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L^p type mapping estimates for oscillatory integrals in higher dimensions

by

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Abstract. We show in two dimensions that if $Kf = \int_{\mathbb{R}^2_+} k(x,y)f(y)\,dy$, $k(x,y) = e^{ix^a\cdot y^b}/|x-y|^\eta$, $p = 4/(2+\eta)$, $a \ge b \ge \bar{1} = (1,1)$, $v_p(y) = y^{(p/p')(\bar{1}-b/a)}$, then $\|Kf\|_p \le C\|f\|_{p,v_p}$ if $\eta+\alpha_1+\alpha_2 < 2$, $\alpha_j = 1-b_j/a_j$, $j = 1,2$. Our methods apply in all dimensions and also for more general kernels.

0. Introduction. Our purpose is to study mapping properties of the operators given by

$$Kf = K_{a,b}f(x) = \int_{\mathbb{R}^d_+} k(x,y)f(y)\,dy \quad \text{with } x \in \mathbb{R}^d_+, \tag{0.1}$$

i.e. $x = (x_1,\dots,x_d)$ with $x_j \ge 0$ for $1 \le j \le d$. Here we are forced to consider mappings from L^p_v into L^p_w where $w(x), v(y)$ are non-negative, measurable functions (generally referred to as *weights*) that are positive a.e., and for the most part the weights will be power weights.

For the kernel, we take

$$\begin{aligned} &k(x,y) = \varphi(x,y)e^{ig(x,y)} \quad \text{with} \\ &g(x,y) = x^a\cdot y^b = \textstyle\sum_{j=1}^d x_j^{a_j}y_j^{b_j}, \quad a_j, b_j \ge 1, \end{aligned} \tag{0.2}$$

and $\varphi(x,y)$ satisfies (0.4) below; we also set $|x|^a\cdot|y|^b = \sum_{j=1}^d |x_j|^{a_j}|y_j|^{b_j}$. The model case occurs when $k(x,y) = |x-y|^{-\eta}e^{ix^a\cdot y^b}$, $0 \le \eta < 2-\alpha_1-\alpha_2$, and $\alpha_j = 1-b_j/a_j, j = 1,2$. We also consider the case where $k(x,y) = |x-y|^{-\eta}e^{ix^a\cdot y^b}\gamma(x-y), 0 \le \eta < 2$, where $\gamma \in C^\infty(\mathbb{R}^2)$ and $\gamma(x) = 0$ if $|x| \le 1$, $\gamma(x) = 1$ if $|x| \ge 2$, and $0 \le \gamma(x) \le 1$.

We mostly work in $d = 2$ dimensions, and since most of the arguments are iteration type these results can be extended to higher dimensions. We shall give a more detailed explanation of this at the end of Section 0.

Our aim is thus to prove an (L^p_v, L^p_w) estimate for the operators defined in (0.1) and (0.2), where $\varphi(x,y)$ satisfies (0.4), $a_1 \ge b_1 \ge 1$, $a_2 \ge b_2 \ge 1$,

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$w(x) = 1$ and $v(y) = v_p(y) = y^{(p/p')(\bar{1}-b/a)}$, where $\bar{1} = (1,1)$. In the case of convolutions, we obtained similar estimates in [JS1].

In Section 1, we do the cases where $p = 2$ and $\varphi(x,y)$ satisfies (0.3); this result appears in Theorem 1.9. In this case, we characterize all the power weights. In Sections 2 and 3, we study the special cases where we place support restrictions on the kernel. The combined result appears in Theorem 3.2 in Section 3. In Theorem 4.1, we prove the first result without any support restrictions on the kernel and we also obtain a necessity result. In the case $p = 2$ ($\eta = 0$) we characterize fully the power weights that are mapped; the case $p \neq 2$ still remains open. See also Corollary 4.2. For similar results in 1-d, see [S].

For the (L^2, L^2) case, the Fourier transform is included among the operators. In this case, the authors of [PhS] obtained an (L^2, L^2) result in d dimensions with $g(x,y) = x \cdot y$ and with ($\mathbb{N} = \{0, 1, \ldots\}$)

$$|\partial_x^\alpha \partial_y^\beta \varphi(x,y)| \le C_{\alpha,\beta}|x-y|^{-(|\alpha|+|\beta|)} \quad \text{for all } \alpha, \beta \in \mathbb{N}^d. \tag{0.3}$$

Recently in [SS] (see also [S1] for simplifications) we studied the (L^p, L^p) mapping problem in dimension $d = 2$, in case $a_1/b_1 = a_2/b_2$ and $a_j, b_j \ge 1$. There we considered φ's so that for some $0 \le \eta < 2$,

$$|\partial_x^\alpha \partial_y^\beta \varphi(x,y)| \le C|x-y|^{-(\eta+|\alpha|+|\beta|)} \quad \text{for all } \alpha, \beta \in \mathbb{N}^2. \tag{0.4}$$

Actually in Proposition 5.1 of [SS], we only needed that $\varphi(x,y)$ satisfies (0.4) for $0 \le |\alpha|, |\beta| \le 3$, rather than for all α, β.

Thus Proposition 5.1 and Theorem 5.2 of [SS] imply the following weaker result:

THEOREM A. *Let $d = 2$, and $a_1/b_1 = a_2/b_2$ and $a, b > \bar{1}$. If $0 \le \eta < 2$ and φ satisfies* (0.4), *then*

$$\|Kf\|_p \le C\|f\|_p \quad \text{if } p \in J = \left[\frac{a_1+b_1}{a_1+b_1\eta/2}, \frac{a_1+b_1}{a_1(1-\eta/2)}\right].$$

Furthermore, if $\varphi(x,y) \ge C|x-y|^{-\eta}$, then

$$\|Kf\|_p \le C\|f\|_p \quad \text{if and only if} \quad p \in J. \tag{0.5}$$

REMARK 1. When $\varphi(x,y) = |x-y|^{-\eta+i\tau}$ (the model case) we get (0.5) if $a_j, b_j \ge 1,\ j = 1,2$, i.e. we can include the cases where a_j or b_j is 1.

REMARK 2. The sufficiency result of Theorem A also applies in case $p = 1$, in that case we get $\|Kf\|_1 \le C\|f\|_{H_E}$ (see Definition 2.1 on p. 1037 in [SS]); but then this holds if $a_1, a_2, b_1, b_2 \ge 1$ and $\varphi(x,y)$ satisfies (0.4) for $\eta = 2$ and $\varphi(x,y)$ is in L_{loc} (see Theorem 2.2 in [SS]). In other words, we can drop the restriction $a_1/b_1 = a_2/b_2$ here.

For the weights $v(y), w(x)$ described above and for any locally integrable function $h(y)$, we set

$$\|h\|_{p,v} = \Big(\int_{\mathbb{R}^2} |h(y)|^p v(y)\, dy \Big)^{1/p}.$$

We suppose that $1 \le p \le 2$ and $1/p + 1/p' = 1$.

We use C as an arbitrary positive constant, and we find it convenient to use the letter M as another positive constant. For $y \in \mathbb{R}^2_+$ and $\sigma \in \mathbb{R}^2$ we set $y^\sigma = y_1^{\sigma_1} y_2^{\sigma_2}$, similarly in case y, σ are in d dimensions (keep in mind the exceptional cases when $y_i = 0$ and $\sigma_i < 0$); also for $a, b \in \mathbb{R}^2$, we write $a \ge b$ to mean that $a_j \ge b_j$ for $j = 1, 2$. For any real number s, we set $\bar{s} = (s, s)$, i.e. $\bar{s} \in \mathbb{R}^2$. If $a > \bar{0}$, we set $b/a = (b_1/a_1, b_2/a_2)$ and also we set $\bar{1} - b/a = (1 - b_1/a_1, 1 - b_2/a_2)$. In d dimensions this same notation is used and it should be clear from the context. Finally, for a given function $h(y)$, we note that $h(y_1^{\sigma_1}, y_2^{\sigma_2}) = h(y^\sigma)$.

To see these results in d dimensions, we proceed as follows. We set up an analytic family of operators, say $S_z f$. In Section 1, we prove the $(2, 2)$ result for the operators $S_{-\eta+i\tau}$. Then in the remainder of the paper, we first decompose (for $d = 2$) $S_{d-\eta+i\tau} = T_1 + T_2$ and we need to prove that T_1 maps L^1 into itself and T_2 maps H_E into L^1. For the operator T_1 we need to show an estimate (2.18). This follows from (2.17) and its counterpart is true in d dimensions. The proof for T_2, which appears in (3.3), essentially follows from Proposition 2.4 in [SS], which was done in 2-d. In quoting from the top of p. 1040 of [SS], the proof of Proposition 2.4 is accomplished by reduction to the one-dimensional case. Similarly the inequality in d dimensions ((2.3) of [SS]) is reduced to the one in dimension $d - 1$.

1. Preliminaries and an L^2-result. Here we study the operator K defined by (0.1) for φ satisfying (0.3). Only in this section do we consider weights that are not power weights. We begin with the following well known result.

PROPOSITION 1.1. *Let $1 \le p \le \infty$ and X, Y be measure spaces. Set*

$$Kf(x) = \int_Y k(x, y) f(y)\, dy, \qquad x \in X.$$

If for some weight $v(y)$,

$$r_1(x) = r_1(x; v) = \int_Y |k(x, y)| (v(y))^{1-p'}\, dy \in L^\infty(X),$$

$$r_2(y) = \int_X |k(x, y)|\, dx \in L^\infty(Y),$$

then for all $1 \le p \le \infty$,

$$\|Kf\|_p \le C_{p,v}\|f\|_{p,v}, \qquad \inf C_{p,v} = \|r_1\|_\infty^{1/p'}\|r_2\|_\infty^{1/p}.$$

For some $M \ge 1$, let $|h(y)| \ge h_1(y_1)h_2(y_2)$ and $h_j(y_j) \ge 0$, $j = 1,2$, with $1/h_j(y_j) \in L_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(|y_j| \ge M)$ for $j = 1,2$. We call such functions $h(y)$ *admissible*; let S be the collection of all such admissible $h(y)$'s.

COROLLARY 1.2. *Let* $k(x,y) \in L^\infty, r_1(x;1), r_2(y) \in L^\infty(\mathbb{R}^2)$. *If*

$$\sup_{x,y_2} \int_1^\infty |k(x,y)|\,dy_1 + \sup_{x,y_1} \int_1^\infty |k(x,y)|\,dy_2 = C < \infty, \tag{1.1}$$

then for admissible $v \in S$,

$$\|Kf\|_2 \le C\|f\|_{2,v}, \qquad \inf C = \|r_1(\cdot,v)\|_\infty^{1/2}\|r_2(\cdot)\|_\infty^{1/2}.$$

REMARK 3. We get this result in all dimensions, and in one dimension it takes the following form. If $1/v(y_1) \in L_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(|y_1| \ge M)$, and S_1 is the collection of all such v's, and

$$\sup_{x_1\in\mathbb{R}} \int_1^\infty |k(x_1,y_1)|\,dy_1 + \sup_{y_1\in\mathbb{R}} \int_1^\infty |k(x_1,y_1)|\,dx_1 < \infty \tag{1.2}$$

then for all $v \in S_1$,

$$\|Kf\|_2 \le C\|f\|_{2,v}, \qquad \inf C = \|r_1\|_\infty^{1/2}\|r_2\|_\infty^{1/2}.$$

Proof of Corollary 1.2. By Proposition 1.1, it suffices to show that $r_1 \in L^\infty(\mathbb{R}^2)$ for each admissible v. But in all these cases we get (we set $[\overline{n},\overline{m}] = [n_1,m_1] \times [n_2,m_2]$)

$$\begin{aligned} r_1(x) \le C\bigg(\int_{[\bar{0},\bar{M}]} \frac{1}{v(y)}\,dy &+ \int_0^M \frac{1}{v_1(y_1)}\Big(\int_M^\infty |k(x,y)|\,dy_2\Big)\,dy_1 \\ &+ \int_0^M \frac{1}{v_2(y_2)}\Big(\int_M^\infty |k(x,y)|\,dy_1\Big)\,dy_2 + \int_{[\bar{M},\bar{\infty})} |k(x,y)|\,dy\bigg) \le C. \ \blacksquare \end{aligned}$$

Let T be the set of $m(x,y)$ supported in $|x-y| \le \beta$ for some $\beta > 0$ and such that $m(x,y)$ satisfies (1.1).

PROPOSITION 1.3. *Let* $\varphi \in L^\infty$, $k \in T$ *and suppose* k *satisfies* (0.2). *Then*

$$\|Kf\|_2 \le C\|f\|_{2,v} \quad \textit{for each } v \in S.$$

Proof. This follows immediately from Corollary 1.2. ■

REMARK 4. We get this result in all dimensions. In one dimension, with

$$T_1 = \{m(x_1,y_1) : m \in L^\infty, m \text{ satisfies (1.2) and has bounded support}\},$$

we get: if

$$\varphi \in L^\infty \quad \text{and} \quad k(x_1, y_1) = e^{ix_1^{a_1} y_1^{b_1}} \varphi(x_1, y_1) \in T_1$$

then

$$\|Kf\|_2 \leq C\|f\|_{2,v} \quad \text{for all } v \in S_1.$$

In [St] the pseudodifferential class $S^0_{0,0}$ is discussed, of all C^∞-functions $\lambda(x,y)$ so that

$$|\partial_x^\alpha \partial_y^\beta \lambda(x,y)| \leq C_{\alpha,\beta} \quad \text{for all } \alpha, \beta.$$

In the Proposition on p. 282 of [St], the following is shown:

THEOREM B. *If $\lambda \in S^0_{0,0}$ and $Tf = \int e^{ix\cdot y}\lambda(x,y) f(y)\,dy$, then*

$$\|Tf\|_2 \leq C\|f\|_2.$$

Set

$$\mu(x_1) = 1 \quad \text{if } |x_1| \geq 2, \qquad \mu(x_1) = 0 \quad \text{if } |x_1| \leq 1,$$
$$0 \leq \mu(x_1) \leq 1 \quad \text{and} \quad \mu \in C^\infty(\mathbb{R}).$$

LEMMA 1.4. *Let $a_1, b_1 \geq 1$ and suppose φ satisfies* (0.3). *Then*

$$\int\limits_{\mathbb{R}_+} \Big| \int\limits_{\mathbb{R}_+} e^{ix_1 y_1} \mu(x_1^{1/a_1} - y_1^{1/b_1}) \varphi(x_1^{1/a_1}, y_1^{1/b_1}, y_2) g(y_1)\, dy_1 \Big|^2 dx_1 \leq C \int\limits_{\mathbb{R}_+} |g(y_1)|^2\, dy_1.$$

Additionally, the variable y_2 (as well as x_2) in the argument of the term φ plays a passive role here.

Proof. We set

$$l(x_1, y_1) = e^{ix_1 y_1} \mu(x_1^{1/a_1} - y_1^{1/b_1}) \varphi(x_1^{1/a_1}, y_1^{1/b_1}, y_2)$$

and notice that

$$\begin{aligned} l(x_1, y_1) &= ((1 - \mu(x_1))(1 - \mu(y_1)) + (1 - \mu(x_1))\mu(y_1) \\ &\quad + \mu(x_1)(1 - \mu(y_1)) + \mu(x_1)\mu(y_1)) l(x_1, y_1) \\ &= I_1(x_1, y) + I_2(\cdots) + I_3(\cdots) + I_4(\cdots). \end{aligned}$$

The estimate for I_1 follows by Proposition 1.1 with $v(y) = 1$, and the estimate for I_4 follows from Theorem B for

$$\mu(x_1^{1/a_1} - y_1^{1/b_1}) \varphi(x_1^{1/a_1}, y_1^{1/b_1}, y_2) \in S^0_{0,0}, \qquad \text{since } x_1, y_1 \geq 1.$$

We are left with the terms I_2, I_3.

We just need to estimate I_3, since the case of I_2 follows by duality. Now,

$$\int_0^\infty (\mu(x_1))^2 \Big| \int_0^\infty (1-\mu(y_1)) e^{ix_1y_1} (\mu(x_1^{1/a_1} - y_1^{1/b_1}) - 1)\varphi(\cdots) g(y_1)\,dy_1\Big|^2 dx_1 + \int_0^\infty (\mu(x_1))^2 \Big| \int_0^\infty (1-\mu(y_1)) e^{ix_1y_1} \varphi(\cdots) g(y_1)\,dy_1\Big|^2 dx_1 = A_{31} + A_{32}.$$

Since $0 \le y_1 \le 2$ that implies $0 \le x_1^{1/a_1} \le 4$ for the term A_{31}; otherwise, the integrand is zero. The estimates of A_{31} follow by Proposition 1.1.

For the term A_{32}, set $u = y_1^{1/b_1}$ and note that

$$\varphi(x_1^{1/a_1}, u, y_2) = \sum_{j=0}^{M-1} \partial_u^j \varphi(x_1^{1/a_1}, 0, y_2)\, \frac{u^j}{j!} + \partial_u^M \varphi(x_1^{1/a_1}, \xi, y_2)\, \frac{u^M}{M!}$$

where M is an integer chosen so that $M/a_1 > 1$. Hence we get

$$\begin{aligned} A_{32} \le C\bigg(&\int_0^\infty (\mu(x_1))^2 \bigg(1 - \Big(\mu\Big(\frac{x_1}{2^{2a_1}}\Big)\Big)^2\bigg) \\ &\times \Big| \int_0^\infty (1-\mu(y_1)) e^{ix_1y_1}\varphi(\cdots) g(y_1)\,dy_1\Big|^2 dx_1 \\ &+ \sum_{j=0}^{M-1} \int_0^\infty \Big(\mu\Big(\frac{x_1}{2^{2a_1}}\Big)\Big)^2 \Big| \int_0^\infty (1-\mu(y_1)) e^{ix_1y_1} \partial^j \varphi(\cdots)\, \frac{u^j}{j!}\, g(y_1)\,dy_1\Big|^2 dx_1 \\ &+ \int_0^\infty \Big(\mu\Big(\frac{x_1}{2^{2a_1}}\Big)\Big)^2 \Big| \int_0^\infty (1-\mu(y_1)) e^{ix_1y_1} \partial^M \varphi(\cdots)\, \frac{u^M}{M!}\, g(y_1)\,dy_1\Big|^2 dx_1\bigg) \\ = A + \sum_{j=0}^{M} A_{32j}. \end{aligned}$$

The estimates for A follow from Proposition 1.1. By (0.3) since $x_1^{1/a_1} \ge 4$, we get

$$A_{32M} \le C \int_1^\infty \frac{1}{x_1^{2M/a_1}} \Big(\int_0^2 |g(y_1)|\,dy_1\Big)^2 dx_1 \le C\|g\|_2^2.$$

For the remaining terms A_{32j}, $\mu(x_1/2^{2a_1})(1-\mu(y_1))\partial^j\varphi(x_1^{1/a_1}, 0, y_2) \in S^0_{0,0}$, as a function of the variables x_1, y_1. This completes the proof. ∎

Proposition 1.5. *With the hypotheses of Lemma* 1.4,

$$\int_0^\infty \Big| \int_0^\infty e^{ix_1^{a_1}y_1^{b_1}} \varphi(x_1, y) f(y)\,dy_1\Big|^2 dx_1 \le C\|f\|^2_{2,v_1}, \text{ where } v_1(y_1) = y_1^{\alpha_1}.$$

For the term φ note the role of the variable y_2 (as well as x_2).

Proof. Using Remark 4 and then changing variables for the remaining expression, we end up considering

$$I = \int_0^\infty x_1^{1/a_1-1}\Big|\int_0^\infty e^{ix_1y_1}\mu(x_1^{1/a_1}-y_1^{1/b_1})\varphi(x_1^{1/a_1},y_1^{1/b_1},y_2) \times f(y_1^{1/b_1},y_2)y_1^{1/b_1-1}\,dy_1\Big|^2\,dx_1.$$

But by Lemma 1.4 the operator

$$Kg(x_1) = \int_0^\infty e^{ix_1y_1}\mu(x_1^{1/a_1}-y_1^{1/b_1})\varphi(x_1^{1/a_1},y_1^{1/b_1},y_2)g(y_1)\,dy_1$$

maps L^2 into itself with a bounded integrand, therefore it satisfies the assumptions of Theorem 1 of [JS3] and so by Theorem 2 of [JS2] it follows that

$$I \le C\int_0^\infty |f(y_1^{1/b_1},y_2)y_1^{1/b_1-1}|^2 y_1^{1-1/a_1}\,dy_1$$

and we get our result after a change of variables. ∎

With $\psi_{\bar{1}}(x) = \mu(x_1)\cdots\mu(x_d)$ in d dimensions (in particular, $\psi_{11}(x) = \mu(x_1)\mu(x_2)$ for $d=2$) we set

$$(1.3)\qquad \begin{aligned} h(x,y) &= \psi_{\bar{1}}(x^{1/a})\psi_{\bar{1}}(y^{1/b})\gamma(x^{1/a}-y^{1/b})\varphi(x^{1/a},y^{1/b})e^{ix\cdot y},\\ Lg(x) &= \int_{\mathbb{R}^d_+} h(x,y)g(y)\,dy. \end{aligned}$$

Proposition 1.6. *Let $a,b \ge \bar{1}$. If $\varphi(x,y)$ satisfies* (0.3), $x,y\in\mathbb{R}^d_+$, *and the operator L* (*and the kernel $h(x,y)$*) *are as defined in* (1.3), *then*

(i) $\|Lf\|_2 \le C\|f\|_2$,

(ii) $\displaystyle\int_{\mathbb{R}^d_+}\Big|\int_{\mathbb{R}^d_+} h(x^a,y^b)f(y)\,dy\Big|^2\,dx \le C\|f\|_2^2.$

Proof. Since $x,y,a,b\ge\bar{1}$, by (0.3) we get

$$\psi_{\bar{1}}(x^{1/a})\psi_{\bar{1}}(y^{1/b})\gamma(x^{1/a}-y^{1/b})\varphi(x^{1/a},y^{1/b}) \in S^0_{0,0},$$

and so (i) follows by Theorem B.

To see (ii), note that $x,y,a,b\ge\bar{1}$, and therefore

$$\begin{aligned} A = \int_{\mathbb{R}^d_+} x^{1/a-\bar{1}}\Big|\int_{\mathbb{R}^d_+} h(x,y)g(y)\,dy\Big|^2\,dx &\le \|Lg\|_2^2\\ &\le C\|\psi_{\bar{1}}(y^{1/b})g(y)\|_2^2 \le C\int_{\mathbb{R}^d_+} y^{\bar{1}-1/b}|g(y)|^2\,dy. \end{aligned}$$

Now set $g(y) = f(y^{1/b})y^{1/b-\bar{1}}$ into A, and after changing variables we get the result. ∎

PROPOSITION 1.7. *Let $a, b \geq \bar{1}$, and let $v(y) = y^{\bar{1}-b/a}$ and $w(x) = 1$ in case $a \geq b \geq \bar{1}$, and $w(x) = x^{a/b-\bar{1}}$ and $v(y) = 1$ in case $b \geq a \geq \bar{1}$, where K is defined by* (0.1) *and* (0.2). *Suppose φ does not depend upon one of the variables x_1, x_2, y_1, y_2 and satisfies* (0.3). *Then*

$$\text{(i) } \|Kf\|_{2,w} \leq C\|f\|_{2,v},$$
$$\text{(ii) } \|\psi_{11}(x)Kf(x)\|_2 \leq C\|f\|_2.$$

Proof. We begin with (i). By duality we need only consider the case where $a \geq b \geq \bar{1}$. Also it is enough to suppose that φ does not depend upon x_2 or y_2.

In case x_2 is missing from φ, from Proposition 1.5 we get

$$\int_0^\infty \Big| \int_0^\infty e^{ix_2^{a_2} y_2^{b_2}} H(x_1, y_2)\, dy_2 \Big|^2 dx_2 \leq C \int_0^\infty |H(x_1, y_2)|^2 y_2^{\alpha_2}\, dy_2$$

with $H(x_1, y_2) = \int_0^\infty e^{ix_1^{a_1} y_1^{b_1}} \varphi(x_1, y) f(y)\, dy_1$. Again Proposition 1.5 yields

$$\|Kf\|_2^2 \leq C \int_0^\infty \Big(\int_0^\infty |H(x_1, y_2)|^2\, dx_1 \Big) y_2^{\alpha_2}\, dy_2 \leq C\|f\|_{2,v}^2.$$

This proves the estimates in case x_2 is missing from φ in (i).

Next suppose that φ does not depend on y_2; then again we deduce from Proposition 1.5 that

$$\int_0^\infty \Big| \int_0^\infty e^{ix_1^{a_1} y_1^{b_1}} \varphi(x, y_1) \Big(\int_0^\infty e^{ix_2^{a_2} y_2^{b_2}} f(y)\, dy_2 \Big) dy_1 \Big|^2 dx_1$$
$$\leq C \int_0^\infty \Big| \int_0^\infty e^{ix_2^{a_2} y_2^{b_2}} f(y)\, dy_2 \Big|^2 y_1^{\alpha_1}\, dy_1,$$

and now proceed as above to complete (i).

To prove (ii), we consider the four terms,

$$\begin{aligned} Kf(x) = \int_{\mathbb{R}_+^2} &((1-\mu(y_1))(1-\mu(y_2)) + (1-\mu(y_1))\mu(y_2) \\ &+ (1-\mu(y_2))\mu(y_1) + \mu(y_1)\mu(y_2))k(x, y)f(y)\, dy \\ = I_1&(x) + I_2(x) + I_3(x) + I_4(x). \end{aligned}$$

By (i) we get (note $a \geq b \geq \bar{1}$)

$$\|\psi_{11}(x)I_1(x)\|_2 \leq \|I_1(x)\|_2 \leq C\|f(y)(1-\mu(y_1))(1-\mu(y_2))\|_{2,v}.$$

Next notice that Proposition 1.6(ii) yields $\|\psi_{11}(x)I_4(x)\|_2 \leq C\|f\|_2$. To estimate the term I_2 (the proof for I_3 is similar) we suppose that φ does not depend upon y_2, thus we notice (as in (i)) that by Proposition 1.5 it follows that

$$\int_0^\infty \psi_{11}(x)|I_2(x)|^2\,dx_1 \le C\int_0^2 y_1^{\alpha_1}\Big|\int_0^\infty \mu(y_2)e^{ix_2^{a_2}y_2^{b_2}}f(y)\,dy_2\Big|^2\,dy_1$$

and then by Proposition 1.6(ii) (in dimension $d=1$) we get

$$\begin{aligned}\int_0^\infty\Big(\int_0^\infty \psi_{11}(x)|I_2(x)|^2\,dx_1\Big)\,dx_2 \\ &\le C\int_0^2 y_1^{\alpha_1}\Big(\int_0^\infty \mu(x_2)\Big|\int_0^\infty \mu(y_2)e^{ix_2^{a_2}y_2^{b_2}}f(y)\,dy_2\Big|^2\,dx_2\Big)\,dy_1\\ &\le C\int_0^2 y_1^{\alpha_1}\Big(\int_1^\infty |f(y)|^2\,dy_2\Big)\,dy_1,\end{aligned}$$

and this completes the proof in case y_2 is missing from φ; if x_2 is missing we first employ Proposition 1.6(ii). ■

Remark 5. In the special case where $\varphi(x,y)=1$, we obtain the same conclusion as in Proposition 1.7.

Proposition 1.8. *Take* $k(x,y)$ *as in* (0.2) *with* $a,b\ge\bar{1}$. *Assume that* φ *satisfies* (0.3). *Let* K *be the operator with any of the kernels* $(1-\mu(x_l))k(x,y)$, $(1-\mu(y_l))k(x,y)$, *or* $k(x,y)(1-\gamma(x-y))$ $(l=1$ *or* $2)$. *Then*

$$\text{(i) } \|Kf\|_{2,w}\le C\|f\|_{2,v},$$
$$\text{(ii) } \|\psi_{11}(x)Kf(x)\|_2\le C\|f\|_2,$$

for $v(y)$ *and* $w(x)$ *as in Proposition* 1.7.

Proof. It suffices to consider the case when $a\ge b\ge\bar{1}$. First consider the kernel $k(x,y)(1-\gamma(x-y))$. Then (i) follows from Proposition 1.3 and also (ii) with $v(y)=1$.

Next by the above we only need to consider the case where the kernel is supported in $|x-y|\ge 1$. Thus, it suffices to take $l=1$ in the x_1 variable,

$$\begin{aligned}(1-\mu(x_1))k(x,y)\gamma(x-y) &= (1-\mu(x_1))\gamma(x-y)e^{ix^a\cdot y^b}\varphi(x,y)\\ &= (1-\mu(x_1))\gamma(x-y)e^{ix^a\cdot y^b}\\ &\quad\times\Big[\varphi(0,x_2,y)+x_1\partial_{x_1}\varphi(0,x_2,y)+\frac12 x_1^2\partial_{x_1}^2\varphi(0,x_2,y)+r(x,y)\Big]\\ &= \sum_{j=0}^3 k_j(x,y)\end{aligned}$$

and let K_j denote the associated operators.

We notice by (0.3) that $|k_3(x,y)| \le C\min(1, |x-y|^{-3})$. Thus we get (i) and (ii) by Corollary 1.2 for the operator K_3.

For the remaining operators K_0, K_1, K_2 we just apply Proposition 1.7(j), j = (i),(ii), respectively for cases (i), (ii). ■

We are in a position to obtain the main result of this section.

THEOREM 1.9. *Suppose $k(x,y)$ satisfies* (0.2) *and $\varphi(x,y)$ satisfies* (0.3) *with $a, b \ge \bar{1}$. Then*

$$\|Kf\|_{2,w} \le C\|f\|_{2,v} \tag{1.3}$$

where $v(y)$ and $w(x)$ are given in Proposition 1.7*.*

Proof. It is enough to assume that $a \ge b \ge \bar{1}$. Suppose that $v(y) = y^{\bar{1}-b/a}$ and $a = b = \bar{1}$. Then (1.4) follows from [PhS].

Putting together Proposition 1.6(ii) and Proposition 1.8(i), we obtain the proof. ■

2. More of the special cases. In this section, we handle the cases where $\varphi(x,y)$ has bounded support in one of the variables y_1, y_2, i.e. we consider $\varphi(x,y)(1-\mu(2y_l/\beta))$ for some $\beta > 0$ for $l = 1$ or 2 and where $\varphi(x,y)$ satisfies (0.4) for some $\eta \in [0,2)$.

We follow the approach we employed in Propositions 1.7 and 1.8 in case $\eta = 0$.

We begin with

PROPOSITION 2.1. *Let $a \ge b \ge \bar{1}$, $v_p(y) = y^{(p/p')(\bar{1}-b/a)}$, $\eta \in [0,2)$ and let the operator K be as in* (0.1)*. Also suppose $\varphi(x,y)$ does not depend upon one of the variables y_1, y_2 and satisfies* (0.4)*. Then for $p = 4/(2+\eta)$,*

$$\begin{cases} \text{(i)} & \Big(\int\limits_{\mathbb{R}^2_+}\Big|\int\limits_{\mathbb{R}^2_+} k_1(x,y)f(y)\,dy\Big|^p dx\Big)^{1/p} \le C\|f\|_{p,v_p}, \\ \text{(ii)} & \Big(\int\limits_{\mathbb{R}^2_+}\Big|\int\limits_{\mathbb{R}^2_+} k_2(x,y)f(y)\,dy\Big|^p dx\Big)^{1/p} \le C\|f\|_{p,v_p}, \end{cases} \tag{2.1}$$

with

$$k_1(x,y) = e^{ix^a\cdot y^b}\mu\Big(\frac{x_1}{3\beta}\Big)\varphi(x,0,y_2)\Big(1-\mu\Big(\frac{2y_1}{\beta}\Big)\Big),$$
$$k_2(x,y) = e^{ix^a\cdot y^b}\mu\Big(\frac{x_2}{3\beta}\Big)\varphi(x,y_1,0)\Big(1-\mu\Big(\frac{2y_2}{\beta}\Big)\Big).$$

Proof. If $\eta = 0$, then $p = 2$ and the result follows from Proposition 1.7(i).

We can suppose that $0 < \eta < 2$, thus $1 < p < 2$, and that φ does not depend upon y_1, as in (2.1)(i), and without any loss we can take $\beta = 1$. As we did in Proposition 1.7, we shall iterate the estimates. We first show that with $\chi_n(x_1) = \chi\big(\frac{1}{2}\cdot 2^n \le x_1 \le 2\cdot 2^n\big)$,

$$\chi_n(x_1)\int_0^\infty \Big|\int_0^\infty e^{ix_2^{a_2}y_2^{b_2}}\varphi(x,0,y_2)h(x_1,y_2)\,dy_2\Big|^p\,dx_2 \le \frac{C}{2^{np\eta/2}}\int_0^\infty |h(x_1,y_2)|^p y_2^{(p/p')\alpha_2}\,dy_2, \tag{2.2}$$

where $h(x_1,y_2) = \int_0^\infty e^{ix_1^{a_1}y_1^{b_1}} f_1(y)\,dy_1$ with $f_1(y) = (1-\mu(2y_1))f(y)$.

To prove (2.2) we set

$$S_zg(x) = \chi_n(x_1)\int_0^\infty e^{ix_2^{a_2}y_2^{b_2}}\varphi(x,0,y_2)|x-(0,y_2)|^{-z}(u(y_2))^{z+\eta-2}g(y_2)\,dy_2$$

where $u(y_2) = y_2^{\alpha_2/4}$; note S_zg is a 1-dimensional integral. We first notice that (the 1-norm is in the x_2-variable)

$$\begin{aligned}\|S_{2-\eta+i\tau}g\|_1 &= \Big\|\chi_n(x_1)\int_0^\infty e^{ix_2^{a_2}y_2^{b_2}}\varphi(x,0,y_2)|x-(0,y_2)|^{-2+\eta-i\tau}(u(y_2))^{i\tau}g(y_2)\,dy_2\Big\|_1\\ &\le C\int_0^\infty \frac{1}{2^{2n}+x_2^2}\,dx_2\,\|g\|_1 \le \frac{C}{2^n}\,\|g\|_1,\end{aligned} \tag{2.3}$$

and notice all that we needed from (0.4) in order to get (2.3) is the size of the function φ.

It follows from Proposition 1.5 that

$$\|S_{-\eta+i\tau}g\|_2 \le C\Big(\int_0^\infty |g(y_2)(u(y_2))^{-2}|^2 y_2^{\alpha_2}\,dy_2\Big)^{1/2} \le C\|g\|_2. \tag{2.4}$$

By interpolation with (2.3) and (2.4) for this analytic family, we get

$$\begin{aligned}\|S_0g\|_p &= \Big\|\chi_n(x_1)\int_0^\infty e^{ix_2^{a_2}y_2^{b_2}}\varphi(x,0,y_2)(u(y_2))^{\eta-2}g(y_2)\,dy_2\Big\|_p\\ &\le \frac{C}{2^{nt}}\,\|g\|_p\end{aligned} \tag{2.5}$$

for $\frac{1}{p} = t + \frac{1-t}{2}$ and $z = tz_0 + (1-t)z_1$ with $z_0 = 2-\eta+i\tau$, $z_1 = -\eta+i\tau$, therefore $0 = (-\eta+i\tau)+t(z_0-z_1)$ or $\eta = 2t$. Thus (2.2) follows from (2.5).

Now we sum up (2.2) to obtain

$$\sum_{n=2}^{\infty}\int_{\frac{1}{2}\cdot 2^n}^{2\cdot 2^n}\Big(\int_0^\infty\Big|\int_0^\infty e^{ix_2^{a_2}y_2^{b_2}}\varphi(x,0,y_2)h(x_1,y_2)\,dy_2\Big|^p\,dx_2\Big)\,dx_1 \tag{2.6}$$
$$\le C\int_0^\infty y_2^{(p/p')\alpha_2}\Big(\int_2^\infty\frac{1}{x_1^{\eta p/2}}\Big|\int_0^\infty e^{ix_1^{a_1}y_1^{b_1}}f_1(y)\,dy_1\Big|^p\,dx_1\Big)\,dy_2.$$

We take

$$p=p_1+p_2,\qquad 1<p_2\le p,\qquad p_2=p \text{ only in case } b_1=a_1. \tag{2.7}$$

Note that if $a_1>b_1\ge 1$, then since $1<p_2<p$ and $p<2$, by (2.7) we get $0<p_1<1$.

We notice that for the inner term on the right side of (2.6) we get

$$\int_2^\infty\frac{1}{x_1^{\eta p/2}}\Big|\int_0^\infty e^{ix_1^{a_1}y_1^{b_1}}f_1(y)\,dy_1\Big|^{p_2}\,dx_1$$
$$=\int_2^\infty\Big|\int_0^\infty e^{ix_1^{a_1}y_1^{b_1}}\widetilde{h}(x_1,y_1)f_1(y)\,dy_1\Big|^{p_2}\,dx_1$$
$$\le C\Big(\int_2^\infty\Big|\int_0^\infty e^{ix_1^{a_1}y_1^{b_1}}\Big(\frac{1}{x_1^{\eta p/(2p_2)}}-\frac{1}{(x_1-y_1)^{\eta p/(2p_2)}}\Big)f_1(y)\,dy_1\Big|^{p_2}\,dx_1$$
$$+\int_2^\infty\Big|\int_0^\infty e^{ix_1^{a_1}y_1^{b_1}}\frac{1}{(x_1-y_1)^{\eta p/(2p_2)}}\,f_1(y)\,dy_1\Big|^{p_2}\,dx_1\Big)=II_1+II_2,$$

with

$$\widetilde{h}(x_1,y_1)=\frac{1}{x_1^{\eta p/(2p_2)}}-\frac{1}{(x_1-y_1)^{\eta p/(2p_2)}}+\frac{1}{(x_1-y_1)^{\eta p/(2p_2)}}.$$

Since the integrand for the term II_1 can be estimated by an L^1 convolution kernel, and we can employ Theorem 4 of [PS] to II_2, we see that

$$II_1+II_2\le C\int_0^1|f(y)|^{p_2}\,dy_1 \tag{2.8}$$

as long as

$$p_2\ge\frac{a_1+b_1}{a_1+\frac{b_1\eta p}{2p_2}}. \tag{2.9}$$

And in order for (2.8) to hold we need that $p_2>1$.

In case $b_1=a_1$ (the exceptional case) we take $p_2=p$ and so by (2.6), (2.8) and (2.9) we get the result.

Now we are left with the cases where $a_1 > b_1 \ge 1$. We first note that

$$\Big|\int_0^\infty e^{ix_1^{a_1}y_1^{b_1}} f_1(y)\,dy_1\Big|^{p_1} \le C\Big(\int_0^1 |f(y)|^p y_1^{(p/p')\alpha_1}\,dy_1\Big)^{p_1/p}. \tag{2.10}$$

It remains to estimate the term on the right side of (2.8). We note that

$$\int_0^1 |f(y)|^{p_2}\,dy_1 \le \Big(\int_0^1 |f(y)|^p y_1^{\alpha_1 p/p'}\,dy_1\Big)^{p_2/p}\Big(\int_0^1 y_1^{-\alpha_1 p_2 p/(p'p_1)}\,dy_1\Big)^{p_1/p}. \tag{2.11}$$

To complete this argument we need to show that $\alpha_1 p_2 p/p'p_1 < 1$ and $1 < p_2 < p$ for each pair (a_1, b_1) with $a_1 > b_1 \ge 1$ and each $0 < \eta < 2$, and that (2.9) holds.

In order to see that (2.9) holds and $p_2 > 1$, note that $p_2 a_1 + b_1\eta p/2 \ge a_1 + b_1$, or

$$p_2 \ge 1 + \frac{b_1}{a_1}\Big(1 - \frac{2\eta}{2+\eta}\Big),$$

and since $\eta < 2$ this implies we can choose $p_2 > 1$ that satisfies (2.9).

In order to see (2.11) we need that $p_2 < p$ and $\alpha_1 p_2 p/p'p_1 < 1$. For the first inequality note that

$$p > 1 + \frac{b_1}{a_1} - \frac{b_1\eta p}{2a_1},$$

or

$$\frac{4}{2+\eta} + \frac{2b_1\eta}{a_1(2+\eta)} > 1 + \frac{b_1}{a_1},$$

since

$$4 + \frac{2b_1\eta}{a_1} > 2 + \eta + \frac{(2+\eta)b_1}{a_1}$$

or

$$2\Big(1 - \frac{b_1}{a_1}\Big) > \eta\Big(1 - \frac{b_1}{a_1}\Big),$$

and this last inequality is always true since $\eta < 2$ and $a_1 > b_1$. To complete this argument we still need to see that $\alpha_1 p_2 p/p'p_1 < 1$, so take $p_1 = p - 1 - \varepsilon$ (this is possible for some $\varepsilon > 0$); then from (2.7) we get $p_2 = 1 + \varepsilon$. Hence

$$\frac{p_2 p}{p'p_1} = \frac{(p-1)(1+\varepsilon)}{p-1-\varepsilon} = 1 + \delta,$$

or $\delta = \frac{\varepsilon p}{p-1-\varepsilon}$ (> 0). Thus

$$\frac{\alpha_1 p_2 p}{p'p_1} = \alpha_1(1+\delta) = (1+\delta) - \frac{b_1}{a_1}(1+\delta) < 1,$$

where the last inequality follows from

$$\delta\Big(1 - \frac{b_1}{a_1}\Big) < \frac{b_1}{a_1} \tag{2.12}$$

for $\delta > 0$ small enough. Thus for each pair (a_1, b_1) we can always find a $\delta > 0$ close enough to zero in order that (2.12) is satisfied. Putting (2.6), (2.10) and (2.11) together we get the result. ∎

PROPOSITION 2.2. *Assume the hypothesis of Proposition* 2.1, *except this time we allow* $\varphi(x,y)$ *to depend upon* y_1 *or* y_2. *Then for* $p = 4/(2+\eta)$,

$$\text{(2.13)}\qquad \begin{cases} \text{(i)}\ \Big(\int\limits_{\mathbb{R}^2_+}\Big|\int\limits_{\mathbb{R}^2_+} k_1(x,y)f(y)\,dy\Big|^p dx\Big)^{1/p} \le C\|f\|_{p,v_p}, \\ \text{(ii)}\ \Big(\int\limits_{\mathbb{R}^2_+}\Big|\int\limits_{\mathbb{R}^2_+} k_2(x,y)f(y)\,dy\Big|^p dx\Big)^{1/p} \le C\|f\|_{p,v_p} \end{cases}$$

with

$$k_l(x,y) = e^{ix^a\cdot y^b}\mu\Big(\frac{x_l}{3\beta}\Big)\varphi(x,y)\Big(1-\mu\Big(\frac{2y_l}{\beta}\Big)\Big), \qquad l = 1,2.$$

Proof. As in the proof of Proposition 1.8 (take $\beta = 1$ with $l = 1$),

$$\text{(2.14)}\qquad \varphi(x,y) = \varphi(x,0,y_2) + y_1\partial^1_{y_1}\varphi(x,0,y_2) + \frac{y_1^2}{2}\,\partial^2_{y_1}\varphi(x,0,y_2) + r(x,y),$$

and so since $x_1 \ge 3$ and $y_1 \le 1$, we get $\mu(x_1/3)(1-\mu(2y_1))|r(x,y)| \le C\min(1,|x-y|^{-3-\eta})$. Thus (2.13)(i) holds by Proposition 1.1 for the operator K_1 with the kernel $k_1(x,y)$, with $r(x,y)$ in place of $\varphi(x,y)$ and $v(y) = v_p(y) = y^{(p/p')(\bar{1}-b/a)}$.

For the three remaining terms in (2.14), if we let K_1 this time be the operator with these terms in place of $\varphi(x,y)$ in $k_1(x,y)$, then we get the result by Proposition 2.1. ∎

PROPOSITION 2.3. *Let* $a \ge b \ge \bar{1}$, $v_p(y) = y^{(p/p')(\bar{1}-b/a)}$, $\eta \in [0,2)$ *and let the operator* K_l *be as in* (0.1), *where*

$$k_l(x,y) = e^{ix^a\cdot y^b}\varphi(x,y)\Big(1-\mu\Big(\frac{2y_l}{\beta}\Big)\Big)\gamma(x-y), \qquad l = 1,2.$$

If $\varphi(x,y)$ *satisfies* (0.4), *then for* $p = 4/(2+\eta)$,

$$\text{(2.15)}\qquad \|K_l f\|_p \le C\|f\|_{p,v_p} \quad \textit{for } l = 1,2.$$

Proof. Because of Proposition 2.2, say with $\beta = 1$ and $l = 1$ (note that $x_1 \ge 3$ and $y_1 \le 1$ implies $\gamma(x-y) = 1$), it suffices to show that

$$\text{(2.16)}\qquad \int\limits_0^6\Big(\int\limits_0^\infty\Big|\int\limits_{\mathbb{R}^2_+} k_1(x,y)f(y)\,dy\Big|^p dx_2\Big)\,dx_1 \le C\|f\|^p_{p,v_p}.$$

First notice that

$$\text{(2.17)}\qquad \int\limits_0^6\Big(\int\limits_0^\infty \frac{\gamma(x-y)}{|x-y|^2}\,dx_2\Big)\,dx_1 + \int\limits_0^6\Big(\int\limits_0^\infty \frac{\gamma(x-y)}{|x-y|^2}\,dx_1\Big)\,dx_2 \le C.$$

We return to (2.17) at the end of this proof. Consider the analytic family given by

$$S_z f(x) = \chi(0 \le x_1 \le 6) \int_{\mathbb{R}^2_+} k_1(x,y)|x-y|^{-z}(w(y))^{z+\eta-2} f(y)\,dy$$

with $w(y) = y^{\frac{1}{4}(\bar{1}-b/a)}$. Then it follows from (2.17) that

$$\|S_{2-\eta+i\tau} f\|_1 \le C\|f\|_1. \tag{2.18}$$

From Proposition 1.8(i) we get

$$\|S_{-\eta+i\tau} f\|_2 \le C\|f\|_2, \tag{2.19}$$

which implies that

$$\|S_0 f\|_p \le C\|f\|_p \quad \text{for } p = \frac{4}{2+\eta}, \tag{2.20}$$

and now (2.16) follows from (2.20).

We are left with showing (2.17). Note that

$$\int_0^\infty \frac{\gamma(x-y)}{|x-y|^2}\,dx_2$$
$$\le \int_0^\infty \frac{\gamma(x-y)}{|x-y|^2}\,(\chi(|x_1-y_1| \le 1/2) + \chi(|x_1-y_1| \ge 1/2))\,dx_2 = I + II.$$

In I we notice that $1/4 + (x_2-y_2)^2 \ge |x-y|^2 \ge 1$, or $|x_2-y_2| \ge \sqrt{3}/2$, therefore

$$I \le \int_{|x_2-y_2|\ge\sqrt{3}/2} \frac{1}{|x_2-y_2|^2}\,dx_2 \le \frac{4\sqrt{3}}{3}.$$

Moreover,

$$II \le 4\int_{-\infty}^{\infty} \frac{1}{1+4|x_2-y_2|^2}\,dx_2 = 2\pi.$$

Therefore,

$$\int_0^6 \biggl(\int_0^\infty \frac{\gamma(x-y)}{|x-y|^2}\,dx_2\biggr)\,dx_1 \le C.$$

The proof in case $0 \le x_2 \le 6$ is similar, and this implies (2.17). ∎

3. The first sufficiency result. In this section we wish to complete the result given in Proposition 2.3 by dropping the support restriction on φ. This is the main result in this section and it appears in Theorem 3.2.

We begin with the operator

$$Sf(x) = \psi_{22}(x) \int_{\mathbb{R}^2_+} e^{ix^a \cdot y^b} \varphi(x,y)\gamma(x-y)\psi_{11}(y)f(y)\,dy, \tag{3.1}$$

where $\psi_{22}(y) = \mu(y_1/2)\mu(y_2/2)$ ($\psi_{11}(y)$ is defined prior to (1.3)) and $\varphi(x,y)$ satisfies (0.4). Notice that the kernel for S is supported in $x_1, x_2, y_1, y_2 \geq 1$.

PROPOSITION 3.1. *Let $a \geq b \geq \bar{1}$, $\eta \in [0,2)$, and suppose $\varphi(x,y)$ satisfies* (0.4)*. Then the operator defined by* (3.1) *satisfies*

$$\|Sf\|_p \leq C\|f\|_{p,v_p} \quad \text{for } p = \frac{4}{2+\eta}. \tag{3.2}$$

Proof. We consider the analytic family

$$\widetilde{S}_z f(x) = \psi_{22}(x) \int_{\mathbb{R}^2} e^{i|x|^a \cdot |y|^b} \varphi(x,y)\gamma(x-y)|x-y|^{-z} f(y)\,dy.$$

From Theorem 2.2 of [SS] (also see Remark 2 in Section 0), we get

$$\|\widetilde{S}_{-\eta+2+i\tau} f\|_1 \leq C\|f\|_{H_E}, \tag{3.3}$$

where H_E is a specially constructed Hardy space related to the operator $\widetilde{S}$.

It suffices to obtain $(2,2)$ estimates for the operator $\widetilde{S}_{-\eta+i\tau}$.

For this operator, since $\varphi(x,y)$ satisfies (0.4), it follows that $\gamma(x-y)$ $\varphi(x,y)|x-y|^{\eta+i\tau}$ satisfies (0.3) and so by Propositions 1.6(ii) and 1.8(ii), we get

$$\|\widetilde{S}_{-\eta+i\tau} f\|_2 \leq C\|f\|_2. \tag{3.4}$$

Thus by (3.3) and (3.4) it follows that

$$\|\widetilde{S}_0 f\|_p \leq C\|f\|_p \quad \text{for } p = \frac{4}{2+\eta}. \tag{3.5}$$

Hence the operator given in (3.1) satisfies

$$\|Sf\|_p \leq C\|f(y)\psi_{11}(y)\|_p \leq C\|f\|_{p,v_p}$$

and this completes the argument. ■

Now we are in a position to state and prove the main result in this section. This completes the result begun in Proposition 2.3, by dropping the support restriction placed on φ there.

THEOREM 3.2. *Let $a \geq b \geq \bar{1}$, $\eta \in [0,2)$, and suppose $\varphi(x,y)$ satisfies* (0.4), *and the operator K as in* (0.1) *has the kernel*

$$k(x,y) = e^{ix^a \cdot y^b} \varphi(x,y)\gamma(x-y).$$

Then for $p = 4/(2+\eta)$,

$$\|Kf\|_p \leq C\|f\|_{p,v_p}. \tag{3.6}$$

Proof. Because of Propositions 2.3 and 3.1, it suffices to prove that

$$\Big(\int_0^6\int_0^\infty + \int_0^\infty\int_0^6\Big)|Kf(x)|^p\,dx_2\,dx_1 \le C\|f\|_{p,v_p}^p. \tag{3.7}$$

To do that, we set up the analytic family defined in the proof of Proposition 2.3 and we notice that the corresponding estimate (2.18) follows from (2.17), and (2.19) follows from Theorem 1.9. This completes the proof of (3.7) and hence the proof of the result. ■

4. Necessity and sufficiency results. In this section, we show (3.6) for the operator K defined in (0.1) and (0.2) where $\varphi(x,y)$ satisfies (0.4). We also obtain a necessity result.

We now state one of the main results of this paper. We recall that $\alpha_j = 1 - b_j/a_j$ for $j = 1, 2$ in case $a \ge b \ge \bar{1}$.

THEOREM 4.1. *Let $a \ge b \ge \bar{1}, 0 \le \eta < 2 - \alpha_1 - \alpha_2$ and let K be as in* (0.1), (0.2) *with φ satisfying* (0.4). *Then for $v_q(y) = y^{(q/p')(\bar{1}-b/a)}$,*

$$\|Kf\|_p \le C\|f\|_{p,v_p} \quad \text{for } p = \frac{4}{2+\eta}. \tag{4.1}$$

Furthermore, if $\varphi(x,y) \ge C|x-y|^{-\eta}$ and (4.1) *holds with $w(x) = 1$ and $v(y) = y^\nu$, then for $j = 1, 2$,*

$$(p-2)\,\frac{b_j}{a_j} - p\eta \le \nu_j - \frac{p}{p'}\,\alpha_j \le \frac{b_j}{a_j}\,(p - 2 + \eta p). \tag{4.2}$$

REMARK 6. If $b \ge a \ge \bar{1}$, $\eta \in [0,2)$, $p' = 4/(2-\eta)$, and

$$Kf(x) = \int_{\mathbb{R}_+^2} e^{ix^a\cdot y^b}\varphi(x,y)\gamma(x-y)f(y)\,dy,$$

then

$$\|Kf\|_{p'} \le C\|f\|_{p',v_{p'}}.$$

Proof of Theorem 4.1. To show the sufficiency, note that

$$\begin{aligned} Kf(x) &= \int_{\mathbb{R}_+^2} e^{ix^a\cdot y^b}\varphi(x,y)(\gamma(x-y) + (1-\gamma(x-y)))f(y)\,dy \\ &= K_1f(x) + K_2f(x). \end{aligned}$$

By Theorem 3.2 we conclude that (4.1) holds for the operator K_1.

To see that the operator K_2 satisfies (4.1) we employ Proposition 1.1. Note that in this case the kernel $k_2(x,y) = e^{ix^a\cdot y^b}\varphi(x,y)(1-\gamma(x-y))$ is in $L^1(dx) \cap L^1(dy)$.

We get

$$r_1(x) = \int_0^\infty \int_0^\infty |\varphi(x,y)| y_1^{-\alpha_1} y_2^{-\alpha_2} (1-\gamma(x-y))\, dy$$
$$= \Big(\int_0^1\int_0^1 + \int_0^1\int_1^\infty + \int_1^\infty\int_0^1 + \int_1^\infty\int_1^\infty \Big) (\cdots)\, dy = \sum_{j=1}^4 I_j.$$

We see that

$$I_4 \le C, \qquad \text{since } 1 \le y_1, y_2.$$

We shall be brief here and just estimate the terms that are the most challenging. We begin with the term I_1 and $0 \le x_1 \le x_2 \le 1$:

$$\int_0^{x_1/2}\int_0^{x_2/2} + \int_0^{x_1/2}\int_{x_2/2}^{x_2} + \int_0^{x_1/2}\int_{x_2}^{1} + \int_{x_1/2}^{x_1}\int_0^{x_2/2}$$
$$+ \int_{x_1/2}^{x_1}\int_{x_2/2}^{x_2} + \int_{x_1/2}^{x_1}\int_{x_2}^{1} + \int_{x_1}^{1}\int_0^{x_2/2} + \int_{x_1}^{1}\int_{x_2/2}^{x_2} + \int_{x_1}^{1}\int_{x_2}^{1} = \sum_{j=1}^9 I_{1j}.$$

First,

$$I_{11} \le \int_0^{x_1/2} \Big(\int_0^{x_2/2} \frac{1}{|x-y|^\eta y_1^{\alpha_1} y_2^{\alpha_2}}\, dy_2 \Big) dy_1$$
$$\le \frac{C}{((x_1/2)^2 + (x_2/2)^2)^{\eta/2}} \Big(\frac{x_1}{2}\Big)^{1-\alpha_1} \Big(\frac{x_2}{2}\Big)^{1-\alpha_2} \le C {x_2}^{2-\alpha_1-\alpha_2-\eta} \le C.$$

Next we estimate

$$I_{19} \le \int_{x_1}^{1} \Big(\int_{x_2}^{1} \frac{1}{|x-y|^\eta y_1^{\alpha_1} y_2^{\alpha_2}}\, dy_2 \Big) dy_1 \le \int_0^1 \Big(\int_0^1 \frac{1}{|y|^\eta y_1^{\alpha_1} y_2^{\alpha_2}}\, dy_2 \Big) dy_1$$
$$\le \int_0^{\pi/2} \Big(\int_0^1 \frac{r}{r^{\eta+\alpha_1+\alpha_2} (\cos\theta)^{\alpha_1} (\sin\theta)^{\alpha_2}}\, dr \Big) d\theta \le C,$$

as long as $1-\eta-\alpha_1-\alpha_2 > -1$ or $\eta < 2-\alpha_1-\alpha_2$.

Next,

$$I_{12} = \int_0^{x_1/2} \Big(\int_{x_2/2}^{x_2} \frac{1}{|x-y|^\eta y_1^{\alpha_1} y_2^{\alpha_2}}\, dy_2 \Big) dy_1$$
$$\le \int_0^{x_1/2} \Big(\int_{x_2/2}^{x_2} \frac{1}{((x_1/2)^2 + (x_2-y_2)^2)^{\eta/2} y_1^{\alpha_1} y_2^{\alpha_2}}\, dy_2 \Big) dy_1$$
$$(\text{set } y_2 = s x_2)$$

$$= \int_0^{x_1/2} \left(\int_{1/2}^1 \frac{x_2^{1-\alpha_2}}{((x_1/2)^2 + x_2^2(1-s)^2)^{\eta/2} y_1^{\alpha_1} s^{\alpha_2}} \, ds \right) dy_1$$

$$\le C \int_{1/2}^1 \frac{x_1^{1-\alpha_1} x_2^{1-\alpha_2}}{(x_1^2 + x_2^2(1-s)^2)^{\eta/2}} \, ds \le C \int_0^1 \frac{x_1^{1-\alpha_1} x_2^{1-\alpha_2}}{(x_1^2 + x_2^2(1-s)^2)^{\eta/2}} \, ds$$

$$\le C \left(\int_0^{x_1/x_2} \frac{x_1^{1-\alpha_1} x_2^{1-\alpha_2}}{x_1^\eta} \, ds + \int_{x_1/x_2}^1 \frac{x_1^{1-\alpha_1} x_2^{1-\alpha_2}}{(s x_2)^\eta} \, ds \right)$$

$$\le C(x_1^{2-\alpha_1-\eta} x_2^{-\alpha_2} + x_1^{1-\alpha_1} x_2^{1-\alpha_2-\eta}(1 + (x_1/x_2)^{1-\eta})) \le C$$

for $\eta \neq 1$, where we used the fact that $x_1 \le x_2$ and $\eta < 2 - \alpha_1 - \alpha_2$. If $\eta = 1$, we get

$$\begin{aligned} I_{12} &\le C(1 + x_1^{1-\alpha_1} x_2^{-\alpha_2} \log(x_2/x_1)) \le C(1 + x_1^{1-\alpha_1} x_2^{-\alpha_2} (x_2/x_1)^{1-\alpha_1}) \\ &\le C(1 + x_2^{1-\alpha_1-\alpha_2}) \le C, \end{aligned}$$

but here $1 = \eta < 2 - \alpha_1 - \alpha_2$ or $\alpha_1 + \alpha_2 < 1$.

Now,

$$I_{13} = \int_0^{x_1/2} \left(\int_{x_2}^1 \frac{1}{((x_1/2)^2 + (x_2 - y_2)^2)^{\eta/2} y_1^{\alpha_1} y_2^{\alpha_2}} \, dy_2 \right) dy_1$$

$$\le \int_0^{x_1/2} \frac{1}{y_1^{\alpha_1}} \left(\int_{|y_2 - x_2| \le x_1/2} \frac{\chi(y_2 \ge x_2)}{x_1^\eta y_2^{\alpha_2}} \, dy_2 \right.$$

$$\left. + \int_{x_1/2 \le |y_2 - x_2| \le 1} \frac{\chi(y_2 \ge x_2)}{|y_2 - x_2|^\eta y_2^{\alpha_2}} \, dy_2 \right) dy_1$$

$$\le C \int_0^{x_1} \frac{1}{y_1^{\alpha_1}} \left(\frac{x_1}{x_1^\eta x_2^{\alpha_2}} + \int_{x_2 + x_1/2}^1 \frac{1}{(y_2 - x_2)^\eta y_2^{\alpha_2}} \, dy_2 \right) dy_1 \le C.$$

Finally,

$$I_{15} = \int_{x_1/2}^{x_1} \left(\int_{x_2/2}^{x_2} \frac{1}{|x - y|^\eta y_1^{\alpha_1} y_2^{\alpha_2}} \, dy_2 \right) dy_1$$

$$(\text{set } y_l = s_l x_l, \ l = 1, 2)$$

$$\le \int_{1/2}^1 \left(\int_{1/2}^1 \frac{x_1^{1-\alpha_1} x_2^{1-\alpha_2}}{((x_1(1-s_1))^2 + (x_2(1-s_2))^2)^{\eta/2}} \, ds_2 \right) ds_1$$

$$\le \int_0^1 \left(\int_0^1 \frac{x_1^{1-\alpha_1} x_2^{1-\alpha_2}}{((x_1 s_1)^2 + (x_2 s_2)^2)^{\eta/2}} \, ds_2 \right) ds_1$$

$$\le \int_0^1 \Bigg(\int_0^{s_1x_1/x_2} \frac{x_1^{1-\alpha_1}x_2^{1-\alpha_2}}{(x_1s_1)^\eta}\,ds_2 + \int_{s_1x_1/x_2}^1 \frac{x_1^{1-\alpha_1}x_2^{1-\alpha_2}}{(x_2s_2)^\eta}\,ds_2\Bigg)\,ds_1$$
$$= I_{151} + I_{152}.$$

We have

$$I_{151} \le C\,\frac{x_1^{2-\alpha_1}x_2^{-\alpha_2}}{x_1^\eta}\int_0^1 s_1^{1-\eta}\,ds_1 \le C, \qquad \eta \ne 1,$$
$$I_{152} \le C\,\frac{x_1^{1-\alpha_1}x_2^{1-\alpha_2}}{x_2^\eta}\int_0^1 \bigg(\Big(\frac{s_1x_1}{x_2}\Big)^{1-\eta} + 1\bigg)\,ds_1$$
$$\le C\,\frac{x_1^{1-\alpha_1}x_2^{1-\alpha_2}}{x_2^\eta}\bigg(\Big(\frac{x_1}{x_2}\Big)^{1-\eta} + 1\bigg) \le C.$$

If $\eta = 1$, then

$$I_{152} \le C\,\frac{x_1^{1-\alpha_1}x_2^{1-\alpha_2}}{x_2}\int_0^1 \bigg(\int_{s_1x_1/x_2}^1 \frac{1}{s_2^{2-\alpha_1}}\,ds_2\bigg)ds_1$$
$$\le C\,\frac{x_1^{1-\alpha_1}x_2^{1-\alpha_2}}{x_2}\,\frac{x_2^{1-\alpha_1}}{x_1^{1-\alpha_1}}\int_0^1 \frac{1}{s_1^{1-\alpha_1}}\,ds_1 \le C.$$

This explains the reason for the hypothesis on η, $0 \le \eta < 2 - \alpha_1 - \alpha_2$.

In order to see the necessity, we take $w(x) = 1$ and $v(y) = y^\nu$. If we assume that our operator maps continuously, then we get, with $\varphi(x,y) \ge C|x-y|^{-\eta}$ and $g(y) = f(y^{1/b})y^{1/b-\bar{1}}$,

$$\int_{[N,2N]} x^{1/a-\bar{1}}\bigg|\int_{[1/(2N),1/N]} \varphi(x^{1/a},y^{1/b})e^{ix\cdot y}g(y)\,dy\bigg|^p\,dx \le C\|f\|_{p,v}^p \tag{4.3}$$

where for $N = (N_1,N_2)$ we take

$$f(y) = \begin{cases} y^{-(\nu+\bar{1})/p} & \text{if } 2^{-M/b} \le y \le 2^{M/b},\\ 0 & \text{elsewhere.}\end{cases}$$

Set

$$G(l,k) = \int_{[N,2N]} x^{1/a-\bar{1}}\bigg|\int_{[1/(2N),1/N]} \varphi(x^{1/a},y^{1/b})e^{ix\cdot y}g(y)\,dy\bigg|^p\,dx$$

with $N_1 = 2^l$ and $N_2 = 2^k$. It follows from (4.3) that, with $-M_1 \le l \le M_1$ and $-M_2 \le k \le M_2$,

$$\sum_{(l,k)} G(l,k) \le C\int_{2^{-(M_1+1)/b_1}}^{2^{M_1/b_1}}\int_{2^{-(M_2+1)/b_2}}^{2^{M_2/b_2}} |f(y)|^p v(y)\,dy \le CM_1M_2.$$

And so we get

(4.4) $I + II + III + IV$

$$= \sum_{l=-M_1}^{1} \sum_{k=-M_2}^{1} + \sum_{l=-M_1}^{1} \sum_{k=2}^{M_2} + \sum_{l=2}^{M_1} \sum_{k=-M_2}^{1} + \sum_{l=2}^{M_1} \sum_{k=2}^{M_2} \le CM_1M_2.$$

In particular,

$$IV_1 = \sum_{l=2}^{M_1} \sum_{k=la_2/a_1}^{M_2} (\cdots) \le CM_1M_2.$$

Hence,

$$\sum_{l=2}^{M_1} \sum_{k=la_2/a_1}^{M_2} 2^{k(\kappa_2 - p\eta/a_2)} 2^{l\kappa_1} \le CM_1M_2,$$

where

$$\kappa_j = 1/a_j + (\nu_j + 1)/b_j - p/b_j \quad \text{for } j = 1, 2.$$

In order to avoid a contradiction, we must have

$$\kappa_2 - p\,\frac{\eta}{a_2} \le 0 \quad \text{or} \quad \nu_2 \le \alpha_2\,\frac{p}{p'} + (p - 2 + \eta p)\,\frac{b_2}{a_2}. \tag{4.5}$$

Considering the piece IV_2 we would get a similar estimate for ν_1.

Estimating the term I_1, i.e.

$$I_1 = \sum_{k=-M_2}^{1} \sum_{l=-M_1}^{(b_1/b_2)k} 2^{k\kappa_2} 2^{l(\kappa_1 + p\eta/b_1)} \le CM_1M_2,$$

in order to avoid a contradiction, we must have

$$\kappa_1 + p\,\frac{\eta}{b_1} \ge 0, \quad \text{or} \quad \nu_1 \ge \frac{p}{p'}\,\alpha_1 + \frac{b_1}{a_1}\,(p-2) - p\eta. \tag{4.6}$$

Putting (4.5) and (4.6) together gets us the result (4.2) and hence the proof is complete. ∎

Remark 7. To see the necessity to Theorem 1.9, if $\varphi(x, y) \ge C$, and (1.4) holds for $v(y) = y^\nu$ with $a \ge b \ge \bar{1}$ and $w(x) = 1$, then by (4.2) it follows ($p = 2$ and $\eta = 0$) that $\nu = \bar{1} - b/a$. Similarly, if $b \ge a \ge \bar{1}$ and $w(x) = x^\nu$ with $v(y) = 1$, then it follows that $\nu = \bar{1} - a/b$. This characterizes fully the power weights case for $p = 2$.

Remark 8. It should be pointed out that from Theorem A (and Remark 1), if $a_1/b_1 = a_2/b_2$ and if $p_0 = 4/(2 + \eta) \in J$, then our operators map L^{p_0} into L^{p_0} (i.e. with constant weights). For example if $a_j/b_j \ge 1$ in Theorem A, then $4/(2 + \eta)$ is in J (if $\eta \ge 2/3$), and so as this example shows, $\nu_1 = \nu_2 = 0$ in (4.2) is possible for this p_0.

But in contrast to Remark 8 we get

Corollary 4.2. *With the hypothesis in Theorem* 4.1, *suppose additionally that* $\varphi(x,y) \geq C|x-y|^{-\eta}$ *and* $\nu_j \geq (p/p')\alpha_j$ *for* $j=1,2$. *Then for* $p = 4/(2+\eta)$,

$$\|Kf\|_p \leq C\|f\|_{p,v_p} \quad \textit{if and only if} \quad \nu_j = \frac{p}{p'}\,\alpha_j \textit{ for } j=1,2.$$

Proof. It follows from (4.3) with

$$f(y) = \begin{cases} 1 & \text{if } (2N)^{-1/b} \leq y \leq N^{-1/b}, \\ 0 & \text{elsewhere,} \end{cases}$$

that for $N_2 = N_1^{\varrho}$ and $\varrho \geq 0$,

$$\frac{N_1^{(1/p)(\kappa_1+\varrho\kappa_2)}}{(N_1^{2/a_1} + N_1^{2\varrho/a_2})^{\eta/2}} \leq C. \tag{4.7}$$

Let $0 \leq \varrho \leq a_2/a_1$ and $a \geq b \geq \bar{1}$. Then we get

$$N_1^{\kappa_1/p-\eta/a_1+\varrho\kappa_2/p} \leq C. \tag{4.8}$$

Thus if we let $N_1 \to \infty$ in (4.8), it follows that

$$\nu_1 \leq p - 1 - \frac{b_1}{a_1}\,(1-p\eta) - b_1\varrho\kappa_2.$$

We select

$$\varrho = \frac{p(1+\eta)-2}{a_1\kappa_2}. \tag{4.9}$$

To check that ϱ in (4.9) is consistent with our previous inequalities, we first show that $\varrho \geq 0$. Since $p = 4/(2+\eta)$, we get $p(1+\eta)-2 \geq 0$. Thus, it suffices to see that $\kappa_2 > 0$. Indeed, since $\nu_2 \geq (p-1)\alpha_2$, it follows that

$$\kappa_2 \geq \frac{(1-p)(b_2/a_2)}{b_2} + 1/a_2 = \frac{2-p}{a_2} > 0 \quad \text{for } 1 \leq p < 2.$$

Because of (4.2) of Theorem 4.1 we need only consider the cases where $p \neq 2$.

To check that $\varrho \leq a_2/a_1$, note that this implies that

$$p(1+\eta) - 2 \leq a_2\kappa_2, \quad \text{or} \quad p((a_2/b_2)+1+\eta) \leq 3 + (\nu_2+1)(a_2/b_2),$$

or

$$(p-(\nu_2+1))(a_2/b_2) \leq 3 - \frac{4(1+\eta)}{2+\eta}, \quad \text{or} \quad p \leq \frac{2-\eta}{2+\eta}\,(b_2/a_2) + \nu_2 + 1.$$

But $\nu_2 + 1 \geq p - (p-1)(b_2/a_2)$, and since

$$p - 1 = \frac{2-\eta}{2+\eta},$$

this finishes the argument because these steps can be reversed.

We showed that if $1 \le p < 2$ and $\nu_2 \ge (p/p')\alpha_2$, then $\nu_1 \le (p/p')\alpha_1$; similarly if $\nu_1 \ge (p/p')\alpha_1$, then $\nu_2 \le (p/p')\alpha_2$. Putting these two results together we get the proof. ∎

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