

**Definition 5.1.** Let \( H \) and \( J \) be open subgroups of \( G \) and \( H \subseteq J \). The maps \( \Lambda_{H,J} : A(H) \to A(J), \Phi_{H,J} : \text{VN}(H) \to \text{VN}(J), \) and \( \Gamma_{H,J} : \text{VN}(H)/\text{WAP}(\hat{H}) \to \text{VN}(J)/\text{WAP}(\hat{J}) \) are defined as follows: for \( u \in A(H) \) and \( T \in \text{VN}(H) \),
\[ \Lambda_{H,J}(u) = u^c, \]
\[ \Phi_{H,J}(T) = r_{H,J}^*(T), \]
\[ \Gamma_{H,J}(T + \text{WAP}(\hat{H})) = \tilde{r}_{H,J}^*(T) \]
\[ = r_{H,J}^*(T) + \text{WAP}(\hat{J}) \quad \text{(as in Proposition 4.5)}, \]
where $u^*$ denotes the trivial extension of $u$ to $J$ (i.e., $u^*(x) = 0$ if $x \in J \setminus H$), and $r_{HJ}^*$ is the adjoint of the restriction map $r_{HJ} : A(J) \to A(H)$. Also, we define $\Lambda_H = \Lambda_{HG}, \Phi_H = \Phi_{HG},$ and $\Gamma_H = \Gamma_{HG}$.

**Lemma 5.2.** Let $H$ and $J$ be open subgroups of $G$ such that $H \subseteq J$. Let $\Lambda_{HJ}, \Phi_{HJ}, \Gamma_{HJ}, \Lambda_H, \Phi_H,$ and $\Gamma_H$ be the maps from Definition 5.1.

(a) $\Lambda_{HJ}$ is an isometric isomorphism from the Banach algebra $A(H)$ onto the Banach subalgebra $A_H(J)$ of $A(J)$, where $A_H(J) = \{f \in A(J) : \text{supp } f \subseteq H\}$. 

(b) $\Phi_{HJ}$ is a $\ast$-isomorphism (and hence an isometry) from the von Neumann algebra $VN(H)$ onto the von Neumann subalgebra $VN_H(J)$ of $VN(J)$. 

(c) $\Gamma_{HJ}$ is a linear isometry with range $VN_H(J)/WAP(J)$. 

(d) If $K$ is an open subgroup of $G$ and $H \subseteq J \subseteq K$, then $\Lambda_{JK} \Lambda_{HJ} = \Lambda_{HK}, \Phi_{JK} \Phi_{HJ} = \Phi_{HK},$ and $\Gamma_{JK} \Gamma_{HJ} = \Gamma_{HK}$. In particular, the maps $\Lambda_H, \Phi_H,$ and $\Gamma_H$ are compatible with $\Lambda_{HJ}, \Phi_{HJ},$ and $\Gamma_{HJ},$ respectively. That is, $\Lambda_J \Lambda_{HJ} = \Lambda_H, \Phi_J \Phi_{HJ} = \Phi_H,$ and $\Gamma_J \Gamma_{HJ} = \Gamma_H$ for all $H \subseteq J$. 

Proof. (a) and (b) follow from [5, (3.21)]. (c) holds by Proposition 4.5. And it is easy to check (d) by Definition 5.1. \[\square\]

Summarizing the above discussion, we are ready to give the following decompositions for the Fourier algebra $A(G)$, the von Neumann algebra $VN(G)$, and the quotient Banach space $VN(G)/WAP(\hat{G})$.

**Theorem 5.3.** Let $G$ be a non-discrete locally compact group with $b(G) < d(G)$ and let $G_0$ be a $\sigma$-compact open subgroup of $G$. Let $B = \{B : B \subseteq G$ and $\text{card}(B) \leq b(G)\}$ and let $\mathcal{H}$ be the family of open subgroups of $G$ generated by $G_0 \cup \bigcup_{g \in B} gG_0$ ($B \in B$). Then:

1. $\mathcal{H}$ is a directed set under the relation “$\subseteq$”, $d(H) \leq b(H)$ for all $H \in \mathcal{H}, G = \bigcup_{H \in \mathcal{H}} H$, and $\text{card}(\mathcal{H}) \leq d(G)^{b(G)}$.
2. For all $H \in \mathcal{H}$, $A(H)$ is extremely non-Arens regular.
3. $A(G) = \bigcup_{H \in \mathcal{H}} A(H)$ is an inductive union of the Banach algebras $\{A(H)\}_{H \in \mathcal{H}}$ via the isometric isomorphisms $\{\Lambda_H\}_{H \in \mathcal{H}}$.
4. $VN(G) = \bigcup_{H \in \mathcal{H}} VN(H)$ is an inductive union of the von Neumann algebras $\{VN(H)\}_{H \in \mathcal{H}}$ via the $\ast$-isomorphisms $\{\Phi_H\}_{H \in \mathcal{H}}$.
5. $VN(G)/WAP(\hat{G}) = \bigcup_{H \in \mathcal{H}} [VN(H)/WAP(\hat{H})]$ is an inductive union of the quotient Banach spaces $\{VN(H)/WAP(\hat{H})\}_{H \in \mathcal{H}}$ via the linear isometries $\{\Gamma_H\}_{H \in \mathcal{H}}$.
6. $A(G) = \bigcup_{H \in \mathcal{H}} A(H)$ is a WAP-strong inductive union of the algebras $\{A(H)\}_{H \in \mathcal{H}}$.
7. $\Lambda_{HJ}, \Phi_{HJ},$ and $\Gamma_{HJ}$ ($H, J \in \mathcal{H}$ and $H \subseteq J$) are the maps compatible with $\{\Lambda_H\}_{H \in \mathcal{H}}, \{\Phi_H\}_{H \in \mathcal{H}},$ and $\{\Gamma_H\}_{H \in \mathcal{H}}$ as in Corollary 3.2 and Corollary 3.5, respectively.
In particular, if $G$ is metrizable, then $H$ is a $\sigma$-compact open subgroup of $G$ for all $H \in \mathcal{H}$ and $A(G)$ is a WAP-strong inductive union of the separable Fourier algebras \{\(A(H)\)\}_{H \in \mathcal{H}}.

**Proof.** Clearly, $\mathcal{H}$ is a directed set under “$\subseteq$”, $d(H) \leq b(H)$ for all $H \in \mathcal{H}$ (see the second paragraph in this section), and $G = \bigcup_{H \in \mathcal{H}} H$. Let $S$ be a complete set of left coset representatives of $G_0$ in $G$ and let $E = \{B \subseteq S : \text{card}(B) \leq b(G)\}$. It can be seen that $\text{card}(S) = d(G)$ and hence $\text{card}(\mathcal{H}) \leq \text{card}(E) \leq d(G)b(G)$. Therefore, (1) holds.

(2) and (4) are true according to the discussion in the second paragraph of this section, Lemma 5.2(b), and Definition 3.1.

Note that $A(G) \cap C_{00}(G)$ is norm dense in $A(G)$. So, if $f \in A(G)$, then $\text{supp } f$ can be covered by countably many cosets $gG_0$ ($g \in G$). Hence, $\text{supp } T \subseteq H$ for some $H \in \mathcal{H}$. Therefore, $f \in A_H(G) = \Lambda_H(A(H))$ for some $H \in \mathcal{H}$. By Lemma 5.2(a) and Definition 3.1, (3) holds.

(5) follows from (4) and Lemma 5.2(c).

Let $H \in \mathcal{H}$ and let $r_H : A(G) \to A(H)$ be the restriction map. Then $r_H A_H = \text{Id}$ and $\Phi_H = r_H^*$. Thus, $\Lambda_H \Phi_H = A_H r_H^* = \text{Id}$. It is easy to see that $\Lambda_H \Phi_H (u \cdot T) = \Lambda_H (u \cdot \Phi_H (T))$ for all $u \in A(H)$ and $T \in VN(H)$ by the fact that $r_H A_H = \text{Id}$ and $\Phi_H = r_H^*$. Clearly, $\Phi_H (\text{WAP}(\hat{H})) = \text{WAP}(\hat{G}) \cap \Phi_H (VN(H))$ and $\Gamma_H : VN(H)/\text{WAP}(\hat{H}) \to VN(G)/\text{WAP}(\hat{G})$ is the linear isometry lifted by $\Phi_H : VN(H) \to VN(G)$. Therefore, $A(G) = \bigcup_{H \in \mathcal{H}} A(H)$ is a WAP-strong inductive union of \{\(A(H)\)\}_{H \in \mathcal{H}} by (4), (5), and Definition 3.3, i.e., (6) is true.

(7) holds by Lemma 5.2(d) and the uniqueness of the maps $\Lambda_{HJ}$, $\Phi_{HJ}$, and $\Gamma_{HJ}$ satisfying Corollary 3.2(a) and Corollary 3.5(a), respectively.

Finally, suppose that $G$ is metrizable. Let $H \in \mathcal{H}$. Then $d(H) \leq b(H) = \aleph_0$ by (2). Therefore, $H$ is $\sigma$-compact and metrizable and hence $A(H)$ is separable. ■

**Remark 5.4.** Let $V$ be any closed $B(G)$-submodule of $VN(G)$ and let $V_H = \Phi_H^{-1}[V \cap VN_H(G)]$. By Remark 4.6, the spaces WAP($\hat{G}$) and \{WAP($\hat{H}$)\}_{H \in \mathcal{H}} in Theorem 5.3(5) can be replaced by $V$ and \{\(V_H\)\}_{H \in \mathcal{H}} respectively. Therefore, the inductive union $A(G) = \bigcup_{H \in \mathcal{H}} A(H)$ in Theorem 5.3 is more than WAP-strong.

Let $G$ be a locally compact abelian group with the dual group $\Gamma$. Then the Fourier algebra $A(G)$ of $G$ is isometrically isomorphic to the group algebra $L^1(\Gamma)$ of $\Gamma$ by the Fourier transform (see Eymard [5, (3.6)]). So, $VN(G)$ is identified with $L^\infty(\Gamma)$. Under these identifications, the module action of $L^1(\Gamma)$ on $L^\infty(\Gamma)$ is given by

$$f \cdot \phi = \hat{f} \ast \phi \quad (f \in L^1(\Gamma) \text{ and } \phi \in L^\infty(\Gamma)),$$
where \( \bar{f}(x) = f(x^{-1}) \ (x \in \Gamma) \) (see Dunkl and Ramirez [4]). This coincides with the module action of the Banach algebra \( L^1(\Gamma) \) (taking the convolution as the multiplication) on \( L^\infty(\Gamma) = L^1(\Gamma)^* \). Also, we have \( b(G) = d(\Gamma) \) (cf. [12, (24.48)]) and hence \( d(G) = b(\Gamma) \) by the Pontryagin duality theorem. In particular, \( G \) is non-discrete if and only if \( \Gamma \) is non-compact. Now, for any open subgroup \( H \) of \( G \), let \( \mathcal{N}_H = \{ \gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in H \} \). Then \( \hat{H} \cong \Gamma/\mathcal{N}_H \) and \( \mathcal{N}_H \) is a compact subgroup of \( \Gamma \). Applying Theorem 5.3, we obtain the following decomposition for the group algebra of any non-compact locally compact abelian group.

**Corollary 5.5.** Let \( G \) be a non-compact locally compact abelian group satisfying \( d(G) < b(G) \). Then there exists a family \( \{\mathcal{N}_i\}_{i \in I} \) of compact subgroups of \( G \) indexed by a directed set \( I \) such that:

1. \( \mathcal{N}_i \supseteq \mathcal{N}_j \neq \{e\} \) for all \( i, j \in I \) with \( i \preceq j \) and \( \bigcap_{i \in I} \mathcal{N}_i = \{e\} \).
2. \( b(G/\mathcal{N}_i) \leq d(G/\mathcal{N}_i) \) for all \( i \in I \) and \( \text{card}(I) \leq b(G)^{d(G)} \).
3. \( L^1(G) = \bigcup_{i \in I} L^1(G/\mathcal{N}_i) \) is a WAP-strong inductive union via the isometric isomorphisms \( \Lambda_i : L^1(G/\mathcal{N}_i) \rightarrow L^1(G) \) given by \( \Lambda_i(f) = f \circ \eta_i \) (\( f \in L^1(G/\mathcal{N}_i) \)), where \( \eta_i \) is the natural homomorphism of \( G \) onto \( G/\mathcal{N}_i \) (\( i \in I \)).

**Remark 5.6.** Under the assumptions of Theorem 5.3, we also have the inductive union \( L^1(G) = \bigcup_{H \in \mathcal{H}} L^1(H) \) of Banach algebras via the isometric isomorphisms \( \{\Omega_H\}_{H \in \mathcal{H}} \), where \( \Omega_H : L^1(H) \rightarrow L^1(G) \) is defined by \( \Omega_H(f) = f^\alpha \) (the trivial extension of \( f \) to \( G \)). However, usually \( L^\infty(G) \) cannot be an inductive union of \( \{L^\infty(H)\}_{H \in \mathcal{H}} \). For example, suppose that \( d(G) = 2^\alpha \) for some \( \alpha \geq b(G) \). Note that \( \text{card}(\mathcal{H}) \leq d(G)^{b(G)} = 2^\alpha \) and \( D(L^1(H)) \leq b(H) = b(G) \) for all \( H \in \mathcal{H} \), where \( D(L^1(H)) \) is the smallest cardinality of a norm dense subset of \( L^1(H) \). It follows that \( \text{card}(\bigcup_{H \in \mathcal{H}} L^\infty(H)) \leq 2^{b(G)} \text{card}(\mathcal{H}) \leq 2^\alpha = d(G) < 2^{d(G)} \leq \text{card}(L^\infty(G)) \), i.e., \( \text{card}(\bigcup_{H \in \mathcal{H}} L^\infty(H)) < \text{card}(L^\infty(G)) \). Therefore, the inductive union \( L^1(G) = \bigcup_{H \in \mathcal{H}} L^1(H) \) is not WAP-strong.

According to Theorem 5.3(2), for each \( H \in \mathcal{H} \), there exists a closed linear subspace \( Z_H \) of \( \text{VN}(H)/\text{WAP}(\hat{H}) \) and a continuous linear map \( \Pi_H : Z_H \rightarrow \text{VN}(H) \) such that \( \Pi_H(Z_H) = \text{VN}(H) \). We will consider whether the family \( \{Z_H, \Pi_H \} : H \in \mathcal{H} \) is compatible with the maps \( \Phi_{H,J} \) and \( \Gamma_{H,J} \) (\( H, J \in \mathcal{H} \) and \( H \subseteq J \)). For this purpose, we will need the following two lemmas.

**Lemma 5.7.** Let \( H \) and \( J \) be open subgroups of \( G \) with \( H \subseteq J \) and let \( \Lambda_{H,J}, \Phi_{H,J}, \) and \( \Gamma_{H,J} \) be the maps defined in Definition 5.1. Let \( \Psi_{H,J} = \Lambda_{H,J}^* \). Then:

(a) \( \Psi_{H,J} : \text{VN}(J) \rightarrow \text{VN}(H) \) is a continuous linear surjection, \( \|\Psi_{H,J}\| = 1 \), and \( \Psi_{H,J} \Phi_{H,J} = \text{Id} \).
(b) \( \Psi_{HJ}(\text{WAP}(\hat{J})) = \text{WAP}(\hat{H}). \)

Define \( \Theta_{HJ} : \text{VN}(J)/\text{WAP}(\hat{J}) \to \text{VN}(H)/\text{WAP}(\hat{H}) \) by \( \Theta_{HJ}(T + \text{WAP}(\hat{J})) = \Psi_{HJ}(T) + \text{WAP}(\hat{H}) \) \((T \in \text{VN}(J))\). Then:

(c) \( \Theta_{HJ} \) is a continuous linear surjection, \( \|\Theta_{HJ}\| = 1 \), and \( \Theta_{HJ}\Gamma_{HJ} = \text{Id} \).

(d) If \( K \) is an open subgroup of \( G \) and \( H \subseteq J \subseteq K \), then \( \Psi_{HK}\Psi_{JK} = \Psi_{HK} \) and \( \Theta_{HK}\Theta_{JK} = \Theta_{HK} \).

Proof. (a) This follows from [5, (3.21)].

(b) Note that \( \Psi_{HJ}\Phi_{HJ} = \text{Id} \) and \( \Phi_{HJ}(\text{WAP}(\hat{H})) \subseteq \text{WAP}(\hat{J}) \) (see [2, Lemma 4.2]). So, \( \text{WAP}(\hat{H}) \subseteq \Psi_{HJ}(\text{WAP}(\hat{J})) \). On the other hand, for \( u \in A(H) \) and \( T \in \text{VN}(J) \), we have \( u \cdot \Psi_{HJ}(T) = \Psi_{HJ}(\Lambda_{HJ}(u) \cdot T) \). Therefore, \( \Psi_{HJ}(\text{WAP}(\hat{J})) \subseteq \text{WAP}(\hat{H}) \) and hence \( \Psi_{HJ}(\text{WAP}(\hat{J})) = \text{WAP}(\hat{H}) \).

(c) By (a) and (b), \( \Theta_{HJ} \) is well-defined, linear, continuous, and onto. And \( \Theta_{HJ}\Gamma_{HJ} = \text{Id} \) since \( \Psi_{HJ}\Phi_{HJ} = \text{Id} \). Note that \( \Theta_{HJ} \) is an isometry. So we have \( \|\Theta_{HJ}\| \geq 1 \). On the other hand, by the definition of \( \Theta_{HJ} \) and by the fact that \( \|\Psi_{HJ}\| = 1 \), we get \( \|\Theta_{HJ}\| \leq 1 \). Therefore, \( \|\Theta_{HJ}\| = 1 \).

(d) Since \( \Lambda_{JK}\Lambda_{HJ} = \Lambda_{HK} \), by taking the adjoint, we have \( \Psi_{HJ}\Psi_{JK} = \Psi_{HK} \) and hence \( \Theta_{HK}\Theta_{JK} = \Theta_{HK} \).

**Remark 5.8.** Comparing to the diagram in Proposition 4.5, we now have the following commutative diagram:

\[
\begin{array}{ccc}
\text{VN}(J) & \xrightarrow{\Psi_{HJ}} & \text{VN}(H) \\
\downarrow{\varrho_J} & & \downarrow{\varrho_H} \\
\text{VN}(J)/\text{WAP}(\hat{J}) & \xrightarrow{\Theta_{HJ}} & \text{VN}(H)/\text{WAP}(\hat{H})
\end{array}
\]

where \( \varrho_H \) and \( \varrho_J \) are the canonical quotient maps.

**Lemma 5.9.** Let \( G, G_0, \) and \( H \) be as in Theorem 5.3. Let \( \mu \) be the initial ordinal with \( |\mu| = b(G_0) = b(G) \) and \( X = \{\alpha : \alpha < \mu\} \). Then there exists a continuous linear surjection \( \omega_H : \text{VN}(H)/\text{WAP}(\hat{H}) \to l^\infty(X) \) for each \( H \in \mathcal{H} \) such that the family \( \{\|\omega_H\| : H \in \mathcal{H}\} \) is bounded by a constant which depends only on \( b(G) \).

Furthermore, if \( H, J \in \mathcal{H} \) and \( H \subseteq J \), then \( \omega_H\Theta_{HJ} = \omega_J \) and we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{VN}(H)/\text{WAP}(\hat{H}) & \xrightarrow{\Gamma_{HJ}} & \text{VN}(J)/\text{WAP}(\hat{J}) \\
\downarrow{\omega_H} & & \downarrow{\omega_J} \\
l^\infty(X)
\end{array}
\]

Proof. Let \( \pi : \text{VN}(G_0) \to l^\infty(X) \) be the map constructed in Hu [9, Theorem 5.1]. According to [9, Theorem 5.1] and its proof, \( \pi \) is a continuous
Thus, \((260 \text{ Z. Hu})\) (Lemma 5.7(c)). It follows that \(mutes\).

\(VN(\quotient{X}{k})\) which depends only on \(\text{card}(X)\). So, there exists a continuous linear surjection \(l : l^\infty(X) / c(X) \rightarrow l^\infty(X)\). Define \(\omega : VN(G_0) / WAP(\widehat{G}_0) \rightarrow l^\infty(X)\) by \(\omega(T + WAP(\widehat{G}_0)) = \tau(\pi(T) + c(X))\) \((T \in VN(G_0))\). Then \(\omega\) is well defined, linear, continuous, onto \(l^\infty(X)\), and \(\|\omega\| \leq \|\tau\|\).

For \(H \in \mathcal{H}\), let \(\omega_H = \omega \Theta_{G_0H}\), where \(\Theta_{G_0H} : VN(H) / WAP(\widehat{H}) \rightarrow VN(G_0) / WAP(\widehat{G}_0)\) is the surjection as defined in Lemma 5.7. Then \(\omega_H\) is continuous, linear, onto \(l^\infty(X)\), and \(\|\omega_H\| = \|\omega \Theta_{G_0H}\| \leq \|\omega\| \leq \|\tau\|\). It turns out that the family \(\{\|\omega_H\| : H \in \mathcal{H}\}\) is bounded by the constant \(\|\tau\|\) which depends only on \(\text{card}(X)\) = \(b(G)\).

Suppose \(H, J \in \mathcal{H}\) and \(H \subseteq J\). Then \(\Theta_{G_0H \Theta_{HJ}} = \Theta_{G_0J}\) (Lemma 5.7(d)). Thus, \(\omega \Theta_{G_0H \Theta_{HJ}} = \omega \Theta_{G_0J}\), i.e., \(\omega_H \Theta_{HJ} = \omega_J\). But \(\Theta_{HJ} \Gamma_{HJ} = \text{Id}\) (Lemma 5.7(c)). It follows that \(\omega_H = \omega_J \Gamma_{HJ}\) and hence the diagram commutes.

Now we have the following inductive compatibility of the pairs \(\{Z_H, \Pi_H\}\) \((H \in \mathcal{H})\) with the maps \(\Phi_{HJ}\) and \(\Gamma_{HJ}\).

**Theorem 5.10.** The following hold under the assumptions of Theorem 5.3:

1. For each \(H \in \mathcal{H}\), there exists a closed linear subspace \(Z_H\) of the quotient \(VN(H) / WAP(\widehat{H})\) and a continuous linear map \(\Pi_H : Z_H \rightarrow VN(H)\) such that \(\Pi_H(Z_H) = VN(H)\).

2. There exists a constant \(M > 0\) (which depends only on \(b(G)\)) such that \(\|\Pi_H\| \leq M\) for all \(H \in \mathcal{H}\).

3. Let \(K \in \mathcal{H}\) and let \(\mathcal{H}_K = \{H \subseteq \mathcal{H} : H \subseteq K\}\). Then, for each \(H \in \mathcal{H}_K\), a pair \(\{Z_H, \Pi_H\}\) as in (1) can be chosen such that the family \(\{\{Z_H, \Pi_H\} : H \in \mathcal{H}_K\}\) is compatible with the maps \(\Phi_{HJ}\) and \(\Gamma_{HJ}\) \((H, J \in \mathcal{H}_K\) and \(H \subseteq J\)\). That is, if \(H, J \in \mathcal{H}_K\) and \(H \subseteq J\), then \(\Gamma_{HJ}(Z_H) \subseteq Z_J\) and the following diagram commutes when \(\Gamma_{HJ}\) is restricted to \(Z_H\):

\[
\begin{array}{ccc}
Z_H & \xrightarrow{\Pi_H} & VN(H) \\
\downarrow{\Gamma_{HJ}} & & \downarrow{\Phi_{HJ}} \\
Z_J & \xrightarrow{\Pi_J} & VN(J)
\end{array}
\]

**Proof.** (1) This follows from Theorem 5.3(2).

(2) Let \(H \in \mathcal{H}\). Then \(d(H) \leq b(H)\) (Theorem 5.3(1)) and hence \(D(A(H)) = b(H) = b(G) = \text{card}(X)\), where \(D(A(H))\) is the smallest cardinality of a norm dense subset of \(A(H)\) and \(X\) is the same set as in Lemma 5.9. Let \(\{u_\alpha : \alpha \in X\}\) be a norm dense subset of the unit ball in \(A(H)\) and let
$t_H : VN(H) \to l^\infty(X)$ be defined by $t_H(T)(\alpha) = \langle T, u_\alpha \rangle$ ($T \in VN(H)$ and $\alpha \in X$). Then $t_H$ is a linear isometry. Let $\omega_H : VN(H)/WAP(\hat{H}) \to l^\infty(X)$ be the surjection as constructed in Lemma 5.9. We take $Z_H = \omega_H^{-1}[t_H(VN(H))] \subseteq VN(H)/WAP(\hat{H})$ and $\Pi_H = t_H^{-1}(\omega_H|_{Z_H})$. Then $Z_H$ is a closed linear subspace of $VN(H)/WAP(\hat{H})$, $\Pi_H : Z_H \to VN(H)$ is a continuous linear map, and $\Pi_H(Z_H) = VN(H)$.

It is clear that $\|\Pi_H\| \leq \|\omega_H\|$. By Lemma 5.9, the family $\{\|\Pi_H\| : H \in \mathcal{H}\}$ is bounded by a constant which depends only on $b(G)$.

(3) Let $K \in \mathcal{H}$. Let $t_K, Z_K$, and $\Pi_K$ be as constructed in (2). Let $H \in \mathcal{H}_K$ and $t'_H = t_K\Phi_{HK}$. Then $t'_H : VN(H) \to l^\infty(X)$ is also a linear isometry since $\Phi_{HK}$ is an isometry. Now we take $Z_H = \omega_H^{-1}[t'_H(VN(H))] \subseteq VN(H)/WAP(\hat{H})$ and $\Pi_H = (t'_H)^{-1}(\omega_H|_{Z_H})$. Then $\Pi_H : Z_H \to VN(H)$ is also a continuous linear surjection and we still have $\|\Pi_H\| \leq \|\omega_H\|$.

Suppose that $H, J \in \mathcal{H}_K$ and $H \subseteq J$. Since $\Phi_{HK} = \Phi_{JK}\Phi_{HJ}$, we have $\Phi_{HK}(VN(H)) = \Phi_{JK}[\Phi_{HJ}(VN(H))] \subseteq \Phi_{JK}(VN(J))$ and hence $t_K\Phi_{HK}(VN(H)) \subseteq t_K\Phi_{JK}(VN(H))$, i.e., $t'_H(VN(H)) \subseteq t'_J(VN(J))$. Note that $\omega_J\Gamma_{HJ} = \omega_H$ (Lemma 5.9). Therefore, we have

$$\Gamma_{H, J}[\omega_H^{-1}(t'_H(VN(H)))] \subseteq \omega_J^{-1}(t'_H(VN(H))) \subseteq \omega_J^{-1}[t'_J(VN(J))]$$

i.e., $\Gamma_{H, J}(Z_H) \subseteq Z_J$. Finally, the construction of $\{Z_H, \Pi_H\}$ and $\{Z_J, \Pi_J\}$ makes the diagram commutative.

**Remark 5.11.** Let $K \in \mathcal{H}$ and $\{Z_K, \Pi_K\}$ be the same as constructed in Theorem 5.10(2). If $H, J \in \mathcal{H}_K$ with $H \subseteq J$ and $\{Z_H, \Pi_H\}, \{Z_J, \Pi_J\}$ are chosen as in the proof of Theorem 5.10(3), then we only have $Z_H \subseteq \Theta_{H, J}(Z_J)$, where $\Theta_{H, J} : VN(J)/WAP(\hat{J}) \to VN(H)/WAP(\hat{H})$ is the surjection as defined in Lemma 5.7. So, generally, we cannot simultaneously have the following commutative diagram when $\Theta_{H, J}$ is restricted to $Z_J$:

\[
\begin{array}{ccc}
Z_H & \xrightarrow{\Pi_H} & VN(H) \\
\downarrow{\Theta_{H, J}} & & \downarrow{\Psi_{H, J}} \\
Z_J & \xrightarrow{\Pi_J} & VN(J)
\end{array}
\]

However, for $H \in \mathcal{H}_K$, if we let $Q_H = \Theta_{HK}(Z_K) \subseteq VN(H)/WAP(\hat{H})$ and let $\Sigma_H : Q_H \to VN(H)$ be defined by

$$\Sigma_H[\Theta_{HK}(T + WAP(\hat{K}))] = \Psi_{HK}\Pi_K(T + WAP(\hat{K})) (T + WAP(\hat{K}) \in Z_K),$$

then it can be seen that $Q_H$ is a linear subspace of $VN(H)/WAP(\hat{H})$, $\Sigma_H : Q_H \to VN(H)$ is well defined, linear, onto $VN(H)$, and $\|\Sigma_H\| \leq \|\tau\|$, where $\tau : l^\infty(X)/c_0(X) \to l^\infty(X)$ is the surjection as appeared in the proof of Lemma 5.9. Let $Y_H$ denote the norm closure of $Q_H$ in $VN(H)/WAP(\hat{H})$ and extend $\Sigma_H$ continuously to $Y_H$. Then $\Sigma_H : Y_H \to VN(H)$ is a continuous
linear surjection. Now, if $H, J \in \mathcal{H}_K$ and $H \subseteq J$, then $\Theta_{H,J}(Q_J) \subseteq Q_H$ and hence $\Theta_{H,J}(Y_J) \subseteq Y_H$. Also, we have $\Sigma_H \Theta_{H,J}[\Theta_{JK}(T + \text{WAP}(\tilde{K}))] = \Psi_{H,J} \Sigma_J[\Theta_{JK}(T + \text{WAP}(\tilde{K}))]$ for $\Theta_{JK}(T + \text{WAP}(\tilde{K})) \in Q_J$ and thus the following diagram commutes when $\Theta_{H,J}$ is restricted to $Y_J$:

\[
\begin{array}{ccc}
Y_H & \xrightarrow{\Sigma_H} & \text{VN}(H) \\
\downarrow{\Theta_{H,J}} & & \uparrow{\Psi_{H,J}} \\
Y_J & \xrightarrow{\Sigma_J} & \text{VN}(J)
\end{array}
\]

But, in this case, we do not have $\Gamma_{H,J}(Y_H) \subseteq Y_J$ and hence we cannot have $\Sigma_J \Gamma_{H,J}|_{Y_H} = \Phi_{H,J} \Sigma_H$, i.e., we do not have the commutative diagram in Theorem 5.10 when $\{Z_H, \Pi_H\}$ and $\{Z_J, \Pi_J\}$ are replaced by $\{Y_H, \Sigma_H\}$ and $\{Y_J, \Sigma_J\}$, respectively.

It is not clear whether in Theorem 5.10 we could choose a family $\{\{Z_H, \Pi_H\} : H \in \mathcal{H}\}$ compatible with all of the maps $\Phi_{H,J}$ and $\Gamma_{H,J}$ ($H, J \in \mathcal{H}$ and $H \subseteq J$). If so, then we would be able to obtain a continuous linear surjection $\Pi : \bigcup_{H \in \mathcal{H}} \Gamma_H(Z_H) \to \text{VN}(G)$ and hence we would be able to conclude that $A(G)$ is extremely non-Arens regular. For this reason, we give the following version of extreme non-Arens regularity.

**Definition 5.12.** Let $A$ be a Banach algebra. $A$ is called *inductively extremely non-Arens regular* if there exists a family $\{A_i\}_{i \in I}$ of Banach algebras such that:

1. For each $i \in I$, $A_i$ is extremely non-Arens regular.
2. $A = \bigsqcup_{i \in I} A_i$ is a WAP-strong inductive union of $\{A_i\}_{i \in I}$ with $A^* = \bigsqcup_{i \in I} A_i^*$ via $\{\Phi_i\}_{i \in I}$ and $A^*/\text{WAP}(A) = \bigsqcup_{i \in I}[A_i^*/\text{WAP}(A_i)]$ via $\{\Gamma_i\}_{i \in I}$.
3. Let $k \in I$ and let $I_k = \{i \in I : i \leq k\}$. Then, for each $i \in I_k$, there exists a closed linear subspace $Z_i$ of $A_i^*/\text{WAP}(A_i)$ and a continuous linear surjection $\Pi_i : Z_i \to A_i^*$ such that $\{\|\Pi_i\| : i \in I_k\}$ is bounded (by a constant independent of $k$) and $\{\{Z_i, \Pi_i\} : i \in I_k\}$ is compatible. That is, if $i, j \in I_k$ and $i \leq j$, then $\Gamma_{ij}(Z_i) \subseteq Z_j$ and $\Phi_{ij}\Pi_i = \Pi_j\Gamma_{ij}|_{Z_i}$, where $\Phi_{ij}$ and $\Gamma_{ij}$ are the same maps as in Corollary 3.5.

Combining Theorem 5.3 and Theorem 5.10 with [10, Corollary 4.2 and Remark 4.7], we are able to deduce the non-Arens regularity of $A(G)$ as follows.

**Corollary 5.13.** Let $G$ be a non-discrete locally compact group. Then:

1. $A(G)$ is extremely non-Arens regular if $b(G) \geq d(G)$.
2. $A(G)$ is inductively extremely non-Arens regular if $b(G) < d(G)$.
As an immediate consequence of Corollary 5.13, we have the following result on the non-Arens regularity of the group algebra $L^1(G)$ of any non-compact locally compact abelian group $G$.

**Corollary 5.14.** Let $G$ be a non-compact locally compact abelian group. Then:

1. $L^1(G)$ is extremely non-Arens regular if $b(G) \leq d(G)$.
2. $L^1(G)$ is inductively extremely non-Arens regular if $b(G) > d(G)$.

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**References**


Department of Mathematics and Statistics
University of Windsor
Windsor, Ontario, Canada N9B 3P4
E-mail: zhiguohu@uwindsor.ca

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