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Perturbations of isometries between Banach spaces

by

Rafał Górak (Warszawa)

Abstract. We prove a very general theorem concerning the estimation of the expression $\|T(\frac{a+b}{2}) - \frac{Ta+Tb}{2}\|$ for different kinds of maps T satisfying some general perturbed isometry condition. It can be seen as a quantitative generalization of the classical Mazur–Ulam theorem. The estimates improve the existing ones for bi-Lipschitz maps. As a consequence we also obtain a very simple proof of the result of Gevirtz which answers the Hyers–Ulam problem and we prove a non-linear generalization of the Banach–Stone theorem which improves the results of Jarosz and more recent results of Dutrieux and Kalton.

1. Introduction. The aim of this paper is to prove a very general theorem (Theorem 2.1) that will allow us to obtain several facts concerning approximate preservation of midpoints by different kinds of maps with a perturbed isometry condition. Let us define the main notion of this paper:

Definition 1.1. Let $T : E \to F$ be a bijection between two metric spaces (E, d_E) and (F, d_F). Assume that there is a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ (where $\mathbb{R}_+ = \{x \in \mathbb{R};\ x \geq 0\}$) which is non-decreasing and such that the following conditions hold:

(i) $d_F(Tx, Ty) \leq \mu(d_E(x, y))$ for all $x, y \in E$.
(ii) $d_E(T^{-1}f, T^{-1}g) \leq \mu(d_F(f, g))$ for all $f, g \in F$.

Then T is called a *μ-isometry.*

In our article we consider (except Corollary 3.4) μ-isometries between Banach spaces only. It should be noticed that following [8] for a given map $T : E \to F$ we can easily find the optimal μ which is $\mu(t) = t + \varepsilon_T(t)$ where

$$\varepsilon_T(t) = \sup\big\{\big|\|Tx - Ty\| - \|x - y\|\big| \ : \|x - y\| \leq t \text{ or } \|Tx - Ty\| \leq t\big\}.$$

Lindenstrauss and Szankowski consider maps T that are surjective but not

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necessarily injective. However they observe that one can easily reduce the considerations to the bijective case when $t \to \infty$:

FACT 1.2. *Let $T : E \to F$ be a surjective map between Banach spaces E and F, respectively. If $\varepsilon_T(t) < \infty$ for all $t \in \mathbb{R}_+$ and $\varepsilon_T(\delta_0)/\delta_0 < 1$ for some $\delta_0 > 0$ then there exists a bijection $\widetilde{T} : E \to F$ such that*

$$\forall x \in E \quad \|Tx - \widetilde{T}x\| \le 2\delta_0 + 2\varepsilon_T(\delta_0).$$

Hence $\widetilde{T}$ is a μ-isometry for $\mu(t) = t + \varepsilon_T(t) + 4\delta_0 + 4\varepsilon_T(\delta_0)$. In particular $\varepsilon_{\widetilde{T}}(t) \sim \varepsilon_T(t)$ as $t \to \infty$ (if only $\varepsilon_T(t) \to \infty$).

Proof. For the sake of completeness we sketch the proof. Let us consider the maximal set $A \subset E$ all of whose points are at a distance of at least δ_0 from each other. Then for every $a \neq b$ in A we have $\delta_0 - \varepsilon_T(\delta_0) \le \|Ta - Tb\|$, hence $T|A$ is injective. Moreover $T(A)$ is $\delta_0 + \varepsilon_T(\delta_0)$-dense in F (that is, the distance of every element of F from $T(A)$ is not greater than $\delta_0 + \varepsilon_T(\delta_0)$). This shows that the density characters of E and F are equal. Now it is easy to construct decompositions $E = \dot{\bigcup}_{a\in A} E_a$ and $F = \dot{\bigcup}_{a\in A} F_a$ such that for all $a \in A$:

(1) $a \in E_a$, $Ta \in F_a$;
(2) $|E_a| = |F_a|$;
(3) $\operatorname{diam} E_a \le \delta_0$ and $\operatorname{diam} F_a \le \delta_0 + \varepsilon_T(\delta_0)$.

By the standard set-theoretical reasoning we can extend $T|A$ to the required μ-isometry $\widetilde{T} : E \to F$. ∎

Below we only consider the notion of μ-isometry since it provides sufficient generality, and by considering bijective maps we avoid some easy but rather technical problems.

When considering μ-isometries one should not think that they are perturbed isometries (since it may easily happen that there is no isometry to perturb) but rather that they satisfy a perturbed isometry condition. Hence the following natural question arises: "How can you perturb the definition of an isometry between Banach spaces so that the existence of a map satisfying the perturbed condition implies the existence of an isometry?". If one has an answer to this question, another one can be asked: "How far is the perturbed isometry from an isometry?" Lindenstrauss and Szankowski [8] answered these questions for the class of all Banach spaces and for all μ-isometries. However one can investigate the above problems for some subclasses of Banach spaces (such as function spaces, which leads to generalizations of the Banach–Stone theorem).

Let us now discuss, in more detail, some examples of μ-isometries for different functions μ, and results related to both questions asked above. Let T be a μ-isometry between Banach spaces E and F. If $\mu(t) = t$ then T is just an isometry. Let us now consider $\mu(t) = t + L$ for some constant

$L \geq 0$. Such maps are called *L-isometries*. More generally an L-isometry T is a surjective map between Banach spaces for which $\varepsilon_T(t) \leq L$. But as we have already noticed, Fact 1.2 allows us to reduce considerations to the bijective case (see Corollary 3.1 where we show how this is done). Hyers and Ulam asked whether L-isometries are close to isometries. The question was answered positively for all pairs of Banach spaces E and F by Gevirtz [4] (let us say that L can be as large as we please).

Szankowski and Lindenstrauss gave a complete characterization of those μ-isometries whose existence implies the existence of an isometry. More precisely:

THEOREM 1.3. *Let $T : E \to F$ be a μ-isometry between Banach spaces E and F where $\mu(t) = t + \varepsilon_T(t)$, $T(0) = 0$ and $\int_1^\infty (\varepsilon_T(t)/t^2)\, dt < \infty$. Then there exists an isometry $I : E \to F$ such that*

$$\|Tx - Ix\| = o(\|x\|) \qquad \text{as } \|x\| \to \infty.$$

Moreover the result is sharp (*see* [8] *for more details*) *in the case when E and F are general Banach spaces.*

Let us now consider $\mu(t) = Mt$. In this case T is a bi-Lipschitz map (or Lipschitz equivalence). This means that distances between points are perturbed according to the inequalities

$$\frac{1}{M}\|x - y\| \leq \|Tx - Ty\| \leq M\|x - y\| \qquad \text{for all } x, y \in E.$$

Obviously if $M = 1$ then T is just an isometry. Let us look at the case when $M \searrow 1$. Unfortunately, no matter how close to one M is, we cannot guarantee the existence of an isometry between general Banach spaces E and F. Clearly $\int_1^\infty ((M-1)t/t^2)\, dt = \infty$ $(M > 1)$ hence you can find in [8] a construction of Banach spaces E and F that are μ-isometric for $\mu(t) = Mt$ but they are not isometric. However, for some particular class of Banach spaces E and F one can obtain interesting positive results even for the more general case of $\mu(t) = Mt + L$ (maps that are bi-Lipschitz for large distances). Indeed let us consider $E = C_0(X)$ and $F = C_0(Y)$, the spaces of continuous real valued functions vanishing at ∞ on locally compact spaces X and Y, respectively. The spaces $C_0(X)$ and $C_0(Y)$ are endowed with the sup norms. It turns out that in this case one can obtain more than Theorem 1.3:

THEOREM 1.4. *Let $T : C_0(X) \to C_0(Y)$ be a μ-isometry, where X and Y are locally compact spaces, $\mu(t) = Mt + L$ $(M \geq 1,\ L \geq 0)$ and $T(0) = 0$. Then there exists an absolute constant $M_0 > 1$ and functions $\delta : [1, \infty) \to \mathbb{R}_+$ and $\Delta : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that whenever $M < M_0$ then there exists an isometry $I : C_0(X) \to C_0(Y)$ such that*

$$\|Tf - If\| \leq \delta(M)\|f\| + \Delta(M, L) \qquad \text{for all } f \in C_0(X). \tag{1.1}$$

Moreover, $\Delta(M,0) = 0$ *and* $\lim_{M\to 1^+}\delta(M) = 0$. *In particular, from the Banach–Stone theorem, the spaces* X *and* Y *are homeomorphic. It is known that* $M_0 \le \sqrt{2}$ *and equality holds if we assume additionally that* T *is linear* (*see* [3] *and* [5] *for the discussion*).

The first of such results was obtained by Jarosz in [6] but for $L = 0$ only. However the value of M_0 which he obtains is very close to 1, and the function δ is far from being optimal ($\delta(M) = O((M-1)^{0.1})$ as $M \searrow 1$ and $\Delta(M,0) = 0$ in his result). Later Dutrieux and Kalton [2] obtained the value of $M_0 = \sqrt{17/16}$ (in their notation the condition $M < M_0$ can be seen as the inequality $d_N(C_0(X), C_0(Y)) < M_0^2$) but they do not provide any estimation like (1.1) (this time L can be positive). Finally the author in [5] improved the constant to $M_0 = \sqrt{6/5}$ and showed that $\delta(M) = 26(M-1)$. Moreover $\Delta(M,0) = 0$, hence the result improved both the constant M_0 obtained in [2] and the function δ obtained in [6] as well as showed the existence of δ and Δ if $L > 0$. However, the proof works only for X and Y *compact* and it is not that easy to extend it to the locally compact case. We will do this in the last section of this paper by applying the main result of Section 2.

It turns out that in the proofs of most of the above results the estimation of $\left\|T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2}\right\|$ is crucial and far from being obvious. Moreover the results estimating this expression can be regarded as generalizations of the Banach–Mazur theorem so in some sense they are of independent interest. We deal with this problem in the next section.

2. Approximate preservation of midpoints by μ-isometries. We present here a very general method of estimating $\left\|T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2}\right\|$ for μ-isometries T. It should be mentioned that some results of this kind have already been obtained in [8] (in fact this is the most demanding part of [8]). However the method presented here has several important advantages. First of all it has an astonishingly simple proof and covers the result of Gevirtz (Corollary 3.1) which solves the famous Hyers–Ulam problem (the proofs in the original paper [4] or in the survey paper of Rassias [9] are clearly more complicated). Secondly, applying our result for μ-isometries where $\mu(t) = Mt + L$, we obtain new and elegant estimates (they are interesting even in the Lipschitz case, that is, when $L = 0$). This will allow us to prove new results concerning the nonlinear version of the Banach–Stone theorem. Finally, although our theorem does not cover the result of Lindenstrauss and Szankowski in full generality, it gives their result for particular functions $\mu(t) = t + \varepsilon(t)$ such as $\mu(t) = t + t^\alpha$ where $\alpha \in [0, 1)$ (see Section 4). It is very tempting (due to the simplicity of the proof below) to investigate whether Theorem 2.1 gives us the result of [8] in full generality.

Before we formulate and prove the main result let us say that the idea of it comes from the beautiful proof of the classical Mazur–Ulam theorem due to Väisälä (see [10]).

THEOREM 2.1. *Let $T : E \to F$ be a μ-isometry between two normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$. Assume that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\mu(t)/2 \le \mu(t/2)$ for all t. Then for all $a, b \in E$ and $n \in \mathbb{Z}_+$,*

$$\left\| T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2} \right\|_F \le \mu^{\circ(2^{n+1}-1)}\left(\frac{\|a-b\|_E}{2^{n+1}}\right),$$

where $\mu^{\circ n} = \mu \circ \cdots \circ \mu$ (n-fold composition).

Proof. Let us consider the set $W_E(\mu)$ consisting of all μ-isometries from E onto some normed space. Fix a, b in E and set $z = (a+b)/2$. Denote

$$\lambda(\mu) = \sup\left\{ \left\| Tz - \frac{Ta+Tb}{2} \right\|_F \,\middle|\, T \in W_E(\mu),\ F = \operatorname{Im} T \right\}.$$

Let us observe that for $T \in W_E(\mu)$ we have

$$\begin{aligned} \left\| Tz - \frac{Ta+Tb}{2} \right\|_F &\le \frac{1}{2}(\|Tz - Ta\|_F + \|Tz - Tb\|_F) \\ &\le \frac{1}{2}\left(2\mu\left(\frac{\|a-b\|_E}{2}\right)\right) = \mu\left(\frac{\|a-b\|_E}{2}\right). \end{aligned}$$

Hence

$$\lambda(\mu) \le \mu\left(\frac{\|a-b\|_E}{2}\right), \tag{2.1}$$

and one can see that $\lambda(\mu)$ is finite. For $T \in W_E(\mu)$ let us define Ψ and Ψ' to be the reflections with respect to z and $(Ta+Tb)/2$, respectively. Consider a new bijection on E defined as the composition $S = \Psi T^{-1} \Psi' T$. It is easy to check that $S \in W_E(\mu \circ \mu)$, $Sa = a$ and $Sb = b$. We have

$$\begin{aligned} 2\left\| Tz - \frac{Ta+Tb}{2} \right\|_F &= \|\Psi' Tz - Tz\|_F \le \mu(\|T^{-1}\Psi' Tz - T^{-1}Tz\|_E) \\ &= \mu(\|Sz - z\|_E) = \mu\left(\left\| Sz - \frac{Sa+Sb}{2} \right\|_E\right). \end{aligned}$$

Consequently,

$$\lambda(\mu) \le \frac{1}{2}\mu(\lambda(\mu \circ \mu)) \le \mu\left(\frac{\lambda(\mu^{\circ 2})}{2}\right).$$

Hence

$$\lambda(\mu^{\circ 2^n}) \le \mu^{\circ 2^n}\left(\frac{\lambda(\mu^{\circ 2^{n+1}})}{2}\right).$$

Applying the above formula recursively we obtain

$$\lambda(\mu) = \lambda(\mu^{\circ 1}) \leq \mu^{\circ 1}\left(\frac{\lambda(\mu^{\circ 2})}{2}\right) \leq \mu^{\circ 1}\left(\frac{1}{2}\mu^{\circ 2}\left(\frac{\lambda(\mu^{\circ 4})}{2}\right)\right)$$
$$\leq \mu^{\circ 1} \circ \mu^{\circ 2}\left(\frac{\lambda(\mu^{\circ 4})}{4}\right) \leq \cdots .$$

Finally,

$$\lambda(\mu) \leq \mu^{\circ 1} \circ \mu^{\circ 2} \circ \cdots \circ \mu^{\circ 2^{n-1}}\left(\frac{\lambda(\mu^{\circ 2^n})}{2^n}\right) = \mu^{\circ (2^n - 1)}\left(\frac{\lambda(\mu^{\circ 2^n})}{2^n}\right).$$

From the estimate (2.1) we have

$$\lambda(\mu) \leq \mu^{\circ (2^{n+1}-1)}\left(\frac{\|a-b\|_E}{2^{n+1}}\right). \ \blacksquare$$

3. Applications. The result of the previous section gives us a very simple proof of the main result from [4], which answers the question of Hyers and Ulam. More precisely:

COROLLARY 3.1. *Let T be an L-isometry between Banach spaces E and F such that $T(0) = 0$. Then there exist constants A and B, depending on L only, such that*

$$\left\| T\left(\frac{a+b}{2}\right) - \frac{Ta + Tb}{2} \right\| \leq A\sqrt{\|a-b\|} + B \quad \textit{for all } a, b \in E.$$

As a corollary of that estimate, Gevirtz easily deduces (relying on a result of Gruber) that the map $I : E \to F$ defined as $Ix = \lim_{n\to\infty} T(2^n x)/2^n$ is an isometry such that $\|Tx - Ix\| \leq 5L$ (later the constant was improved to $2L$, which turns out to be optimal).

Proof. Let us first assume that T is a μ-isometry for $\mu(t) = t + L$. Applying Theorem 2.1 for $\mu(t) = t + L$, we obtain

$$\left\| T\left(\frac{a+b}{2}\right) - \frac{Ta + Tb}{2} \right\| \leq \frac{\|a-b\|}{2^{n+1}} + 2^{n+1}L.$$

Taking $n = \lfloor \log_2 \sqrt{\|a-b\|} \rfloor - 1$ we have

$$\left\| T\left(\frac{a+b}{2}\right) - \frac{Ta + Tb}{2} \right\| = O(\sqrt{\|a-b\|})$$

as $\|a-b\| \to \infty$. By applying Fact 1.2, we easily get the estimate for all L-isometries, not only the bijective ones. $\blacksquare$

For further applications of Theorem 2.1 we need the following simple observation:

LEMMA 3.2. *Let $\mu(t) = t + \varepsilon(t)$ where $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ is a non-decreasing function. Then*

$$\int_t^{\mu^{\circ n}(t)} \frac{1}{\varepsilon(x)}\,dx \le n.$$

Proof. Since $1/\varepsilon$ is a non-increasing function,

$$\int_t^{\mu^{\circ n}(t)} \frac{1}{\varepsilon(x)}\,dx \le \sum_{k=0}^{n-1} \frac{1}{\varepsilon(\mu^{\circ k}(t))}(\mu^{\circ k+1}(t) - \mu^{\circ k}(t)) = n. \ \blacksquare$$

COROLLARY 3.3. *Let $T : E \to F$ be a μ-isometry for $\mu(t) = (1+\varepsilon)t + L$ where $0 < \varepsilon < 0.2$. Then*

$$\left\| T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2} \right\| \le 3\varepsilon\|a-b\| + \frac{4}{\varepsilon}L \quad \textit{for all } a, b \in E.$$

Let us remark that for $\varepsilon \ge 0.2$, we easily obtain

$$\left\| T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2} \right\| \le \frac{1+\varepsilon}{2}\|a-b\| + \frac{L}{2},$$

a better estimate than in the above corollary when $\|a-b\| \to \infty$. The above result is most interesting when ε is close to 0 and $\|a-b\| \to \infty$.

Proof. From Theorem 2.1 we obtain

$$\left\| T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2} \right\| \le \mu^{\circ(k-1)}\left(\frac{d}{k}\right) \le \mu^{\circ k}\left(\frac{d}{k}\right),$$

where $k = 2^{n+1}$ and $d = \|a-b\|$. From Lemma 3.2 we get

$$\int_{d/k}^{\mu^{\circ k}(d/k)} \frac{1}{\varepsilon x + L}\,dx \le k.$$

Hence

$$\mu^{\circ k}\left(\frac{d}{k}\right) \le \frac{e^{\varepsilon k}}{k}d + \frac{1}{\varepsilon}(e^{\varepsilon k} - 1)L.$$

The function $k \mapsto e^{\varepsilon k}/k$ has its minimum at $k = 1/\varepsilon$, which is $e\varepsilon$. Since in our application $k = 2^{n+1}$, we have to find n so that 2^{n+1} is as close to $1/\varepsilon$ as possible. For $\varepsilon < 0.2 < 1/\sqrt{2}$ there exists $n \in [\log_2(1/\varepsilon) - 1.5; \log_2(1/\varepsilon) - 0.5] \cap \mathbb{Z}_+$. Hence $2^{n+1} = k \in [1/\sqrt{2}\varepsilon; \sqrt{2}/\varepsilon]$ and this interval contains $1/\varepsilon$. Checking the values of $e^{\varepsilon k}/k$ at the endpoints we obtain $\mu^{\circ k}(d/k) \le 3\varepsilon d + (4/\varepsilon)L$. $\blacksquare$

For the Lipschitz case ($L = 0$) similar estimates can be found in [11]. Vestfrid obtains the inequality $\left\|T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2}\right\| \le 6\varepsilon\|a-b\|$. So one can see that the above result improves the estimate and extends it to maps that are not necessarily continuous ($L > 0$). By applying the above corollary we can also obtain some interesting estimates for bi-Lipschitz maps between ξ-dense subspaces of Banach spaces (nets in particular):

COROLLARY 3.4. *Let $T : A \to B$ be a μ-isometry from a ξ_E-dense set in Banach space E onto a ξ_F-dense set in F, where $\mu(t) = (1+\varepsilon)t$ and $0 < \varepsilon < 0.2$. Then for every $a, b \in A$ and every $z \in A$ such that $\|(a+b)/2 - z\| \le \xi_E$ we have*

$$\left\| Tz - \frac{Ta + Tb}{2} \right\| \le 3\varepsilon\|a - b\| + \frac{34(\xi_E + \xi_F)}{\varepsilon}.$$

Proof. Using a simple Fact 1.5 from [5] (or reasoning as in the proof of Fact 1.2) we obtain a map $\widetilde{T} : E \to F$ which is a μ-isometry for $\mu(t) = (1+\varepsilon)t + 4\xi_F + 3\xi_E$ and $\|\widetilde{T}x - Tx\| \le 2\xi_F + 2\xi_E$ for all $x \in A$. Let us take any $z \in A$ such that $\|(a+b)/2 - z\| \le \xi_E$. Applying Corollary 3.3 to the map $\widetilde{T}$, we obtain the desired estimate. ∎

We will now show how Corollary 3.3 allows us to obtain improvements on the constant M_0 and the function δ (defined as in Theorem 1.4) for all *locally compact* spaces.

THEOREM 3.5. *Let X and Y be locally compact spaces. Consider a μ-isometry $T : C_0(X) \to C_0(Y)$ where $\mu(t) = Mt + L$ ($M \ge 1$, $L \ge 0$). If $M < M_0 = \sqrt{16/15}$ then there exists a homeomorphism $\varphi : X \to Y$ and a continuous map $\lambda : X \to \{-1, 1\}$ such that for every $f \in C_0(X)$,*

$$\|Tf - If\| \le 76(M-1)\|f\| + \Delta \tag{3.1}$$

where I is the isometry defined by $If(y) = \lambda(\varphi^{-1}(y))f(\varphi^{-1}(y))$. The constant Δ depends on M and L only. Moreover, for $L = 0$ we have $\Delta = 0$.

As we can see, the constant M_0 improves the result obtained by Dutrieux and Kalton. However, more important is the estimate $\delta(M) \le 76(M-1)$ that is far better than that previously known, due to Jarosz [6].

Proof. Let us assume that $1 < M < \sqrt{16/15}$. If $M = 1$ then the conclusion easily follows from the above-mentioned solution of the Hyers–Ulam problem (Corollary 3.1) and from the Banach–Stone theorem. Let us first recall the construction of the homeomorphism φ and the function λ from [5].

In the construction, when dealing with the topology of general topological spaces, we use the notion of net convergence (Moore–Smith convergence). Σ will always denote a directed set and whenever we write $a_\sigma \to a$ we always mean $\lim_{\sigma\in\Sigma} a_\sigma = a$.

DEFINITION 3.6. $(f_\sigma)_{\sigma\in\Sigma} \subset C(X)$ is an *m-peak net* at $x \in X$ for some directed set Σ if

- $\|f_\sigma\| = |f_\sigma(x)| = m$ for all $\sigma \in \Sigma$,
- $\lim_{\sigma\in\Sigma} f_\sigma|(X \setminus U) \equiv 0$ uniformly for all open neighborhoods U of x.

We denote by $P_m^X(x)$ the set of m-peak nets at x.

DEFINITION 3.7. Let $D > 0$ and $m > 0$. We define

$$S_m^D(x) = \{y \in Y : \exists (f_\sigma)_{\sigma\in\Sigma} \in P_m^X(x)\ \exists y_\sigma \to y\ \forall \sigma \in \Sigma,\\ Tf_\sigma(y_\sigma) \ge Dm \text{ and } T(-f_\sigma)(y_\sigma) \le -Dm\}.$$

In [5] the author proves that for suitably chosen D and m we can define φ by $\{\varphi(x)\} = S_m^D(x)$, and φ turns out to be a homeomorphism between X and Y. In all the steps in [5] where we prove that φ is a homeomorphism the only place were compactness is crucial is [5, Fact 2.4]. We will modify its proof using Corollary 3.3 so that it works for the locally compact case.

FACT 3.8. *Let $D = 14 - 13M$. There exists m_0 (depending on M and L) such that for all $m > m_0$ we have $S_m^D(x) \neq \emptyset$ for all $x \in X$. Moreover, if $L = 0$ then $m_0 = 0$.*

Proof. Let us take any $(\widetilde{f}_\sigma)_{\sigma\in\Sigma} \in P_m^X(x)$ such that $\widetilde{f}_\sigma(x) = m$ for all $\sigma \in \Sigma$, and pick one $\sigma_0 \in \Sigma$. Let us define $\widetilde{g}_\sigma = (\widetilde{f}_\sigma + \widetilde{f})/2$ where $\widetilde{f} = \widetilde{f}_{\sigma_0}$. We have

$$\forall \sigma \in \Sigma \quad \|T\widetilde{g}_\sigma - T(-\widetilde{g}_\sigma)\| \ge \frac{2}{M}m - L.$$

Hence for every $\sigma \in \Sigma$ there exists $y_\sigma \in Y$ such that $|T\widetilde{g}_\sigma(y_\sigma) - T(-\widetilde{g}_\sigma)(y_\sigma)| \ge (2/M)m - L$. Let us observe that the numbers $T\widetilde{g}_\sigma(y_\sigma)$ and $T(-\widetilde{g}_\sigma)(y_\sigma)$ must be of different signs. Assume the contrary. Since $\|T(\pm\widetilde{g}_\sigma)\| \le Mm + L$ we have $Mm + L \ge (2/M)m - L$, which is impossible for m large enough, say $m > m_0'$ (or for all $m > 0$ if $L = 0$), provided $2/M > M$ (that is, if $M < \sqrt{2}$). We can and do assume that $T\widetilde{g}_\sigma(y_\sigma) \ge 0$ for all $\sigma \in \Sigma$ or $T\widetilde{g}_\sigma(y_\sigma) \le 0$ for all $\sigma \in \Sigma$. Let us define:

- If $T\widetilde{g}_\sigma(y_\sigma) \ge 0$ for all $\sigma \in \Sigma$ then $f_\sigma = \widetilde{f}_\sigma$, $f = \widetilde{f}$ and $g_\sigma^m = (f_\sigma + f)/2$.
- If $T\widetilde{g}_\sigma(y_\sigma) \le 0$ for all $\sigma \in \Sigma$ then $f_\sigma = -\widetilde{f}_\sigma$, $f = -\widetilde{f}$ and $g_\sigma = (f_\sigma + f)/2$.

Hence $Tg_\sigma(y_\sigma) - T(-g_\sigma)(y_\sigma) \ge (2/M)m - L$. Because $\|T(\pm g_\sigma)\| \le Mm + L$ we have

$$Tg_\sigma(y_\sigma) \ge \left(\frac{2}{M} - M\right)m - 2L,$$

$$T(-g_\sigma)(y_\sigma) \le -\left(\frac{2}{M} - M\right)m + 2L.$$

Since $g_\sigma = (f_\sigma + f)/2$, by Corollary 3.3 we obtain

$$\left\| T(\pm g_\sigma) - \frac{T(\pm f_\sigma) + T(\pm f)}{2} \right\| \le 3(M-1)m + \frac{4}{M-1}L.$$

Hence

$$Tf_\sigma(y_\sigma) \ge \left(\frac{4}{M} - 9M + 6\right)m - \left(5 + \frac{8}{M-1}\right)L,$$
$$T(-f_\sigma)(y_\sigma) \le -\left(\frac{4}{M} - 9M + 6\right)m + \left(5 + \frac{8}{M-1}\right)L,$$
$$Tf(y_\sigma) \ge \left(\frac{4}{M} - 9M + 6\right)m - \left(5 + \frac{8}{M-1}\right)L.$$

Let $m_0 \ge m_0'$ be such that

$$\left(\frac{4}{M} - 9M + 6\right)m_0 - \left(5 + \frac{8}{M-1}\right)L \ge (14 - 13M)m$$

for all $m > m_0$ (such an m_0 exists since $4/M - 9M + 6 > 14 - 13M > 0$ for all $M \in (1, \sqrt{16/15})$). By the compactness of the set

$$\{y \in Y : Tf(y) \ge (14 - 13M)m\}$$

for $m > m_0$ we can assume that $y_\sigma \to y \in S_m^D(x)$. Let us notice that for $L = 0$ we have $m_0 = 0$. ∎

Now the proof of Theorem 3.5 is exactly the same as the proof of Theorems 2.1 and Corollary 3.4 from [5]. Firstly, it is proven in [5, Section 2] that $\varphi(x) = S_m^D(x)$ is a homeomorphism for suitably chosen $m > m_2$ (where $m_2 = 0$ if $L = 0$) if

(i) D is so that $S_m^D(x) \ne \emptyset$ for all $x \in X$;
(ii) $1 - \varepsilon(M)M - \varepsilon(M) > 0$ where $\varepsilon(M) = 2M - 1 - D$.

In the compact case condition (i) means that it is enough to take $D = 4 - 3M < 2/M - M$ (see Fact 2.4 in [5]). This, together with (ii), leads to the conclusion that indeed $M < \sqrt{6/5}$. In the locally compact case we have already shown (Fact 3.8) that we can take $D = 14 - 13M$. Now (ii) leads to the inequality $M < \sqrt{16/15}$.

For every $x \in X$ and $m > m_0$ let us define (following [5, Section 3]) $\lambda_m(x) = f_\sigma(x)/|f_\sigma(x)|$ where the family $(f_\sigma)_{\sigma\in\Sigma} \in P_m^X(x)$ is such that:

- $f_{\sigma_0}(x)/|f_{\sigma_0}(x)| = f_{\sigma_1}(x)/|f_{\sigma_1}(x)|$ for all $\sigma_0, \sigma_1 \in \Sigma$ ($\lambda_m(x)$ does not depend on σ).
- There exist $y_\sigma \to y \in S_m^D(x)$ such that for every $\sigma \in \Sigma$ we have $Tf_\sigma(y_\sigma) \ge Dm$ and $T(-f_\sigma)(y_\sigma) \le -Dm$.

The existence of the above family for every $x \in X$ is exactly what was shown in the proof of Fact 3.8. The function λ from the formulation of Theorem 3.5 is defined as λ_m for m sufficiently large.

In order to prove (3.1) it is enough to notice that Fact 2.7 in [5] works also for X locally compact and hence gives the estimate

$$\big||Tf(\varphi(x))| - |f(x)|\big| \le \varepsilon(M)M\|f\| + \Delta = 15(M^2 - M)\|f\| + \Delta$$

for all $f \in C_0(X)$, $x \in X$ and some constant Δ depending on M, L and such that $\Delta = 0$ if $L = 0$.

Repeating the reasoning from Section 3 of [5] for $D = 14 - 13M$ we obtain a slight modification of [5, Fact 3.1] (only one constant is changed):

FACT 3.9. *Assume that $|f(x)| > 30(M-1)\|f\|$ and let $\|f\| = m$. Then for $m > m_3$ ($m_3 \ge 0$ depends on M and L only), the sign of $Tf(\varphi(x))$ is the same as the sign of $\lambda_m(x)f(x)$. If $L = 0$ then $m_3 = 0$.*

As a consequence, reasoning as in the proof of [5, Corollary 3.4], we get (3.1) where $\lambda \equiv \lambda_m$ for $m > m_3$. Summarizing the proof, let us just mention that having Fact 3.8 at hand it is very easy to modify the reasoning from [5]. One should only keep in mind that this time $D = 14 - 13M$. ∎

4. Final remarks. Natural directions of further investigations and some open problems arise from both of the above sections. First of all, as already mentioned, it would be interesting to see how the result of Szankowski and Lindenstrauss follows from Theorem 2.1. For instance, if we consider $\varepsilon_T(t) \le t^\alpha$ for $\alpha \in [0,1)$, by using Fact 3.2 one can obtain

$$\left\|T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2}\right\| = O(\|a-b\|^{1/(2-\alpha)}) \quad \text{as } \|a-b\| \to \infty,$$

which is sufficient to show that $Ix = \lim_{n\to\infty} T(2^n x)/2^n$ is the required isometry in Theorem 1.3.

Another interesting question concerns the expression $\left\|T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2}\right\|$ and its optimal estimation when T is a μ-isometry for $\mu(t) = (1+\varepsilon)t$ and $\varepsilon \to 0$. We have already seen that $\left\|T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2}\right\| = O(\varepsilon\|a-b\|)$ as $\varepsilon \to 0$. It is easy to show that this is all one can obtain in the general case. Indeed, as noticed by Vestfrid [11], it suffices to consider $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} (1+\varepsilon)x & \text{if } x \ge 0, \\ \dfrac{1}{1+\varepsilon}x & \text{if } x < 0. \end{cases}$$

However the exact value of the constant

$$K = \limsup_{\varepsilon\to 0} K_\varepsilon, \quad \text{where} \quad K_\varepsilon = \sup \frac{\left\|T\left(\frac{a+b}{2}\right) - \frac{Ta+Tb}{2}\right\|}{\varepsilon\|a-b\|},$$

remains unknown. Here the supremum is taken over all μ-isometries T between Banach spaces, where $\mu(t) = (1+\varepsilon)t$, and over all pairs of points $a \ne b$ from the domain of T. The above example shows that $K \ge 0.5$ and

Corollary 3.3 shows that $K \leq 3$. It is worth noting that a simple analysis of the proof of Corollary 3.3 gives $\liminf_{\varepsilon\to 0} K_\varepsilon \leq e$.

Finally it is of interest to find the optimal constant M_0 and the optimal estimate of δ in Theorem 1.4. In particular it is still unknown whether the constant $M_0 = \sqrt{2}$ is optimal or not. However we skip the detailed discussion of this problem and direct the reader to the final section of [5].

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Rafał Górak
Technical University of Warsaw
Pl. Politechniki 1
00-661 Warszawa, Poland
E-mail: rafalgorak@gmail.com

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