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Inductive extreme non-Arens regularity of the Fourier algebra $A(G)$

by

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Abstract. Let G be a non-discrete locally compact group, $A(G)$ the Fourier algebra of G, $\mathrm{VN}(G)$ the von Neumann algebra generated by the left regular representation of G which is identified with $A(G)^*$, and $\mathrm{WAP}(\widehat{G})$ the space of all weakly almost periodic functionals on $A(G)$. We show that there exists a directed family $\mathcal{H}$ of open subgroups of G such that: (1) for each $H \in \mathcal{H}$, $A(H)$ is extremely non-Arens regular; (2) $\mathrm{VN}(G) = \bigcup_{H\in\mathcal{H}} \mathrm{VN}(H)$ and $\mathrm{VN}(G)/\mathrm{WAP}(\widehat{G}) = \bigcup_{H\in\mathcal{H}}[\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})]$; (3) $A(G) = \bigcup_{H\in\mathcal{H}} A(H)$ and it is a *WAP-strong inductive union* in the sense that the unions in (2) are strongly compatible with it. Furthermore, we prove that the family $\{A(H) : H \in \mathcal{H}\}$ of Fourier algebras has a kind of inductively compatible extreme non-Arens regularity.

1. Introduction. For a Banach algebra A, there exist two Banach algebra multiplications on A^{**} (known as *Arens products*) which extend the multiplication of A (see Arens [1]). When these two multiplications coincide on A^{**}, the algebra A is said to be *Arens regular*. Every C^*-algebra is Arens regular. If A is a commutative Banach algebra, then A is Arens regular if and only if A^{**} is commutative with respect to either (and hence both) of the Arens products. Let $\mathrm{WAP}(A)$ be the space of all weakly almost periodic functionals on A, i.e., $\mathrm{WAP}(A) = \{T \in A^* : \{u \cdot T : u \in A \text{ and } \|u\| \le 1\}$ is relatively weakly compact in $A^*\}$, where $\langle u\cdot T, v\rangle = \langle T, uv\rangle$ for $v \in A$. It is known that A is Arens regular if and only if $\mathrm{WAP}(A) = A^*$ (see Pym [15], and also Duncan and Hosseinium [3]). Hence, the quotient Banach space $A^*/\mathrm{WAP}(A)$ measures the non-Arens regularity of A in some sense. In particular, Granirer introduced the concept of "extreme non-Arens regularity". A is called *extremely non-Arens regular* if $A^*/\mathrm{WAP}(A)$ contains a closed linear subspace which has A^* as a continuous linear image (see [7]).

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Let G be a locally compact group and $A(G)$ the Fourier algebra of G. Lau proved that if G is amenable then $A(G)$ is Arens regular if and only if G is finite (see [13, Proposition 3.3]). Generally, Forrest showed that if $A(G)$ is Arens regular then G must be discrete (he even showed this for the Figà-Talamanca Herz algebra $A_p(G)$; see [6]). It is still open whether Lau's result is true for non-amenable groups G or for algebras $A_p(G)$ with $p \neq 2$. Recently, Granirer investigated the non-Arens regularity of quotients of $A(G)$. A special case of his Corollary 7 in [7] implies that $A(G)$ is extremely non-Arens regular if G is non-discrete and second countable. Let $b(G)$ be the smallest cardinality of an open basis at the unit e of G, and $d(G)$ the smallest cardinality of a covering of G by compact sets. It is proved that Granirer's result holds for all non-discrete locally compact groups G satisfying $b(G) \geq d(G)$ (see Hu [10, Corollary 4.2 and Remark 4.7]). In particular, $A(G)$ is extremely non-Arens regular if G is a σ-compact non-discrete locally compact group.

In this paper we will investigate the non-Arens regularity of $A(G)$ when $b(G) < d(G)$. Let $\mathrm{VN}(G)$ be the von Neumann algebra generated by the left regular representation of G. It is well known that $A(G)$ can be identified with the predual of $\mathrm{VN}(G)$, i.e., $\mathrm{VN}(G) = A(G)^*$. Let $\mathrm{WAP}(\widehat{G})$ denote the space of all weakly almost periodic functionals on $A(G)$ (i.e., $\mathrm{WAP}(\widehat{G}) = \mathrm{WAP}(A(G))$). We show (Theorem 5.3) that, for any non-discrete locally compact group G satisfying $b(G) < d(G)$, there exists a directed family $\mathcal{H}$ of open subgroups of G such that:

(1) For each $H \in \mathcal{H}$, $A(H)$ is extremely non-Arens regular, i.e., for each $H \in \mathcal{H}$, there exists a closed linear subspace Z_H of $\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$ and a continuous linear map $\Pi_H : Z_H \to \mathrm{VN}(H)$ such that $\Pi_H(Z_H) = \mathrm{VN}(H)$.

(2) $\mathrm{VN}(G) = \bigcup_{H\in\mathcal{H}} \mathrm{VN}(H)$ is an *inductive union* of von Neumann algebras and $\mathrm{VN}(G)/\mathrm{WAP}(\widehat{G}) = \bigcup_{H\in\mathcal{H}}[\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})]$ is an inductive union of Banach spaces (see Definition 3.1).

(3) $A(G) = \bigcup_{H\in\mathcal{H}} A(H)$ is an inductive union of Banach algebras and it is a *WAP-strong inductive union* (see Definition 3.3) in the sense that the two inductive unions in (2) are strongly compatible with the inductive union $A(G) = \bigcup_{H\in\mathcal{H}} A(H)$.

In particular, if G is metrizable, then H is a σ-compact open subgroup of G for all $H \in \mathcal{H}$, and $A(G)$ is a WAP-strong inductive union of the separable Fourier algebras $\{A(H)\}_{H\in\mathcal{H}}$. Furthermore, we obtain the inductive extreme non-Arens regularity of $A(G)$ by showing that $\{\|\Pi_H\| : H \in \mathcal{H}\}$ is bounded and the pairs $\{Z_H, \Pi_H\}$ ($H \in \mathcal{H}$) are inductively compatible (Theorem 5.10).

The analysis of the relation between open subgroups of G and the support of operators in $\mathrm{VN}(G)$ plays a key role in our discussion of the inductive extreme non-Arens regularity of $A(G)$. We show that if H is an open subgroup of a non-discrete locally compact group G, then, for any operator T in $\mathrm{VN}(G)$, the support of T can be covered by no more than $b(G)$ cosets of H in G (Proposition 4.1).

Motivated by the inductive limits of C^*-algebras, in Section 3 we introduce the concept of "inductive union", which provides a natural mechanism to relate the Fourier algebra of a locally compact group to the Fourier algebras of its open subgroups.

2. Preliminaries and notations. Let G be a locally compact group with unit e and a fixed left Haar measure. The *Fourier–Stieltjes algebra* $B(G)$ is the linear span of positive-definite continuous functions on G and is identified with the Banach dual of the group C^*-algebra $C^*(G)$ of G. With the dual norm and the pointwise multiplication, $B(G)$ is a commutative Banach algebra. Let $C_{00}(G)$ be the space of all continuous functions on G with compact support. Then the *Fourier algebra* $A(G)$ is the closed ideal in $B(G)$ generated by elements in $B(G) \cap C_{00}(G)$. Let $\mathrm{VN}(G)$ be the von Neumann algebra generated by the left regular representation of G. Then $A(G)$ can be identified with the predual of $\mathrm{VN}(G)$ (i.e., $\mathrm{VN}(G) = A(G)^*$) and $\mathrm{VN}(G)$ becomes a $B(G)$-module under the action $\langle u\cdot T, v\rangle = \langle T, uv\rangle$ for $u \in B(G)$, $v \in A(G)$, and $T \in \mathrm{VN}(G)$. Also, $\mathrm{VN}(G)$ coincides with the space of all bounded linear operators on $L^2(G)$ which satisfy $T(f * g) = T(f) * g$ for all $f \in L^2(G)$ and $g \in C_{00}(G)$. See Eymard [5] for more information on $A(G)$, $B(G)$, and $\mathrm{VN}(G)$.

The space $\{T \in \mathrm{VN}(G) : u \mapsto u \cdot T$ is a weakly compact operator from $A(G)$ into $\mathrm{VN}(G)\}$ is called the space of *weakly almost periodic* functionals on $A(G)$ and is denoted by $\mathrm{WAP}(\widehat{G})$. It turns out that $\mathrm{WAP}(\widehat{G})$ is a self-adjoint closed $B(G)$-submodule of $\mathrm{VN}(G)$. When G is a locally compact abelian group, $\mathrm{WAP}(\widehat{G})$ is identified with the space of weakly almost periodic functions on the dual group of G. See Dunkl and Ramirez [4] for more details on $\mathrm{WAP}(\widehat{G})$.

The support of a function f in $L^2(G)$ is defined by saying that $x \notin \operatorname{supp} f$ if and only if there exists a neighbourhood V of x such that $\int_G f(x)v(x)\,dx = 0$ for all $v \in C_{00}(G)$ with $\operatorname{supp} v \subseteq V$. The support of an operator T in $\mathrm{VN}(G)$ is defined by saying that $x \notin \operatorname{supp} T$ if and only if there exists a neighbourhood U of e such that $x \notin \operatorname{supp}(Tu)$ for all $u \in C_{00}(G)$ with $\operatorname{supp} u \subseteq U$. An equivalent description for $\operatorname{supp} T$ is that $x \in \operatorname{supp} T$ if and only if $u \cdot T = 0$ implies $u(x) = 0$ for all $u \in A(G)$ (see Eymard [5] and Herz [8]).

Let $b(G)$ be the smallest cardinality of an open basis at e and $d(G)$ denote the smallest cardinality of a covering of G by compact sets. It is known that $b(G) = d(\widehat{G})$ when G is abelian with dual group $\widehat{G}$ (see Hewitt and Ross [12, (24.48)]). Clearly, G is metrizable if and only if $b(G) \leq \aleph_0$.

3. Inductive unions. Inspired by the inductive limits of C^*-algebras, we introduce the concept of "inductive union", which is of importance for our investigation on the non-Arens regularity of the Fourier algebra $A(G)$.

Definition 3.1. Let A be a Banach space (Banach algebra, C^*-algebra, respectively) and let $\{A_i\}_{i\in I}$ be a family of Banach spaces (Banach algebras, C^*-algebras, respectively) indexed by a directed set I. We say that A is an *inductive union* of $\{A_i\}_{i\in I}$ (denoted by $A=\bigsqcup_{i\in I} A_i$) if there exists a linear isometry (isometric isomorphism, $*$-isomorphism, respectively) $\Lambda_i : A_i \to A$ for each $i\in I$ such that $\Lambda_i(A_i)\subseteq \Lambda_j(A_j)$ for all $i,j\in I$ with $i\preceq j$ and $A=\bigcup_{i\in I}\Lambda_i(A_i)$.

Immediately, we can show the existence of maps Λ_{ij} $(i\preceq j)$ compatible with $\{\Lambda_i\}_{i\in I}$.

Corollary 3.2. *Let $A=\bigsqcup_{i\in I} A_i$ be an inductive union of the family $\{A_i\}_{i\in I}$ of Banach spaces* (*Banach algebras*, *C^*-algebras*, *respectively*) *via the linear isometries* (*isometric isomorphisms*, *$*$-isomorphisms*, *respectively*) *$\{\Lambda_i\}_{i\in I}$. Then*, *for all i, $j\in I$ with $i\preceq j$*, *there exists a unique linear isometry* (*isometric isomorphism*, *$*$-isomorphism*, *respectively*) *$\Lambda_{ij} : A_i \to A_j$ such that*:

(a) *$\Lambda_j\Lambda_{ij}=\Lambda_i$ for all $i,j\in I$ with $i\preceq j$.*
(b) *$\Lambda_{jk}\Lambda_{ij}=\Lambda_{ik}$ if $i,j,k\in I$ and $i\preceq j\preceq k$.*

Proof. Let $i,j\in I$ and $i\preceq j$. Note that $\Lambda_i(A_i)\subseteq\Lambda_j(A_j)$ and hence $\Lambda_i(A_i)$ is a closed linear subspace (subalgebra, C^*-subalgebra, respectively) of $\Lambda_j(A_j)$. Define $\Lambda_{ij}=(\Lambda_j)^{-1}|_{\Lambda_i(A_i)}\Lambda_i$. Then $\Lambda_{ij} : A_i\to A_j$ is a linear isometry (isometric isomorphism, $*$-isomorphism, respectively). By the definition of Λ_{ij}, it can be seen that (a) holds and the map Λ_{ij} satisfying (a) is unique.

Suppose that $i,j,k\in I$ and $i\preceq j\preceq k$. By (a), we have $\Lambda_k(\Lambda_{jk}\Lambda_{ij}) = \Lambda_j\Lambda_{ij}=\Lambda_i=\Lambda_k\Lambda_{ik}$, i.e., $\Lambda_{jk}\Lambda_{ij}=\Lambda_{ik}$ since Λ_k is one-to-one. Therefore, (b) is true. ∎

When A is an inductive union of $\{A_i\}_{i\in I}$, it is interesting to know if A^* is an inductive union of $\{A_i^*\}_{i\in I}$ and if a quotient space of A^* is an inductive union of the corresponding quotient spaces of A_i^* $(i\in I)$, etc. For our purpose, we only consider the following "WAP" strongly compatible inductive unions of Banach algebras. Recall that, for a Banach algebra A, $\mathrm{WAP}(A)$ denotes the space of all weakly almost periodic functionals on A.

Definition 3.3. Let A be a Banach algebra and let $A=\bigsqcup_{i\in I} A_i$ be an inductive union of the Banach algebras $\{A_i\}_{i\in I}$ via the isometric isomorphisms $\{\Lambda_i\}_{i\in I}$. We say that A is a *WAP-strong inductive union* of $\{A_i\}_{i\in I}$ if the following hold.

(1) $A^* = \bigsqcup_{i\in I} A_i^*$ is an inductive union of the Banach spaces $\{A_i^*\}_{i\in I}$ via some linear isometries $\{\Phi_i\}_{i\in I}$ such that, for all $i \in I$, $\Lambda_i^*\Phi_i = \mathrm{Id}$ and $\Phi_i(u\cdot T) = \Lambda_i(u)\cdot\Phi_i(T)$ for $u \in A_i$ and $T \in A_i^*$.

(2) For all $i \in I$, $\Phi_i(\mathrm{WAP}(A_i)) = \mathrm{WAP}(A) \cap \Phi_i(A_i^*)$ and Φ_i lifts a linear isometry $\Gamma_i : A_i^*/\mathrm{WAP}(A_i) \to A^*/\mathrm{WAP}(A)$.

It is easy to see that (1) and (2) in Definition 3.3 are equivalent to the following two conditions.

Corollary 3.4. *Let $A = \bigsqcup_{i\in I} A_i$ be an inductive union of the Banach algebras $\{A_i\}_{i\in I}$ via $\{\Lambda_i\}_{i\in I}$. Then A is a WAP-strong inductive union of $\{A_i\}_{i\in I}$ if and only if the following conditions are satisfied*:

(1)′ *$A^* = \bigsqcup_{i\in I} A_i^*$ is an inductive union of $\{A_i^*\}_{i\in I}$ via $\{\Phi_i\}_{i\in I}$ such that, for all $i \in I$, $\Phi_i\Lambda_i^* : A^* \to A^*$ is a $\Lambda_i(A_i)$-invariant projection (i.e., $(\Phi_i\Lambda_i^*)^2 = \Phi_i\Lambda_i^*$ and $\Phi_i\Lambda_i^*(v\cdot T) = v\cdot[\Phi_i\Lambda_i^*(T)]$ for all $v \in \Lambda_i(A_i)$ and $T \in A^*$).*

(2)′ *$\mathrm{WAP}(A) = \bigsqcup_{i\in I}\mathrm{WAP}(A_i)$ is an inductive union of the Banach spaces $\{\mathrm{WAP}(A_i)\}_{i\in I}$ via the restrictions $\{\Phi_i|_{\mathrm{WAP}(A_i)}\}_{i\in I}$ and $A^*/\mathrm{WAP}(A) = \bigsqcup_{i\in I}[A_i^*/\mathrm{WAP}(A_i)]$ is an inductive union of the quotient Banach spaces $\{A_i^*/\mathrm{WAP}(A_i)\}_{i\in I}$ via $\{\Gamma_i\}_{i\in I}$ such that $\Gamma_i\,\varrho_i = \varrho\Phi_i$ for all $i \in I$, where $\varrho_i : A_i^* \to A_i^*/\mathrm{WAP}(A_i)$ and $\varrho : A^* \to A^*/\mathrm{WAP}(A)$ are the canonical quotient maps.*

Analogously to Corollary 3.2, we are able to get maps Φ_{ij} and Γ_{ij} $(i \preceq j)$ which are compatible with $\{\Phi_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$, respectively.

Corollary 3.5. *Let $A = \bigsqcup_{i\in I} A_i$ be a WAP-strong inductive union of the Banach algebras $\{A_i\}_{i\in I}$ via the maps $\{\Lambda_i\}_{i\in I}$ with $A^* = \bigsqcup_{i\in I} A_i^*$ via $\{\Phi_i\}_{i\in I}$ and $A^*/\mathrm{WAP}(A) = \bigsqcup_{i\in I}[A_i^*/\mathrm{WAP}(A_i)]$ via $\{\Gamma_i\}_{i\in I}$. Then, for all $i, j \in I$ with $i \preceq j$, there exist unique linear isometries $\Phi_{ij} : A_i^* \to A_j^*$ and $\Gamma_{ij} : A_i^*/\mathrm{WAP}(A_i) \to A_j^*/\mathrm{WAP}(A_j)$ such that the following hold*:

(a) *$\Phi_j\Phi_{ij} = \Phi_i$ and $\Gamma_j\Gamma_{ij} = \Gamma_i$ for all $i, j \in I$ with $i \preceq j$.*

(b) *$\Phi_{jk}\Phi_{ij} = \Phi_{ik}$ and $\Gamma_{jk}\Gamma_{ij} = \Gamma_{ik}$ if $i, j, k \in I$ and $i \preceq j \preceq k$.*

(c) *$\Lambda_{ij}^*\Phi_{ij} = \mathrm{Id}$ and $\Phi_{ij}(u\cdot T) = \Lambda_{ij}(u)\cdot\Phi_{ij}(T)$ for all $i, j \in I$ with $i \preceq j$, $u \in A_i$ and $T \in A_i^*$, where $\Lambda_{ij} : A_i \to A_j$ is the same map as in Corollary* 3.2.

(d) *$\Phi_{ij}(\mathrm{WAP}(A_i)) = \mathrm{WAP}(A_j) \cap \Phi_{ij}(A_i^*)$ and $\Gamma_{ij}\varrho_i = \varrho_j\Phi_{ij}$ if $i, j \in I$ and $i \preceq j$ (i.e., Γ_{ij} is the map lifted by Φ_{ij}).*

Proof. It can be seen that (a) and (b) hold by the same argument as in the proof of Corollary 3.2. Clearly, the maps Φ_{ij} and Γ_{ij} satisfying (a) are unique.

Let $i, j \in I$ and $i \preceq j$. Note that $\Lambda_i^* \Phi_i = \mathrm{Id}$, $\Lambda_i^* = \Lambda_{ij}^* \Lambda_j^*$ (by Corollary 3.2), and $\Phi_i = \Phi_j \Phi_{ij}$. Therefore, $\Lambda_{ij}^* \Phi_{ij} = \Lambda_{ij}^* (\Lambda_j^* \Phi_j) \Phi_{ij} = (\Lambda_{ij}^* \Lambda_j^*)(\Phi_j \Phi_{ij}) = \Lambda_i^* \Phi_i = \mathrm{Id}$, i.e., $\Lambda_{ij}^* \Phi_{ij} = \mathrm{Id}$. Suppose that $u \in A_i$ and $T \in A_i^*$. Then

$$\begin{aligned}\Phi_j[\Phi_{ij}(u \cdot T)] &= \Phi_i(u \cdot T) = \Lambda_i(u) \cdot \Phi_i(T) \\ &= \Lambda_j[\Lambda_{ij}(u)] \cdot \Phi_j[\Phi_{ij}(T)] = \Phi_j[\Lambda_{ij}(u) \cdot \Phi_{ij}(T)].\end{aligned}$$

We conclude that $\Phi_{ij}(u \cdot T) = \Lambda_{ij}(u) \cdot \Phi_{ij}(T)$ since the map Φ_j is one-to-one. Therefore, (c) is true.

Note that $\Phi_i(\mathrm{WAP}(A_i)) \subseteq \Phi_j(\mathrm{WAP}(A_j)) \subseteq \mathrm{WAP}(A)$ and hence we have $\Phi_i(\mathrm{WAP}(A_i)) = \Phi_j(\mathrm{WAP}(A_j)) \cap \Phi_i(A_i^*)$, that is, $\Phi_j[\Phi_{ij}(\mathrm{WAP}(A_i))] = \Phi_j[\mathrm{WAP}(A_j) \cap \Phi_{ij}(A_i^*)]$. Therefore, $\Phi_{ij}(\mathrm{WAP}(A_i)) = \mathrm{WAP}(A_j) \cap \Phi_{ij}(A_i^*)$. Finally, by using the facts that $\Gamma_j \Gamma_{ij} = \Gamma_i$, $\Gamma_i\, \varrho_i = \varrho\, \Phi_i$, and $\Phi_i = \Phi_j \Phi_{ij}$, we have $\Gamma_j(\Gamma_{ij}\, \varrho_i) = \Gamma_i\, \varrho_i = \varrho\, \Phi_i = \varrho\, \Phi_j \Phi_{ij} = \Gamma_j(\varrho_j\, \Phi_{ij})$. It follows that $\Gamma_{ij}\, \varrho_i = \varrho_j\, \Phi_{ij}$ since Γ_j is one-to-one. Therefore, (d) holds. ■

4. Open subgroups, support of T in $\mathrm{VN}(G)$, and isometric embeddings. In this section, G is a locally compact group and H is an open subgroup of G. Let $\mathrm{VN}_H(G)$ denote the von Neumann subalgebra of $\mathrm{VN}(G)$ generated by $\{\lambda_G(x) : x \in H\}$, where λ_G is the left regular representation of G. Then $\mathrm{VN}_H(G) = \{T \in \mathrm{VN}(G) : \operatorname{supp} T \subseteq H\}$ (see Chou [2, Lemma 4.2]). Let $1_H \in B(G)$ be the characteristic function of H. Then $1_H \cdot T \in \mathrm{VN}_H(G)$ for all $T \in \mathrm{VN}(G)$ and $T = 1_H \cdot T$ if $T \in \mathrm{VN}_H(G)$. Therefore, $\mathrm{VN}_H(G) = 1_H \cdot \mathrm{VN}(G)$.

It is known that if an element T of $\mathrm{VN}(G)$ is the left convolution operator by a bounded complex-valued regular Borel measure μ on G, then the support of T is just the support of the measure μ and hence it is a countable union of compact sets in G by the regularity of μ.

Generally, for an arbitrary operator T in $\mathrm{VN}(G)$, we are concerned with the question of how many cosets gH we will need at least to cover the support of T. If G is discrete, then every element T of $\mathrm{VN}(G)$ is identified with a left convolution operator by a function in $l^2(G)$ and so the support of T is a countable subset of G. In the following, we will consider the case when G is non-discrete.

PROPOSITION 4.1. *Let G be a non-discrete locally compact group and let H be an open subgroup of G. Then, for any $T \in \mathrm{VN}(G)$, there are at most $b(G)$ cosets gH $(g \in G)$ such that* $\operatorname{supp} T \cap gH \neq \emptyset$.

Proof. Replacing H by a σ-compact open subgroup of H, we may assume that H is a σ-compact open subgroup of G.

Let $\mathcal{U}$ be a compact neighbourhood system at e such that $\mathrm{card}(\mathcal{U}) = b(G)$. Then $\mathcal{U}$ is a directed set under the relation $U \preceq V$ if and only if

$V \subseteq U$. For each $U \in \mathcal{U}$, let $h_U = (1/|U|)1_U$ and $T_U = T(h_U) \in L^2(G)$, where $|U|$ is the left Haar measure of U and 1_U denotes the characteristic function of U. By [12, (20.15)], for all $f \in L^2(G)$, $\lim_U \|h_U * f - f\|_2 = 0$. If $f \in C_{00}(G)$, then $T(h_U * f) = T(h_U) * f = T_U * f$ and hence $T(f) = \lim_U (T_U * f)$ in the $\|\cdot\|_2$-norm. Therefore, T is completely determined by the net $(T_U)_{U\in\mathcal{U}}$ in $L^2(G)$ since $C_{00}(G)$ is $\|\cdot\|_2$-norm dense in $L^2(G)$. For each $U \in \mathcal{U}$, since $T_U \in L^2(G)$, there exists a sequence $\{g_U^n\}_n$ in G such that $\operatorname{supp} T_U \subseteq \bigcup_{n=1}^{\infty} g_U^n H$.

Fix a compact neighbourhood V of e. Since H is σ-compact, HV and hence $\bigcup_{n=1}^{\infty} g_U^n HV$ is a countable union of compact sets. Therefore, $\bigcup_{n=1}^{\infty} g_U^n HV$ can be covered by countably many cosets gH. Note that $\operatorname{card}(\mathcal{U}) = b(G) \geq \aleph_0$. It follows that there exists a subset B of G such that $\operatorname{card}(B) \leq b(G) = \operatorname{card}(\mathcal{U})$ and $\bigcup_{U\in\mathcal{U}} \bigcup_{n=1}^{\infty} g_U^n HV \subseteq \bigcup_{g\in B} gH$.

To complete the proof, we only need to show that $\operatorname{supp} T \subseteq \bigcup_{g\in B} gH$.

Suppose $x \in G \setminus \bigcup_{g\in B} gH$. In the following, we will prove that $x \notin \operatorname{supp} T(f)$ for all $f \in C_{00}(G)$ with $\operatorname{supp} f \subseteq V$ and it follows that $x \notin \operatorname{supp} T$.

Let $f \in C_{00}(G)$ and $\operatorname{supp} f \subseteq V$. Then $T(f) = \lim_{U\in\mathcal{U}} (T_U * f)$ in the $\|\cdot\|_2$-norm. Recall that, for each $U \in \mathcal{U}$, $\operatorname{supp} T_U \subseteq \bigcup_{n=1}^{\infty} g_U^n H$ and hence $\operatorname{supp}(T_U * f) \subseteq \bigcup_{n=1}^{\infty} g_U^n HV \subseteq \bigcup_{g\in B} gH$. Also note that $\bigcup_{g\in B} gH$ is closed in G. Therefore, $\operatorname{supp} T(f) \subseteq \bigcup_{g\in B} gH$ and we have $x \notin \operatorname{supp} T(f)$. ∎

COROLLARY 4.2. *Let G be a metrizable locally compact group and let H be an open subgroup of G. Then, for any $T \in \mathrm{VN}(G)$, there exists a sequence $\{g_n\}_n$ in G such that* $\operatorname{supp} T \subseteq \bigcup_{n=1}^{\infty} g_n H$.

REMARK 4.3. Let G be a locally compact group and let H be an open subgroup of G. If $T \in \overline{\operatorname{span}}\,[\lambda_G(G)\mathrm{VN}_H(G)]$ (the norm closed linear span generated by the translates of elements in $\mathrm{VN}_H(G)$), then the support of T can be covered by countably many cosets gH. However, it is possible that the support of any operator in $\mathrm{VN}(G)$ can be covered by countably many cosets gH (e.g., when G is metrizable or σ-compact) but $\mathrm{VN}(G) \neq \overline{\operatorname{span}}\,[\lambda_G(G)\mathrm{VN}_H(G)]$. For example, let G be a non-compact metrizable locally compact group containing a compact open subgroup H. Then $\mathrm{VN}(H) = UC(\widehat{H})$ (the C^*-algebra of uniformly continuous functionals on $A(H)$ introduced by Granirer) and thus $\overline{\operatorname{span}}\,[\lambda_G(G)\mathrm{VN}_H(G)] = \mathrm{UC}(\widehat{G})$ (see Hu [11, Proposition 3.5]). Now $\overline{\operatorname{span}}\,[\lambda_G(G)\mathrm{VN}_H(G)] = \mathrm{UC}(\widehat{G}) \subsetneqq \mathrm{VN}(G)$ because G is non-compact.

COROLLARY 4.4. *Let G be a metrizable locally compact group. Then, for any $T \in \mathrm{VN}(G)$, there exists a σ-compact open subgroup H of G such that* $\operatorname{supp} T \subseteq H$.

Proof. Let G_0 be a σ-compact open subgroup of G. Let $T \in \mathrm{VN}(G)$. By Corollary 4.2, there exists a sequence $\{g_n\}_n$ in G such that $\operatorname{supp} T \subseteq$

$\bigcup_{n=1}^{\infty} g_n G_0$. Let H be the open subgroup of G generated by $G_0 \cup \bigcup_{n=1}^{\infty} g_n G_0$. Then H is a σ-compact open subgroup of G and $\operatorname{supp} T \subseteq H$. ∎

Let $r : A(G) \to A(H)$ be the restriction map. According to Eymard [5], r is a linear contractive surjection and its adjoint r^* is a $*$-isomorphism of the von Neumann algebra $\mathrm{VN}(H)$ onto the von Neumann subalgebra $\mathrm{VN}_H(G)$ of $\mathrm{VN}(G)$ (see [5, (3.21)], where $r^*(T)$ is denoted as T° for $T \in \mathrm{VN}(H)$). It is known that $r^*(\mathrm{WAP}(\widehat{H})) = \mathrm{WAP}(\widehat{G}) \cap \mathrm{VN}_H(G)$ (see Chou [2, Lemma 4.2]). Therefore, the $*$-isomorphism r^* lifts a linear map from the quotient Banach space $\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$ into the quotient Banach space $\mathrm{VN}(G)/\mathrm{WAP}(\widehat{G})$. Let $\mathrm{VN}_H(G)/\mathrm{WAP}(\widehat{G})$ denote the linear subspace $\{T + \mathrm{WAP}(\widehat{G}) : T \in \mathrm{VN}_H(G)\}$ of $\mathrm{VN}(G)/\mathrm{WAP}(\widehat{G})$. In the following we will show that in fact r^* lifts a linear isometry between $\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$ and $\mathrm{VN}_H(G)/\mathrm{WAP}(\widehat{G})$.

Proposition 4.5. *For $T \in \mathrm{VN}(H)$, define $\widetilde{r}^*(T + \mathrm{WAP}(\widehat{H})) = r^*(T) + \mathrm{WAP}(\widehat{G})$. Then $\widetilde{r}^* : \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) \to \mathrm{VN}(G)/\mathrm{WAP}(\widehat{G})$ is a linear isometry with range $\mathrm{VN}_H(G)/\mathrm{WAP}(\widehat{G})$ and the following diagram commutes*:

$$\begin{array}{ccc} \mathrm{VN}(H) & \xrightarrow{\quad r^* \quad} & \mathrm{VN}(G) \\ \Big\downarrow{\scriptstyle \varrho_H} & & \Big\downarrow{\scriptstyle \varrho} \\ \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) & \xrightarrow{\widetilde{r}^*} & \mathrm{VN}(G)/\mathrm{WAP}(\widehat{G}) \end{array}$$

where ϱ_H and ϱ are the canonical quotient maps.

Proof. Since $r^*(\mathrm{VN}(H)) = \mathrm{VN}_H(G)$ and $r^*(\mathrm{WAP}(\widehat{H})) = \mathrm{WAP}(\widehat{G}) \cap \mathrm{VN}_H(G)$, by the definition, $\widetilde{r}^* : \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) \to \mathrm{VN}(G)/\mathrm{WAP}(\widehat{G})$ is well defined, linear, and onto the linear subspace $\mathrm{VN}_H(G)/\mathrm{WAP}(\widehat{G})$ of $\mathrm{VN}(G)/\mathrm{WAP}(\widehat{G})$. According to the definition of $\widetilde{r}^*$, it is clear that the diagram is commutative. To complete the proof, we only need to show that $\widetilde{r}^*$ is an isometry.

Let $T \in \mathrm{VN}(H)$. Obviously, $\|\widetilde{r}^*(T + \mathrm{WAP}(\widehat{H}))\| \le \|T + \mathrm{WAP}(\widehat{H})\|$ since $\|\widetilde{r}^*\| \le \|r^*\| = 1$. Conversely, let $W \in \mathrm{WAP}(\widehat{G})$. Then $W = W_1 + W_2$, where $W_1 = 1_H \cdot W$ and hence $W_1 \in \mathrm{WAP}(\widehat{G}) \cap \mathrm{VN}_H(G)$, and $W_2 = W - W_1 \in \mathrm{WAP}(\widehat{G})$ with $\operatorname{supp} W_2 \subseteq G \backslash H$. Thus, $W_1 = r^*(V_1)$ for some $V_1 \in \mathrm{WAP}(\widehat{H})$. So,

$$\begin{aligned} \|r^*(T) + W\| &= \|r^*(T) + r^*(V_1) + W_2\| \\ &\ge \|1_H \cdot (r^*(T) + r^*(V_1) + W_2)\| \\ &= \|r^*(T) + r^*(V_1)\| \quad \text{(since } 1_H \cdot W_2 = 0) \\ &= \|T + V_1\| \\ &\ge \|T + \mathrm{WAP}(\widehat{H})\|. \end{aligned}$$

Since $W \in \mathrm{WAP}(\widehat{G})$ is arbitrary, it follows that

$$\|r^*(T) + \mathrm{WAP}(\widehat{G})\| \geq \|T + \mathrm{WAP}(\widehat{H})\|,$$

i.e., $\|\widetilde{r}^*(T + \mathrm{WAP}(\widehat{H}))\| \geq \|T + \mathrm{WAP}(\widehat{H})\|$. Therefore, $\widetilde{r}^*$ is a linear isometry. ∎

REMARK 4.6. Let V be any closed $B(G)$-submodule of $\mathrm{VN}(G)$ and let $V_H = (r^*)^{-1}[V \cap \mathrm{VN}_H(G)]$. Then V_H is a closed $B(H)$-submodule of $\mathrm{VN}(H)$ and $r^*(V_H) = V \cap \mathrm{VN}_H(G)$. From the proof it can be seen that Proposition 4.5 holds if $\mathrm{WAP}(\widehat{G})$ and $\mathrm{WAP}(\widehat{H})$ are replaced by V and V_H, respectively. In particular, if we take $V = \mathrm{AP}(\widehat{G})$, $\mathrm{UC}(\widehat{G})$, $C^*_{\mathrm{r}}(G)$, and $C^*_\delta(G)$ (the space of almost periodic functionals on $A(G)$, the space of uniformly continuous functionals on $A(G)$, the reduced group C^*-algebra of G, and the C^*-algebra generated by $\{\lambda_G(x) : x \in G\}$, respectively), then we will get $V_H = \mathrm{AP}(\widehat{H})$, $\mathrm{UC}(\widehat{H})$, $C^*_{\mathrm{r}}(H)$, and $C^*_\delta(H)$, respectively (cf. [11]).

5. Inductive extreme non-Arens regularity of $A(G)$. Throughout this section, we assume that G is a non-discrete locally compact group and G_0 is a σ-compact open subgroup of G.

Let $T \in \mathrm{VN}(G)$. By Proposition 4.1, there exists a subset B of G such that $\mathrm{card}(B) \leq b(G)$ and $\operatorname{supp} T \cap gG_0 = \emptyset$ for all $g \in G \setminus B$. Hence, $\operatorname{supp} T \subseteq \bigcup_{g\in B} gG_0$. Let H_B be the open subgroup of G generated by $G_0 \cup \bigcup_{g\in B} gG_0$, i.e.,

$$H_B = \bigcup_{n=1}^{\infty} \Big\{ \Big[G_0 \cup \bigcup_{g\in B} gG_0\Big] \cup \Big[G_0 \cup \bigcup_{g\in B} gG_0\Big]^{-1} \Big\}^n.$$

Then we have $T \in VN_{H_B}(G)$ and H_B can be covered by no more than $b(G)$ compact sets (since G_0 is σ-compact and $b(G) \geq \aleph_0$). Therefore, $d(H_B) \leq b(H_B)$ $(= b(G))$. According to the result of Hu [10, Corollary 4.2 and Remark 4.7], $A(H_B)$ is extremely non-Arens regular.

To obtain the inductive extreme non-Arens regularity of $A(G)$, we need to consider the following maps.

DEFINITION 5.1. Let H and J be open subgroups of G and $H \subseteq J$. The maps $\Lambda_{HJ} : A(H) \to A(J)$, $\Phi_{HJ} : \mathrm{VN}(H) \to VN(J)$, and $\Gamma_{HJ} : \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) \to \mathrm{VN}(J)/\mathrm{WAP}(\widehat{J})$ are defined as follows: for $u \in A(H)$ and $T \in \mathrm{VN}(H)$,

$$\begin{aligned}
\Lambda_{HJ}(u) &= u^\circ,\\
\Phi_{HJ}(T) &= r^*_{HJ}(T),\\
\Gamma_{HJ}(T + \mathrm{WAP}(\widehat{H})) &= \widetilde{r}^*_{HJ}(T)\\
&= r^*_{HJ}(T) + \mathrm{WAP}(\widehat{J}) \quad \text{(as in Proposition 4.5)},
\end{aligned}$$

where u° denotes the trivial extension of u to J (i.e., $u^\circ(x) = 0$ if $x \in J\setminus H$), and r^*_{HJ} is the adjoint of the restriction map $r_{HJ} : A(J) \to A(H)$. Also, we define $\Lambda_H = \Lambda_{HG}$, $\Phi_H = \Phi_{HG}$, and $\Gamma_H = \Gamma_{HG}$.

LEMMA 5.2. *Let H and J be open subgroups of G such that $H \subseteq J$. Let Λ_{HJ}, Φ_{HJ}, Γ_{HJ}, Λ_H, Φ_H, and Γ_H be the maps from Definition* 5.1.

(a) *Λ_{HJ} is an isometric isomorphism from the Banach algebra $A(H)$ onto the Banach subalgebra $A_H(J)$ of $A(J)$, where $A_H(J) = \{f \in A(J) : \operatorname{supp} f \subseteq H\}$.*

(b) *Φ_{HJ} is a $*$-isomorphism (and hence an isometry) from the von Neumann algebra $\mathrm{VN}(H)$ onto the von Neumann subalgebra $\mathrm{VN}_H(J)$ of $\mathrm{VN}(J)$.*

(c) *Γ_{HJ} is a linear isometry with range $\mathrm{VN}_H(J)/\mathrm{WAP}(\widehat{J})$.*

(d) *If K is an open subgroup of G and $H \subseteq J \subseteq K$, then $\Lambda_{JK}\Lambda_{HJ} = \Lambda_{HK}$, $\Phi_{JK}\Phi_{HJ} = \Phi_{HK}$, and $\Gamma_{JK}\Gamma_{HJ} = \Gamma_{HK}$. In particular, the maps Λ_H, Φ_H, and Γ_H are compatible with Λ_{HJ}, Φ_{HJ}, and Γ_{HJ}, respectively. That is, $\Lambda_J\Lambda_{HJ} = \Lambda_H$, $\Phi_J\Phi_{HJ} = \Phi_H$, and $\Gamma_J\Gamma_{HJ} = \Gamma_H$ for all $H \subseteq J$.*

Proof. (a) and (b) follow from [5, (3.21)]. (c) holds by Proposition 4.5. And it is easy to check (d) by Definition 5.1. ■

Summarizing the above discussion, we are ready to give the following decompositions for the Fourier algebra $A(G)$, the von Neumann algebra $\mathrm{VN}(G)$, and the quotient Banach space $\mathrm{VN}(G)/\mathrm{WAP}(\widehat{G})$.

THEOREM 5.3. *Let G be a non-discrete locally compact group with $b(G) < d(G)$ and let G_0 be a σ-compact open subgroup of G. Let $\mathcal{B} = \{B : B \subseteq G$ and $\operatorname{card}(B) \le b(G)\}$ and let $\mathcal{H}$ be the family of open subgroups of G generated by $G_0 \cup \bigcup_{g\in B} gG_0$ $(B \in \mathcal{B})$. Then:*

(1) *$\mathcal{H}$ is a directed set under the relation "$\subseteq$", $d(H) \le b(H)$ for all $H \in \mathcal{H}$, $G = \bigcup_{H\in\mathcal{H}} H$, and $\operatorname{card}(\mathcal{H}) \le d(G)^{b(G)}$.*

(2) *For all $H \in \mathcal{H}$, $A(H)$ is extremely non-Arens regular.*

(3) *$A(G) = \bigsqcup_{H\in\mathcal{H}} A(H)$ is an inductive union of the Banach algebras $\{A(H)\}_{H\in\mathcal{H}}$ via the isometric isomorphisms $\{\Lambda_H\}_{H\in\mathcal{H}}$.*

(4) *$\mathrm{VN}(G) = \bigsqcup_{H\in\mathcal{H}} \mathrm{VN}(H)$ is an inductive union of the von Neumann algebras $\{\mathrm{VN}(H)\}_{H\in\mathcal{H}}$ via the $*$-isomorphisms $\{\Phi_H\}_{H\in\mathcal{H}}$.*

(5) *$\mathrm{VN}(G)/\mathrm{WAP}(\widehat{G}) = \bigsqcup_{H\in\mathcal{H}}[\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})]$ is an inductive union of the quotient Banach spaces $\{\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})\}_{H\in\mathcal{H}}$ via the linear isometries $\{\Gamma_H\}_{H\in\mathcal{H}}$.*

(6) *$A(G) = \bigsqcup_{H\in\mathcal{H}} A(H)$ is a WAP-strong inductive union of the algebras $\{A(H)\}_{H\in\mathcal{H}}$.*

(7) *Λ_{HJ}, Φ_{HJ}, and Γ_{HJ} $(H, J \in \mathcal{H}$ and $H \subseteq J)$ are the maps compatible with $\{\Lambda_H\}_{H\in\mathcal{H}}$, $\{\Phi_H\}_{H\in\mathcal{H}}$, and $\{\Gamma_H\}_{H\in\mathcal{H}}$ as in Corollary 3.2 and Corollary 3.5, respectively.*

In particular, if G is metrizable, then H is a σ-compact open subgroup of G for all $H \in \mathcal{H}$ and $A(G)$ is a WAP-strong inductive union of the separable Fourier algebras $\{A(H)\}_{H\in\mathcal{H}}$.

Proof. Clearly, $\mathcal{H}$ is a directed set under "$\subseteq$", $d(H) \leq b(H)$ for all $H \in \mathcal{H}$ (see the second paragraph in this section), and $G = \bigcup_{H\in\mathcal{H}} H$. Let S be a complete set of left coset representatives of G_0 in G and let $\mathcal{E} = \{B \subseteq S : \mathrm{card}(B) \leq b(G)\}$. It can be seen that $\mathrm{card}(S) = d(G)$ and hence $\mathrm{card}(\mathcal{H}) \leq \mathrm{card}(\mathcal{E}) \leq d(G)^{b(G)}$. Therefore, (1) holds.

(2) and (4) are true according to the discussion in the second paragraph of this section, Lemma 5.2(b), and Definition 3.1.

Note that $A(G) \cap C_{00}(G)$ is norm dense in $A(G)$. So, if $f \in A(G)$, then $\mathrm{supp}\, f$ can be covered by countably many cosets gG_0 ($g \in G$). Hence, $\mathrm{supp}\, T \subseteq H$ for some $H \in \mathcal{H}$. Therefore, $f \in A_H(G) = \Lambda_H(A(H))$ for some $H \in \mathcal{H}$. By Lemma 5.2(a) and Definition 3.1, (3) holds.

(5) follows from (4) and Lemma 5.2(c).

Let $H \in \mathcal{H}$ and let $r_H : A(G) \to A(H)$ be the restriction map. Then $r_H\Lambda_H = \mathrm{Id}$ and $\Phi_H = r_H^*$. Thus, $\Lambda_H^*\Phi_H = \Lambda_H^* r_H^* = \mathrm{Id}$. It is easy to see that $\Phi_H(u\cdot T) = \Lambda_H(u)\cdot\Phi_H(T)$ for all $u \in A(H)$ and $T \in \mathrm{VN}(H)$ by the fact that $r_H\Lambda_H = \mathrm{Id}$ and $\Phi_H = r_H^*$. Clearly, $\Phi_H(\mathrm{WAP}(\widehat{H})) = \mathrm{WAP}(\widehat{G}) \cap \Phi_H(\mathrm{VN}(H))$ and $\Gamma_H : \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) \to \mathrm{VN}(G)/\mathrm{WAP}(\widehat{G})$ is the linear isometry lifted by $\Phi_H : \mathrm{VN}(H) \to \mathrm{VN}(G)$. Therefore, $A(G) = \bigsqcup_{H\in\mathcal{H}} A(H)$ is a WAP-strong inductive union of $\{A(H)\}_{H\in\mathcal{H}}$ by (4), (5), and Definition 3.3, i.e., (6) is true.

(7) holds by Lemma 5.2(d) and the uniqueness of the maps Λ_{HJ}, Φ_{HJ}, and Γ_{HJ} satisfying Corollary 3.2(a) and Corollary 3.5(a), respectively.

Finally, suppose that G is metrizable. Let $H \in \mathcal{H}$. Then $d(H) \leq b(H) = \aleph_0$ by (2). Therefore, H is σ-compact and metrizable and hence $A(H)$ is separable. ■

REMARK 5.4. Let V be any closed $B(G)$-submodule of $\mathrm{VN}(G)$ and let $V_H = \Phi_H^{-1}[V \cap \mathrm{VN}_H(G)]$. By Remark 4.6, the spaces $\mathrm{WAP}(\widehat{G})$ and $\{\mathrm{WAP}(\widehat{H})\}_{H\in\mathcal{H}}$ in Theorem 5.3(5) can be replaced by V and $\{V_H\}_{H\in\mathcal{H}}$, respectively. Therefore, the inductive union $A(G) = \bigsqcup_{H\in\mathcal{H}} A(H)$ in Theorem 5.3 is more than WAP-strong.

Let G be a locally compact abelian group with the dual group Γ. Then the Fourier algebra $A(G)$ of G is isometrically isomorphic to the group algebra $L^1(\Gamma)$ of Γ by the Fourier transform (see Eymard [5, (3.6)]). So, $\mathrm{VN}(G)$ is identified with $L^\infty(\Gamma)$. Under these identifications, the module action of $L^1(\Gamma)$ on $L^\infty(\Gamma)$ is given by

$$f \cdot \phi = \check{f} * \phi \quad (f \in L^1(\Gamma) \text{ and } \phi \in L^\infty(\Gamma)),$$

where $\check{f}(x) = f(x^{-1})$ $(x \in \Gamma)$ (see Dunkl and Ramirez [4]). This coincides with the module action of the Banach algebra $L^1(\Gamma)$ (taking the convolution as the multiplication) on $L^\infty(G) = L^1(G)^*$. Also, we have $b(G) = d(\Gamma)$ (cf. [12, (24.48)]) and hence $d(G) = b(\Gamma)$ by the Pontryagin duality theorem. In particular, G is non-discrete if and only if Γ is non-compact. Now, for any open subgroup H of G, let $N_H = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in H\}$. Then $\widehat{H} \cong \Gamma/N_H$ and N_H $(\cong \widehat{G/H})$ is a compact subgroup of Γ. Applying Theorem 5.3, we obtain the following decomposition for the group algebra of any non-compact locally compact abelian group.

Corollary 5.5. *Let G be a non-compact locally compact abelian group satisfying $d(G) < b(G)$. Then there exists a family $\{N_i\}_{i\in I}$ of compact subgroups of G indexed by a directed set I such that*:

(1) *$N_i \supseteq N_j \neq \{e\}$ for all $i, j \in I$ with $i \preceq j$ and $\bigcap_{i\in I} N_i = \{e\}$.*

(2) *$b(G/N_i) \le d(G/N_i)$ for all $i \in I$ and* $\operatorname{card}(I) \le b(G)^{d(G)}$.

(3) *$L^1(G) = \bigsqcup_{i\in I} L^1(G/N_i)$ is a WAP-strong inductive union via the isometric isomorphisms $\Lambda_i : L^1(G/N_i) \to L^1(G)$ given by $\Lambda_i(f) = f \circ \eta_i$ $(f \in L^1(G/N_i))$, where η_i is the natural homomorphism of G onto G/N_i $(i \in I)$.*

Remark 5.6. Under the assumptions of Theorem 5.3, we also have the inductive union $L^1(G) = \bigsqcup_{H\in\mathcal{H}} L^1(H)$ of Banach algebras via the isometric isomorphisms $\{\Omega_H\}_{H\in\mathcal{H}}$, where $\Omega_H : L^1(H) \to L^1(G)$ is defined by $\Omega_H(f) = f^\circ$ (the trivial extension of f to G). However, usually $L^\infty(G)$ cannot be an inductive union of $\{L^\infty(H)\}_{H\in\mathcal{H}}$. For example, suppose that $d(G) = 2^\alpha$ for some $\alpha \ge b(G)$. Note that $\operatorname{card}(\mathcal{H}) \le d(G)^{b(G)} = 2^\alpha$ and $D(L^1(H)) \le b(H) = b(G)$ for all $H \in \mathcal{H}$, where $D(L^1(H))$ is the smallest cardinality of a norm dense subset of $L^1(H)$. It follows that $\operatorname{card}(\bigcup_{H\in\mathcal{H}} L^\infty(H)) \le 2^{b(G)} \operatorname{card}(\mathcal{H}) \le 2^\alpha = d(G) < 2^{d(G)} \le \operatorname{card}(L^\infty(G))$, i.e., $\operatorname{card}(\bigcup_{H\in\mathcal{H}} L^\infty(H)) < \operatorname{card}(L^\infty(G))$. Therefore, the inductive union $L^1(G) = \bigsqcup_{H\in\mathcal{H}} L^1(H)$ is not WAP-strong.

According to Theorem 5.3(2), for each $H \in \mathcal{H}$, there exists a closed linear subspace Z_H of $\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$ and a continuous linear map $\Pi_H : Z_H \to \mathrm{VN}(H)$ such that $\Pi_H(Z_H) = \mathrm{VN}(H)$. We will consider whether the family $\{\{Z_H, \Pi_H\} : H \in \mathcal{H}\}$ is compatible with the maps Φ_{HJ} and Γ_{HJ} $(H, J \in \mathcal{H}$ and $H \subseteq J)$. For this purpose, we will need the following two lemmas.

Lemma 5.7. *Let H and J be open subgroups of G with $H \subseteq J$ and let Λ_{HJ}, Φ_{HJ}, and Γ_{HJ} be the maps defined in Definition* 5.1. *Let $\Psi_{HJ} = \Lambda_{HJ}^*$. Then*:

(a) *$\Psi_{HJ} : \mathrm{VN}(J) \to \mathrm{VN}(H)$ is a continuous linear surjection, $\|\Psi_{HJ}\| = 1$, and $\Psi_{HJ}\Phi_{HJ} = \mathrm{Id}$.*

(b) $\Psi_{HJ}(\mathrm{WAP}(\widehat{J})) = \mathrm{WAP}(\widehat{H})$.

Define $\Theta_{HJ} : \mathrm{VN}(J)/\mathrm{WAP}(\widehat{J}) \to \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$ *by* $\Theta_{HJ}(T + \mathrm{WAP}(\widehat{J})) = \Psi_{HJ}(T) + \mathrm{WAP}(\widehat{H})$ $(T \in VN(J))$. *Then*:

(c) Θ_{HJ} *is a continuous linear surjection*, $\|\Theta_{HJ}\| = 1$, *and* $\Theta_{HJ}\Gamma_{HJ} = \mathrm{Id}$.

(d) *If K is an open subgroup of G and $H \subseteq J \subseteq K$, then* $\Psi_{HJ}\Psi_{JK} = \Psi_{HK}$ *and* $\Theta_{HJ}\Theta_{JK} = \Theta_{HK}$.

Proof. (a) This follows from [5, (3.21)].

(b) Note that $\Psi_{HJ}\Phi_{HJ} = \mathrm{Id}$ and $\Phi_{HJ}(\mathrm{WAP}(\widehat{H})) \subseteq \mathrm{WAP}(\widehat{J})$ (see [2, Lemma 4.2]). So, $\mathrm{WAP}(\widehat{H}) \subseteq \Psi_{HJ}(\mathrm{WAP}(\widehat{J}))$. On the other hand, for $u \in A(H)$ and $T \in \mathrm{VN}(J)$, we have $u \cdot \Psi_{HJ}(T) = \Psi_{HJ}(\Lambda_{HJ}(u) \cdot T)$. Therefore, $\Psi_{HJ}(\mathrm{WAP}(\widehat{J})) \subseteq \mathrm{WAP}(\widehat{H})$ and hence $\Psi_{HJ}(\mathrm{WAP}(\widehat{J})) = \mathrm{WAP}(\widehat{H})$.

(c) By (a) and (b), Θ_{HJ} is well-defined, linear, continuous, and onto. And $\Theta_{HJ}\Gamma_{HJ} = \mathrm{Id}$ since $\Psi_{HJ}\Phi_{HJ} = \mathrm{Id}$. Note that Γ_{HJ} is an isometry. So we have $\|\Theta_{HJ}\| \geq 1$. On the other hand, by the definition of Θ_{HJ} and by the fact that $\|\Psi_{HJ}\| = 1$, we get $\|\Theta_{HJ}\| \leq 1$. Therefore, $\|\Theta_{HJ}\| = 1$.

(d) Since $\Lambda_{JK}\Lambda_{HJ} = \Lambda_{HK}$, by taking the adjoint, we have $\Psi_{HJ}\Psi_{JK} = \Psi_{HK}$ and hence $\Theta_{HJ}\Theta_{JK} = \Theta_{HK}$. ∎

REMARK 5.8. Comparing to the diagram in Proposition 4.5, we now have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{VN}(J) \arrow[r, "\Psi_{HJ}"] \arrow[d, "\varrho_J"] & \mathrm{VN}(H) \arrow[d, "\varrho_H"] \\
\mathrm{VN}(J)/\mathrm{WAP}(\widehat{J}) \arrow[r, "\Theta_{HJ}"] & \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})
\end{tikzcd}$$

where ϱ_H and ϱ_J are the canonical quotient maps.

LEMMA 5.9. *Let G, G_0, and $\mathcal{H}$ be as in Theorem* 5.3. *Let μ be the initial ordinal with $|\mu| = b(G_0)$ $(= b(G))$ and $X = \{\alpha : \alpha < \mu\}$. Then there exists a continuous linear surjection* $\omega_H : \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) \to l^\infty(X)$ *for each $H \in \mathcal{H}$ such that the family $\{\|\omega_H\| : H \in \mathcal{H}\}$ is bounded by a constant which depends only on $b(G)$.*

Furthermore, if $H, J \in \mathcal{H}$ and $H \subseteq J$, then $\omega_H\Theta_{HJ} = \omega_J$ and we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) \arrow[rr, "\Gamma_{HJ}"] \arrow[dr, "\omega_H"] & & \mathrm{VN}(J)/\mathrm{WAP}(\widehat{J}) \arrow[dl, "\omega_J"] \\
& l^\infty(X) &
\end{tikzcd}$$

Proof. Let $\pi : \mathrm{VN}(G_0) \to l^\infty(X)$ be the map constructed in Hu [9, Theorem 5.1]. According to [9, Theorem 5.1] and its proof, π is a continuous linear surjection, $\|\pi\| = 1$, and $\pi(\mathrm{WAP}(\widehat{G}_0)) \subseteq c(X)$, where $c(X) = \{f \in l^\infty(X) : \lim_\alpha f(\alpha) \text{ exists}\}$. Note that $l^\infty(X)/c(X)$ contains an isomorphic copy of $l^\infty(X)$ (see [9, Lemma 3.2]) and $l^\infty(X)$ is an injective Banach space (see [14]). So, there exists a continuous linear surjection $\tau : l^\infty(X)/c(X) \to l^\infty(X)$. Define $\omega : \mathrm{VN}(G_0)/\mathrm{WAP}(\widehat{G}_0) \to l^\infty(X)$ by $\omega(T + \mathrm{WAP}(\widehat{G}_0)) = \tau(\pi(T) + c(X))$ $(T \in \mathrm{VN}(G_0))$. Then ω is well defined, linear, continuous, onto $l^\infty(X)$, and $\|\omega\| \le \|\tau\|$.

For $H \in \mathcal{H}$, let $\omega_H = \omega\,\Theta_{G_0H}$, where $\Theta_{G_0H} : \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) \to \mathrm{VN}(G_0)/\mathrm{WAP}(\widehat{G}_0)$ is the surjection as defined in Lemma 5.7. Then ω_H is continuous, linear, onto $l^\infty(X)$, and $\|\omega_H\| = \|\omega\Theta_{G_0H}\| \le \|\omega\| \le \|\tau\|$. It turns out that the family $\{\|\omega_H\| : H \in \mathcal{H}\}$ is bounded by the constant $\|\tau\|$ which depends only on $\mathrm{card}(X) = b(G)$.

Suppose $H, J \in \mathcal{H}$ and $H \subseteq J$. Then $\Theta_{G_0H}\Theta_{HJ} = \Theta_{G_0J}$ (Lemma 5.7(d)). Thus, $\omega\Theta_{G_0H}\Theta_{HJ} = \omega\,\Theta_{G_0J}$, i.e., $\omega_H\Theta_{HJ} = \omega_J$. But $\Theta_{HJ}\Gamma_{HJ} = \mathrm{Id}$ (Lemma 5.7(c)). It follows that $\omega_H = \omega_J\Gamma_{HJ}$ and hence the diagram commutes. ■

Now we have the following inductive compatibility of the pairs $\{Z_H, \Pi_H\}$ $(H \in \mathcal{H})$ with the maps Φ_{HJ} and Γ_{HJ}.

THEOREM 5.10. *The following hold under the assumptions of Theorem* 5.3:

(1) *For each $H \in \mathcal{H}$, there exists a closed linear subspace Z_H of the quotient* $\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$ *and a continuous linear map* $\Pi_H : Z_H \to \mathrm{VN}(H)$ *such that* $\Pi_H(Z_H) = \mathrm{VN}(H)$.

(2) *There exists a constant $M > 0$* (*which depends only on $b(G)$*) *such that $\|\Pi_H\| \le M$ for all $H \in \mathcal{H}$.*

(3) *Let $K \in \mathcal{H}$ and let $\mathcal{H}_K = \{H \subseteq \mathcal{H} : H \subseteq K\}$. Then*, *for each $H \in \mathcal{H}_K$, a pair $\{Z_H, \Pi_H\}$ as in* (1) *can be chosen such that the family $\{\{Z_H, \Pi_H\} : H \in \mathcal{H}_K\}$ is compatible with the maps Φ_{HJ} and Γ_{HJ}* ($H, J \in \mathcal{H}_K$ *and* $H \subseteq J$). *That is*, *if $H, J \in \mathcal{H}_K$ and $H \subseteq J$*, *then $\Gamma_{HJ}(Z_H) \subseteq Z_J$ and the following diagram commutes when Γ_{HJ} is restricted to Z_H*:

$$\begin{array}{ccc} Z_H & \xrightarrow{\Pi_H} & \mathrm{VN}(H) \\ \Big\downarrow{\scriptstyle \Gamma_{HJ}} & & \Big\downarrow{\scriptstyle \Phi_{HJ}} \\ Z_J & \xrightarrow{\Pi_J} & \mathrm{VN}(J) \end{array}$$

Proof. (1) This follows from Theorem 5.3(2).

(2) Let $H \in \mathcal{H}$. Then $d(H) \le b(H)$ (Theorem 5.3(1)) and hence $\mathcal{D}(A(H)) = b(H) = b(G) = \mathrm{card}(X)$, where $\mathcal{D}(A(H))$ is the smallest cardinality of a norm dense subset of $A(H)$ and X is the same set as in Lemma 5.9. Let $\{u_\alpha : \alpha \in X\}$ be a norm dense subset of the unit ball in $A(H)$ and let

$t_H : \mathrm{VN}(H) \to l^\infty(X)$ be defined by $t_H(T)(\alpha) = \langle T, u_\alpha\rangle$ ($T \in \mathrm{VN}(H)$ and $\alpha \in X$). Then t_H is a linear isometry. Let $\omega_H : \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H}) \to l^\infty(X)$ be the surjection as constructed in Lemma 5.9. We take $Z_H = \omega_H^{-1}[t_H(\mathrm{VN}(H))]$ ($\subseteq \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$) and $\Pi_H = t_H^{-1}(\omega_H|_{Z_H})$. Then Z_H is a closed linear subspace of $\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$, $\Pi_H : Z_H \to \mathrm{VN}(H)$ is a continuous linear map, and $\Pi_H(Z_H) = \mathrm{VN}(H)$.

It is clear that $\|\Pi_H\| \le \|\omega_H\|$. By Lemma 5.9, the family $\{\|\Pi_H\| : H \in \mathcal{H}\}$ is bounded by a constant which depends only on $b(G)$.

(3) Let $K \in \mathcal{H}$. Let t_K, Z_K, and Π_K be as constructed in (2). Let $H \in \mathcal{H}_K$ and $t'_H = t_K\Phi_{HK}$. Then $t'_H : \mathrm{VN}(H) \to l^\infty(X)$ is also a linear isometry since Φ_{HK} is an isometry. Now we take $Z_H = \omega_H^{-1}[t'_H(\mathrm{VN}(H))]$ ($\subseteq \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$) and $\Pi_H = (t'_H)^{-1}(\omega_H|_{Z_H})$. Then $\Pi_H : Z_H \to \mathrm{VN}(H)$ is also a continuous linear surjection and we still have $\|\Pi_H\| \le \|\omega_H\|$.

Suppose that $H, J \in \mathcal{H}_K$ and $H \subseteq J$. Since $\Phi_{HK} = \Phi_{JK}\Phi_{HJ}$, we have $\Phi_{HK}(\mathrm{VN}(H)) = \Phi_{JK}[\Phi_{HJ}(\mathrm{VN}(H))] \subseteq \Phi_{JK}(\mathrm{VN}(J))$ and hence $t_K\Phi_{HK}(\mathrm{VN}(H)) \subseteq t_K\Phi_{JK}(\mathrm{VN}(J))$, i.e., $t'_H(\mathrm{VN}(H)) \subseteq t'_J(\mathrm{VN}(J))$. Note that $\omega_J\Gamma_{HJ} = \omega_H$ (Lemma 5.9). Therefore, we have

$$\Gamma_{HJ}[\omega_H^{-1}(t'_H(\mathrm{VN}(H)))] \subseteq \omega_J^{-1}(t'_H(\mathrm{VN}(H))) \subseteq \omega_J^{-1}[t'_J(\mathrm{VN}(J))],$$

i.e., $\Gamma_{HJ}(Z_H) \subseteq Z_J$. Finally, the construction of $\{Z_H, \Pi_H\}$ and $\{Z_J, \Pi_J\}$ makes the diagram commutative. ■

Remark 5.11. Let $K \in \mathcal{H}$ and $\{Z_K, \Pi_K\}$ be the same as constructed in Theorem 5.10(2). If $H, J \in \mathcal{H}_K$ with $H \subseteq J$ and $\{Z_H, \Pi_H\}$, $\{Z_J, \Pi_J\}$ are chosen as in the proof of Theorem 5.10(3), then we only have $Z_H \subseteq \Theta_{HJ}(Z_J)$, where $\Theta_{HJ} : \mathrm{VN}(J)/\mathrm{WAP}(\widehat{J}) \to \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$ is the surjection as defined in Lemma 5.7. So, generally, we cannot simultaneously have the following commutative diagram when Θ_{HJ} is restricted to Z_J:

$$\begin{tikzcd} Z_H \arrow[r, "\Pi_H"] & \mathrm{VN}(H) \\ Z_J \arrow[u, "\Theta_{HJ}"'] \arrow[r, "\Pi_J"] & \mathrm{VN}(J) \arrow[u, "\Psi_{HJ}"'] \end{tikzcd}$$

However, for $H \in \mathcal{H}_K$, if we let $Q_H = \Theta_{HK}(Z_K)$ ($\subseteq \mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$) and let $\Sigma_H : Q_H \to \mathrm{VN}(H)$ be defined by

$$\Sigma_H[\Theta_{HK}(T + \mathrm{WAP}(\widehat{K}))] = \Psi_{HK}\Pi_K(T + \mathrm{WAP}(\widehat{K}))\ (T + \mathrm{WAP}(\widehat{K}) \in Z_K),$$

then it can be seen that Q_H is a linear subspace of $\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$, $\Sigma_H : Q_H \to \mathrm{VN}(H)$ is well defined, linear, onto $\mathrm{VN}(H)$, and $\|\Sigma_H\| \le \|\tau\|$, where $\tau : l^\infty(X)/c(X) \to l^\infty(X)$ is the surjection as appeared in the proof of Lemma 5.9. Let Y_H denote the norm closure of Q_H in $\mathrm{VN}(H)/\mathrm{WAP}(\widehat{H})$ and extend Σ_H continuously to Y_H. Then $\Sigma_H : Y_H \to \mathrm{VN}(H)$ is a continuous linear surjection. Now, if $H, J \in \mathcal{H}_K$ and $H \subseteq J$, then $\Theta_{HJ}(Q_J) \subseteq Q_H$ and hence $\Theta_{HJ}(Y_J) \subseteq Y_H$. Also, we have $\Sigma_H \Theta_{HJ}[\Theta_{JK}(T + \mathrm{WAP}(\widehat{K}))] = \Psi_{HJ}\Sigma_J[\Theta_{JK}(T + \mathrm{WAP}(\widehat{K}))]$ for $\Theta_{JK}(T + \mathrm{WAP}(\widehat{K})) \in Q_J$ and thus the following diagram commutes when Θ_{HJ} is restricted to Y_J:

$$\begin{array}{ccc} Y_H & \xrightarrow{\Sigma_H} & \mathrm{VN}(H) \\ \uparrow{\scriptstyle \Theta_{HJ}} & & \uparrow{\scriptstyle \Psi_{HJ}} \\ Y_J & \xrightarrow{\Sigma_J} & \mathrm{VN}(J) \end{array}$$

But, in this case, we do not have $\Gamma_{HJ}(Y_H) \subseteq Y_J$ and hence we cannot have $\Sigma_J \Gamma_{HJ}|_{Y_H} = \Phi_{HJ}\Sigma_H$, i.e., we do not have the commutative diagram in Theorem 5.10 when $\{Z_H, \Pi_H\}$ and $\{Z_J, \Pi_J\}$ are replaced by $\{Y_H, \Sigma_H\}$ and $\{Y_J, \Sigma_J\}$, respectively.

It is not clear whether in Theorem 5.10 we could choose a family $\{\{Z_H, \Pi_H\} : H \in \mathcal{H}\}$ compatible with all of the maps Φ_{HJ} and Γ_{HJ} ($H, J \in \mathcal{H}$ and $H \subseteq J$). If so, then we would be able to obtain a continuous linear surjection $\Pi : \bigcup_{H\in\mathcal{H}} \Gamma_H(Z_H) \to \mathrm{VN}(G)$ and hence we would be able to conclude that $A(G)$ is extremely non-Arens regular. For this reason, we give the following version of extreme non-Arens regularity.

DEFINITION 5.12. Let A be a Banach algebra. A is called *inductively extremely non-Arens regular* if there exists a family $\{A_i\}_{i\in I}$ of Banach algebras such that:

(1) For each $i \in I$, A_i is extremely non-Arens regular.

(2) $A = \bigsqcup_{i\in I} A_i$ is a WAP-strong inductive union of $\{A_i\}_{i\in I}$ with $A^* = \bigsqcup_{i\in I} A_i^*$ via $\{\Phi_i\}_{i\in I}$ and $A^*/\mathrm{WAP}(A) = \bigsqcup_{i\in I}[A_i^*/\mathrm{WAP}(A_i)]$ via $\{\Gamma_i\}_{i\in I}$.

(3) Let $k \in I$ and let $I_k = \{i \in I : i \preceq k\}$. Then, for each $i \in I_k$, there exists a closed linear subspace Z_i of $A_i^*/\mathrm{WAP}(A_i)$ and a continuous linear surjection $\Pi_i : Z_i \to A_i^*$ such that $\{\|\Pi_i\| : i \in I_k\}$ is bounded (by a constant independent of k) and $\{\{Z_i, \Pi_i\} : i \in I_k\}$ is compatible. That is, if $i, j \in I_k$ and $i \preceq j$, then $\Gamma_{ij}(Z_i) \subseteq Z_j$ and $\Phi_{ij}\Pi_i = \Pi_j \Gamma_{ij}|_{Z_i}$, where Φ_{ij} and Γ_{ij} are the same maps as in Corollary 3.5.

Combining Theorem 5.3 and Theorem 5.10 with [10, Corollary 4.2 and Remark 4.7], we are able to deduce the non-Arens regularity of $A(G)$ as follows.

COROLLARY 5.13. *Let G be a non-discrete locally compact group. Then*:

(1) *$A(G)$ is extremely non-Arens regular if $b(G) \geq d(G)$.*

(2) *$A(G)$ is inductively extremely non-Arens regular if $b(G) < d(G)$.*

As an immediate consequence of Corollary 5.13, we have the following result on the non-Arens regularity of the group algebra $L^1(G)$ of any non-compact locally compact abelian group G.

COROLLARY 5.14. *Let G be a non-compact locally compact abelian group. Then*:

(1) $L^1(G)$ *is extremely non-Arens regular if* $b(G) \leq d(G)$.
(2) $L^1(G)$ *is inductively extremely non-Arens regular if* $b(G) > d(G)$.

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