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Normed algebras of differentiable functions on compact plane sets: completeness and semisimple completions

by

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Abstract. We continue the study of the completeness and completions of normed algebras of differentiable functions $D^n(K)$ (where K is a perfect, compact plane set), initiated by Bland, Dales and Feinstein [Studia Math. 170 (2005) and Indian J. Pure Appl. Math. 41 (2010)]. We prove new characterizations of the completeness of $D^1(K)$ and results concerning the semisimplicity of the completion of $D^1(K)$. In particular, we prove that semi-rectifiability is necessary for the completion of $D^1(K)$ to be semisimple in the case where K lies on a rectifiable, injective curve. Furthermore, we answer a question posed by Dales and Feinstein and show that another question posed by them has an affirmative answer in some special cases. As compared with the approach taken by Bland, Dales and Feinstein, which comes from the theory of function algebras, we move within an operator-theoretic framework by investigating the mapping properties of certain derivation operators.

1. Introduction. Throughout this paper, let $K \subseteq \mathbb{C}$ be a non-empty, compact, perfect set, i.e., a set without isolated points. Then one can define differentiability of a function defined on K at a point $z_0 \in K$ in the usual way via the difference quotient: $f : K \to \mathbb{C}$ is *differentiable at* z_0 if the limit

$$f'(z_0) := \lim_{K\setminus\{z_0\}\ni z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists. Now it is clear how to define n-times and infinitely continuously differentiable functions. We denote the space of n-times continuously differentiable functions by $D^n(K)$. We endow it with the norm defined by

$$\|\cdot\|_{D^n(K)} : D^n(K) \to [0,\infty);\ f \mapsto \sum_{k=0}^{n} \frac{1}{k!}\|f^{(k)}\|_K$$

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(where $\|\cdot\|_K$ is the uniform norm on K). It is easy to show that $(D^n(K), \|\cdot\|_{D^n(K)})$ is a unital, commutative, normed algebra. These algebras, together with certain algebras of infinitely differentiable functions in the above sense, were first introduced by Dales and Davie in [7], and are therefore also known as Dales–Davie algebras. However, there are many examples showing that these algebras are not necessarily complete (see, e.g., Theorem 2.3 and Example 2.4 in [6], and also Theorem 10.12, Example 10.13 and Theorem 10.16 in [8]).

If this is the case, we can consider the completion $\widetilde{D}^n(K)$ of the normed algebra $(D^n(K), \|\cdot\|_{D^n(K)})$ and ask whether or not it is possible to regard $\widetilde{D}^n(K)$ as a Banach function algebra over K. Here by a *Banach function algebra over* K we mean a subalgebra of $\mathcal{C}(K)$ (the space of continuous functions on K) endowed with a complete algebra norm and containing at least the constant functions and separating the points of K. Unfortunately, this is not possible in general since an example due to Bishop shows that the completion need not be semisimple (see [4] combined with Example 6.2 in [8] as well as Theorems 3.2 and 3.5 below).

As a result, we are confronted with the task of finding necessary and sufficient conditions for $(D^n(K), \|\cdot\|_{D^n(K)})$ to be either complete or for its completion to be semisimple. Moreover, several results in [1], [9]–[15] and [22] rely on completeness assumptions imposed on certain Dales–Davie algebras, which gives, especially in view of Theorem 2.2 in [6], an additional impetus to examine these algebras and their properties. However, unlike the approach in [6] and in [8], we shall adopt a mainly operator-theoretical point of view in order to attend to these questions.

Now we give an overview of the article and its organization. In the second section, we shall provide some auxiliary results used later; these are mostly from measure and integration theory and the theory of unbounded linear operators. In the short third section, we shall give an operator-theoretical interpretation of the questions we are interested in; we extensively use this approach in what follows. The fourth section is devoted to the examination of the semisimplicity of $\widetilde{D}^1(K)$. In the fifth section, we shall consider the problem of the completeness of $(D^1(K), \|\cdot\|_{D^1(K)})$. Some applications are given in the last sixth section.

Some notations and definitions. Let (X,τ) be a topological space and $A \subseteq X$. Then $\overline{A}$, $\operatorname{int} A$, ∂A and $\mathbb{1}_A$ denote the closure, the interior, the boundary and the indicator function of A, respectively. If (X,τ) is Hausdorff, let $\operatorname{rca}(X)$ denote the space of all regular, countably additive Borel measures on (X,τ) equipped with the total variation norm.

By $\mathbb{D}$ we denote the open unit disk in the complex plane. For a non-empty, compact set $L \subseteq \mathbb{C}$, we denote by $\mathcal{C}(L)$ the space of continuous functions on L, usually endowed with the uniform norm $\|\cdot\|_L$. By $\mathcal{A}(L)$ we denote the space of continuous functions whose restriction to $\operatorname{int} L$ is holomorphic. We denote the set of polynomial functions on L by $\mathcal{P}_0(L)$ and its uniform closure in $\mathcal{C}(L)$ by $\mathcal{P}(L)$. Furthermore, $\mathcal{R}(L)$ denotes the uniform closure of the set of rational functions on L with poles off L. Recall that a compact set L is *polynomially convex* if it coincides with its polynomially convex hull $\mathrm{PCH}(L)$ given by $\{z \in \mathbb{C};\ |p(z)| \leq \|p\|_L \text{ for all } p \in \mathcal{P}_0(L)\}$. Furthermore, recall that $\mathrm{PCH}(L)$ is the union of L and all bounded components of $\mathbb{C} \setminus L$. Thus, L is polynomially convex if and only if $\mathbb{C} \setminus L$ is connected. If U is a non-empty, open subset of $\mathbb{C}$, let $\mathcal{O}(U)$ denote the set of holomorphic functions on U.

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathfrak{D}(T)$ a subspace of E and $T : \mathfrak{D}(T) \to F$ a linear map. By $\ker(T)$, $\operatorname{ran}(T)$ and $\mathrm{G}(T)$ we denote the kernel, the range and the graph of T, respectively. By $\widehat{T}$ we denote the operator induced by T on the quotient space $\mathfrak{D}(T)/\ker(T)$. Moreover, $\|\cdot\|_T$ denotes the graph norm on $\mathfrak{D}(T)$ induced by T. For $A \subseteq E$, let $\mathrm{LH}\, A$ denote the linear hull of A. The symbol E' denotes the (topological) dual space of E (endowed with the norm topology).

Recall that the (Jacobson) radical $\operatorname{rad}(\mathcal{A})$ of a Banach algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is the intersection of all maximal (left or respectively right) ideals of $\mathcal{A}$, and it coincides in the commutative case with the intersection of the kernels of all multiplicative functionals on $\mathcal{A}$, and thus with the set of all elements with spectral radius zero. Recall that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is semisimple if $\operatorname{rad}(\mathcal{A})$ consists only of the null vector.

Let (X, d) be a metric space. If $\gamma : [a, b] \to X$ is a continuous path (where a and b are real numbers with $a \leq b$), then we denote by $\gamma^- := \gamma(a)$ its starting point and by $\gamma^+ := \gamma(b)$ its end point. As usual, we call γ *closed* if $\gamma^+ = \gamma^-$. The image $\gamma([a, b])$ will be denoted by γ^*. An injective path is a *Jordan path.* A path γ in X such that $\gamma|_{[a,b)}$ is injective and with $\gamma(a) = \gamma(b)$ is a *simple closed path.* In addition, a path $\gamma : [a, b] \to X$ with $a < b$ is called *admissible* if for all $c, d \in [a, b]$ with $c < d$ the path $\gamma|_{[c,d]}$ is not constant (see also Definition 2.4 in [8]).

The length $L(\gamma) \in [0, \infty]$ of a path γ is defined in the usual way and γ is called *rectifiable* if $L(\gamma) < \infty$.

If γ is a closed path in $\mathbb{C}$ and $z \in \mathbb{C} \setminus \gamma^*$, then $\operatorname{ind}_\gamma(z)$ denotes the winding number of γ with respect to the point z. If $f \in \mathcal{C}(\gamma^*)$ for a rectifiable path $\gamma : [a, b] \to \mathbb{C}$, then we define the path integral $\int_\gamma f := \int_\gamma f(z)\, dz$ of f along γ to be the Riemann–Stieltjes integral $\int_a^b f(\gamma(t))\, d\gamma(t)$ as introduced in Chapter 7 of [3].

A non-empty, compact subset K of the complex plane is *rectifiably connected* if K contains at least two points and any two points of K can be joined by a rectifiable path in K.

If K is rectifiably connected, we define the *geodesic metric* δ_K on K by

$$\delta_K(z,w) = \inf\{L(\gamma);\ \gamma \text{ is a path in } K \text{ joining } z \text{ to } w\}$$

for all $z, w \in K$. It can be shown that this infimum is indeed a minimum (see, e.g., 1.4.11 and 1.4.12 in [24]), actually attained on a Jordan path as one can see by using 1.2.2 in [24]. Such a path of minimum length is called a *geodesic path.*

The set K is *geodesically bounded* if K is rectifiably connected and the metric space (K, δ_K) has finite diameter.

If K is rectifiably connected and $z \in K$ is such that there exists a constant $C_z > 0$ with

$$\delta_K(z,w) \le C_z|z-w|$$

for all $w \in K$, then we call K *regular at* z. We say that K is *pointwise regular* (see also [6] and [8]) if K is regular at each of its points. If the Euclidean and the geodesic metric are strongly equivalent on a rectifiably connected, compact plane set K, i.e., $\delta_K(z,w) \le C|z-w|$ for some $C > 0$ and all $z, w \in K$, then K is said to be *uniformly regular* or *quasi-convex.*

We mention that all continuously differentiable Jordan paths with nowhere vanishing derivatives, all circles and all convex sets with at least two points are uniformly regular and that all connected unions of finitely many continuously differentiable Jordan paths with nowhere vanishing derivatives are pointwise regular. Examples of uniformly regular, compact plane sets more sophisticated than the ones just listed are provided by (generalized) Sierpiński carpets (see, e.g., Lemma 7.2 in [3] and 3.25 in [21]).

In what follows, K is always a non-empty, perfect, compact plane set.

2. Auxiliary results

2.1. Results from measure and integration theory. We first record a simple lemma, whose proof is omitted since it is an immediate consequence of the Jordan decomposition of complex measures and a well-known uniqueness result from measure theory.

LEMMA 2.1. *Let* $(X, \mathfrak{M})$ *be a measurable space and* $\mu : \mathfrak{M} \to \mathbb{C}$ *a complex measure. Furthermore, let* $\mathfrak{E}$ *be a generator of* $\mathfrak{M}$ *stable with respect to intersection such that:*

(i) $\mu|_{\mathfrak{E}} = 0$,
(ii) *there is a sequence* $(E_n)_{n=1}^{\infty}$ *in* $\mathfrak{E}$ *such that* $\bigcup_{n=1}^{\infty} E_n = X$.

Then $\mu = 0$. ∎

Next, we recall the classical Riesz representation theorem for the dual space of the space of continuous functions on a compact interval. For this, we need some notation.

For $a < b$, denote by $\mathrm{NBV}([a,b])$ the space of normalized functions of bounded variation on $[a,b]$, i.e., the set

$$\{f \in \mathrm{BV}([a,b]);\ f(a) = 0 \text{ and } f \text{ is continuous from the right on } (a,b)\},$$

where $\mathrm{BV}([a,b])$ denotes the space of functions of bounded variation on $[a,b]$. As usual we endow $\mathrm{NBV}([a,b])$ with the variation norm $\mathrm{var}(\cdot,[a,b])$. Finally, we put $\mathrm{BVC}([a,b]) := \mathcal{C}([a,b]) \cap \mathrm{BV}([a,b])$.

THEOREM 2.2 (see, e.g., [2, U4.10]). *The mapping*

$$J : \mathrm{rca}([a,b]) \to \mathrm{NBV}([a,b]);\ \mu \mapsto J\mu,$$

where

$$J\mu : [a,b] \to \mathbb{C};\ x \mapsto \begin{cases} 0 & \text{if } x = a, \\ \mu([a,x]) & \text{if } a < x \le b, \end{cases}$$

is a well-defined isometric isomorphism, and

$$\int_{[a,b]} f(x)\, d\mu(x) = \int_a^b f(x)\, d(J\mu)(x)$$

for all $\mu \in \mathrm{rca}([a,b])$ and all $f \in \mathcal{C}([a,b])$. ■

We now prove some useful corollaries of this version of the Riesz representation theorem.

COROLLARY 2.3. *Let $a < b$ and $\alpha \in \mathrm{BVC}([a,b])$. Then there exists exactly one $\mu \in \mathrm{rca}([a,b])$ such that*

$$\int_c^d f(x)\, d\alpha(x) = \int_{[c,d]} f(x)\, d\mu(x)$$

for all $a \le c < d \le b$ and all $f \in \mathcal{C}([a,b])$. In addition, this complex measure satisfies $\mu(A) = 0$ for all countable subsets A of $[a,b]$ and it is identically 0 if and only if α is constant.

Proof. The uniqueness is an immediate consequence of the Riesz representation theorem. Furthermore, one easily verifies that the measure $\mu := (J_{[a,b]})^{-1}(\alpha - \alpha(a))$ has the desired properties. Finally, the bijectivity of $J_{[a,b]}$ implies that μ is the trivial measure if and only if $\alpha - \alpha(a)$ is identically 0, i.e., if and only if α is constant. ■

Applying the last result to path integrals, we obtain the following corollary.

Corollary 2.4. *Let $\gamma : [a,b] \to \mathbb{C}$ be an admissible rectifiable path. Furthermore, let $f \in \mathcal{C}(\gamma^*)$ be such that*

$$\int_{\gamma|_{[c,d]}} f(z)\,dz = 0$$

for all $a \le c < d \le b$. Then f vanishes on the whole of γ^*.

Proof. Assume, for contradiction, that $f \neq 0$. Since f is continuous, we can find $t_1 < t_2$ belonging to $[a,b]$ such that f does not vanish anywhere on $\gamma([t_1,t_2])$. Then for all $t_1 \le c < d \le t_2$,

$$0 = \int_{\gamma|_{[c,d]}} f(z)\,dz = \int_c^d f(\gamma(t))\,d\gamma(t) = \int_c^d f(\gamma(t))\,d\mu(t),$$

where μ is the measure associated to $\gamma|_{[t_1,t_2]} \in \mathrm{BVC}([t_1,t_2])$ as in Corollary 2.3. Lemma 2.1 implies that the complex measure $f \circ \gamma|_{[t_1,t_2]}\,d\mu$ is the zero measure, which implies by a standard argument that μ itself is the zero measure. However, this shows by Corollary 2.3 that γ must be constant on $[t_1,t_2]$, contradicting the premise that γ is admissible. ∎

Another application of the Riesz representation theorem is given by the following result.

Lemma 2.5. *Let $\emptyset \neq L \subseteq \mathbb{C}$ be compact and let $\gamma : [a,b] \to L$ (with $a < b$) be a rectifiable path. For the mapping*

$$\int_\gamma \cdot\,dz : \mathcal{C}(L) \to \mathbb{C};\ f \mapsto \int_\gamma f,$$

the following assertions hold:

(i) *$\int_\gamma \cdot\,dz$ is in $\mathcal{C}(L)'$ and $\|\int_\gamma \cdot\,dz\|_{\mathcal{C}(L)'} \le L(\gamma)$.*
(ii) *If γ is a Jordan path, then $\|\int_\gamma \cdot\,dz\|_{\mathcal{C}(L)'} = L(\gamma)$.*

Proof. Part (i) is well-known, so let us turn to (ii). Clearly, $\gamma : [a,b] \to \gamma^*$ is a homeomorphism inducing an isometric isomorphism T from $\mathcal{C}(\gamma^*)$ onto $\mathcal{C}([a,b])$ by dint of $f \mapsto f \circ \gamma$. The transpose operator T' is also an isometric isomorphism. Now we consider the following element of $\mathcal{C}([a,b])'$:

$$\int_a^b \cdot\,d\gamma : \mathcal{C}([a,b]) \to \mathbb{C};\ g \mapsto \int_a^b g(t)\,d\gamma(t).$$

A straightforward calculation yields $T'(\int_a^b \cdot\,d\gamma) = \int_\gamma \cdot\,dz$. Consequently, the isometry property of T' implies that

$$\Big\|\int_\gamma \cdot\,dz\Big\|_{\mathcal{C}(\gamma^*)'} = \Big\|\int_a^b \cdot\,d\gamma\Big\|_{\mathcal{C}([a,b])'} = \mathrm{var}(\gamma,[a,b]) = L(\gamma)$$

by 2.2 and the Riesz representation theorem on compact Hausdorff spaces.

Now let $\epsilon > 0$. By the above, there exists an $f \in \mathcal{C}(\gamma^*)$ with $\|f\|_{\gamma^*} \leq 1$ such that $|\int_\gamma f| > L(\gamma) - \epsilon$. Clearly, there exists $F \in \mathcal{C}(L)$ with $F|_{\gamma^*} = f$ and $\|F\|_L = \|f\|_{\gamma^*}$. Then $|\int_\gamma F| = |\int_\gamma f| > L(\gamma) - \epsilon$. Hence, $\|\int_\gamma \cdot\, dz\|_{\mathcal{C}(L)'} > L(\gamma) - \epsilon$. Letting $\epsilon \to 0$ gives $\|\int_\gamma \cdot\, dz\|_{\mathcal{C}(L)'} \geq L(\gamma)$. ∎

To end this subsection, we mention the fundamental theorem of calculus for rectifiable paths stated in [6] and [8]. A proof can be found in [21, Section 4 of Anhang A], where the idea to use a bisection argument, as indicated in [6], is carried out based on appropriately adopting the scheme of the proof in [11].

THEOREM 2.6. *Let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable path and $f \in D^1(\gamma^*)$. Then*

$$\int_\gamma f'(z)\, dz = f(\gamma(b)) - f(\gamma(a)). \ \blacksquare$$

2.2. Unbounded operators. In this subsection, let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathfrak{D}(T)$ a subspace of E and $T : \mathfrak{D}(T) \to F$ a linear map.

We have the following characterization of the closability of T.

THEOREM 2.7. *The following assertions are equivalent:*

(i) *The operator T is closable.*
(ii) *There is a set $\mathcal{A} \subseteq F'$ separating the points of F such that the linear map $\varphi T : (\mathfrak{D}(T), \|\cdot\|_E) \to \mathbb{C}$ is continuous for each $\varphi \in \mathcal{A}$.*

Proof. (i)⇒(ii): See the proof of Theorem II.2.11 in [19] and observe that this proof does not use the density assumption made in the formulation of the cited theorem in order to show the implication in question.

(ii)⇒(i): Let $\mathcal{A}$ be as in (ii). Using the Hahn–Banach theorem, we choose for each $\varphi \in \mathcal{A}$ a functional $\psi_\varphi \in E'$ that extends φT. Next, we define

$$\mathfrak{D}(T_\mathcal{A}) := \{x \in E;\ \exists y \in F\ \forall \varphi \in \mathcal{A} : \psi_\varphi(x) = \varphi(y)\}.$$

One easily sees that for every $x \in E$ there is at most one $y \in F$ such that $\psi_\varphi(x) = \varphi(y)$ for all $\varphi \in \mathcal{A}$. For $x \in \mathfrak{D}(T_\mathcal{A})$, we denote by $T_\mathcal{A} x$ this unique element of F. One easily verifies that $\mathfrak{D}(T_\mathcal{A})$ is a linear subspace of E and that $T_\mathcal{A} : \mathfrak{D}(T_\mathcal{A}) \to F;\ x \mapsto T_\mathcal{A} x$ is a closed linear extension of T. ∎

3. Operator-theoretical characterizations. We start by endowing $\mathcal{C}(K)^{n+1}$ $(n \in \mathbb{N})$ with a suitable norm and multiplication to make it a commutative Banach algebra with unit. For this, we define

$$\|\cdot\|_{n+1} : \mathcal{C}(K)^{n+1} \to [0, \infty);\ (f_0, \ldots, f_n) \mapsto \sum_{j=0}^{n} \frac{1}{j!} \|f_j\|_K.$$

For $(f_0,\dots,f_n)$ and $(g_0,\dots,g_n)$ in $\mathcal{C}(K)^{n+1}$, we set

$$(f_0,\dots,f_n)\cdot(g_0,\dots,g_n) := \Bigl(\sum_{k=0}^{j}\binom{j}{k} f_k g_{j-k}\Bigr)_{j=0}^{n}.$$

It is easy to verify that with this norm and multiplication, $\mathcal{C}(K)^{n+1}$ becomes a commutative Banach algebra with unit $(1,0,\dots,0)$.

Next, we determine the radical of the Banach algebra $(\mathcal{C}(K)^{n+1}, \|\cdot\|_{n+1})$ in the following obvious lemma.

Lemma 3.1.

$$\operatorname{rad}(\mathcal{C}(K)^{n+1}) = \{(g_0,\dots,g_n)\in\mathcal{C}(K)^{n+1};\ g_0=0\}.$$

In addition, all elements belonging to $\operatorname{rad}(\mathcal{C}(K)^{n+1})$ *are nilpotent with nilpotency index at most* $n+1$. ∎

For $n\in\mathbb{N}$, we consider the mappings

$$\iota_n : (D^n(K), \|\cdot\|_{D^n(K)}) \to (\mathcal{C}(K)^{n+1}, \|\cdot\|_{n+1});\ f\mapsto (f^{(0)}, f^{(1)},\dots,f^{(n)})$$

and the derivation operators

$$T_n : (D^n(K), \|\cdot\|_K) \to (\mathcal{C}(K)^{n+1}, \|\cdot\|_{n+1});\ f\mapsto (0, f^{(1)},\dots,f^{(n)}).$$

We define $\mathfrak{D}(T_n) := D^n(K)$ as well as $D := \pi_2\circ T_1$, where π_2 is the canonical projection onto the second component.

It is immediate that ι_n is an isometrical algebra homomorphism. Therefore, we can identify (up to an isometrical algebra isomorphism) the completion $\widetilde{D}^n(K)$ of $(D^n(K), \|\cdot\|_{D^n(K)})$ with the closure of $\operatorname{ran}(\iota_n)$ in $(\mathcal{C}(K)^{n+1}, \|\cdot\|_{n+1})$. Moreover, $\mathrm{G}(T_n)$ endowed with the sum norm arising from $\|\cdot\|_K$ and $\|\cdot\|_{n+1}$ is isometrically isomorphic to $(\operatorname{ran}(\iota_n), \|\cdot\|_{n+1})$ via $(f, T_n f)\mapsto \iota_n(f)$ $(f\in\mathfrak{D}(T_n))$. We finally note that $(D^n(K), \|\cdot\|_{T_n}) = (D^n(K), \|\cdot\|_{D^n(K)})$ with equal norms.

These simple observations are crucial for the proof of the following theorem that completely characterizes those cases in which $\widetilde{D}^n(K)$ is a Banach function algebra over K.

Theorem 3.2. *The following assertions are equivalent:*

(i) *The operator*

$$T_n : (D^n(K), \|\cdot\|_K) \to (\mathcal{C}(K)^{n+1}, \|\cdot\|_{n+1});\ f\mapsto (0, f^{(1)},\dots,f^{(n)})$$

is closable.

(ii) *There is a Banach function algebra* $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *on* K *such that* $D^n(K)$ *is dense in* $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *and* $\|\cdot\|_{\mathcal{A}}$ *extends* $\|\cdot\|_{D^n(K)}$. *In particular,* $\widetilde{D}^n(K)$ *is a Banach function algebra over* K.

(iii) *The completion* $\widetilde{D}^n(K)$ *of* $(D^n(K), \|\cdot\|_{D^n(K)})$ *is semisimple.*

Proof. (i)⇒(ii): Let $\overline{T_n}$ be the minimal closed extension of T_n. Then the space $\mathrm{G}(\overline{T_n}) = \overline{\mathrm{G}(T_n)}$ is isometrically isomorphic to $\overline{\mathrm{ran}(\iota_n)}$, which is a closed subalgebra of $(\mathcal{C}(K)^{n+1}, \|\cdot\|_{n+1})$; on the other hand, it is isometrically isomorphic to the Banach space $(\mathfrak{D}(\overline{T_n}), \|\cdot\|_{\overline{T_n}})$. Furthermore, $\mathfrak{D}(\overline{T_n}) = \pi_1(\mathrm{G}(\overline{T_n})) \subseteq \mathcal{C}(K)$. Hence, $(\mathfrak{D}(\overline{T_n}), \|\cdot\|_{\overline{T_n}})$ is a Banach algebra with the usual multiplication of continuous functions. Since this algebra is the completion of $\mathfrak{D}(T_n)$ with respect to the graph norm, we see that the algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathfrak{D}(\overline{T_n}), \|\cdot\|_{\overline{T_n}})$ fulfills the desired conditions.

(ii)⇒(iii): As we can identify $\widetilde{D}^n(K)$ with $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$, it suffices to show that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is semisimple—but every Banach function algebra is semisimple.

(iii)⇒(i): Assume to the contrary that T_n is not closable. Then there exists $(f_1, \ldots, f_n) \in \mathcal{C}(K)^n \setminus \{(0, \ldots, 0)\}$ such that $(0, f_1, \ldots, f_n) \in \overline{\mathrm{ran}(\iota_n)} \cong \widetilde{D}^n(K)$ (with an isometrical algebra isomorphism). However, by 3.1, $(0, f_1, \ldots, f_n)$ is a nilpotent element of $\mathcal{C}(K)^{n+1}$ different from $(0, \ldots, 0)$. Thus $\mathrm{rad}(\widetilde{D}^n(K))$ cannot be trivial, contrary to (iii). ∎

Similarly, we have the following simple, but useful observation, whose proof we omit.

Lemma 3.3. *The following assertions are equivalent:*

(i) $(D^n(K), \|\cdot\|_{D^n(K)})$ *is complete.*
(ii) *The operator*

$$T_n : (D^n(K), \|\cdot\|_K) \to (\mathcal{C}(K)^{n+1}, \|\cdot\|_{\mathcal{C}(K)^{n+1}});\ f \mapsto (0, f^{(1)}, \ldots, f^{(n)})$$

is closed. ∎

Clearly, the preceding two results also hold for $n = 1$ with T_1 replaced by D.

As an immediate consequence of these two results, we obtain the following corollary, whose second assertion has been stated, e.g., in [6] and [8]. Since the proof is routine, we leave it to the reader.

Corollary 3.4.

(i) *If there exists an* $n \in \mathbb{N}$ *such that* $\widetilde{D}^n(K)$ *is semisimple, then* $\widetilde{D}^m(K)$ *is semisimple for all* $m \geq n$.
(ii) *If there exists an* $n \in \mathbb{N}$ *such that* $(D^n(K), \|\cdot\|_{D^n(K)})$ *is complete, then* $(D^m(K), \|\cdot\|_{D^m(K)})$ *is complete for all* $m \geq n$. ∎

To close this section, we give a class of compact plane sets for which $\widetilde{D}^n(K)$ is not semisimple (and thus $(D^n(K), \|\cdot\|_{D^n(K)})$ is not complete) for any $n \in \mathbb{N}$. Even stronger results have been stated in [4] and [5] and proved in the latter paper. Here we state a weaker result, but sufficient for our purposes. An elementary proof can be found in [21, 2.11].

Theorem 3.5. *Suppose that K is a totally disconnected set. Then $\widetilde{D}^n(K) \cong \mathcal{C}(K)^{n+1}$ (with an isometrical algebra isomorphism) for each $n \in \mathbb{N}$. Hence, $\widetilde{D}^n(K)$ is not semisimple.* ∎

4. Semisimplicity of $\widetilde{D}^1(K)$. From now on, we are interested only in $D^1(K)$ and $\widetilde{D}^1(K)$. Thus we frequently write just $\|\cdot\|$ instead of $\|\cdot\|_{D^1(K)}$.

Applying 3.2 in combination with 2.7 and with the Riesz representation theorem on compact Hausdorff spaces, we obtain the following result.

Theorem 4.1. *The completion $\widetilde{D}^1(K)$ of $(D^1(K), \|\cdot\|)$ is semisimple if and only if there is a family $\mathcal{A} \subseteq \mathrm{rca}(K)$ such that:*

(i) $\bigcap_{\mu\in\mathcal{A}} \ker(\mathcal{C}(K) \to \mathbb{C};\ f \mapsto \int_K f\,d\mu) = \{0\}$.
(ii) $\forall \mu \in \mathcal{A}\ \exists C > 0\ \forall f \in D^1(K) : |\int_K f'(z)\,d\mu(z)| \le C\|f\|_K$. ∎

Next, we turn to a broad class of non-empty, perfect, compact plane sets K introduced in [8] for which $\widetilde{D}^1(K)$ is semisimple. For this, we need a definition.

Definition 4.2. We denote by $\mathcal{F}_K$ the set of all rectifiable, admissible paths in K. Moreover, we put $\mathcal{F}_K(K) := \bigcup_{\gamma\in\mathcal{F}_K} \gamma^*$. We call K *semi-rectifiable* if $\mathcal{F}_K(K)$ is dense in K.

Now we give an alternative proof of Theorem 6.3 in [8] within the operator-theoretic framework that we have developed (see also Remark 4.5).

Theorem 4.3. *Suppose that K is semi-rectifiable. Then $\widetilde{D}^1(K)$ is semisimple.*

Proof. We consider the family $\mathcal{A} := \{\int_\gamma \cdot\,dz\}_{\gamma\in\mathcal{F}_K}$ of continuous, linear functionals on $\mathcal{C}(K)$. The set $\mathcal{A}$ separates the points of $\mathcal{C}(K)$. Indeed, if $f \in \mathcal{C}(K)$ with $\int_\gamma f(z)\,dz = 0$ for all $\gamma \in \mathcal{F}_K$, then 2.4 tells us that f must vanish on the whole of γ^* for all $\gamma \in \mathcal{F}_K$, i.e., $f = 0$ on $\mathcal{F}_K(K)$. The continuity of f and the premise $\overline{\mathcal{F}_K(K)} = K$ thus imply that f must be the zero function. Moreover, 2.6 immediately shows that for all $\varphi \in \mathcal{A}$ the composition $\varphi \circ D : (D^1(K), \|\cdot\|_K) \to \mathbb{C}$ is continuous. Because of 2.7, D is closable, so the assertion follows from 3.2. ∎

Obviously, this theorem applies, e.g., to the closure of bounded open sets and to the image of non-constant paths $\gamma : [a, b] \to \mathbb{C}$ with $L(\gamma|_{[a,t]}) < \infty$ for all $a < t < b$. Furthermore, we have the following corollary, where we let $\mathcal{H}^1$ denote the one-dimensional Hausdorff measure on $\mathbb{C}$.

Corollary 4.4. *If K is connected with $\mathcal{H}^1(K) < \infty$, then $\widetilde{D}^1(K)$ is semisimple.*

Proof. Due to [11, 3.12], the set K is pathwise connected. The proof of [11, 3.13] now shows that K is semi-rectifiable. ∎

REMARK 4.5. We want to relate our proof of 4.3 to the approach in [8] (for details on this approach see [6] and [8]). In view of the proof of 2.7, we see that Dales and Feinstein indeed construct a closed extension of D by using the family $\{\int_\gamma \cdot\, dz\}_{\gamma\in\mathcal{F}_K}$. However, whereas in [8] the semisimplicity primarily looks like a technical premise to make sure that the so called $\mathcal{F}_K$-derivatives are unique, we now see that the applicability of families similar to $\{\int_\gamma \cdot\, dz\}_{\gamma\in\mathcal{F}_K}$ and implicitly used in [6] and in [8] is probably only useful for semi-rectifiable sets. If K is not semi-rectifiable, the family $\{\int_\gamma \cdot\, dz\}_{\gamma\in\mathcal{F}_K}$ of continuous, linear functionals does not separate the points of $\mathcal{C}(K)$ any longer; e.g., the function

$$K \to \mathbb{C};\ z \mapsto \frac{\inf\{|z-a|;\ a\in A\}}{\inf\{|z-a|;\ a\in A\}+|z-z_0|},$$

where $A := \overline{\mathcal{F}_K}$ and $z_0 \in K\setminus\overline{\mathcal{F}_K}$, cannot be distinguished from the zero function by means of the family $\{\int_\gamma \cdot\, dz\}_{\gamma\in\mathcal{F}_K}$.

To close this section, we address the question whether semi-rectifiability is not only sufficient, but also necessary for $\widetilde{D}^1(K)$ to be semisimple. In a special situation, the answer is affirmative.

Before stating the corresponding theorem, note that for a Jordan path γ all connected subsets of γ^* have the form $(\gamma|_I)^*$, where $I\subseteq[0,1]$ is any kind of interval (including the empty set and singletons). This can be easily seen, using that $\gamma:[0,1]\to\gamma^*$ is a homeomorphism. Furthermore, observe that every subset X of γ^* has at most countably many components with more than one point. Indeed, consider

$$\mathfrak{K} := \{C\subseteq X;\ C \text{ is a component of } X \text{ with } \sharp C\geq 2\}$$

first for the special case $X\subseteq[0,1]$ and let λ be the one-dimensional Lebesgue measure on $\mathbb{R}$. Since $\lambda(C)>0$ for each $C\in\mathfrak{K}$, the set $\mathfrak{K}$ must be countable; otherwise, by the pairwise disjointness of components, we would arrive at the contradiction

$$1=\lambda([0,1])\geq \sup\Big\{\sum_{C\in\mathcal{K}}\lambda(C);\ \mathcal{K}\subseteq\mathfrak{K},\ \sharp\mathcal{K}<\infty\Big\}=\infty.$$

Since $\gamma:[0,1]\to\gamma^*$ is a homeomorphism, we obtain the general statement for $X\subseteq\gamma^*$.

THEOREM 4.6. *Let $K\subseteq\gamma^*$ for a rectifiable Jordan path $\gamma:[0,1]\to\mathbb{C}$. The Banach algebra $\widetilde{D}^1(K)$ is semisimple if and only if the union of all components of K having more than one point is dense in K, and this occurs if and only if the interior of K relative to γ^* is dense in K, which is equivalent to K being semi-rectifiable.*

Proof. The second and third equivalence statements are easy. The sufficiency in the first equivalence statement follows from 4.3. Now we turn to the necessity and assume that the interior of K relative to γ^* is not dense in K. We shall show that $\widetilde{D}^1(K)$ cannot be semisimple. Set as above

$$\mathfrak{K} := \{C \subseteq K;\, C \text{ is a component of } K \text{ with } \sharp C \geq 2\}.$$

For $\mathfrak{K} = \emptyset$, K is totally disconnected and the assertion follows from 3.5. Now suppose that $\mathfrak{K} \neq \emptyset$ and put $K_0 := \overline{\bigcup_{C\in\mathfrak{K}} C}$.

By assumption, $K \setminus K_0 \neq \emptyset$. Fix $z_0 \in K \setminus K_0$. Let $C(z_0)$ be the connected component of $\gamma^* \setminus K_0$ that contains z_0. Since $\gamma^* \setminus K_0$ is open in γ^*, one easily sees (once again using that $\gamma : [0,1] \to \gamma^*$ is a homeomorphism) that $C(z_0)$ itself is open in γ^*. Since this set is also connected with $C(z_0) \subsetneq \gamma^*$, only three cases are possible:

- $\exists 0 < t_1 < t_2 < 1 : C(z_0) = \gamma((t_1, t_2))$.
- $\exists 0 < \widetilde{t} < 1 : C(z_0) = \gamma((\widetilde{t}, 1])$.
- $\exists 0 < \widetilde{t} < 1 : C(z_0) = \gamma([0, \widetilde{t}))$.

Let $\partial_{\gamma^*} C(z_0)$ denote the boundary of $C(z_0)$ relative to γ^*. We then have $\partial_{\gamma^*} C(z_0) \subseteq K_0$, which can be rapidly verified by taking into consideration the maximality of components and that $C(z_0)$ is open in γ^*.

Next, we define $\widetilde{K} := \overline{C(z_0) \cap K}$. The set $\widetilde{K}$ is non-empty, compact and perfect. The first two properties are obvious and the third one holds since $C(z_0) \cap K$ is (relatively) open in K.

Now $C(z_0) \cap K \subseteq \widetilde{K} \subseteq \overline{C(z_0)} \cap \overline{K} = \overline{C(z_0)} \cap K$. As a result, we have $\widetilde{K} \setminus (C(z_0) \cap K) \subseteq \partial_{\gamma^*} C(z_0)$.

Let $\widetilde{C}$ be a component of $\widetilde{K}$. Then $\sharp\widetilde{C} = 1$. Otherwise, we could find $0 \leq a < b \leq 1$ with $\widetilde{C} = (\gamma|_{[a,b]})^*$. This would lead to

$$\emptyset \neq (\gamma|_{[a,b]})^* \setminus \partial_{\gamma^*} C(z_0) \subseteq C(z_0) \cap K.$$

Since $\sharp\partial_{\gamma^*} C(z_0) \leq 2$, this implies the existence of a $C \in \mathfrak{K}$ with $C \cap C(z_0) \neq \emptyset$, contradicting the definition of $C(z_0)$.

Therefore, $\widetilde{K}$ is totally disconnected. But then it has a base of clopen sets. Consequently, there is a clopen subset U of $\widetilde{K}$ with $z_0 \in U$ and $U \cap K_0 = \emptyset$. Clearly, U is closed in K. We infer $U \subseteq C(z_0) \cap K$ from $\partial_{\gamma^*} C(z_0) \subseteq K_0$ and $\widetilde{K} \setminus (C(z_0) \cap K) \subseteq \partial_{\gamma^*} C(z_0)$. Thus, it is easy to see that U is also open in K.

By 3.5, there exists a sequence $(\widetilde{f}_n)_{n=1}^\infty$ in $D^1(\widetilde{K})$ such that $(\widetilde{f}_n)_{n=1}^\infty$ converges uniformly on $\widetilde{K}$ to the zero function and $(\widetilde{f}_n')_{n=1}^\infty$ converges uniformly on $\widetilde{K}$ to $\mathbb{1}_U|_{\widetilde{K}}$. Since U is a clopen subset of K, the functions

$$f_n : K \to \mathbb{C};\ z \mapsto \begin{cases} \widetilde{f}_n(z) & \text{if } z \in U, \\ 0 & \text{if } z \notin U, \end{cases}$$

are obviously differentiable on K with continuous derivatives

$$f_n'(z) = \begin{cases} \widetilde{f}_n'(z) & \text{if } z \in U, \\ 0 & \text{if } z \notin U, \end{cases}$$

for all $z \in K$. The sequence $(f_n)_{n=1}^{\infty}$ clearly converges uniformly on K to the zero function, whereas $(f_n')_{n=1}^{\infty}$ converges uniformly on K to the indicator function of U. By 3.2, this shows that $\widetilde{D}^1(K)$ is not semisimple. ∎

We have the following immediate corollaries.

Corollary 4.7. *The conclusion of* 4.6 *also holds for a simple closed, rectifiable path* γ *instead of a Jordan path.*

Proof. For $K = \gamma^*$, there is nothing to show. For $K \subsetneq \gamma^*$, the set K lies on a Jordan path running in γ^*. ∎

Theorem 4.8. *Let* $K \subseteq \gamma^*$ *for a* (*not necessarily rectifiable*) *Jordan path* $\gamma : [0,1] \to \mathbb{C}$. *If the Banach algebra* $\widetilde{D}^1(K)$ *is semisimple, then the union of all components of* K *having more than one point is dense in* K, *and the latter occurs if and only if the interior of* K *relative to* γ^* *is dense in* K.

Proof. Observe that in the proof of 4.6 we needed the rectifiability of γ only to establish the third equivalence and the sufficiency in the first equivalence statement. ∎

Remark 4.9. If X is any non-empty, compact subset of $\mathbb{C}$, one can consider the operator

$$T : (\mathcal{P}_0(X), \|\cdot\|_X) \to (\mathcal{C}(X), \|\cdot\|_X);\ \sum_{n=0}^{N} a_n Z^n \mapsto \sum_{n=1}^{N} n a_n Z^{n-1}$$

and ask for a characterization of the closability of T. In Theorem 3.1 of [2], this question is completely answered for $X \subseteq \mathbb{R}$. In this case, T is closable if and only if the interior of X relative to $\mathbb{R}$ is dense in X. Thus, 4.6 can be considered as an analogous result for D and non-empty, perfect, compact subsets of rectifiable Jordan paths. In addition, if X is a non-empty, perfect, compact subset of $\mathbb{R}$, a consequence is that $\widetilde{D}^1(X)$ is semisimple if and only if T is closable.

5. Completeness of $D^1(K)$. Our main tool to examine the completeness of $D^1(K)$ is the next theorem. It gives a new operator-theoretic characterization of the completeness in terms of the mapping properties of the derivation operator.

Theorem 5.1. *The following assertions are equivalent:*

(i) *The normed algebra* $D^1(K)$ *is complete.*
(ii) *The operator* $D : (D^1(K), \|\cdot\|_K) \to \mathcal{C}(K)$ *is closed.*

(iii) *The operator* $D : (D^1(K), \|\cdot\|_K) \to \mathcal{C}(K)$ *is closed,* $\operatorname{ran} D$ *is closed in* $\mathcal{C}(K)$ *and* $\dim\ker D < \infty$ *(i.e., D is an upper semi-Fredholm operator).*

(iv) *The subspaces* $\ker D$ *and* $\operatorname{ran} D$ *are closed in* $C(K)$ *and the operator*

$$\widehat{D} : (D^1(K)/\ker D, \|\cdot\|_Q) \to (\operatorname{ran} D, \|\cdot\|_K)$$

is continuously invertible, where $\|\cdot\|_Q$ *denotes the quotient norm on* $D^1(K)/\ker D$ *arising from* $\|\cdot\|_K$.

Proof. The equivalence (i)⇔(ii) has already been shown in 3.3 in a more general setting. The implication (iii)⇒(ii) is trivial.

The implication (iv)⇒(ii) can be deduced from IV.1.6 and IV.1.7 in [19].

To complete the proof, we show that (i) implies both (iii) and (iv). Clearly, D is closed, and hence $\ker D$ is closed in $\mathcal{C}(K)$. Since $D^1(K)$ is complete, K has only finitely many components (see 2.3 in [6]), say $K_1, \dots, K_n$, and all of them are connected, compact and perfect. Moreover, it is easy to establish the completeness of $(D^1(K_j), \|\cdot\|_{D^1(K_j)})$ for all $j \in \{1, \dots, n\}$. As a result, $\ker D = \mathrm{LH}\{\mathbb{1}_{K_j};\, j \in \{1, \dots, n\}\}$, by Lemma 9.2 in [8]; in particular, $\ker D$ is finite-dimensional.

Now we fix a tuple $(z_1, \dots, z_n) \in K_1 \times \cdots \times K_n$. For each $f \in \operatorname{ran} D$, there exists, because of $\ker D = \mathrm{LH}\{\mathbb{1}_{K_j};\, j \in \{1, \dots, n\}\}$, exactly one function $Tf \in D^1(K)$ with $(Tf)' = f$ and $(Tf)(z_j) = 0$ for all $j \in \{1, \dots, n\}$. This uniqueness implies that the map $T : \operatorname{ran} D \to D^1(K)$ defined this way is linear. Theorem 9.3 in [8] gives constants $c_1, \dots, c_n \in (0, \infty)$ (each only depending on z_j and K_j) such that

$$\begin{aligned}\|Tf\|_K &= \max_{j=1,\dots,n} \|(Tf)|_{K_j}\|_{K_j} \le \max_{j=1,\dots,n} c_j \|((Tf)|_{K_j})'\|_{K_j} \\ &= \max_{j=1,\dots,n} c_j \|f|_{K_j}\|_{K_j} \le \max\{c_j;\, j = 1, \dots, n\} \cdot \|f\|_K.\end{aligned}$$

Consequently, $T : (\operatorname{ran} D, \|\cdot\|_K) \to (D^1(K), \|\cdot\|_K)$ is continuous. Let π denote the canonical epimorphism $\mathcal{C}(K) \to \mathcal{C}(K)/\ker D$. Then

$$I := \pi \circ T : (\operatorname{ran} D, \|\cdot\|_K) \to (D^1(K)/\ker D, \|\cdot\|_Q)$$

is continuous and linear. For all $f \in \operatorname{ran} D$, we have

$$\widehat{D}(If) = \widehat{D}(Tf + \ker D) = (Tf)' = f,$$

and for all $f \in D^1(K)$, we have

$$I(\widehat{D}(f + \ker D)) = Tf' + \ker D = f + \ker D,$$

in view of $(Tf')' = f'$. This means that I is the inverse of $\widehat{D}$. Thus, $\widehat{D}$ is continuously invertible. Note that $\widehat{D}$ is closed whenever D is closed (see, e.g., II.4.7(i) in [19]). As a consequence, I is itself closed as the inverse of a closed operator. However, since I is both continuous and closed, $(\operatorname{ran} D, \|\cdot\|_K)$

must be a Banach space, i.e., $\operatorname{ran} D$ is closed in $\mathcal{C}(K)$. Hence, the theorem is established. ∎

Next, we deduce some corollaries from the theorem just proved.

Once the conditions of Theorem 5.1 have been checked for certain well-behaved compact sets K, the characterization of the completeness simplifies for certain subsets of K, as shown in the following statements.

First, we need some notation. Consider a non-empty, perfect, compact subset X of K as well as a set $\mathcal{F}$ of functions on K. We then denote by $\mathcal{F}|_X$ the set $\{f|_X;\ f \in \mathcal{F}\}$ and by D_X the derivation operator on $D^1(X)$, i.e., the mapping

$$D_X : D^1(X) \to \mathcal{C}(X);\ f \mapsto f'.$$

Lemma 5.2. *Let $(D^1(K), \|\cdot\|)$ be complete, and let X be a non-empty, perfect, compact subset of K such that there is a constant $M > 0$ with the property that for every $f \in \operatorname{ran} D_X$ we can find $\widehat{f} \in \operatorname{ran} D$ satisfying $\widehat{f}|_X = f$ and $\|\widehat{f}\|_K \le M\|f\|_X$. Then:*

(i) *The induced operator*

$$\widehat{D_X} : D^1(X)/\ker D_X \to \operatorname{ran} D_X$$

is continuously invertible, where $D^1(X)/\ker(D_X)$ is endowed with the quotient seminorm arising from $\|\cdot\|_X$ and denoted by $\|\cdot\|_Q$ and $\operatorname{ran} D_X$ carries the norm topology of $\|\cdot\|_X$.

(ii) *The following assertions are equivalent:*

(a) *$D^1(X)$ is complete.*

(b) *$\dim\ker D_X < \infty$ and $\operatorname{ran} D_X$ is closed in $\mathcal{C}(X)$.*

(c) *Both $\ker D_X$ and $\operatorname{ran} D_X$ are closed in $\mathcal{C}(X)$.*

Proof. Part (ii) follows easily from (i) and 5.1. To prove (i), let $K_1, \dots, K_n$ and $(z_1, \dots, z_n) \in K_1 \times \dots \times K_n$ be as in the proof of 5.1. As in that proof we see that for each $f \in \operatorname{ran} D_X$ there exists a unique $Tf \in D^1(K)$ with $(Tf)' = \widehat{f}$ and $Tf(z_j) = 0$ for all $j \in \{1, \dots, n\}$, where $\widehat{f}$ is as in the statement. In particular, $(Tf)|_X \in D^1(X)$ with $((Tf)|_X)' = f$. As in the proof of 5.1, one gets a constant $C > 0$ (only depending on $z_1, \dots, z_n$) such that

$$\begin{aligned}\|(Tf)|_X + \ker D_X\|_Q \le \|(Tf)|_X\|_X \le \|Tf\|_K \le C\|(Tf)'\|_K \\ = C\|\widehat{f}\|_K \le CM\|f\|_X\end{aligned}$$

for all $f \in \operatorname{ran} D_X$. As a consequence,

$$\operatorname{ran} D_X \to D^1(X)/\ker D_X;\ f \mapsto (Tf)|_X + \ker D_X$$

is the required continuous inverse to $\widehat{D_X}$. ∎

COROLLARY 5.3. *Let $D : (D^1(K), \|\cdot\|_K) \to \mathcal{C}(K)$ be closed and surjective and let X be a non-empty, perfect, compact subset of K. Then the following assertions are equivalent:*

(i) *$D^1(X)$ is complete.*
(ii) *D_X has a kernel of finite dimension.*

Proof. The implication from (i) to (ii) is clear in view of 5.1. So assume conversely that condition (ii) holds. Extend any $f \in \mathcal{C}(X)$ to a function $\widehat{f} \in \mathcal{C}(K)$ with $\|\widehat{f}\|_K = \|f\|_X$. Our assumption implies the existence of $F \in D^1(K)$ with $F' = \widehat{f}$. In particular, $F|_X \in D^1(X)$ and $(F|_X)' = f$. Thus, $\operatorname{ran} D_X = \mathcal{C}(X)$. Now, apply 5.2. ∎

REMARK 5.4. Assume that K is polynomially convex and $D^1(K)$ is complete. Let $\emptyset \neq X \subseteq \partial K$ be perfect and compact. If $\operatorname{ran} D_X$ is closed and coincides with $\mathcal{A}(K)|_X$, then all conditions of 5.2 are fulfilled.

Proof. Mergelyan's theorem (see, e.g., 9.1 in Chapter II of [17]) implies that $\mathcal{A}(K) = \mathcal{P}(K)$. We know that $\mathcal{P}(K)$ is a Dirichlet algebra on ∂K (see, e.g., 3.4 in Chapter II of [17]). If $\operatorname{ran} D_X = \mathcal{A}(K)|_X$ is closed, Glicksberg's interpolation theorem (see 4.6 in [12]) shows that for every $f \in \operatorname{ran} D_X$ we can find $\widehat{f} \in \mathcal{A}(K)$ with $\widehat{f}|_X = f$ and $\|\widehat{f}\|_K = \|f\|_X$. Since $D^1(K)$ is assumed to be complete, we conclude that $\mathcal{P}(K) \subseteq \operatorname{ran} D \subseteq \mathcal{A}(K)$ by means of 5.1. As a result, $\operatorname{ran} D = \mathcal{A}(K)$. ∎

We now turn to sets K satisfying the rather mild conditions of rectifiable connectedness and geodesical boundedness. In this context, Theorem 5.1 also allows simplified characterizations of the completeness of $D^1(K)$.

COROLLARY 5.5. *If K is rectifiably connected and geodesically bounded, then the following assertions are equivalent:*

(i) *$D^1(K)$ is complete.*
(ii) $\operatorname{ran} D$ *is closed in $\mathcal{C}(K)$.*

Proof. The implication (i)⇒(ii) is clear by now. For the converse, observe that in the case of a rectifiably connected K the fundamental theorem of calculus on rectifiable paths implies that $\ker D$ consists only of constant functions. Because of 5.1 and the premise that $\operatorname{ran} D$ is closed in $\mathcal{C}(K)$ it is only left to show that

$$\widehat{D} : (D^1(K)/\ker D, \|\cdot\|_Q) \to (\operatorname{ran} D, \|\cdot\|_K)$$

is continuously invertible (where we use the notation of 5.1). For this purpose we fix a $z_0 \in K$ as well as a geodesic path γ_z joining z_0 to z for each $z \in K$. For $f \in \operatorname{ran} D$, Theorem 2.6 shows that

$$Tf : K \to \mathbb{C};\ z \mapsto \int_{\gamma_z} f(\zeta)\, d\zeta$$

is a primitive function for f. The mapping

$$T : (\operatorname{ran} D, \|\cdot\|_K) \to (D^1(K), \|\cdot\|_K)$$

defined in this way is obviously linear and satisfies

$$\|Tf\|_K \le \sup_{z\in K} L(\gamma_z)\|f\|_K = \sup_{z\in K} \delta_K(z, z_0)\|f\|_K \le \sup_{z,w\in K} \delta_K(z, w)\|f\|_K$$

for every $f \in \operatorname{ran} D$, i.e., T is continuous. As in the proof of 5.1, one now concludes that the map $\pi\circ T$ (where $\pi : \mathcal{C}(K) \to \mathcal{C}(K)/\ker D$ is the canonical epimorphism) is the continuous inverse to $\widehat{D}$. ■

We now assume that the compact set K is rectifiably connected and geodesically bounded, that $\mathbb{C} \setminus K$ has finitely many bounded components $G_1, \dots, G_n$ and that for each $j \in \{1, \dots, n\}$ there exists a rectifiable closed path γ_j in K such that

$$\operatorname{ind}_{\gamma_j}(G_k) \begin{cases} \neq 0 & \text{if } k = j, \\ = 0 & \text{if } k \neq j. \end{cases}$$

Then $\mathcal{A}(K) = \mathcal{R}(K)$ (see II.10.4 in [17]). Furthermore, we put

$$\mathfrak{A} := \{A \subseteq \mathbb{C} \setminus K;\ \forall j \in \{1, \dots, n\} : A \cap G_j \neq \emptyset\}.$$

and $\mathfrak{A} = \{\emptyset\}$ if K is polynomially convex. For $A \in \mathfrak{A} \setminus \{\emptyset\}$, we define

$$X_A := \mathrm{LH}(\mathcal{P}_0(K) \cup \{K \ni z \mapsto (z - a)^{-m};\ m \in \mathbb{N} \text{ with } m \ge 2,\ a \in A\}).$$

Furthermore, we set $X_\emptyset = \mathcal{P}_0(K)$. If $(\alpha_1, \dots, \alpha_n)$ is any tuple of rectifiable closed paths in K with

$$\operatorname{ind}_{\alpha_j}(G_k) \begin{cases} \neq 0 & \text{if } k = j, \\ = 0 & \text{if } k \neq j, \end{cases}$$

we denote by $\varphi_{(\alpha_1,\dots,\alpha_n)} : \mathcal{C}(K) \to \mathbb{C}^n$ the continuous linear map defined via

$$\varphi_{(\alpha_1,\dots,\alpha_n)}(f) := \Big(\int_{\alpha_1} f(z)\, dz, \dots, \int_{\alpha_n} f(z)\, dz\Big)$$

for all $f \in \mathcal{C}(K)$.

The proof of the following statement is similar to that of 5.9 in [6] and therefore omitted.

LEMMA 5.6. *For any tuple $(\alpha_1, \dots, \alpha_n)$ of rectifiable closed paths in K as described above and any tuple $(a_1, \dots, a_n) \in \prod_{j=1}^n G_j$, we have the equality* $\overline{X_{\{a_1,\dots,a_n\}}} = \ker(\varphi_{(\alpha_1,\dots,\alpha_n)}|_{\mathcal{R}(K)})$. ■

We have the following characterization of the completeness of $D^1(K)$ in terms of the range of D only.

THEOREM 5.7. *In the above circumstances, the following assertions are equivalent:*

(i) $D^1(K)$ *is complete.*
(ii) $\operatorname{ran} D = \overline{X_A}$ *for some (or each) set* $A \in \mathfrak{A}$.
(iii) *For some (or each) tuple* $(\alpha_1, \dots, \alpha_n)$ *of rectifiable closed paths in* K *as described above, we have* $\operatorname{ran} D = \ker(\varphi_{(\alpha_1,\dots,\alpha_n)}|_{\mathcal{R}(K)})$.

Proof. The universally quantified statement in (ii) implies the corresponding statement in (iii) by 5.6. The universally quantified statement in (iii) trivially implies the existence assertion, which gives us the existence assertion in (ii) because of 5.6. Furthermore, the existence assertion in (ii) implies (i) due to 5.5.

To complete the proof, we show that (i) yields the universally quantified assertion in (ii). Let $A \in \mathfrak{A}$. We choose an $a_j \in A \cap G_j$ for each $j \in \{1, \dots, n\}$. We then conclude

$$X_{\{a_1,\dots,a_n\}} \subseteq \operatorname{ran} D \subseteq \ker(\varphi_{(\alpha_1,\dots,\alpha_n)}|_{\mathcal{R}(K)}) = \overline{X_{\{a_1,\dots,a_n\}}},$$

by means of 5.6, thus arriving at $\operatorname{ran} D = \overline{X_{\{a_1,\dots,a_n\}}}$ by using 5.5. Clearly, $X_{\{a_1,\dots,a_n\}} \subseteq X_A$. Mergelyan's theorem yields $\mathcal{A}(\mathrm{PCH}(K)) = \mathcal{P}(\mathrm{PCH}(K))$. As a result, all functions of the type $K \to \mathbb{C}$; $z \mapsto 1/(z-a)$ with $a \in \mathbb{C} \setminus \mathrm{PCH}(K)$ can be uniformly approximated on K by holomorphic polynomials. Thus

$$X_A \subseteq \mathrm{LH}\Big(\bigcup_{(b_1,\dots,b_n)\in\prod_{j=1}^n (A\cap G_j)} X_{\{b_1,\dots,b_n\}} \cup \mathcal{P}(K)\Big).$$

The right hand side is clearly a subset of $\ker(\varphi_{(\alpha_1,\dots,\alpha_n)}|_{\mathcal{R}(K)}) = \overline{X_{\{a_1,\dots,a_n\}}}$. Summarizing, we now obtain the asserted equality $\overline{X_A} = \overline{X_{\{a_1,\dots,a_n\}}} = \operatorname{ran} D$. ∎

In particular, we can apply the above characterization to the case where K is rectifiably connected, geodesically bounded and polynomially convex:

THEOREM 5.8. *If* K *is rectifiably connected, geodesically bounded and polynomially convex, then the following assertions are equivalent:*

(i) $D^1(K)$ *is complete.*
(ii) $\operatorname{ran} D = \mathcal{A}(K)$. ∎

Note that in this theorem the implication (i)⇒(ii) even holds if we drop the rectifiable connectedness and geodesical boundedness of K (cf. Theorem 9.6 in [8]); indeed, this follows immediately from 5.1 and Mergelyan's theorem (see the proof for (i)⇒(ii) of Theorem 5.14 below).

In Section 2, we defined the notion of a pointwise and uniformly regular, compact plane set K. In the following results we explore the relation of these properties to the completeness of $D^1(K)$, thus indicating the importance of these sets.

LEMMA 5.9. *If K is pointwise regular, then a function $f \in \mathcal{C}(K)$ has a primitive function in $D^1(K)$ if and only if $\int_\gamma f(z)\,dz = 0$ for every rectifiable closed path γ contained in K.*

Proof. The necessity is an immediate consequence of the fundamental theorem of calculus for rectifiable paths. We fix an $a \in K$. For $z \in K$, let γ_z be any rectifiable path in K joining a to z. We define $F(z) := \int_{\gamma_z} f(\zeta)\,d\zeta$ for $z \in K$. Furthermore, we fix a $z_0 \in K$. Let $\epsilon > 0$. Then there is an $M > 0$ such that $\delta_K(z_0, w) \le M|z_0 - w|$ for all $w \in K$. Moreover, there exists a $\delta > 0$ such that $|f(z_0) - f(w)| < \epsilon/M$ for all $w \in K$ with $|z_0 - w| < \delta$. Consider any $z \in K$ with $0 < |z_0 - z| < \delta/M$. In addition, let $\gamma_{z_0,z}$ be an injective geodesic path in K joining z_0 to z. Then $F(z) - F(z_0) = \int_{\gamma_{z_0,z}} f(\zeta)\,d\zeta$. This yields, by straightforward estimates, $|(z - z_0)^{-1} \cdot (F(z) - F(z_0)) - f(z_0)| < \epsilon$. In summary, $F \in D^1(K)$ is a primitive for f. ∎

Clearly, the preceding result is inspired by an analogous result well-known from complex analysis.

The following statement, for which different proofs have already been given, e.g., in [7], [22] and [8], flows from the theory developed so far in a very natural way.

THEOREM 5.10 ([8, 2.7]). *If K is the union of finitely many pointwise regular, compact sets, then $D^1(K)$ is complete.*

Proof. With a standard argument, one reduces the assertion to the case where K itself is pointwise regular. Furthermore, the triangle inequality for δ_K and the pointwise regularity immediately imply that K is geodesically bounded. Due to 5.5, it only remains to show that $\operatorname{ran} D$ is closed in $\mathcal{C}(K)$. Let $(f_n)_{n=1}^\infty$ be a sequence in $\operatorname{ran} D$ converging to some function f in $\mathcal{C}(K)$. Then for each rectifiable closed path γ in K,

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0,$$

which implies that $f \in \operatorname{ran} D$ by 5.9. ∎

In particular, all examples of pointwise and uniformly regular, compact plane sets provided in the first section also serve as examples of perfect compact sets K such that $D^1(K)$ is complete.

In [8], Dales and Feinstein asked whether the converse of 5.10 is also true. Despite some positive results they obtained, the general question remained open. Although we cannot decide this question either (even for the specific class of sets discussed in §10 of [8]), our next aim is to continue these studies. We start with a characterization of pointwise regularity under suitable circumstances.

Definition 5.11. Let K be rectifiably connected and fix $z \in K$. For each $w \in K$, choose a rectifiable path γ_w joining z to w and define (with the notation of Lemma 2.5)

$$q_K(z) := \inf_{w \in K \setminus \{z\}} \frac{1}{\delta_K(z,w)} \Big\| \int_{\gamma_w} \cdot \, d\zeta \Big\|_{\operatorname{ran}(D)'}.$$

Here, we denote by $\operatorname{ran}(D)'$ the topological dual space of $(\operatorname{ran}(D), \|\cdot\|_K)$ and by $\|\cdot\|_{\operatorname{ran}(D)'}$ the corresponding norm of linear functionals. Theorem 2.6 guarantees that this definition is legitimate, i.e., independent of the chosen path γ_w.

Lemma 5.12. *If K is rectifiably connected and $D^1(K)$ is complete, then the following assertions are equivalent for each $z \in K$:*

(i) *K is regular at z;*
(ii) $q_K(z) > 0$.

Proof. (i)⇒(ii): By assumption, there is a $C > 0$ such that for all $w \in K$ the inequality $\delta_K(z,w) \le C|z-w|$ holds. This yields

$$\frac{1}{\delta_K(z,w)} \Big\| \int_{\gamma_w} \cdot \, d\zeta \Big\|_{\operatorname{ran}(D)'} \ge \frac{1}{\delta_K(z,w)} \Big| \int_{\gamma_w} 1 \, d\zeta \Big| = \frac{|z-w|}{\delta_K(z,w)} \ge \frac{1}{C}$$

for all $w \in K \setminus \{z\}$, which implies that $q_K(z) > 0$.

(ii)⇒(i): Since K is connected and $D^1(K)$ is complete, there exists an $A > 0$ such that $|f(z) - f(w)| \le A|z-w| \cdot \|f'\|_K$ for all $w \in K$ and $f \in D^1(K)$ (see 9.3 in [8]). This leads to

$$\Big\| \int_{\gamma_w} \cdot \, d\zeta \Big\|_{\operatorname{ran}(D)'} = \sup_{f \in D^1(K) \setminus \mathbb{C}} \frac{1}{\|f'\|_K} \Big| \int_{\gamma_w} f'(\zeta) \, d\zeta \Big| \le A|z-w|$$

for all $w \in K$. As a result, for all $w \in K \setminus \{z\}$,

$$q_K(z)\delta_K(z,w) \le \Big\| \int_{\gamma_w} \cdot \, d\zeta \Big\|_{\operatorname{ran}(D)'} \le A|z-w|$$

and thus

$$\delta_K(z,w) \le \frac{A}{q_K(z)} |z-w|$$

because $q_K(z) > 0$. But this means precisely that K is regular at z. ∎

Theorem 5.13. *If K is rectifiably connected such that $D^1(K)$ is complete and* $\operatorname{ran} D = \mathcal{C}(K)$, *then K is pointwise regular.*

Proof. Fix $z \in K$. For an arbitrary $w \in K \setminus \{z\}$ let γ_w be an injective geodesic path joining z to w. Then 2.5 yields

$$\frac{1}{\delta_K(z,w)} \Big\| \int_{\gamma_w} \cdot \, d\zeta \Big\|_{\mathrm{ran}(D)'} = \frac{1}{\delta_K(z,w)} \Big\| \int_{\gamma_w} \cdot \, d\zeta \Big\|_{\mathcal{C}(K)'} = \frac{L(\gamma_w)}{\delta_K(z,w)} = 1,$$

which implies the assertion by means of 5.12. ■

THEOREM 5.14. *If K is polynomially convex and rectifiably connected with* $\mathrm{int}(K) = \emptyset$, *then the following assertions are equivalent:*

(i) $D^1(K)$ *is complete.*
(ii) K *is pointwise regular.*
(iii) K *is geodesically bounded and* $\mathrm{ran}\, D = \mathcal{C}(K)$.

Proof. (i)⇒(ii): Mergelyan's theorem and 5.1 yield

$$\mathcal{P}(K) \subseteq \mathrm{ran}\, D \subseteq \mathcal{A}(K) = \mathcal{P}(K),$$

which implies that $\mathrm{ran}\, D = \mathcal{C}(K)$ due to $\mathrm{int}(K) = \emptyset$. This gives us (ii) by 5.13.

(ii)⇒(iii): As indicated in the proof of 5.10, pointwise regularity always implies geodesical boundedness. Moreover, if (ii) holds, then so does (i) by 5.10. This leads to $\mathrm{ran}\, D = \mathcal{A}(K) = \mathcal{C}(K)$ by means of 5.8, taking into consideration $\mathrm{int}(K) = \emptyset$.

(iii)⇒(i): This follows from 5.8 as $\mathcal{A}(K) = \mathcal{C}(K)$. ■

The implication (i)⇒(ii) in the preceding theorem can also be found in [8, Theorem 10.2]. Although the proof given there shares some important ingredients with ours, it is indirect and cannot easily be turned into a direct one, while we have given here a direct proof.

By applying 5.14 in the case where K is the image of a rectifiable Jordan path and by observing that γ^* is automatically geodesically bounded for a rectifiable path γ, we obtain the following corollary.

COROLLARY 5.15 (see also [8]). *If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable Jordan path, then the following statements are equivalent:*

(i) $D^1(\gamma^*)$ *is complete.*
(ii) γ^* *is pointwise regular.*
(iii) $D : D^1(\gamma^*) \to \mathcal{C}(\gamma^*)$ *is surjective.*

REMARK 5.16. Using Theorem 2.3 in [6], Corollary 5.15 and Theorem 5.10 as well as Theorem 4.6, we are now able to completely answer the questions asked in the introduction in the case of compact, perfect subsets K of a rectifiable injective curve γ. Moreover, we could even describe $D^1(K)$ and $\widetilde{D}^1(K)$ quite explicitly in the case of completeness respectively semisimplicity by applying Theorem 7.2 in [8] respectively Theorem 5.7 in [6]. In addition, note that for a non-empty, perfect, compact subset X of γ^* with γ^* being pointwise regular, the space $D^1(X)$ is complete if and only if X has only finitely many components.

We note at this point that the implication (i)⇒(ii) in 5.15 is even true for non-rectifiable paths (see 10.5 in [8]). Dales and Feinstein show that this implication also holds if K is starshaped (see 10.14 in [8]) and even for another kind of sets (see the remarks subsequent to 10.12 in [8]).

However, all these compact sets are polynomially convex, and so far there seems to be no general result concerning classes of compact plane sets that are not polynomially convex. We next prove such a result by using very simple considerations.

Let $\mathcal{K}(\mathbb{C})$ be the set of all non-empty, perfect, compact subsets of $\mathbb{C}$, and denote by $\mathfrak{R}(\mathbb{C})$ the set

$$\{K \in \mathcal{K}(\mathbb{C});\ K \text{ is a finite union of pointwise regular, compact plane sets}\}.$$

We call a set $\mathfrak{A} \subseteq \mathcal{K}(\mathbb{C})$ *admissible* if

$$\forall K \in \mathfrak{A}: \quad (D^1(K) \text{ is complete} \Rightarrow K \in \mathfrak{R}(\mathbb{C})).$$

We set $\mathfrak{C} := \bigcup\{\mathfrak{A};\ \mathfrak{A} \subseteq \mathcal{K}(\mathbb{C}) \text{ is admissible}\}$, the biggest admissible subset of $\mathcal{K}(\mathbb{C})$. The question about the converse of 5.10 can now be formulated in the following way: Does $\mathfrak{C} = \mathcal{K}(\mathbb{C})$ hold? Clearly, $\mathfrak{C}$ contains all $K \in \mathcal{K}(\mathbb{C})$ such that $D^1(K)$ is not complete, and $\mathfrak{C}$ is a superset of $\mathfrak{R}(\mathbb{C})$. It is equally apparent that $\mathfrak{C}$ is stable with respect to finite disjoint unions. In fact, a slightly more general result holds.

THEOREM 5.17. *Let $K_1, \dots, K_n$ be sets in $\mathfrak{C}$ satisfying*

$$\sharp\Big(K_k \cap \bigcup_{\substack{j=1 \\ j\neq k}}^{n} K_j\Big) < \infty$$

for all $k \in \{1, \dots, n\}$. Then $K := \bigcup_{j=1}^n K_j \in \mathfrak{C}$.

Proof. If $D^1(K)$ is not complete, there is nothing to show. So assume that $D^1(K)$ is complete. Obviously, it is sufficient to prove that all K_j belong to $\mathfrak{R}(\mathbb{C})$ (as $\mathfrak{R}(\mathbb{C})$ is stable with respect to finite unions). Assume to the contrary that, say, $K_1 \notin \mathfrak{R}(\mathbb{C})$. As $K_1 \in \mathfrak{C}$, the space $D^1(K_1)$ cannot be complete. Accordingly, there exists a Cauchy sequence $(f_m)_{m=1}^\infty$ in $D^1(K_1)$ with no limit in $D^1(K_1)$. If $K_1 \cap \bigcup_{j=2}^n K_j = \emptyset$, one easily deduces a contradiction to the completeness of $D^1(K)$. Thus, assume $K_1 \cap \bigcup_{j=2}^n K_j \neq \emptyset$. By the premise, there are pairwise distinct points $a_1, \dots, a_r \in \mathbb{C}$ such that $K_1 \cap \bigcup_{j=2}^n K_j = \{a_1, \dots, a_r\}$. Hermite interpolation yields (unique) polynomials

$p_{1,0}, p_{1,1}, p_{2,0}, p_{2,1}, \ldots, p_{r,0}, p_{r,1}$ with degree less than or equal to $2r$ such that

$$p_{k,l}^{(j)}(a_\nu) = \begin{cases} 1 & \text{if } j = l \text{ and } \nu = k, \\ 0 & \text{else,} \end{cases}$$

for all $((j,l),(k,\nu)) \in \{0,1\}^2 \times \{1,\ldots,r\}^2$.

For $m \in \mathbb{N}$ and $z \in K$, we now define

$$g_m(z) := \sum_{j=1}^{r} \left(f_m(a_j) p_{j,0}(z) + f_m'(a_j) p_{j,1}(z) \right).$$

Clearly $g_m \in D^1(K)$ with $g_m(a_j) = f_m(a_j)$ and $g_m'(a_j) = f_m'(a_j)$ for all $m \in \mathbb{N}$ and all $j \in \{1,\ldots,r\}$. Next, we set

$$F_m : K \to \mathbb{C};\ z \mapsto \begin{cases} f_m(z) & \text{if } z \in K_1, \\ g_m(z) & \text{if } z \notin K_1, \end{cases}$$

for each $m \in \mathbb{N}$. A standard argument shows that $F_m \in D^1(K)$ with

$$F_m'(z) = \begin{cases} f_m'(z) & \text{if } z \in K_1, \\ g_m'(z) & \text{if } z \notin K_1, \end{cases}$$

for all $z \in K$ and all $m \in \mathbb{N}$. Now put $M := \max_{l,j=0,1} \max_{k=1,\ldots,r} \|p_{k,l}^{(j)}\|_K$. Then for $m_1, m_2 \in \mathbb{N}$, $j \in \{0,1\}$ and $z \in K$,

$$\begin{aligned} |g_{m_1}^{(j)}(z) - g_{m_2}^{(j)}(z)| &\le M \sum_{j=1}^{r} (|f_{m_1}(a_j) - f_{m_2}(a_j)| + |f_{m_1}'(a_j) - f_{m_2}'(a_j)|) \\ &\le Mr \|f_{m_1} - f_{m_2}\|_{D^1(K_1)}. \end{aligned}$$

Therefore,

$$\|F_{m_1} - F_{m_2}\|_{D^1(K)} \le \max\{1, rM\} \|f_{m_1} - f_{m_2}\|_{D^1(K_1)}.$$

As a result, $(F_m)_{m=1}^\infty$ is a Cauchy sequence in $D^1(K)$ and thus converges to some F in $D^1(K)$ by the completeness of $D^1(K)$. But then $(f_m)_{m=1}^\infty$ tends to $F|_{K_1}$ in $D^1(K_1)$, contradicting the choice of the sequence. ∎

From this simple result one easily obtains interesting and in general not polynomially convex compact plane sets for which the converse of 5.10 is true. The following corollary gives us a class of such examples.

COROLLARY 5.18. *Let $\gamma : [0,1] \to \mathbb{C}$ be a path such that:*

(i) $\sharp\gamma^{-1}(\{z\}) < \infty$ *for every $z \in \mathbb{C}$ ("each point in the plane is passed through at most finitely many times").*
(ii) $\sharp\{z \in \mathbb{C};\ \sharp\gamma^{-1}(\{z\}) > 1\} < \infty$ *("there are only finitely many points passed through more than once").*

Then $D^1(\gamma^*)$ is complete if and only if γ^* is pointwise regular.

Proof. First, we treat the special case where γ is a simple closed path. Then $\gamma|_{[0,1/2]}$ and $\gamma|_{[1/2,1]}$ are Jordan paths and $(\gamma|_{[0,1/2]})^* \cap (\gamma|_{[1/2,1]})^* = \{\gamma(0), \gamma(1/2)\}$. In this situation, the assertion follows from Theorem 10.5 in [8] and from 5.15 above combined with Theorem 5.17. Let us now turn to the general case. Without much effort, one can see that the conditions imposed on γ yield the existence of $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma^{-1}(\{\gamma(t)\}) = \{t\}$ for all $t \in [0,1] \setminus \{t_0, \ldots, t_n\}$. Consequently, each path $\gamma|_{[t_{j-1},t_j]}$, where $j = 1, \ldots, n$, is either a simple closed path or a Jordan path and we have

$$(\gamma|_{[t_{j-1},t_j]})^* \cap \bigcup_{\substack{k=1\\k\neq j}}^{n} (\gamma|_{[t_{k-1},t_k]})^* \subseteq \{\gamma(t_{j-1}), \gamma(t_j)\}$$

for all $j \in \{1, \ldots, n\}$. Hence, the special case gives the assertion by applying 10.5 in [8] and 5.15 in combination with 5.17. ∎

6. Applications. This section concerns some applications of the results obtained in the preceding section, especially to properties of certain holomorphic functions.

First, we assume K is rectifiably connected and geodesically bounded, the space $D^1(K)$ is complete, $\mathbb{C} \setminus K$ has finitely many bounded components $G_1, \ldots, G_n$, the set $G := \operatorname{int}(K)$ is connected and dense in K and for each $j \in \{1, \ldots, n\}$ there is a rectifiable closed path γ_j in G with

$$\operatorname{ind}_{\gamma_j}(G_k) \begin{cases} \neq 0 & \text{if } k = j, \\ = 0 & \text{if } k \neq j. \end{cases}$$

We recall once again that in this situation $\mathcal{A}(K) = \mathcal{R}(K)$ (see II.10.4 in [17]).

Theorem 6.1. *If $f \in \mathcal{O}(G)$ (with G as above) and if there exists an $n \in \mathbb{N}$ such that $f^{(n)}$ can be continuously extended to $K = \overline{G}$, then every function $f^{(j)}$ for $j = 0, \ldots, n-1$ can be extended to a continuously differentiable function on K.*

Proof. A proof by induction reduces the assertion to the case $n = 1$. Let $\widetilde{f}$ be the continuous extension of f' to the whole of K. Fix $z_0 \in G$. Since $\gamma_j^* \subseteq G$ for all $j \in \{1, \ldots, n\}$, as well as $f' = \widetilde{f}|_G$ and $\widetilde{f} \in \mathcal{A}(K) = \mathcal{R}(K)$, we deduce that

$$\widetilde{f} \in \ker(\varphi_{(\gamma_1,\ldots,\gamma_n)}|_{\mathcal{R}(K)})$$

(notation as in 5.6). Then 5.7 yields $\widetilde{f} \in \operatorname{ran} D$. Let $F \in D^1(K)$ be the (unique) primitive of $\widetilde{f}$ fulfilling $F(z_0) = f(z_0)$. Because of $(F|_G)' = \widetilde{f}|_G = f'$, $\overline{G} = K$ and the connectedness of G, we conclude that F is the desired extension of f. ∎

For a non-empty, perfect, compact set $L \subseteq \mathbb{C}$ satisfying $\overline{\mathrm{int}(L)} = L$, we set (as in [8])

$$\mathcal{A}^1(L) := \{f \in \mathcal{A}(L);\ (f|_{\mathrm{int}(L)})' \text{ can be continuously extended to } L\}.$$

In [8], Dales and Feinstein give an example of a compact, uniformly regular, polynomially convex set K with dense interior such that $D^1(K)$ and $\mathcal{A}^1(K)$ fail to coincide. The only flaw of this surprising example is that the interior of K is not connected. This gives rise to the question whether there is such an example with connected interior. We now have the following result.

Corollary 6.2. *$D^1(K) = \mathcal{A}^1(K)$ for K as described at the beginning of this section.*

Proof. The inclusion "$\subseteq$" always holds. Conversely, let $f \in \mathcal{A}^1(K)$. Then $f|_G \in \mathcal{O}(G)$ has an extension $F \in D^1(K)$ thanks to 6.1. Due to $\overline{G} = K$, a continuity argument implies that $f = F$, i.e., $f \in D^1(K)$. ∎

As a result, question 7 of [8] has a negative answer even under more general conditions.

Next, we want to deduce a further corollary from 6.1. For this purpose, we consider a simple closed path $\gamma : [0,1] \to \mathbb{C}$.

Owing to the famous Jordan curve theorem, we have

$$\mathbb{C} = G \,\dot{\cup}\, \gamma^* \,\dot{\cup}\, \mathrm{Ext}(\gamma) \quad \text{and} \quad \partial G = \gamma^* = \partial\, \mathrm{Ext}(\gamma),$$

where $G := \mathrm{Int}(\gamma)$ is the interior and $\mathrm{Ext}(\gamma)$ the exterior of γ. Moreover, G and $\mathrm{Ext}(\gamma)$ are connected. Consequently, $\mathbb{C} \setminus G = \overline{\mathrm{Ext}(\gamma)}$ is also connected, implying that G is even simply connected. The Riemann mapping theorem thus gives a conformal mapping $\Phi : \mathbb{D} \to G$. Since $\mathbb{C} \setminus \overline{G} = \mathrm{Ext}(\gamma)$ is connected, $\overline{G} = \gamma^* \cup G$ is polynomially convex. In this situation, the following statement holds.

Corollary 6.3. *If Φ' can be continuously extended to $\overline{\mathbb{D}}$, then each function $f \in \mathcal{O}(G)$ for which f' is continuously extendable to $\overline{G}$, has itself a continuous extension to $\overline{G}$.*

Proof. By the Carathéodory–Osgood theorem (see Chapter IX, 4.9 in [23]), Φ has a continuous extension to a bijective function $\widehat{\Phi} : \overline{\mathbb{D}} \to \overline{G}$. The mapping $\widehat{\Phi}$ is then automatically a homeomorphism. In particular, Φ^{-1} has a continuous extension to $\overline{G}$ with values in $\overline{\mathbb{D}}$, namely $\widehat{\Phi}^{-1}$. For f as in the assertion, the function $f \circ \Phi$ is an element of $\mathcal{O}(\mathbb{D})$ whose derivative $(f \circ \Phi)' = (f' \circ \Phi) \cdot \Phi'$ can be continuously extended to $\overline{\mathbb{D}}$. Now Theorem 6.1 tells us that $f \circ \Phi$ even has a differentiable extension to $\overline{\mathbb{D}}$. Hence, $f = (f \circ \Phi) \circ \Phi^{-1}$ is continuously extendable to $\overline{G}$. ∎

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