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Orlicz–Morrey spaces and the Hardy–Littlewood maximal function

by

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Dedicated to Professor Mikihiro Hayashi on his sixtieth birthday

Abstract. We prove basic properties of Orlicz–Morrey spaces and give a necessary and sufficient condition for boundedness of the Hardy–Littlewood maximal operator M from one Orlicz–Morrey space to another. For example, if $f \in L(\log L)(\mathbb{R}^n)$, then Mf is in a (generalized) Morrey space (Example 5.1). As an application of boundedness of M, we prove the boundedness of generalized fractional integral operators, improving earlier results of the author.

1. Introduction. Orlicz spaces, introduced in [29, 30], are generalizations of Lebesgue spaces L^p. They are useful tools in harmonic analysis and its applications. For example, the Hardy–Littlewood maximal operator is bounded on L^p for $1 < p \le \infty$, but not on L^1. Using Orlicz spaces, we can investigate the boundedness of the operator near $p = 1$ precisely (see Kita [14, 15] and Cianchi [4]). It is known that the fractional integral operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < q < \infty$ and $-n/p+\alpha = -n/q$ (the Hardy–Littlewood–Sobolev theorem). Trudinger [40] investigated the boundedness of I_α near $q = \infty$. The Hardy–Littlewood–Sobolev theorem and Trudinger's result have been generalized by several authors: [28, 37, 38, 5, 4, 23, 24, 25], etc. For the theory of Orlicz spaces, see [18, 16, 33].

On the other hand, Morrey spaces were introduced in [19] to estimate solutions of partial differential equations, and studied in many papers. For the boundedness of the Hardy–Littlewood maximal operator and fractional integral operators, see [31, 1, 3, 20].

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The author introduced Orlicz–Morrey spaces in [26] to investigate the boundedness of generalized fractional integral operators. Orlicz–Morrey spaces unify Orlicz and Morrey spaces. Recently, Orlicz–Morrey spaces were used by Sawano, Sobukawa and Tanaka [34] to prove a Trudinger type inequality for Morrey spaces.

In this paper we prove basic properties of Orlicz–Morrey spaces and give a necessary and sufficient condition for boundedness of the Hardy–Littlewood maximal operator M from one Orlicz–Morrey space to another. It is known that, on a finite ball $B \subset \mathbb{R}^n$, if $f \in L(\log L)(B)$, then $Mf \in L^1(B)$ (see also [35]). However, on $\mathbb{R}^n$ this relation does not hold. We show, for example, that if $f \in L(\log L)(\mathbb{R}^n)$, then Mf is in a (generalized) Morrey space (see Example 5.1).

Moreover, we give a sufficient condition for weak boundedness of the Hardy–Littlewood maximal operator M. As an application of boundedness of M, we show the boundedness of generalized fractional integral operators. In the proof, we use a pointwise estimate by $Mf(x)$ and the boundedness of M. This method was introduced by Hedberg [13] to give a simple proof of the Hardy–Littlewood–Sobolev theorem. Our results improve those in [26]. For generalized fractional integral operators, see also [32, 23, 24, 25, 6, 7, 11, 8].

Our definition of Orlicz–Morrey spaces is different from that of Kokilashvili and Krbec [16, p. 2].

We recall the definitions of Orlicz and Morrey spaces in the next section, and give the definition of Orlicz–Morrey spaces in Section 3. In Section 4, we give generalized Hölder's inequality and inclusion relations for Orlicz–Morrey spaces. The results on boundedness of the Hardy–Littlewood maximal operator and of generalized fractional integral operators are stated in Sections 5, 6 and 7, and proved in the remaining sections.

2. Orlicz and Morrey spaces. First we recall the definition of Young functions. A function $\Phi : [0,+\infty] \to [0,+\infty]$ is called a *Young function* if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = \Phi(+\infty) = +\infty$. Any Young function is neither identically zero nor identically infinite on $(0,+\infty)$. From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing.

If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \ge s$. Let

$$r_0 = \inf\{s > 0 : \Phi(s) = +\infty\}.$$

Then $r_0 > 0$, since $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$. If $\Phi(r_0) < +\infty$, then Φ is absolutely continuous on $[0, r_0]$ by convexity and monotonicity. If $\Phi(r_0) = +\infty$, then Φ is absolutely continuous on any closed interval in $[0, r_0)$ and $\lim_{r\to r_0-0} \Phi(r) = +\infty$ by left-continuity. Note that, if $\Phi(r_0) < +\infty$, then we can find a Young function Ψ such that $\Psi(\delta r) \le \Phi(r) \le \Psi(r)$ for some $0 < \delta < 1$, $\Psi(r) < +\infty$ for $0 \le r < r_0$, and $\lim_{r \to r_0 - 0} \Psi(r) = \Psi(r_0) = +\infty$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \quad \text{for } 0 < r < +\infty. \tag{2.1}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on any closed interval in $[0, +\infty)$ and bijective from $[0, +\infty)$ to itself.

Definition 2.1 (Orlicz space). For a Young function Φ, let

$$L^\Phi(\mathbb{R}^n) = \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)\, dx < +\infty \text{ for some } k > 0 \Big\},$$

$$\|f\|_{L^\Phi} = \inf \Big\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi(|f(x)|/\lambda)\, dx \le 1 \Big\}.$$

Then $\|f\|_{L^\Phi}$ is a norm and $L^\Phi(\mathbb{R}^n)$ is a Banach space. This norm was introduced by Nakano [27] and Luxemburg [17]. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$ $(0 \le r \le 1)$ and $\Phi(r) = +\infty$ $(r > 1)$, then $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

We note that

$$\int_{\mathbb{R}^n} \Phi(|f(x)|/\|f\|_{L^\Phi})\, dx \le 1.$$

For Young functions Φ and Ψ, we write $\Phi \approx \Psi$ if there exists a constant $C \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \quad \text{for all } r \ge 0.$$

If $\Phi \approx \Psi$, then $L^\Phi(\mathbb{R}^n) = L^\Psi(\mathbb{R}^n)$ with equivalent norms. We note that, for Young functions Φ and Ψ, if there exist $C, R \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \quad \text{for } r \in (0, R^{-1}) \cup (R, +\infty),$$

then $\Phi \approx \Psi$.

For a Young function Φ and for $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \quad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < +\infty.$$

The following is due to O'Neil [28] (see also Ando [2]).

Theorem 2.1 ([28, Theorem 2.3]). *If there exists a constant $c > 0$ such that*

$$\Phi_1^{-1}(r)\Phi_3^{-1}(r) \le c\Phi_2^{-1}(r) \quad \textit{for all } r \ge 0,$$

then

$$\|fg\|_{L^{\Phi_2}} \le 2c\|f\|_{L^{\Phi_1}}\|g\|_{L^{\Phi_3}}.$$

A Young function Φ is said to satisfy the Δ_2-*condition*, denoted $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \quad \text{for } r > 0,$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-*condition*, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\,\Phi(kr), \quad r \ge 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

For a Young function Φ, the *complementary function* is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,+\infty)\}, & r \in [0,+\infty), \\ +\infty, & r = +\infty. \end{cases} \tag{2.2}$$

Then $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ $(0 \le r \le 1)$ and $\widetilde{\Phi}(r) = +\infty$ $(r > 1)$. If $1 < p < \infty$, $1/p+1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \quad \text{for } r \ge 0. \tag{2.3}$$

Let $\mathcal{Y}_1$ be the set of all $\Phi \in \mathcal{Y}$ such that $\int_0^1 \Phi(t)t^{-2}\,dt < +\infty$. For $\Phi \in \mathcal{Y}_1$, let

$$\Phi^+(r) = r\int_0^r \frac{\Phi(t)}{t^2}\,dt, \quad r \ge 0. \tag{2.4}$$

Then $\Phi^+ \in \mathcal{Y}$ and $\Phi(r) \le \Phi^+(2r)$ for all $r \ge 0$.

THEOREM 2.2 ([16, Theorem 1.2.1]). *Let $\Phi \in \mathcal{Y}$. Then the following are equivalent*:

(i) $\Phi \in \nabla_2$ (*that is*, $\widetilde{\Phi} \in \Delta_2$).
(ii) $\Phi \in \mathcal{Y}_1$ *and* $\Phi^+ \approx \Phi$.
(iii) *The Hardy–Littlewood maximal operator is bounded on* $L^\Phi(\mathbb{R}^n)$.

Next we recall the definition of Morrey spaces. Let $B(a,r)$ be the ball $\{x \in \mathbb{R}^n : |x-a| < r\}$ with center a and radius $r > 0$.

DEFINITION 2.2 (Morrey space). For $1 \le p < \infty$ and $0 \le \lambda \le n$, let

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < +\infty\},$$

$$\|f\|_{L^{p,\lambda}} = \sup_{B=B(a,r)} \left(\frac{1}{r^\lambda}\int_B |f(x)|^p\,dx\right)^{1/p}.$$

Then $L^{p,\lambda}(\mathbb{R}^n)$ is a Banach space. If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\lambda = n$, then $L^{p,\lambda}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

If $1/p_1 + 1/p_3 = 1/p_2$ and $\lambda_1/p_1 + \lambda_3/p_3 = \lambda_2/p_2$, then by Hölder's inequality we get

$$\|fg\|_{L^{p_2,\lambda_2}} \le \|f\|_{L^{p_1,\lambda_1}} \|g\|_{L^{p_3,\lambda_3}}. \tag{2.5}$$

It is known that, if $1 \le p < q < \infty$ and $0 \le \lambda < n$, then there exists a function $f \in L^{p,\lambda}(\mathbb{R}^n)$ such that $f \notin L^{q,\mu}(\mathbb{R}^n)$ for all $0 \le \mu \le n$ (for example [10, p. 67] and [22, Remark 2.3]). We will extend this fact to Orlicz–Morrey spaces (Theorem 4.9).

3. Definition of Orlicz–Morrey spaces. For a measurable set Ω in $\mathbb{R}^n$, we denote the characteristic function of Ω by χ_Ω and the Lebesgue measure of Ω by $|\Omega|$. For a ball $B = B(a,r)$ and $k > 0$, we shall denote $B(a,kr)$ by kB.

A function $\theta : (0,+\infty) \to (0,+\infty)$ is said to be *almost increasing* (resp. *almost decreasing*) if there exists a constant $C > 0$ such that

$$\theta(r) \le C\theta(s) \quad (\text{resp. } \theta(r) \ge C\theta(s)) \quad \text{for } r \le s.$$

A function $\theta : (0,+\infty) \to (0,+\infty)$ is said to satisfy the *doubling condition* if there exists a constant $C > 0$ such that

$$C^{-1} \le \theta(r)/\theta(s) \le C \quad \text{for } 1/2 \le r/s \le 2.$$

For functions $\theta, \kappa : (0,+\infty) \to (0,+\infty)$, we write $\theta(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r) \quad \text{for } r > 0.$$

Let $\mathcal{G}$ be the set of all functions $\phi : (0,+\infty) \to (0,+\infty)$ such that ϕ is almost decreasing and $\phi(r)r$ is almost increasing. If $\phi \in \mathcal{G}$, then ϕ satisfies the doubling condition. Let $\psi : (0,+\infty) \to (0,+\infty)$ and $\psi \sim \phi$ for some $\phi \in \mathcal{G}$. Then $\psi \in \mathcal{G}$.

For a Young function Φ, $\phi \in \mathcal{G}$ and a ball B, let

$$\|f\|_{\Phi,\phi,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|\phi(|B|)} \int_B \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

Definition 3.1 (Orlicz–Morrey space). For a Young function Φ and $\phi \in \mathcal{G}$, let

$$L^{(\Phi,\phi)}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{(\Phi,\phi)}} < +\infty\},$$
$$\|f\|_{L^{(\Phi,\phi)}} = \sup_B \|f\|_{\Phi,\phi,B}.$$

Then $\|\cdot\|_{L^{(\Phi,\phi)}}$ is a norm and $L^{(\Phi,\phi)}(\mathbb{R}^n)$ is a Banach space, since

$$\|f\|_{\Phi,\phi,B} = \|f\|_{L^\Phi(B,dx/(|B|\phi(|B|)))},$$

which is a norm on the Orlicz space $L^\Phi(B, dx/(|B|\phi(|B|)))$.

Definition 3.2 (generalized Morrey space). If $\Phi(r) = r^p$, $1 \le p < \infty$, then

$$\|f\|_{\Phi,\phi,B} = \left(\frac{1}{|B|\phi(|B|)} \int_B |f(x)|^p \, dx\right)^{1/p}.$$

In this case we denote $L^{(\Phi,\phi)}(\mathbb{R}^n)$ by $L^{(p,\phi)}(\mathbb{R}^n)$.

By the definition we have the following.

Proposition 3.1. *If $\phi(r) = 1/r$, then $L^{(\Phi,\phi)}(\mathbb{R}^n)$ coincides with the Orlicz space $L^\Phi(\mathbb{R}^n)$. If $\Phi(r) = r^p$ and $\phi(r) = r^{-1+\lambda/n}$ $(0 \le \lambda \le n)$, then $L^{(\Phi,\phi)}(\mathbb{R}^n)$ coincides with the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$.*

From the next proposition, if $\Phi \approx \Psi$ and $\phi \sim \psi$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{(\Psi,\psi)}(\mathbb{R}^n)$ with equivalent norms.

Proposition 3.2. *Let Φ, Ψ be Young functions and let $\phi, \psi \in \mathcal{G}$.*

(1) *If $\Phi(r) \le \Psi(Cr)$, then*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \supset L^{(\Psi,\phi)}(\mathbb{R}^n), \qquad \|f\|_{L^{(\Phi,\phi)}} \le C\|f\|_{L^{(\Psi,\phi)}}.$$

(2) *If $\phi(r) \le C\psi(r)$, then*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \subset L^{(\Phi,\psi)}(\mathbb{R}^n), \qquad \max(1,C)\|f\|_{L^{(\Phi,\phi)}} \ge \|f\|_{L^{(\Phi,\psi)}}.$$

Proof. We note that

$$\int_B \Phi(|f(x)|/\|f\|_{L^{(\Phi,\phi)}}) \, dx \le |B|\phi(|B|) \quad \text{for all balls } B.$$

Conversely, if there exists $\lambda > 0$ such that

$$\int_B \Phi(|f(x)|/\lambda) \, dx \le |B|\phi(|B|) \qquad \text{for all balls } B,$$

then $\|f\|_{L^{(\Phi,\phi)}} \le \lambda$.

By the inequality

$$\Phi\left(\frac{|f(x)|}{C\|f\|_{L^{(\Psi,\phi)}}}\right) \le \Psi\left(\frac{|f(x)|}{\|f\|_{L^{(\Psi,\phi)}}}\right),$$

we have (1). By the convexity of Φ we have

$$\Phi\left(\frac{|f(x)|}{\max(1,C)\|f\|_{L^{(\Phi,\phi)}}}\right) \le \frac{1}{\max(1,C)}\,\Phi\left(\frac{|f(x)|}{\|f\|_{L^{(\Phi,\phi)}}}\right) \le \frac{1}{C}\,\Phi\left(\frac{|f(x)|}{\|f\|_{L^{(\Phi,\phi)}}}\right),$$

which yields (2). ∎

By the definition and Lebesgue's differentiation theorem we have the following.

Proposition 3.3. *Let Φ be a Young function and $\phi \in \mathcal{G}$.*

(1) *If $c_0 = \sup_{u>0} \phi(u) < +\infty$, then*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{L^\infty} \le \Phi^{-1}(c_0)\|f\|_{L^{(\Phi,\phi)}}.$$

(2) *If* $c_1 = \inf_{u>0} \phi(u) > 0$, *then*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{L^\infty} \geq \Phi^{-1}(c_1)\|f\|_{L^{(\Phi,\phi)}}.$$

Therefore, if $\phi \sim 1$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ with equivalent norms.

By the next proposition we may assume that ϕ is continuous and strictly decreasing in the definition of $L^{(\Phi,\phi)}(\mathbb{R}^n)$.

PROPOSITION 3.4. *If* $\phi \in \mathcal{G}$, *then there exists* $\bar\phi \in \mathcal{G}$ *such that* $\bar\phi \sim \phi$ *and* $\bar\phi$ *is continuous and strictly decreasing.*

Proof. Let

$$\underline{c}_\phi = \sup_{0<t\leq r<+\infty} \frac{\phi(r)}{\phi(t)} \quad \text{and} \quad \overline{c}_\phi = \sup_{0<t\leq r<+\infty} \frac{t\phi(t)}{r\phi(r)}. \tag{3.1}$$

Then $1 \leq \underline{c}_\phi, \overline{c}_\phi < \infty$ by the definition of $\mathcal{G}$. Let

$$\phi_1(r) = \inf_{t\leq r} \phi(t).$$

Then ϕ_1 is decreasing, $\phi_1(r) \leq \phi(r) \leq \underline{c}_\phi \phi_1(r)$, and so $\phi_1 \in \mathcal{G}$.

If $\inf_{r>0} \phi(r) = c_0 > 0$, then $\lim_{r\to+\infty} \phi_1(r) = c_0$. We choose a strictly increasing function $\theta : (0,+\infty) \to (0,+\infty)$ so that $\lim_{r\to0} \theta(r) = 0$ and $\lim_{r\to+\infty} \theta(r) = c_0/2$, and let $\phi_2 = \phi_1 - \theta$. Then ϕ_2 is strictly decreasing and $\phi_2 \leq \phi_1 \leq (3/2)\phi_2$.

If $\inf_{r>0} \phi(r) = 0$, then $\lim_{r\to+\infty} \phi_1(r) = 0$. In this case we let $\phi_2 = \phi_1$.

Let

$$\bar\phi(r) = r \int_r^{+\infty} \frac{\phi_2(t)}{t^2}\, dt.$$

Then $\bar\phi$ is continuous and strictly decreasing. Indeed, for $r < s$,

$$r \int_r^{+\infty} \frac{\phi_2(t)}{t^2}\, dt = r \int_s^{+\infty} \frac{\phi_2((r/s)t)}{(r/s)t^2}\, dt = s \int_s^{+\infty} \frac{\phi_2((r/s)t)}{t^2}\, dt$$
$$> s \int_s^{+\infty} \frac{\phi_2(t)}{t^2}\, dt.$$

Moreover,

$$r \int_r^{+\infty} \frac{\phi_2(t)}{t^2}\, dt < r \int_r^{+\infty} \frac{\phi_2(r)}{t^2}\, dt = \phi_2(r) = 2r \int_r^{2r} \frac{\phi_2(r)}{t^2}\, dt$$
$$\leq 4\overline{c}_\phi r \int_r^{2r} \frac{\phi_2(t)}{t^2}\, dt < 4\overline{c}_\phi r \int_r^{+\infty} \frac{\phi_2(t)}{t^2}\, dt.$$

Therefore $\bar\phi \sim \phi$ and $\bar\phi \in \mathcal{G}$. ∎

4. Generalized Hölder's inequality and inclusion relations

THEOREM 4.1. *Let Φ_i be Young functions and $\phi_i \in \mathcal{G}$, $i = 1, 2, 3$. Assume that there exists a constant $c > 0$ such that*

$$\Phi_1^{-1}(r\phi_1(s))\Phi_3^{-1}(r\phi_3(s)) \le c\,\Phi_2^{-1}(r\phi_2(s)) \quad \text{for } r, s > 0.$$

If $f \in L^{(\Phi_1,\phi_1)}(\mathbb{R}^n)$ and $g \in L^{(\Phi_3,\phi_3)}(\mathbb{R}^n)$, then $fg \in L^{(\Phi_2,\phi_2)}(\mathbb{R}^n)$ and

$$\|fg\|_{L^{(\Phi_2,\phi_2)}} \le 2c\|f\|_{L^{(\Phi_1,\phi_1)}}\|g\|_{L^{(\Phi_3,\phi_3)}}.$$

Proof. We follow the proof of [28, Theorem 2.3]. We may assume that $\|f\|_{L^{(\Phi_1,\phi_1)}} = \|g\|_{L^{(\Phi_3,\phi_3)}} = 1$. For any ball B and $x \in B$, let

$$r = \max\biggl(\frac{\Phi_1(|f(x)|)}{\phi_1(|B|)}, \frac{\Phi_3(|g(x)|)}{\phi_3(|B|)}\biggr).$$

We note that $r < +\infty$ for a.e. x, since $\int_B \Phi_1(|f(x)|)\,dx \le |B|\phi_1(|B|)$ and $\int_B \Phi_3(|g(x)|)\,dx \le |B|\phi_3(|B|)$. From $\Phi_1(|f(x)|) \le r\phi_1(|B|)$ it follows that

$$|f(x)| \le \Phi_1^{-1}(\Phi_1(|f(x)|)) \le \Phi_1^{-1}(r\phi_1(|B|)).$$

In the same way we have

$$|g(x)| \le \Phi_3^{-1}(\Phi_3(|g(x)|)) \le \Phi_3^{-1}(r\phi_3(|B|)).$$

Hence

$$|f(x)g(x)| \le \Phi_1^{-1}(r\phi_1(|B|))\Phi_3^{-1}(r\phi_3(|B|)) \le c\,\Phi_2^{-1}(r\phi_2(|B|)),$$

and

$$\begin{aligned}\Phi_2\biggl(\frac{|f(x)g(x)|}{2c}\biggr) &\le \frac{1}{2}\,\Phi_2\biggl(\frac{|f(x)g(x)|}{c}\biggr) \le \frac{1}{2}\,\Phi_2(\Phi_2^{-1}(r\phi_2(|B|))) \le \frac{1}{2}\,r\phi_2(|B|)\\ &\le \frac{1}{2}\biggl(\frac{\Phi_1(|f(x)|)}{\phi_1(|B|)} + \frac{\Phi_3(|g(x)|)}{\phi_3(|B|)}\biggr)\phi_2(|B|).\end{aligned}$$

Therefore

$$\begin{aligned}\int_B \Phi_2\biggl(\frac{|f(x)g(x)|}{2c}\biggr)dx &\le \frac{1}{2}\biggl(\int_B \frac{\Phi_1(|f(x)|)}{\phi_1(|B|)}\,dx + \int_B \frac{\Phi_3(|g(x)|)}{\phi_3(|B|)}\,dx\biggr)\phi_2(|B|)\\ &\le |B|\phi_2(|B|).\end{aligned}$$

This shows

$$\|fg\|_{\Phi_2,\phi_2,B} \le 2c,$$

and the conclusion. ∎

COROLLARY 4.2. *Let Φ_i be Young functions, $i = 1, 2, 3$, and $\phi \in \mathcal{G}$. Assume that there exists a constant $c > 0$ such that*

$$\Phi_1^{-1}(r)\Phi_3^{-1}(r) \le c\,\Phi_2^{-1}(r) \quad \text{for } r > 0.$$

If $f \in L^{(\Phi_1,\phi)}(\mathbb{R}^n)$ and $g \in L^{(\Phi_3,\phi)}(\mathbb{R}^n)$, then $fg \in L^{(\Phi_2,\phi)}(\mathbb{R}^n)$ and

$$\|fg\|_{L^{(\Phi_2,\phi)}} \le 2c\|f\|_{L^{(\Phi_1,\phi)}}\|g\|_{L^{(\Phi_3,\phi)}}.$$

COROLLARY 4.3 ([21, 22]). *Let $1 \le p_i < \infty$ and $\phi_i \in \mathcal{G}$, $i = 1, 2, 3$. Assume that $1/p_1 + 1/p_3 = 1/p_2$ and that there exists a constant $c > 0$ such that*

$$\phi_1^{1/p_1}(r)\phi_3^{1/p_3}(r) \le c\phi_2^{1/p_2}(r) \quad \text{for } r > 0.$$

If $f \in L^{(p_1,\phi_1)}(\mathbb{R}^n)$ and $g \in L^{(p_3,\phi_3)}(\mathbb{R}^n)$, then $fg \in L^{(p_2,\phi_2)}(\mathbb{R}^n)$ and

$$\|fg\|_{L^{(p_2,\phi_2)}} \le 2c\|f\|_{L^{(p_1,\phi_1)}}\|g\|_{L^{(p_3,\phi_3)}}.$$

THEOREM 4.4. *Let Φ_i be Young functions and $\phi_i \in \mathcal{G}$, $i = 1, 2$. Assume that*

$$\Phi_2(r)\Phi_2(s) \le c_0\Phi_2(rs) \quad \text{for } r, s > 0,$$

and there exists $\Phi_3 \in \mathcal{Y}$ such that

$$\Phi_1^{-1}(r)\Phi_3^{-1}(r) \le c_1\Phi_2^{-1}(r), \quad \phi_1(r)/\Phi_2(\Phi_3^{-1}(\phi_1(r))) \le c_2\phi_2(r) \quad \text{for } r > 0.$$

Then

$$L^{(\Phi_1,\phi_1)}(\mathbb{R}^n) \subset L^{(\Phi_2,\phi_2)}(\mathbb{R}^n),$$
$$\|f\|_{L^{(\Phi_2,\phi_2)}} \le 2\max(1, c_0)\, c_1 \max(1, c_2)\|f\|_{L^{(\Phi_1,\phi_1)}}.$$

By elementary calculations we have the following.

LEMMA 4.5. *Let Φ be a Young function and $\phi \in \mathcal{G}$. Then*

$$\|1\|_{\Phi,\phi,B} = 1/\Phi^{-1}(\phi(|B|)).$$

Proof of Theorem 4.4. By Theorem 4.1 and Lemma 4.5 we have

$$\|f\|_{\Phi_2,\phi_1,B} \le 2c_1\|f\|_{\Phi_1,\phi_1,B}\|1\|_{\Phi_3,\phi_1,B} \le 2c_1\|f\|_{\Phi_1,\phi_1,B}/\Phi_3^{-1}(\phi_1(|B|)).$$

Let $c_0' = \max(1, c_0)$ and $c_2' = \max(1, c_2)$. By the assumption we have

$$\begin{aligned}\Phi_2\biggl(\frac{|f(x)|}{2c_0'c_1c_2'\|f\|_{\Phi_1,\phi_1,B}}\biggr) &\le \frac{1}{c_0'c_2'}\,\Phi_2\biggl(\frac{|f(x)|}{\Phi_3^{-1}(\phi_1(|B|))\|f\|_{\Phi_2,\phi_1,B}}\biggr)\\ &\le \frac{1}{c_2'}\,\Phi_2\biggl(\frac{|f(x)|}{\|f\|_{\Phi_2,\phi_1,B}}\biggr)\frac{1}{\Phi_2(\Phi_3^{-1}(\phi_1(|B|)))} \le \Phi_2\biggl(\frac{|f(x)|}{\|f\|_{\Phi_2,\phi_1,B}}\biggr)\frac{\phi_2(|B|)}{\phi_1(|B|)}.\end{aligned}$$

Hence

$$\int_B \Phi_2\biggl(\frac{|f(x)|}{2c_0'c_1c_2'\|f\|_{\Phi_1,\phi_1,B}}\biggr)dx \le \frac{\phi_2(|B|)}{\phi_1(|B|)}\int_B \Phi_2\biggl(\frac{|f(x)|}{\|f\|_{\Phi_2,\phi_1,B}}\biggr)dx \le |B|\phi_2(|B|).$$

This shows

$$\|f\|_{\Phi_2,\phi_2,B} \le 2c_0'c_1c_2'\|f\|_{\Phi_1,\phi_1,B} \quad \text{for all balls } B,$$

and the conclusion. ∎

COROLLARY 4.6. *Let $1 \le q \le p < \infty$ and $\phi \in \mathcal{G}$. Then*

$$L^{(p,\phi)}(\mathbb{R}^n) \subset L^{(q,\phi^{q/p})}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^{(q,\phi^{q/p})}} \le \|f\|_{L^{(p,\phi)}}.$$

Corollary 4.7. *Let Φ be a Young function and $\phi \in \mathcal{G}$. Then $\Phi^{-1}(\phi) \in \mathcal{G}$ and*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \subset L^{(1,\Phi^{-1}(\phi))}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^{(1,\Phi^{-1}(\phi))}} \le 4\|f\|_{L^{(\Phi,\phi)}}.$$

Proof. Note that $\Phi^{-1}(cr) \le c\Phi^{-1}(r)$ for $c \ge 1$ and $r > 0$, since Φ^{-1} is concave and nonnegative. Let $\underline{c}_\phi$ and $\overline{c}_\phi$ be the constants defined by (3.1). Then, for $0 < t < r < +\infty$,

$$\Phi^{-1}(\phi(r)) \le \Phi^{-1}(\underline{c}_\phi \phi(t)) \le \underline{c}_\phi \Phi^{-1}(\phi(t)),$$
$$t\Phi^{-1}(\phi(t)) \le t\Phi^{-1}(\overline{c}_\phi r\phi(r)/t) \le \overline{c}_\phi r\Phi^{-1}(\phi(r)).$$

Hence $\Phi^{-1}(\phi) \in \mathcal{G}$. Let $\widetilde{\Phi}$ be the complementary function of Φ. Then it follows from (2.3) that

$$\Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r, \qquad \phi(r)/\widetilde{\Phi}^{-1}(\phi(r)) \le \Phi^{-1}(\phi(r)).$$

By Theorem 4.4 we have the conclusion. ∎

Corollary 4.8 ([25]). *Let Φ be a Young function and $\phi(r) = \Phi^{-1}(1/r)$. Then $\phi \in \mathcal{G}$ and*

$$L^{\Phi}(\mathbb{R}^n) \subset L^{(1,\phi)}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^{(1,\phi)}} \le 4\|f\|_{L^{\Phi}}.$$

Theorem 4.9. *Let $\Phi, \Psi \in \mathcal{Y}$, $\phi \in \mathcal{G}$ and $\phi(r) \to +\infty$ as $r \to 0$. If $\lim_{r\to+\infty} \Phi^{-1}(r)/\Psi^{-1}(r) = +\infty$, then there exists a function $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ with compact support such that $f \notin L^{(\Psi,\psi)}(\mathbb{R}^n)$ for all $\psi \in \mathcal{G}$.*

To prove Theorem 4.9 we state a lemma, whose proof is in Section 8.

Lemma 4.10. *Let Φ be a Young function, $\phi \in \mathcal{G}$ and*

$$c_\phi = \sup_{0<t\le r<+\infty} t\phi(t)/(r\phi(r)).$$

Assume that ϕ is continuous and strictly decreasing. For $0 < t < r$, there exists a function $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ and a ball B_0 such that

$$\begin{cases} \|f\|_{L^{(\Phi,\phi)}} \le C, \\ \operatorname{supp} f \subset B_0, \\ |B_0| = (2\sqrt{n})^n c_\phi\, r, \\ |\operatorname{supp} f| = [c_\phi r\phi(r)/(t\phi(t))]\, t, \\ f(x) = \Phi^{-1}(\phi(t)) \quad \text{for } x \in \operatorname{supp} f, \end{cases} \tag{4.1}$$

where the constant $C > 0$ depends only on n and c_ϕ, and the notation $[s]$ represents the greatest integer less than or equal to the real number s.

Proof of Theorem 4.9. By Proposition 3.4, we may assume that ϕ is continuous and strictly decreasing. Let $0 < t_k \le 1/2^k$ and

$$\frac{\Phi^{-1}(\phi(t_k))}{\Psi^{-1}(\phi(t_k))} \ge 8^k \quad \text{for } k = 1, 2, \ldots.$$

Then

$$\Psi\bigg(\frac{\Phi^{-1}(\phi(t_k))}{8^k}\bigg) \geq \phi(t_k).$$

Using Lemma 4.10, for every k, there exists a function f_k such that (4.1) holds for $t = t_k$ and $r = 1$. Since the radius of B_0 is independent of $t = t_k$, we may assume that every $\operatorname{supp} f_k$ is included in the same B_0, i.e. $\bigcup_k \operatorname{supp} f_k \subset B_0$. Let

$$f = \sum_{k=1}^{\infty} 2^{-k} f_k.$$

Then $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ and $\operatorname{supp} f$ is compact. On the other hand, for all $\lambda > 0$, there exists k_0 such that $\lambda \leq 2^{k_0}$. Then, for $k \geq k_0$, we have

$$\begin{aligned} 2^{-k} \int_{B_0} \Psi(|2^{-k} f_k(x)|/\lambda)\, dx &\geq \int_{B_0} \Psi(|f_k(x)|/8^k)\, dx \\ &= \Psi(\Phi^{-1}(\phi(t_k))/8^k)[c_\phi \phi(1)/(t_k\phi(t_k))]t_k \geq c_\phi\phi(1)/2, \end{aligned}$$

i.e. $\int_{B_0} \Psi(|f(x)|/\lambda)\, dx = +\infty$. This shows that $f \notin L^{(\Psi,\psi)}(\mathbb{R}^n)$ for all $\psi \in \mathcal{G}$. ∎

Corollary 4.11. *Let $1 \leq p < q < \infty$, $\phi \in \mathcal{G}$ and $\phi(r) \to +\infty$ as $r \to 0$. Then there exists a function $f \in L^{p,\phi}(\mathbb{R}^n)$ with compact support such that $f \notin L^{q,\psi}(\mathbb{R}^n)$ for all $\psi \in \mathcal{G}$.*

5. A necessary and sufficient condition for the boundedness of the Hardy–Littlewood maximal operator. The *Hardy–Littlewood maximal function* of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x.

In this section we give a necessary and sufficient condition for the boundedness of the operator M from one Orlicz–Morrey space to another.

Theorem 5.1. *Let $\Phi, \Psi \in \mathcal{Y}$ and $\phi, \psi \in \mathcal{G}$. Then the following are equivalent*:

(i) *There exists a constant $A \geq 1$ such that*

$$\Phi^{-1}(\phi(r)) \leq A\Psi^{-1}(\psi(r)) \quad \text{for } r > 0, \tag{5.1}$$

and

$$\int_{\Psi^{-1}(\psi(r))}^{s/A} \frac{\Psi(t)}{t^2}\, dt \leq A\, \frac{\Phi(s)}{s}\, \frac{\psi(r)}{\phi(r)} \quad \text{for } (r,s) \in E, \tag{5.2}$$

where

$$E = \{(r,s) \in (0,+\infty)^2 : 2A\Psi^{-1}(\psi(r)) < s < \sup_{u>0} \Phi^{-1}(\phi(u))\}.$$

(ii) *The operator M is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\psi)}(\mathbb{R}^n)$.*

REMARK 5.1. By Proposition 3.4 we may assume that ϕ is continuous and strictly decreasing. Indeed, in the proof of Proposition 3.4, we have $\bar{\phi} \le \phi$. If Φ and ϕ satisfy (5.1) and (5.2), then so do Φ and $\bar{\phi}$.

EXAMPLE 5.1. For $0 < \alpha \le 1$, let

$$\Phi(r) = \begin{cases} r, & r < e, \\ r\log r, & r \ge e, \end{cases} \qquad \phi(r) = \frac{1}{r^\alpha},$$
$$\Psi(r) = r, \qquad \psi(r) = \begin{cases} 1/r^\alpha, & r < e, \\ (\log r)/r^\alpha, & r \ge e \end{cases}$$

Then (5.1) and (5.2) hold. Therefore, the operator M is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(1,\psi)}(\mathbb{R}^n)$, where $L^{(1,\psi)}(\mathbb{R}^n)$ is a generalized Morrey space defined in Definition 3.2. In the case $\alpha = 1$, the operator M is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $L^{(1,\psi)}(\mathbb{R}^n)$.

EXAMPLE 5.2. For $0 < \alpha < 1$, let

$$\Phi(r) = \begin{cases} r, & r < e, \\ r\log r, & r \ge e, \end{cases} \qquad \phi(r) = \begin{cases} 1/r^\alpha, & r < e, \\ 1/(r^\alpha \log r), & r \ge e, \end{cases}$$
$$\Psi(r) = r, \qquad \psi(r) = \frac{1}{r^\alpha}.$$

Then (5.1) and (5.2) hold. In the case $\alpha = 1-\lambda/n$ $(0 < \lambda < n)$, the operator M is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{1,\lambda}(\mathbb{R}^n)$, where $L^{1,\lambda}(\mathbb{R}^n)$ is the Morrey space defined in Definition 2.2.

For $\phi = \psi$, Theorem 5.1 yields the following.

COROLLARY 5.2. *Let $\Phi, \Psi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. Then the following are equivalent:*

(i) *There exists a constant $A \ge 1$ such that*

$$\Psi(s/A) \le \Phi(s) \quad \text{for } \inf_{u>0} \Phi^{-1}(\phi(u)) < s < \sup_{u>0} \Phi^{-1}(\phi(u)), \tag{5.3}$$

and

$$\int_{\inf_{u>0} \Psi^{-1}(\phi(u))}^{s/A} \frac{\Psi(t)}{t^2}\,dt \le A\,\frac{\Phi(s)}{s} \tag{5.4}$$
$$\text{for } 2A \inf_{u>0} \Psi^{-1}(\phi(u)) < s < \sup_{u>0} \Phi^{-1}(\phi(u)).$$

(ii) *The operator M is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\phi)}(\mathbb{R}^n)$.*

For $\Phi = \Psi$, Corollary 5.2 and Theorem 2.2 give the following.

Corollary 5.3. *Let $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. If $\phi(r) \to +\infty$ as $r \to 0$ and $\phi(r) \to 0$ as $r \to +\infty$, then the following are equivalent*:

(i) $\Phi \in \nabla_2$ *(that is, $\widetilde{\Phi} \in \Delta_2$).*
(ii) $\Phi \in \mathcal{Y}_1$ *and $\Phi^+ \approx \Phi$, where Φ^+ is defined by* (2.4).
(iii) *The operator M is bounded from $L^{\Phi}(\mathbb{R}^n)$ to itself.*
(iv) *The operator M is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to itself.*

From Corollary 5.2 we have the following.

Corollary 5.4. *Let $\Phi \in \mathcal{Y}_1$ and $\phi \in \mathcal{G}$. Then the operator M is bounded from $L^{(\Phi^+,\phi)}(\mathbb{R}^n)$ to $L^{(\Phi,\phi)}(\mathbb{R}^n)$.*

Example 5.3. For $\varepsilon > 0$ and $\delta \geq 0$, let $\Phi \in \mathcal{Y}_1$ with

$$\Phi(r) = \begin{cases} r(\log(1/r))^{-\varepsilon-1} & \text{for small } r > 0, \\ r(\log r)^{\delta} & \text{for large } r > 0. \end{cases}$$

Then

$$\Phi^+(r) \approx \begin{cases} r(\log(1/r))^{-\varepsilon} & \text{for small } r > 0, \\ r(\log r)^{\delta+1} & \text{for large } r > 0. \end{cases}$$

Example 5.4. For $1 < p < \infty$, $\varepsilon \in \mathbb{R}$ and $\delta \in \mathbb{R}$, let $\Phi \in \mathcal{Y}_1$ with

$$\Phi(r) = \begin{cases} r^p(\log(1/r))^{-\varepsilon} & \text{for small } r > 0, \\ r^p(\log r)^{\delta} & \text{for large } r > 0. \end{cases}$$

Then $\Phi \in \nabla_2$ and $\Phi^+ \approx \Phi$ (see Theorem 2.2).

Example 5.5. Let $\phi \in \mathcal{G}$ and $\phi(r) \geq 1$. For $\beta \geq 0$, let

$$\Phi(r) = \begin{cases} r & \text{for small } r, \\ r(\log r)^{\beta+1} & \text{for large } r, \end{cases} \qquad \Psi(r) = \begin{cases} r & \text{for small } r, \\ r(\log r)^{\beta} & \text{for large } r. \end{cases}$$

Then (5.3) and (5.4) in Corollary 5.2 hold.

Let $\Phi(r) = r$ in Theorem 5.1. If $\sup_{u>0} \Phi^{-1}(\phi(u)) = +\infty$, then (5.2) does not hold for any $\Psi \in \mathcal{Y}$ or for any $\psi \in \mathcal{G}$. Thus we have the following.

Corollary 5.5. *Let $\phi \in \mathcal{G}$ and $\phi(r) \to +\infty$ as $r \to 0$. Then the operator M is not bounded from $L^{(1,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\psi)}(\mathbb{R}^n)$ for any $\Psi \in \mathcal{Y}$ or for any $\psi \in \mathcal{G}$.*

Example 5.6. For $0 < \alpha < 1$, let

$$\Phi(r) = \Psi(r) = r, \qquad \phi(r) = \begin{cases} 1/e^{\alpha}, & r < e, \\ 1/(r^{\alpha} \log r), & r \geq e, \end{cases} \qquad \psi(r) = \min(1, 1/r^{\alpha}).$$

Then (5.1) and (5.2) hold. In this case the operator M is bounded from $L^{(1,\phi)}(\mathbb{R}^n)$ to $L^{(1,\psi)}(\mathbb{R}^n)$.

For generalized Morrey spaces we have the following.

Corollary 5.6. *Let $1 \le p, q < \infty$, $\phi, \psi \in \mathcal{G}$ and $\phi(r) \to +\infty$ as $r \to 0$. Then the following are equivalent*:

(i) *$p \ge q$, $p > 1$ and there exists a constant $A \ge 1$ such that*

$$\phi(r)^{1/p} \le A\psi(r)^{1/q} \quad \text{for } r > 0. \tag{5.5}$$

(ii) *The operator M is bounded from $L^{(p,\phi)}(\mathbb{R}^n)$ to $L^{(q,\psi)}(\mathbb{R}^n)$.*

Remark 5.2. Let $1 < p, q < \infty$, $\phi, \psi \in \mathcal{G}$ and $\phi(r) \to +\infty$ as $r \to 0$. By the corollary the operator M is bounded from $L^{(p,\phi)}(\mathbb{R}^n)$ to itself. From $p \ge q$ and (5.5) it follows that $L^{(p,\phi)}(\mathbb{R}^n) \subset L^{(q,\psi)}(\mathbb{R}^n)$ (see Proposition 3.2 and Corollary 4.6).

Proof of Corollary 5.6. Assume that (i) in the corollary holds. Then (5.1) holds in Theorem 5.1. Case 1: $p \ge q > 1$. If $\psi(r)^{1/q} < s$, then $1 \le (s\psi(r)^{-1/q})^{p-q}$, and

$$s^{q-1} \le s^{p-1}\psi(r)^{1-p/q} \le A^p s^{p-1}\frac{\psi(r)}{\phi(r)}.$$

Hence we have (5.2). Case 2: $p > q = 1$. There exists a constant $C \ge 1$ such that if $C\psi(r) < s$, then $\log(s\psi(r)^{-1}) \le (s\psi(r)^{-1})^{p-1}$, and so

$$\log(s\psi(r)^{-1}) \le s^{p-1}\psi(r)^{1-p} \le A^p s^{p-1}\frac{\psi(r)}{\phi(r)}.$$

Hence we have (5.2).

Conversely, assume that (ii) in the corollary holds. Fix r and let $s \to +\infty$ in (5.2) in Theorem 5.1. Then $p \ge q > 1$ or $p > q = 1$ is needed. ∎

6. Weak boundedness of the Hardy–Littlewood maximal operator. In this section we consider weak boundedness of the Hardy–Littlewood maximal operator.

For a measurable set $\Omega \subset \mathbb{R}^n$, we denote the Lebesgue measure of Ω by $|\Omega|$. For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let

$$m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = \mathbb{R}^n$, we briefly denote it by $m(f, t)$. For $\Phi \in \mathcal{Y}$, $\phi \in \mathcal{G}$ and a ball B, let

$$\|f\|_{\Phi,\phi,B,\mathrm{weak}} = \inf\left\{\lambda > 0 : \sup_{t>0}\frac{tm(B, \Phi(|f|/\lambda), t)}{|B|\phi(|B|)} \le 1\right\}.$$

We note that $\|f\|_{\Phi,\phi,B,\mathrm{weak}} \le \|f\|_{\Phi,\phi,B}$ and

$$\sup_{t>0}\Phi(t)m(\Omega, f, t) = \sup_{t>0} tm(\Omega, f, \Phi^{-1}(t)) = \sup_{t>0} tm(\Omega, \Phi(|f|), t).$$

DEFINITION 6.1 (weak Orlicz–Morrey space). For $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$, let

$$L^{(\Phi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}} < +\infty\},$$
$$\|f\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}} = \sup_B \|f\|_{\Phi,\phi,B,\mathrm{weak}}.$$

Then $\|\cdot\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}}$ is a quasi-norm and $L^{(\Phi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n)$ is a complete quasi-normed space. We note that

$$\|f+g\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}} \le 2(\|f\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}} + \|g\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}}).$$

THEOREM 6.1. *Let $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. Then the operator M is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Phi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then M is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to itself.*

We shall prove this theorem in Section 10.

COROLLARY 6.2 ([20]). *Let $\phi \in \mathcal{G}$. Then the operator M is bounded from $L^{(1,\phi)}(\mathbb{R}^n)$ to $L^{(1,\phi)}_{\mathrm{weak}}(\mathbb{R}^n)$. If $1 < p < \infty$, then M is bounded from $L^{(p,\phi)}(\mathbb{R}^n)$ to itself.*

COROLLARY 6.3 ([3]). *Let $0 \le \lambda < n$. Then the operator M is bounded from $L^{1,\lambda}(\mathbb{R}^n)$ to $L^{1,\lambda}_{\mathrm{weak}}(\mathbb{R}^n)$. If $1 < p < \infty$, then M is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to itself.*

7. Generalized fractional integral operators. As an application of the results for the Hardy–Littlewood maximal operator, we give a sufficient condition for the boundedness of generalized fractional integral operators. The results in this section improve those in [26].

For a function $\varrho : (0,+\infty) \to (0,+\infty)$, let

$$I_\varrho f(x) = \int_{\mathbb{R}^n} f(y)\, \frac{\varrho(|x-y|^n)}{|x-y|^n}\, dy.$$

We consider the following conditions on ϱ:

$$\int_0^1 \frac{\varrho(t)}{t}\, dt < +\infty, \tag{7.1}$$

$$\frac{1}{A_1} \le \frac{\varrho(s)}{\varrho(r)} \le A_1 \quad \text{for } 1/2 \le s/r \le 2, \tag{7.2}$$

$$\frac{\varrho(r)}{r} \le A_2\, \frac{\varrho(s)}{s} \quad \text{for } s \le r. \tag{7.3}$$

If $\varrho(r) = r^{\alpha/n}$, $0 < \alpha < n$, then I_ϱ is the fractional integral operator denoted by I_α.

For a function $\theta : (0,+\infty) \to (0,+\infty)$, let

$$\theta^*(r) = \int_0^r \frac{\theta(t)}{t}\,dt, \qquad \theta_*(r) = \int_r^{+\infty} \frac{\theta(t)}{t}\,dt.$$

THEOREM 7.1. *Let $\Phi, \Psi \in \mathcal{Y}$ and $\phi, \psi \in \mathcal{G}$. If there exist $\Theta \in \mathcal{Y}$ and a constant $A \ge 1$ such that*

$$\Phi^{-1}(\phi(r)) \le A\Theta^{-1}(\psi(r)) \qquad \text{for } r > 0, \tag{7.4}$$

$$\int_{\Theta^{-1}(\psi(r))}^{s} \frac{\Theta(t)}{t^2}\,dt \le A\,\frac{\Phi(As)}{s}\,\frac{\psi(r)}{\phi(r)} \qquad \text{for } (r,s) \in E, \tag{7.5}$$

where

$$E = \{(r,s) \in (0,+\infty)^2 : 2A\Theta^{-1}(\psi(r)) < s < \sup_{u>0} \Phi^{-1}(\phi(u))\},$$

and

$$\Psi\left(\frac{\Theta^{-1}\circ\phi(r)\varrho^*(r) + ((\Phi^{-1}\circ\phi)\varrho)_*(r)}{A}\right) \le \phi(r) \qquad \text{for } r > 0, \tag{7.6}$$

then the operator I_ϱ is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\psi)}(\mathbb{R}^n)$.

We shall prove this theorem in Section 11. In the proof we use a pointwise estimate by $Mf(x)$ and boundedness of the operator M. This method was introduced by Hedberg [13] to give a simple proof of the Hardy–Littlewood–Sobolev theorem.

If, in Theorem 7.1, we use Φ^+ and Φ instead of Φ and Θ, respectively, we obtain the following.

COROLLARY 7.2. *Let $\Phi \in \mathcal{Y}_1$, $\Psi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. If there exists a constant $A \ge 1$ such that*

$$\Psi\left(\frac{\Phi^{-1}\circ\phi(r)\varrho^*(r) + (((\Phi^+)^{-1}\circ\phi)\varrho)_*(r)}{A}\right) \le \phi(r), \qquad r > 0,$$

then the operator I_ϱ is bounded from $L^{(\Phi^+,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\phi)}(\mathbb{R}^n)$.

THEOREM 7.3. *Let $\Phi, \Psi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. If there exists a constant $A \ge 1$ such that*

$$\Psi\left(\frac{\Phi^{-1}\circ\phi(r)\varrho^*(r) + ((\Phi^{-1}\circ\phi)\varrho)_*(r)}{A}\right) \le \phi(r), \qquad r > 0, \tag{7.7}$$

then the operator I_ϱ is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_ϱ is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\phi)}(\mathbb{R}^n)$.

We shall prove this theorem in Section 11.

EXAMPLE 7.1 ([1]). Let $0 < \alpha < n$, $1 < p < q < \infty$, $-n/p + \alpha n/(n-\lambda) = -n/q$, and

$$\varrho(r) = r^{\alpha/n}, \qquad \Phi(r) = r^p, \qquad \Psi(r) = r^q, \qquad \phi(r) = r^{-1+\lambda/n}.$$

Then $\Phi \in \nabla_2$ and

$$\Phi^{-1} \circ \phi(r)\varrho^*(r) + ((\Phi^{-1} \circ \phi)\varrho)_*(r) \sim r^{(-1+\lambda/n)/p+\alpha} = r^{(-1+\lambda/n)/q}.$$

Therefore the operator I_α is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. This is the result of Adams [1] (1975).

EXAMPLE 7.2. Let $\ell : (0,+\infty) \to (0,+\infty)$ satisfy the doubling condition and

$$\ell(r) = \begin{cases} (\log(1/r))^{-1} & \text{for small } r > 0, \\ \log r & \text{for large } r > 0. \end{cases}$$

For $\beta > 0$, let

$$\varrho(r) = \begin{cases} (\log(1/r))^{-\beta-1} & \text{for small } r > 0, \\ (\log r)^{\beta-1} & \text{for large } r > 0. \end{cases}$$

Then ϱ satisfies (7.1)–(7.3) and

$$\varrho^*(r) = \int_0^r \frac{\varrho(t)}{t}\,dt \sim \ell^\beta(r).$$

Let

$$\begin{aligned} &\Phi(r) = r^p, \qquad \Psi(r) = r^p \ell^{p\beta}(r), \qquad (1 \le p < \infty) \\ &\phi(r) = r^{-1+\lambda/n} \qquad (0 \le \lambda < n). \end{aligned}$$

Then we have the following boundedness:

$$\begin{aligned} &I_\varrho : L^{1,\lambda}(\mathbb{R}^n) = L^{(1,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n) \quad (p = 1), \\ &I_\varrho : L^{p,\lambda}(\mathbb{R}^n) = L^{(p,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}(\mathbb{R}^n) \quad (1 < p < \infty). \end{aligned}$$

EXAMPLE 7.3. Let ℓ and ϱ be as in Example 7.2. For $p > 0$, let

$$e_p(r) = \begin{cases} 1/\exp(1/r^p) & \text{for small } r > 0, \\ \exp(r^p) & \text{for large } r > 0. \end{cases}$$

Let

$$\begin{aligned} &\Phi(r) = e_p(r), \qquad \Psi(r) = e_q(r) \qquad (-1/p + \beta = -1/q < 0), \\ &\phi(r) = r^{-1+\lambda/n} \qquad (0 \le \lambda < n). \end{aligned}$$

Then the operator I_ϱ is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\phi)}(\mathbb{R}^n)$.

EXAMPLE 7.4. Let ℓ and ϱ be as in Example 7.2. For $\varepsilon > 0$, $\delta \ge 0$ and $\beta > 0$, let

$$\begin{aligned} \Phi(r) &= \begin{cases} r(\log(1/r))^{-\varepsilon} & \text{for small } r > 0, \\ r(\log r)^{\delta+1} & \text{for large } r > 0. \end{cases} \\ \Theta(r) &= \begin{cases} r(\log(1/r))^{-\varepsilon-1} & \text{for small } r > 0, \\ r(\log r)^{\delta} & \text{for large } r > 0, \end{cases} \\ \Psi(r) &= \begin{cases} r(\log(1/r))^{-\varepsilon-\beta} & \text{for small } r > 0, \\ r(\log r)^{\delta+\beta} & \text{for large } r > 0, \end{cases} \\ \phi(r) &= r^{-1+\lambda/n} \quad (0 \le \lambda < n). \end{aligned}$$

Then the operator I_ϱ is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Psi,\phi)}(\mathbb{R}^n)$.

8. Proof of Lemma 4.10. Let $k = [c_\phi r\phi(r)/(t\phi(t))]$, i.e.

$$k \le \frac{c_\phi \phi(r) r}{\phi(t) t} < k+1.$$

Then

$$kt \le c_\phi \frac{\phi(r)}{\phi(t)}\, r \le c_\phi r.$$

Let κ be the positive integer such that $\kappa^n \le k < (\kappa+1)^n$. We denote the measure of the unit ball in $\mathbb{R}^n$ by σ_n. First, we choose a cube Q_0 and a ball B_0 so that

$$Q_0 \subset B_0, \quad |Q_0| = 4^n c_\phi r/\sigma_n, \quad |B_0| = (2\sqrt{n})^n c_\phi r.$$

In this case the side length of Q_0 is $4(c_\phi r/\sigma_n)^{1/n}$ and the radius of B_0 is $2\sqrt{n}(c_\phi r/\sigma_n)^{1/n}$. We divide Q_0 into $(\kappa+1)^n$ cubes Q_j $(j = 1, \dots, (\kappa+1)^n)$ with side length $4(c_\phi r/\sigma_n)^{1/n}/(\kappa+1)$. Let $\tau = (t/\sigma_n)^{1/n}$. Then

$$2\tau = 2(t/\sigma_n)^{1/n} \le 2\,\frac{k^{1/n}}{\kappa}\,(t/\sigma_n)^{1/n} \le 4\,\frac{k^{1/n}}{\kappa+1}\,(t/\sigma_n)^{1/n} \le \frac{4(c_\phi r/\sigma_n)^{1/n}}{\kappa+1}.$$

So we can choose balls $B_j \subset Q_j$, $j = 1, \dots, k < (\kappa+1)^n$ of radius τ each. Then

$$|B_j| = t \quad \text{for } j = 1, \dots, k, \qquad B_j \cap B_{j'} = \emptyset \quad \text{for } j \ne j', \qquad \bigcup_{j=1}^{k} B_j \subset B_0.$$

Let

$$f = \sum_{j=1}^{k} \Phi^{-1}(\phi(t))\chi_{B_j},$$

where χ_{B_j} is the characteristic function of B_j.

Next, we show $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ and $\|f\|_{L^{(\Phi,\phi)}} \le C$. For all balls B, if $|B| \le t$, then

$$\int_B \Phi(|f(x)|)\,dx \le |B|\phi(t) \le |B|\phi(|B|).$$

If $t < |B| \le r$, then the number of B_j which intersect B is less than or comparable to $k|B|/r$, and so

$$\int_B \Phi(|f(x)|)\,dx \le (c_n k|B|/r)t\phi(t) \le (c_n|B|/r)c_\phi r\phi(r) = c_n c_\phi |B|\phi(r) \le c_n c_\phi |B|\phi(|B|),$$

where c_n depends only on n. If $r < |B|$, then

$$\int_B \Phi(|f(x)|)\,dx \le kt\phi(t) \le c_\phi r\phi(r) \le (c_\phi)^2|B|\phi(|B|).$$

Therefore $\|f\|_{L^{(\Phi,\phi)}} \le \max(c_n c_\phi, (c_\phi)^2)$. ■

9. Proofs of Theorem 5.1. First, we note that, for $\Phi \in \mathcal{Y}$, its left and right derivatives exist for all $r > 0$ and are both increasing. Then Φ can be expressed by

$$\Phi(r) = \int_0^r a(t)\,dt$$

for some increasing function $a : [0,+\infty) \to [0,+\infty)$ such that $a(r) > 0$ for $r > 0$. In this case $a(r) = \Phi'(r)$ for a.e. $r > 0$ and

$$\Phi(r) \le r\Phi'(r) \le \Phi(2r) \quad \text{for a.e. } r > 0, \tag{9.1}$$

since

$$\Phi(r) = \int_0^r a(t)\,dt \le ra(r) = \int_r^{2r} a(r)\,dt \le \int_0^{2r} a(t)\,dt \le \Phi(2r).$$

The following is known.

Theorem 9.1 ([41]). *For a Young function Φ and its complementary function $\widetilde{\Phi}$,*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le 2\|f\|_{L^\Phi}\|g\|_{L^{\widetilde{\Phi}}}.$$

Remark 9.1. Theorem 9.1 is valid for any measure space instead of $\mathbb{R}^n$.

To prove Theorem 5.1, we state five lemmas. The first three are in [26]. We give the proofs for convenience.

Lemma 9.2. *For a Young function Φ, $\phi \in \mathcal{G}$ and $B = B(a,r)$,*

$$\int_B f(x)g(x)\,dx \le 2|B|\phi(r)\|f\|_{\Phi,\phi,B}\|g\|_{\widetilde{\Phi},\phi,B},$$

where $\widetilde{\Phi}$ is the complementary function of Φ.

Proof. For $L^{\Phi}(B, dx/(|B|\phi(r)))$ and $L^{\widetilde{\Phi}}(B, dx/(|B|\phi(r)))$, Theorem 9.1 gives us

$$\begin{aligned}\int_B f(x)g(x)\,\frac{dx}{|B|\phi(r)} &\le 2\|f\|_{L^{\Phi}(B,dx/(|B|\phi(r)))}\|g\|_{L^{\widetilde{\Phi}}(B,dx/(|B|\phi(r)))}\\ &= 2\|f\|_{\Phi,\phi,B}\|g\|_{\widetilde{\Phi},\phi,B}. \ \blacksquare\end{aligned}$$

LEMMA 9.3. *For a Young function Φ, $\phi \in \mathcal{G}$ and $B = B(a,r)$,*

$$\|1\|_{\widetilde{\Phi},\phi,B} \le \Phi^{-1}(\phi(r))/\phi(r),$$

where $\widetilde{\Phi}$ is the complementary function of Φ.

Proof. Apply Lemma 4.5 and (2.3). ■

LEMMA 9.4. *For a Young function Φ, $\phi \in \mathcal{G}$ and a ball B, if $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ and* $\operatorname{supp} f \cap 2B = \emptyset$, *then*

$$Mf(x) \le C\Phi^{-1}(\phi(|B|))\|f\|_{L^{(\Phi,\phi)}} \quad \text{for } x \in B,$$

where C is a constant depending only on Φ and ϕ.

Proof. Let $r > 0$ be the radius of B. For all balls $B' \ni x$, if the radius of B' is less than or equal to $r/2$, then $\int_{B'} |f(x)|\,dx = 0$, and if it is greater than $r/2$, then using Lemmas 9.2 and 9.3, we have

$$\begin{aligned}\frac{1}{|B'|}\int_{B'} |f(x)|\,dx &\le 2\phi(|B'|)\|f\|_{\Phi,\phi,B'}\|1\|_{\widetilde{\Phi},\phi,B'}\\ &\le 2\phi(|B'|)\|f\|_{L^{(\Phi,\phi)}}\Phi^{-1}(\phi(|B'|))/\phi(|B'|)\\ &\le 2\Phi^{-1}(\phi(|B'|))\|f\|_{L^{(\Phi,\phi)}} \le C\Phi^{-1}(\phi(|B|))\|f\|_{L^{(\Phi,\phi)}},\end{aligned}$$

since ϕ is almost decreasing, and Φ^{-1} and ϕ satisfy the doubling condition. ■

LEMMA 9.5 ([39, p. 92]). *If $f \in L^1(\mathbb{R}^n)$, then*

$$m(Mf,t) \le \frac{c_n}{t}\int_{t/2}^{+\infty} m(f,s)\,ds \quad \text{for all } t > 0,$$

where c_n is a constant depending only on n.

LEMMA 9.6 ([12, p. 57], [9, p. 144]). *If $f \in L^1(\mathbb{R}^n)$, then*

$$m(Mf,t) \ge \frac{c_n}{t}\int_{|f|>t} |f(x)|\,dx \quad \text{for all } t > 0,$$

where c_n is a constant depending only on n.

Proof of Theorem 5.1(i)⇒(ii). Let $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$. For all balls B, let

$$f = f_1 + f_2, \quad f_1 = f\chi_{2B}.$$

Then

$$\int_B \Psi(Mf_1(x)/\lambda)\,dx = \int_0^\infty m(B, Mf_1/\lambda, t)\Psi'(t)\,dt.$$

Let $u = \Psi^{-1}(\psi(|B|))$ and $\lambda = 4A\|f\|_{L^{(\Phi,\phi)}}$. Then

$$\int_0^u m(B, Mf_1/\lambda, t)\Psi'(t)\,dt \le |B|\int_0^u \Psi'(t)\,dt = |B|\Psi(u) = |B|\psi(|B|).$$

Using Lemma 9.5 and (9.1), we have

$$\begin{aligned}
\int_u^\infty m(B, Mf_1/\lambda, t)\Psi'(t)\,dt &\le c_n \int_u^\infty \frac{\Psi'(t)}{t}\,dt \int_{t/2}^\infty m(f_1/\lambda, s)\,ds \\
&= c_n \int_u^\infty \frac{\Psi'(t)}{t}\,dt \int_{t/2}^\infty m(4Af_1/\lambda, 4As)\,ds \\
&= \frac{c_n}{4A} \int_u^\infty \frac{\Psi'(t)}{t}\,dt \int_{2At}^\infty m(4Af_1/\lambda, s)\,ds \\
&= \frac{c_n}{4A} \int_{2Au}^\infty \bigg(\int_u^{s/(2A)} \frac{\Psi'(t)}{t}\,dt\bigg) m(4Af_1/\lambda, s)\,ds \\
&\le \frac{c_n}{4A} \int_{2Au}^\infty \bigg(\int_u^{s/(2A)} \frac{\Psi(2t)}{t^2}\,dt\bigg) m(4Af_1/\lambda, s)\,ds \\
&= \frac{c_n}{4A} \int_{2Au}^\infty \bigg(2\int_{2u}^{s/A} \frac{\Psi(t)}{t^2}\,dt\bigg) m(4Af_1/\lambda, s)\,ds.
\end{aligned}$$

Let $\omega = \sup_{u>0} \Phi^{-1}(\phi(u))$. If $\omega < +\infty$, then $m(4Af_1/\lambda, s) = 0$ for $s > \omega$ by Proposition 3.3. Using (5.2) and (9.1), we have

$$\begin{aligned}
\int_u^\infty m(B, Mf_1/\lambda, t)\Psi'(t)\,dt &\le \frac{c_n}{2A} \int_{2Au}^\omega \bigg(\int_{2u}^{s/A} \frac{\Psi(t)}{t^2}\,dt\bigg) m(4Af_1/\lambda, s)\,ds \\
&\le \frac{c_n}{2}\,\frac{\psi(|B|)}{\phi(|B|)} \int_{2Au}^\omega \frac{\Phi(s)}{s} m(4Af_1/\lambda, s)\,ds \\
&\le \frac{c_n}{2}\,\frac{\psi(|B|)}{\phi(|B|)} \int_{2Au}^\omega \Phi'(s) m(4Af_1/\lambda, s)\,ds
\end{aligned}$$

$$= \frac{c_n}{2} \frac{\psi(|B|)}{\phi(|B|)} \int_{2B} \Phi\left(4A \frac{|f(x)|}{\lambda}\right) dx$$
$$\le \frac{c_n}{2} \frac{\psi(|B|)}{\phi(|B|)} |2B|\phi(|2B|) \le C|B|\psi(|B|).$$

Thus we have

$$\int_B \Psi(Mf_1(x)/\lambda)\, dx \le (1+C)|B|\psi(|B|),$$

and

$$\int_B \Psi\left(\frac{Mf_1(x)}{(1+C)\lambda}\right) dx \le |B|\psi(|B|).$$

Hence

$$\|Mf_1\|_{\Psi,\psi,B} \le 4A(1+C)\|f\|_{L^{(\Phi,\phi)}}. \tag{9.2}$$

Since $\operatorname{supp} f_2 \cap 2B = \emptyset$, using Lemma 9.4, we have

$$Mf_2(x) \le C\Phi^{-1}(\phi(|B|))\|f\|_{L^{(\Phi,\phi)}}.$$

Hence, by (5.1),

$$\int_B \Psi\left(\frac{Mf_2(x)}{AC\|f\|_{L^{(\Phi,\phi)}}}\right) dx \le \int_B \Psi\left(\frac{\Phi^{-1}(\phi(|B|))}{A}\right) dx$$
$$\le \int_B \psi(|B|)\, dx = |B|\psi(|B|),$$

and

$$\|Mf_2\|_{\Psi,\psi,B} \le AC\|f\|_{L^{(\Phi,\phi)}}. \tag{9.3}$$

Now (9.2) and (9.3) yield the conclusion. ∎

Proof of Theorem 5.1(ii)⇒(i). By Proposition 3.4 and Remark 5.1, we may assume that ϕ is continuous and strictly decreasing. Since $r\phi(r)$ is almost increasing, there exists a constant $c_\phi \ge 1$ such that $r\phi(r) \le c_\phi s\phi(s)$ for $r < s$.

CASE 1. Assume that (5.1) does not hold. Then there exists a positive sequence $\{r_k\}$ such that

$$\Phi^{-1}(\phi(r_k)) > k\Psi^{-1}(\psi(r_k)) \quad \text{for } k = 1, 2, \ldots.$$

We choose a sequence $\{B_k\}$ of balls so that $|B_k| = r_k$. Let

$$f_k(x) = \Phi^{-1}(\phi(|B_k|))\chi_{B_k} \quad \text{for } k = 1, 2, \ldots.$$

Then, for all balls B,

$$\int_B \Phi(|f_k(x)|)\,dx = |B\cap B_k|\phi(|B_k|) \le \begin{cases} |B|\phi(|B_k|) \le |B|\phi(|B|) & \text{if } |B|\le |B_k|,\\ |B_k|\phi(|B_k|) \le c_\phi|B|\phi(|B|) & \text{if } |B|\ge |B_k|.\end{cases}$$

Hence $f_k \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ and $\|f_k\|_{L^{(\Phi,\phi)}} \le c_\phi$. On the other hand,

$$\int_{B_k} \Psi\biggl(\frac{Mf_k(x)}{k}\biggr)\,dx = \int_{B_k} \Psi\biggl(\frac{\Phi^{-1}(\phi(|B_k|))}{k}\biggr)\,dx \ge \int_{B_k} \Psi(\Psi^{-1}(\psi(|B_k|)))\,dx = |B_k|\psi(|B_k|).$$

This shows that $\|Mf_k\|_{L^{(\Psi,\psi)}} \ge k$. Therefore M is not bounded.

Case 2. Assume that (5.1) holds and (5.2) does not. Then there are positive sequences $\{r_k\}$ and $\{s_k\}$ such that

$$\int_{\Psi^{-1}(\psi(r_k))}^{s_k/k} \frac{\Psi(t)}{t^2}\,dt > k\,\frac{\Phi(s_k)}{s_k}\,\frac{\psi(r_k)}{\phi(r_k)}, \tag{9.4}$$

$$2k\Psi^{-1}(\psi(r_k)) < s_k < \sup_{u>0}\Phi^{-1}(\phi(u)), \qquad k=1,2,\ldots. \tag{9.5}$$

In this case we have

$$\Phi^{-1}(\phi(r_k)) \le A\Psi^{-1}(\psi(r_k)) < 2k\Psi^{-1}(\psi(r_k)) < s_k < \sup_{u>0}\Phi^{-1}(\phi(u))$$

for $k > A/2$. Then, for $k>A/2$, we can choose t_k with $0<t_k<r_k$ so that $s_k = \Phi^{-1}(\phi(t_k))$ by the continuity and strict decreasingness of ϕ.

By Lemma 4.10, for every k, there exists a function $f_k\in L^{(\Phi,\phi)}(\mathbb{R}^n)$ and a ball B_k such that (4.1) holds for $t=t_k$, $r=r_k$ and $B_0=B_k$.

In the following we show $\|Mf_k\|_{L^{(\Psi,\psi)}} \ge ck$ for $k\ge c_\phi A$, where c is a constant independent of k. We note that $\Phi^{-1}(r)/r$ is decreasing, since $\Phi^{-1}(0)=0$ and Φ^{-1} is concave. Then, for $x\notin 3B_k$, we have

$$Mf_k(x) \le \frac{[c_\phi r_k\phi(r_k)/(t_k\phi(t_k))]t_k s_k}{r_k} \le \frac{c_\phi\phi(r_k)}{\phi(t_k)}\,\Phi^{-1}(\phi(t_k)) \le \frac{c_\phi\phi(r_k)}{\phi(r_k)}\,\Phi^{-1}(\phi(r_k)) = c_\phi\Phi^{-1}(\phi(r_k)) \le c_\phi A\Psi^{-1}(\psi(r_k)).$$

Therefore, for $k\ge c_\phi A$, we have

$$m(Mf_k/k,t) = m(3B_k, Mf_k/k,t) \qquad \text{for } t>\Psi^{-1}(\psi(r_k)).$$

By Lemma 9.6, (9.1) and (9.4) we have

$$\begin{aligned}
&\int_{3B_k} \Psi(Mf_k(x)/k)\,dx \\
&\quad\ge \int_{\Psi^{-1}(\psi(r_k))}^{\infty} m(3B_k, Mf_k/k, t)\Psi'(t)\,dt = \int_{\Psi^{-1}(\psi(r_k))}^{\infty} m(Mf_k/k, t)\Psi'(t)\,dt \\
&\quad\ge \int_{\Psi^{-1}(\psi(r_k))}^{\infty} \bigg(\frac{c_n}{t} \int_{|f_k|/k>t} \frac{|f_k(x)|}{k}\,dx\bigg)\Psi'(t)\,dt \\
&\quad= c_n \int_{|f_k|/k>\Psi^{-1}(\psi(r_k))} \frac{|f_k(x)|}{k}\bigg(\int_{\Psi^{-1}(\psi(r_k))}^{|f_k(x)|/k} \frac{\Psi'(t)}{t}\,dt\bigg)\,dx \\
&\quad= c_n \int_{\operatorname{supp} f_k} \frac{s_k}{k}\bigg(\int_{\Psi^{-1}(\psi(r_k))}^{s_k/k} \frac{\Psi'(t)}{t}\,dt\bigg)\,dx \\
&\quad\ge c_n[c_\phi r_k\phi(r_k)/(t_k\phi(t_k))]t_k\,\frac{s_k}{k} \int_{\Psi^{-1}(\psi(r_k))}^{s_k/k} \frac{\Psi(t)}{t^2}\,dt \\
&\quad\ge c_n[c_\phi r_k\phi(r_k)/(t_k\phi(t_k))]t_k\Phi(s_k)\,\frac{\psi(r_k)}{\phi(r_k)} \\
&\quad= c_n[c_\phi r_k\phi(r_k)/(t_k\phi(t_k))]t_k\phi(t_k)\,\frac{\psi(r_k)}{\phi(r_k)} \ge \frac{c_n}{2}\,r_k\psi(r_k).
\end{aligned}$$

Since $|3B_k|$ is comparable to r_k and $|3B_k| > r_k$, we have

$$\int_{3B_k} \Psi(Mf_k(x)/k)\,dx \ge c|3B_k|\psi(|3B_k|).$$

If $c \ge 1$, then $\|Mf_k\|_{L^{(\Psi,\psi)}} \ge k$. If $c < 1$, then

$$\int_{3B_k} \Psi\bigg(\frac{Mf_k(x)}{ck}\bigg)\,dx \ge \frac{1}{c}\int_{3B_k} \Psi\bigg(\frac{Mf_k(x)}{k}\bigg)\,dx \ge |3B_k|\psi(|3B_k|).$$

Hence $\|Mf_k\|_{L^{(\Psi,\psi)}} \ge ck$. ■

10. Proof of Theorem 6.1. By Corollary 5.3, if $\Phi \in \nabla_2$, then the operator M is bounded from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to itself. So we only prove weak boundedness.

Let $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$. For all balls B, let

$$f = f_1 + f_2, \qquad f_1 = f\chi_{2B}.$$

Let $\lambda = 2\|f\|_{L^{(\Phi,\phi)}}$. Then, by Lemma 9.5 and (9.1), we have

$$\begin{aligned}\Phi(t)m(Mf_1/\lambda,t) &\le \frac{c_n\Phi(t)}{t}\int_{t/2}^{+\infty} m(f_1/\lambda,s)\,ds \le c_n\int_{t/2}^{+\infty} m(f_1/\lambda,s)\Phi'(t)\,ds\\ &\le c_n\int_{t/2}^{+\infty} m(f_1/\lambda,s)\Phi'(2s)\,ds \le c_n\int_{2B}\Phi(2|f(x)|/\lambda)\,dx\\ &\le c_n|2B|\phi(|2B|) \le C|B|\phi(|B|).\end{aligned}$$

We may assume $C \ge 1$. Then

$$\Phi(t)m(Mf_1/(C\lambda),t) \le |B|\phi(|B|) \quad \text{for all } t>0.$$

Hence

$$\|Mf_1\|_{\Phi,\phi,B,\mathrm{weak}} \le 2C\|f\|_{L^{(\Phi,\phi)}}. \tag{10.1}$$

Since $\operatorname{supp} f_2 \cap 2B = \emptyset$, using Lemma 9.4, we have

$$Mf_2(x) \le C\Phi^{-1}(\phi(|B|))\|f\|_{L^{(\Phi,\phi)}} \quad \text{for } x\in B,$$

i.e.

$$\Phi\left(\frac{Mf_2(x)}{C\|f\|_{L^{(\Phi,\phi)}}}\right) \le \phi(|B|) \quad \text{for } x \in B.$$

Then

$$tm(B,\Phi(|f|/(C\|f\|_{L^{(\Phi,\phi)}})),t) \le tm(B,\phi(|B|),t) \le |B|\phi(|B|).$$

Hence

$$\|Mf_2\|_{\Phi,\phi,B,\mathrm{weak}} \le C\|f\|_{L^{(\Phi,\phi)}}. \tag{10.2}$$

By (10.1) and (10.2) we have the conclusion. ∎

11. Proof of Theorems 7.1 and 7.3. To prove Theorems 7.1 and 7.3, we state a lemma. For the proof, see [36, p. 63].

Lemma 11.1. *Let g be a function on $\mathbb{R}^n$ which is nonnegative, radial, decreasing (as a function on $(0,\infty)$) and integrable. Then*

$$\int_{\mathbb{R}^n} f(y)g(x-y)\,dy \le Mf(x)\|g\|_{L^1}, \quad x\in\mathbb{R}^n.$$

Proof of Theorem 7.1. By Theorem 5.1 we have the boundedness of M from $L^{(\Phi,\phi)}(\mathbb{R}^n)$ to $L^{(\Theta,\psi)}(\mathbb{R}^n)$, i.e. $\|Mf\|_{L^{(\Theta,\psi)}} \le C_0\|f\|_{L^{(\Phi,\phi)}}$. If we prove the pointwise estimate

$$\Psi\left(\frac{|I_\varrho f(x)|}{C_1\|f\|_{L^{(\Phi,\phi)}}}\right) \le \Theta\left(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi,\phi)}}}\right), \tag{11.1}$$

then we have, for all balls B,

$$\int_B \Psi\left(\frac{|I_\varrho f(x)|}{C_1\|f\|_{L^{(\Phi,\phi)}}}\right) dx \le \int_B \Theta\left(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi,\phi)}}}\right) dx \le |B|\psi(|B|).$$

This shows $\|I_\varrho f\|_{L^{(\Psi,\psi)}} \le C_1\|f\|_{L^{(\Phi,\phi)}}$.

To prove (11.1), for arbitrary $r > 0$, let $B_k = B(x,(2^k r)^{1/n})$, $k = 0,1,\ldots$. Then

$$\begin{aligned} I_\varrho f(x) &= \int_{\mathbb{R}^n} f(y)\,\frac{\varrho(|x-y|^n)}{|x-y|^n}\,dy \\ &= \int_{B_0} f(y)\,\frac{\varrho(|x-y|^n)}{|x-y|^n}\,dy + \sum_{k=0}^{+\infty}\int_{B_{k+1}\setminus B_k} f(y)\,\frac{\varrho(|x-y|^n)}{|x-y|^n}\,dy \\ &= J(x) + \sum_{k=0}^{+\infty} J_k(x), \qquad \text{say.} \end{aligned}$$

Let

$$h(t) = \inf\{\varrho(s^n)/s^n : s \le t\}, \qquad t > 0.$$

Then h is decreasing, $h(t) \sim \varrho(t^n)/t^n$ and

$$\|h(|\cdot|)\|_{L^1(B(0,r^{1/n}))} = \int_{B(0,r^{1/n})} h(|x|)\,dx \le C\int_0^{r^{1/n}} \frac{\varrho(t^n)}{t^n}\,t^{n-1}\,dt = C'\int_0^r \frac{\varrho(t)}{t}\,dt.$$

By Lemma 11.1 we have

$$|J(x)| \le C\int_{B_0} |f(y)|h(|x-y|)\,dy \le CMf(x)\int_0^r \frac{\varrho(t)}{t}\,dt.$$

We note that $\Phi^{-1}(\phi(r))$ satisfies the doubling condition, since ϕ does and Φ^{-1} is concave. By Lemmas 9.2 and 9.3 we have

$$\begin{aligned} |J_k(x)| &\le \int_{B_{k+1}\setminus B_k} \left| f(y)\,\frac{\varrho(|x-y|^n)}{|x-y|^n}\right| dy \\ &\sim \frac{\varrho(|B_k|)}{|B_k|}\int_{B_{k+1}\setminus B_k} |f(y)|\,dy \le \frac{\varrho(|B_k|)}{|B_k|}\int_{B_{k+1}} |f(y)|\,dy \\ &\le 2\,\frac{\varrho(|B_k|)}{|B_k|}\,|B_{k+1}|\phi(|B_{k+1}|)\|f\|_{\Phi,\phi,B_{k+1}}\|1\|_{\widetilde{\Phi},\phi,B_{k+1}} \\ &\le 2\,\frac{\varrho(|B_k|)}{|B_k|}\,|B_{k+1}|\phi(|B_{k+1}|)\|f\|_{L^{(\Phi,\phi)}}\Phi^{-1}(\phi(|B_{k+1}|))/\phi(|B_{k+1}|) \end{aligned}$$

$$\sim \Phi^{-1}(\phi(|B_k|))\varrho(|B_k|)\|f\|_{L^{(\Phi,\phi)}} \sim \Phi^{-1}(\phi(2^k r))\varrho(2^k r)\|f\|_{L^{(\Phi,\phi)}}$$
$$= (\log 2)^{-1} \int_{2^k r}^{2^{k+1} r} \Phi^{-1}(\phi(2^k r))\varrho(2^k r)\,\frac{dt}{t}\,\|f\|_{L^{(\Phi,\phi)}}$$
$$\sim \int_{2^k r}^{2^{k+1} r} \Phi^{-1}(\phi(t))\varrho(t)\,\frac{dt}{t}\,\|f\|_{L^{(\Phi,\phi)}}.$$

Thus

$$|I_\varrho f(x)| \le C_2\biggl(Mf(x)\int_0^r \frac{\varrho(t)}{t}\,dt + \|f\|_{L^{(\Phi,\phi)}} \int_r^{+\infty} \frac{\Phi^{-1}(\phi(t))\varrho(t)}{t}\,dt\biggr). \tag{11.2}$$

Choose r so that $\Theta^{-1}(\phi(r)) = Mf(x)/(C_0\|f\|_{L^{(\Phi,\phi)}})$. Then

$$|I_\varrho f(x)| \le C_2 C_0\|f\|_{L^{(\Phi,\phi)}}\biggl(\Theta^{-1}(\phi(r))\int_0^r \frac{\varrho(t)}{t}\,dt + \int_r^{+\infty} \frac{\Phi^{-1}(\phi(t))\varrho(t)}{t}\,dt\biggr).$$

Let $C_1 = AC_2C_0$, where A is the constant in (7.6). Then

$$\Psi\biggl(\frac{|I_\varrho f(x)|}{C_1\|f\|_{L^{(\Phi,\phi)}}}\biggr) \le \Psi\biggl(\frac{\Theta^{-1}\circ\phi(r)\varrho^*(r) + ((\Phi^{-1}\circ\phi)\varrho)_*(r)}{A}\biggr)$$
$$\le \phi(r) = \Theta\biggl(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi,\phi)}}}\biggr).$$

This is (11.1). ∎

Proof of Theorem 7.3. Theorem 6.1 implies $\|Mf\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}} \le C_0\|f\|_{L^{(\Phi,\phi)}}$. Moreover, if $\Phi \in \nabla_2$, then $\|Mf\|_{L^{(\Phi,\phi)}} \le C_0\|f\|_{L^{(\Phi,\phi)}}$.

We use (11.2). Choose r so that $\Phi^{-1}(\phi(r)) = Mf(x)/(C_0\|f\|_{L^{(\Phi,\phi)}})$. Then

$$|I_\varrho f(x)| \le C_2 C_0\|f\|_{L^{(\Phi,\phi)}}\biggl(\Phi^{-1}(\phi(r))\int_0^r \frac{\varrho(t)}{t}\,dt + \int_r^{+\infty} \frac{\Phi^{-1}(\phi(t))\varrho(t)}{t}\,dt\biggr).$$

Let $C_1 = AC_2C_0$, where A is the constant in (7.7). Then

$$\Psi\biggl(\frac{|I_\varrho f(x)|}{C_1\|f\|_{L^{(\Phi,\phi)}}}\biggr) \le \Phi\biggl(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi,\phi)}}}\biggr). \tag{11.3}$$

Since $\|Mf\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}} \le C_0\|f\|_{L^{(\Phi,\phi)}}$ we find that, for all balls B,

$$\sup_{t>0} tm(B, \Psi(I_\varrho f/(C_1\|f\|_{L^{(\Phi,\phi)}})), t)$$
$$\le \sup_{t>0} tm(B, \Phi(Mf/(C_0\|f\|_{L^{(\Phi,\phi)}})), t) \le |B|\phi(|B|).$$

This shows $\|I_\varrho f\|_{L^{(\Psi,\phi)}_{\mathrm{weak}}} \le C_1\|f\|_{L^{(\Phi,\phi)}}$.

Since $\|Mf\|_{L^{(\Phi,\phi)}} \le C_0\|f\|_{L^{(\Phi,\phi)}}$ we see that, for all balls B,

$$\int_B \Psi\left(\frac{|I_\varrho f(x)|}{C_1\|f\|_{L^{(\Phi,\phi)}}}\right) dx \le \int_B \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi,\phi)}}}\right) dx \le |B|\phi(|B|).$$

This shows $\|I_\varrho f\|_{L^{(\Psi,\phi)}} \le C_1\|f\|_{L^{(\Phi,\phi)}}$. ■

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