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On λ-commuting operators

by

JOHN B. CONWAY (Knoxville, TN) and
GABRIEL PRĂJITURĂ (Brockport, NY)

Abstract. For a scalar λ, two operators T and S are said to λ-*commute* if $TS = \lambda ST$. In this note we explore the pervasiveness of the operators that λ-commute with a compact operator by characterizing the closure and the interior of the set of operators with this property.

For a scalar λ, two operators T and S are said to λ-*commute* if $TS = \lambda ST$. This notion has received attention in the past. In particular, both [B] and [CC] examined the concept. [B] shows that if T λ-commutes with a compact operator, then T has a non-trivial hyperinvariant subspace. In [CC] it is shown that for an integer n, the commutants of T and T^n are different if and only if there is a non-zero operator Y and an nth root of unity, $\lambda \neq 1$, such that $TY = \lambda YT$. In [L] some additional properties and examples of λ-commuting operators are explored. In this note we explore the pervasiveness of the operators that λ-commute with a compact operator by characterizing the closure and the interior of the set of operators with this property.

Throughout we will denote by $\mathcal{B}(\mathcal{H})$ the set of all operators on the Hilbert space $\mathcal{H}$, and by $\mathcal{B}_0(\mathcal{H})$ and $\mathcal{B}_{00}(\mathcal{H})$, respectively, the set of all compact and finite rank operators on $\mathcal{H}$. For a complex number λ define the following classes of operators:

$$\mathcal{C}_1(\lambda) = \{T \in \mathcal{B}(\mathcal{H}) : \text{there is } S \in \mathcal{B}_{00}(\mathcal{H}) \setminus \{0\} \text{ such that } TS = \lambda ST\},$$
$$\mathcal{C}_2(\lambda) = \{T \in \mathcal{B}(\mathcal{H}) : \text{there is } S \in \mathcal{B}_{0}(\mathcal{H}) \setminus \{0\} \text{ such that } TS = \lambda ST\},$$
$$\mathcal{C}_3(\lambda) = \{T \in \mathcal{B}(\mathcal{H}) : \text{there is } S \in \mathcal{B}(\mathcal{H}) \setminus \{0\} \text{ such that } TS = \lambda ST\}.$$

Note that when $\lambda = 1$, these classes and their topological properties were examined in [CP]. If the scalar λ is understood, write $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$.

It is clear that $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3$ and that all three sets are invariant under similarities. Every operator in $\mathcal{C}_1$ has a finite-dimensional invariant subspace

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and, if $\lambda \neq 0$, so does its adjoint. Therefore if T is in $\mathcal{C}_1$, then $\sigma_{\mathrm{p}}(T) \neq \emptyset$ and, for a non-zero λ, $\sigma_{\mathrm{c}}(T) \neq \emptyset$. Here σ_{p} and σ_{c} denote the point spectrum and the compression spectrum of the operator.

1. EXAMPLE. Let T be an operator with the property that there is a complex number α such that $\alpha \in \sigma_{\mathrm{c}}(T)$ and $\lambda\alpha \in \sigma_{\mathrm{p}}(T)$. Let x be a norm-one eigenvector corresponding to the eigenvalue $\lambda\alpha$ of T and y a norm-one eigenvector corresponding to the eigenvalue $\overline{\alpha}$ of T^*(here the bar stands for complex conjugate). For two non-zero vectors u and v, we will use $u \otimes v$ to denote the rank-one operator that maps v into u. Then $Tx \otimes y = (Tx) \otimes y = (\lambda\alpha x) \otimes y = \lambda\alpha x \otimes y$, while $(x \otimes y)T = x \otimes (T^* y) = x \otimes (\overline{\alpha} y) = \alpha x \otimes y$. Thus $T(x \otimes y) = \lambda(x \otimes y)T$, so $T \in \mathcal{C}_1(\lambda)$.

In a similar way we can construct operators that λ-commute with a rank n operator for every $n \geq 1$. The previous example, as simple as it is, is nevertheless characteristic for the operators in $\mathcal{C}_1$. Next we show an example of an operator that belongs to $\mathcal{C}_2$ but does not belong to $\mathcal{C}_1$.

2. EXAMPLE. Let S be the unilateral shift with respect to the basis $\{e_n\}_{n\geq 1}$. For $|\lambda| > 1$, let K_λ be the operator defined by $K_\lambda e_n = \lambda^{-n} e_n$ for $n \geq 1$. Then K_λ is a compact operator and S λ-commutes with K_λ. Thus S belongs to $\mathcal{C}_2$, but, since S does not have eigenvalues, it does not belong to $\mathcal{C}_1$.

Finally, we give an example of an operator that belongs to $\mathcal{C}_3$ but does not belong to $\mathcal{C}_2$.

3. EXAMPLE. Let S be the unilateral shift with respect to the basis $\{e_n\}_{n\geq 1}$. Let T be the operator defined by $Te_n = (-1)^n e_n$ for $n \geq 1$. Then S (-1)-commutes with T. To see that S does not (-1)-commute with any non-zero compact operator, notice first that $SK = -KS$ implies that $S^2 K = KS^2$. Since S^2 is unitarily equivalent to $S \oplus S$, it does not commute with any non-zero compact operator.

4. EXAMPLE. Let M be the Bergman shift (the operator of multiplication by z on L^2_{a} of the open unit disc, $\mathbb{D}$). The relation $MX = \lambda XM$ means in this case $zX(f) = \lambda X(zf)$ or, since λ cannot be equal to 0, $X(zf) = z\lambda^{-1} X(f)$ for every function f in the Bergman space. In particular, if $g = X(1)$, then this implies that $X(z^n) = (z/\lambda)^n g$. Now $z^n \to 0$ weakly in $L^2_{\mathrm{a}}(\mathbb{D})$, the Hilbert space of the analytic functions on $\mathbb{D}$ which are square integrable with respect to the area measure; hence $X(z^n) \to 0$ weakly. But if $|\lambda| < 1$, then $\lim_n \|(z/\lambda)^n g\| = \infty$. Thus there can be no such operator X when $|\lambda| < 1$.

If $|\lambda| \geq 1$, then the above equation gives

$$X(f) = gf(z/\lambda), \tag{5}$$

when f is a polynomial. By taking limits of polynomials we deduce that (5) holds for every function f in the Bergman space.

If $|\lambda| = 1$, let U_λ be the unitary operator defined by $(U_\lambda f)(z) = f(\lambda z)$. Then $XU_\lambda f = gf$ and therefore g must be a bounded analytic function. In this case no such compact operator X, other than 0, can exist.

Recall that if a sequence in $L^2_{\mathrm{a}}(\mathbb{D})$ converges weakly, then it converges uniformly on compact subsets of $\mathbb{D}$. Therefore if $|\lambda| > 1$, g can be any function in the Bergman space and all operators X of this type are compact.

For a normal operator we can characterize the operators that λ-commute with it. We start with a more general result that does not seem to be readily accessible in the literature, though it is surely known.

6. THEOREM. *For $j = 1, 2$ let N_j be a normal operator on $\mathcal{H}_j$ with scalar-valued spectral measure μ_j. There is a non-trivial bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that $TN_1 = N_2T$ if and only if there is a Borel set Δ with $\mu_1(\Delta) > 0$ and $\mu_1|\Delta \ll \mu_2|\Delta$.*

Proof. Assume that such a Borel set Δ exists. By passing to a subset of Δ we may assume that $[\mu_1|\Delta] = [\mu_2|\Delta]$. That is, $\mu_1|\Delta$ and $\mu_2|\Delta$ have the same sets of zero measure. If $N_j = \int z\, dE_j(z)$ is the spectral decomposition of N_j, there is a separating vector e_j for $W^*(N_j)$ such that $\mu_j(S) = \langle E_j(S)e_j, e_j\rangle$ for all Borel sets S [C, Corollary IX.7.9]. Thus we can identify the reducing subspace $\mathcal{K}_j = \mathrm{cl}[W^*(N_j)e_j]$ with $L^2(\mu_j)$ in such a way that the action of N_j on $\mathcal{K}_j$ is identified with multiplication by z on $L^2(\mu_j)$. If we can find a non-trivial bounded operator $T_0 : L^2(\mu_1) \to L^2(\mu_2)$ such that $T_0M_z = M_zT_0$, then this can be extended to a bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ by letting $T = 0$ on $\mathcal{H}_{e_1}^\perp$, completing the proof of this half of the theorem.

Let $d(\mu_2|\Delta)/d(\mu_1|\Delta) = h^2$, h a positive function in $L^2(\mu_1|\Delta)$. Choose a positive scalar c such that $\Delta_c = \{z \in \Delta : h(z) \le c\}$ has $\mu_1(\Delta_c) > 0$. If $f \in L^2(\mu|\Delta_c)$, then

$$\int |f|^2\, d\mu_2(z) = \int_{\Delta_c} |f|^2 h^2\, d\mu_1 \le c^2 \int |f|^2\, d\mu_1.$$

Thus if $T_0 : L^2(\mu_1|\Delta_c) \to L^2(\mu_2)$ is defined by $(T_0f)(z) = f(z)$, then T_0 is a bounded operator. Also $(T_0N_1f)(z) = (T_0(zf))(z) = zf(z) = (N_2T_0f)(z)$. If we set $T_0 = 0$ on $L^2(\mu_1) \ominus L^2(\mu_1|\Delta_c)$, we have the desired operator that intertwines N_1 and N_2.

For the converse, assume that there is a non-trivial bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that $TN_1 = N_2T$. According to Proposition IX.6.10 of [C], $\mathcal{K}_2 = \mathrm{cl}[\mathrm{ran}\, T]$ reduces N_2, $\mathcal{K}_1 = (\ker T)^\perp$ reduces N_1, and $M_1 = N_1|\mathcal{K}_1$ is unitarily equivalent to $M_2 = N_2|\mathcal{K}_2$. If ν_j is the scalar-valued spectral measure for M_j, then ν_1 and ν_2 are equivalent. But there is a Borel set Δ such that $[\nu_1|\Delta] = [\mu_1|\Delta]$. Thus $\mu_2|\Delta \ll \mu_1|\Delta$.

For any compactly supported measure μ on $\mathbb{C}$ and scalar $\lambda \neq 0$, define the measure μ^λ by $\mu^\lambda(S) = \mu(\lambda^{-1}S)$. If μ is a scalar-valued spectral measure for the normal operator N and λ is a non-zero scalar, then a scalar-valued spectral measure for the normal operator λN is μ^λ. Also note that for any function f in $L^1(\mu^\lambda)$, $\int f\,d\mu^\lambda = \int f(\lambda z)\,d\mu(z)$. The next result is immediate from the preceding theorem.

7. Corollary. *If N is a normal operator on $\mathcal{H}$ with scalar-valued spectral measure μ and $\lambda \neq 0$, then $N \in \mathcal{C}_3(\lambda)$ if and only if there is a Borel set Δ with $\mu(\Delta) > 0$ such that $\mu|\Delta \ll \mu^\lambda|\Delta$.*

8. Corollary. *If N is a normal operator with scalar-valued spectral measure μ and $\lambda \neq 0$, then the following statements are equivalent*:

(a) $N \in \mathcal{C}_1(\lambda)$.
(b) $N \in \mathcal{C}_2(\lambda)$.
(c) *μ has an atom at some a_0 such that $\lambda^{-1}a_0$ is also an atom.*

Proof. Assume that T is a compact operator such that $TN = \lambda NT$. Again Proposition IX.6.10 of [C] implies that $\mathcal{K}_1 = (\ker T)^\perp$ and $\mathcal{K}_2 = \mathrm{cl}[\mathrm{ran}\, T]$ reduce N and $N|\mathcal{K}_1$ is unitarily equivalent to $\lambda N|\mathcal{K}_2$. By the Fuglede–Putnam Theorem $T^*N = \lambda NT^*$ and it follows that T^*T commutes with N. But $\ker T = \ker T^*T$. Since T^*T is a non-zero, positive compact operator, this implies there is a finite-dimensional reducing subspace for N contained in $\mathcal{K}_1$. Thus there is an eigenvalue a_0 for N with an eigenvector contained in $\mathcal{K}_1$. Since $N|\mathcal{K}_1 \cong \lambda N|\mathcal{K}_2$, this implies that $\lambda^{-1}a_0 \in \sigma_{\mathrm{p}}(N)$, proving (c).

Now assume that (c) holds. So $b_0 = \lambda^{-1}a_0 \in \sigma_{\mathrm{p}}(N)$ and $\lambda b_0 = a_0 \in \sigma_{\mathrm{p}}(N) \subseteq \sigma_{\mathrm{c}}(N)$. By Example 1, $N \in \mathcal{C}_1$. ∎

With a bit more work using standard tools from operator theory, the operators that λ-commute with a cyclic normal operator can be characterized. As in the characterization of the normal operators in $\mathcal{C}_2(\lambda)$ it is easier to start with a more general result, which, again, is probably known but does not seem to have a reference.

9. Proposition. *For $j = 1, 2$ let μ_j be a compactly supported measure on the plane and let N_j be the normal operator defined on $L^2(\mu_j)$ as multiplication by the independent variable. If $T : L^2(\mu_1) \to L^2(\mu_2)$ is a bounded operator such that $TN_1 = N_2T$, then there is a Borel set Δ, a positive function u in $L^\infty(\mu_1|\Delta)$, and a function ψ in $L^2(\mu_2|\Delta)$ such that $T = 0$ on $L^2(\mu_1|\mathbb{C}\backslash\Delta)$ and for f in $L^2(\mu_1|\Delta)$,*

$$Tf = u\psi f.$$

Conversely, if Δ, u, and ψ are as described and $Tf = u\psi f$ defines a bounded operator on $L^2(\mu_1|\Delta)$, then $TN_1 = N_2T$.

Proof. Let $T : L^2(\mu_1) \to L^2(\mu_2)$ be a bounded operator such that $TN_1 = N_2T$. By the Fuglede–Putnam Theorem, $N_1T^* = T^*N_2$. Thus $T^*T \in \{N_1\}'$. So there is a positive function u in $L^\infty(\mu_1)$ such that $T^*T = u(N_1)^2 = M_{u^2}$ on $L^2(\mu_1)$ ([C, Theorem IX.6.6]). It follows that there are Borel sets Δ_j such that

$$(\ker T)^\perp = L^2(\mu_1|\Delta_1), \qquad \mathrm{cl}[\mathrm{ran}\, T] = L^2(\mu_2|\Delta_2).$$

If $T = VM_u$ is the polar decomposition of T, then $V : L^2(\mu_1|\Delta_1) \to L^2(\mu_2|\Delta_2)$ is an isomorphism that intertwines multiplication by the independent variable. By Proposition IX.6.10 of [C], if $\psi = V(\chi_{\Delta_1})$, then

(a) $Vf = f\psi$ for all f in $L^2(\mu_1|\Delta_1)$,
(b) $[\mu_1|\Delta_1] = [\mu_2|\Delta_2]$,
(c) $\mu_1|\Delta_1 = |\psi|^2\mu_2|\Delta_2$.

Because of (b), $\mu_j(\Delta_1\backslash\Delta_2) = \mu_j(\Delta_2\backslash\Delta_1) = 0$. Thus we can assume that $\Delta_1 = \Delta_2 = \Delta$, a Borel set. If $f \in L^2(\mu_1|\Delta)$, then $Tf = VM_uf = \psi uf$, as desired.

The converse is clear. ■

As before the next result is immediate from the preceding one since λN is unitarily equivalent to M_z on $L^2(\mu^\lambda)$ when N is M_z on $L^2(\mu)$.

10. COROLLARY. *If $\lambda \neq 0$, N is multiplication by the independent variable on $L^2(\mu)$, and T λ-commutes with N, then there is a Borel set Δ, there is a function u in $L^\infty(\mu|\Delta)$ with $u \geq 0$, and there is a function ψ in $L^2(\mu^\lambda|\Delta)$ such that $T = 0$ on $L^2(\mu|\mathbb{C}\backslash\Delta)$ and for f in $L^2(\mu|\Delta)$,*

$$(Tf)(z) = u(\lambda z)\psi(\lambda z)f(\lambda z).$$

Conversely, if T is so defined and bounded, then $TN = \lambda NT$.

The next result shows that all classes considered have the same closure and gives a spectral description of this closure. Let $\sigma_{\mathrm{r}}(T)$ and $\sigma_{\mathrm{l}}(T)$ denote the right and the left spectrum of the operator T.

11. THEOREM. *For an operator T the following are equivalent.*

(a) $T \in \mathrm{cl}\,\mathcal{C}_1(\lambda)$.
(b) $T \in \mathrm{cl}\,\mathcal{C}_2(\lambda)$.
(c) $T \in \mathrm{cl}\,\mathcal{C}_3(\lambda)$.
(d) $\sigma_{\mathrm{r}}(T) \cap \sigma_{\mathrm{l}}(\lambda T) \neq \emptyset$.

Proof. It is clear that (a)⇒(b)⇒(c).

(c)⇒(d). Let $\mathcal{C}(\lambda)$ be the set of all operators with the property in (d). Recall from [DR] that if $\sigma_{\mathrm{r}}(A) \cap \sigma_{\mathrm{l}}(B) = \emptyset$, then the operator $S \to AS - BS$ defined on $\mathcal{B}(\mathcal{H})$ is bounded below. In particular it is one-to-one. This implies that $\mathcal{C}_3(\lambda) \subset \mathcal{C}(\lambda)$. The conclusion follows because $\mathcal{C}(\lambda)$ is a closed set.

(d)⇒(a). Let T be an operator such that $\sigma_{\mathrm{r}}(T)\cap\sigma_{\mathrm{l}}(\lambda T)\neq\emptyset$. Thus there is $\mu\in\sigma_{\mathrm{l}}(T)$ such that $\lambda\mu\in\sigma_{\mathrm{r}}(T)$. We have $\mu\in\sigma_{\mathrm{lre}}(T)\cup\sigma_{\mathrm{p}}^0(T)\cup\varrho_{\mathrm{sF}}(T)$, where $\sigma_{\mathrm{lre}}(T)$ denotes the intersection of left and right essential spectrum of the operator, $\sigma_{\mathrm{p}}^0(T)$ is the set of isolated eigenvalues of finite multiplicity, and $\varrho_{\mathrm{sF}}(T)$ is the semi-Fredholm domain. Since $\varrho_{\mathrm{sF}}(T)\cap\sigma_{\mathrm{r}}(T)\subset\sigma_{\mathrm{c}}(T)$, we conclude that, in fact, $\mu\in\sigma_{\mathrm{lre}}(T)\cup\sigma_{\mathrm{c}}(T)$. In a similar way it follows that $\lambda\mu\in\sigma_{\mathrm{lre}}(T)\cup\sigma_{\mathrm{p}}(T)$.

If $\mu\in\sigma_{\mathrm{c}}(T)$ and $\lambda\mu\in\sigma_{\mathrm{p}}(T)$, then, from Example 1, we deduce that $T\in\mathcal{C}_1(\lambda)$. If $\mu\in\sigma_{\mathrm{c}}(T)$ and $\lambda\mu\in\sigma_{\mathrm{lre}}(T)$, then let x be a non-zero vector in $\ker(\mu-T)^*$ and let $\mathcal{M}$ be the one-dimensional subspace spanned by x. Since $\mathcal{M}^\perp$ is invariant for T, we obtain

$$T=\begin{bmatrix} T_1 & A\\ 0 & \mu\end{bmatrix}$$

with respect to the decomposition $\mathcal{H}=\mathcal{M}^\perp\oplus\mathcal{M}$. Since $\mathcal{M}$ has dimension 1, $\lambda\mu\in\sigma_{\mathrm{lre}}(T_1)$. By Theorem 2.2 of [AFV], for every $\varepsilon>0$, there is a compact operator K_ε such that $\|K_\varepsilon\|<\varepsilon$ and $\lambda\mu\in\sigma_{\mathrm{p}}(T_1-K_\varepsilon)$. Let

$$T_\varepsilon=\begin{bmatrix} T_1-K_\varepsilon & A\\ 0 & \mu\end{bmatrix}.$$

Then $\|T-T_\varepsilon\|<\varepsilon$ and as $\mu\in\sigma_{\mathrm{c}}(T_\varepsilon)$ and $\lambda\mu\in\sigma_{\mathrm{p}}(T_\varepsilon)$, we conclude that $T\in\operatorname{cl}\mathcal{C}_1(\lambda)$.

If $\mu\in\sigma_{\mathrm{lre}}(T)$ and $\lambda\mu\in\sigma_{\mathrm{p}}(T)$, an argument similar to the previous one will lead to the same conclusion.

If $\mu\in\sigma_{\mathrm{lre}}(T)$ and $\lambda\mu\in\sigma_{\mathrm{lre}}(T)$, then, by Corollary 3.50 in [H], for every $\varepsilon>0$, there is an operator L_ε such that $\|T-L_\varepsilon\|<\varepsilon$, $\mu\in\sigma_{\mathrm{c}}(L_\varepsilon)$, and $\lambda\mu\in\sigma_{\mathrm{p}}(L_\varepsilon)$. Therefore T belongs to $\operatorname{cl}\mathcal{C}_1(\lambda)$. ∎

We will give next a spectral description of the interior of the first class. But first we need to introduce some notation. If T is an operator, then use $P_+(T)$ and $P_-(T)$ to denote the semi-Fredholm domain of T where the index is positive and negative, respectively.

12. Theorem. *If T is an operator and $\lambda\neq0$, then $T\in\operatorname{int}\mathcal{C}_1(\lambda)$ if and only if one of the following three conditions holds*:

(a) $\lambda P_-(T)\cap P_+(T)\neq\emptyset$;
(b) $\lambda\sigma_{\mathrm{p}}^0(T)\cap P_+(T)\neq\emptyset$;
(c) $\lambda P_-(T)\cap\sigma_{\mathrm{p}}^0(T)\neq\emptyset$.

Proof. It is easy to see that the set of all operators satisfying the above conditions is open and included in $\mathcal{C}_1(\lambda)$. Therefore the conditions are sufficient.

For necessity, suppose that there is an operator T in $\operatorname{int}\mathcal{C}_1(\lambda)$ such that $\lambda P_-(T)\cap P_+(T)=\emptyset$, $\lambda\sigma_{\mathrm{p}}^0(T)\cap P_+(T)=\emptyset$, and $\lambda P_-(T)\cap\sigma_{\mathrm{p}}^0(T)=\emptyset$. Let

$\varepsilon > 0$ such that $\|T - S\| < \varepsilon$ implies that S belongs to $\operatorname{int}\mathcal{C}_1(\lambda)$. By Proposition 2.1 in [AM], there is an operator T_1 with the following properties: $\|T - T_1\| < \varepsilon$; $\sigma_{\mathrm{lre}}(T_1)$ is the closure of a finite number of Cauchy domains, $\{D_j\}_{j=1}^m$, with $\sigma_{\mathrm{lre}}(T) \subset \bigcup_{j=1}^m D_j$; $\sigma(T_1) = \sigma(T) \cup \bigcup_{j=1}^m \operatorname{cl} D_j$; $\sigma_{\mathrm{p}}^0(T_1) \subset \sigma_{\mathrm{p}}^0(T)$ and is finite; $\operatorname{cl}(\varrho_{\mathrm{sF}}(T_1) \cap \sigma(T_1)) \subset \varrho_{\mathrm{sF}}(T) \cap \sigma(T)$; $\varrho_{\mathrm{sF}}(T_1) \cap \sigma(T_1)$ has a finite number of components; $\operatorname{ind}(\lambda - T_1) = \operatorname{ind}(\lambda - T)$ and $\dim\ker(\lambda - T_1) = \dim\ker(\lambda - T)$ for every λ in $\varrho_{\mathrm{sF}}(T_1) \cap \sigma(T_1)$.

Let $\sigma_{\mathrm{p}}^0(T_1) = \{\lambda_1, \ldots, \lambda_r\}$. There are operators A and $B_1, \ldots, B_r$ such that T_1 is similar to $T_2 = A \oplus \bigoplus_{j=1}^r B_j$, where $\sigma(B_j) = \{\lambda_j\}$, each B_j is in the Jordan form, and $\sigma(A) = \sigma(T_1) \setminus \sigma_{\mathrm{p}}^0(T_1)$. Since $\mathcal{C}_1(\lambda)$ is invariant under similarities, so is the interior of $\mathcal{C}_1(\lambda)$. Thus T_2 is in the interior of $\mathcal{C}_1(\lambda)$. Hence there is a $\delta_1 > 0$ such that $\|T_2 - S\| < \delta_1$ implies that S belongs to the interior of $\mathcal{C}_1(\lambda)$. By assumption, for $1 \le j \le r$, $\lambda\lambda_j \notin \operatorname{cl} P_+(T_2)$ and $\lambda_j \notin \lambda \operatorname{cl} P_-(T_2)$. Therefore there is a $\delta_2 > 0$ such that $\lambda D_{\delta_2}(\lambda_j) \cap \operatorname{cl} P_+(T_2) = \emptyset$ and $D_{\delta_2}(\lambda_j) \cap \lambda \operatorname{cl} P_-(T_2) = \emptyset$ for $1 \le j \le r$. (Here $D_\delta(\alpha)$ denotes the disk of radius δ centered at α.) Let $\delta = \min\{\delta_1, \delta_2\}$ and let $\{\beta_1, \ldots, \beta_r\}$ be such that $|\beta_j - \lambda_j| < \delta$ and $\lambda\beta_j \ne \beta_k$ for $1 \le j, k \le r$. Let $C_j = B_j + (\beta_j - \lambda_j)I_j$, $C = \bigoplus_{j=1}^r C_j$ and $T_3 = A \oplus C$. Since $\|T_2 - T_3\| < \delta$, T_3 is in the interior of $\mathcal{C}_1(\lambda)$. For $1 \le j, k \le r$, $\sigma(C_j) \cap \sigma(\lambda C_k) = \emptyset$. So there is no non-zero operator F such that $C_j F = \lambda F C_k$. This implies that C does not λ-commute with any non-zero operator.

Let $U_1, \ldots, U_p$ be the components of $P_-(T_3)$. For $1 \le j \le p$, let M_j be the Bergman operator on $L_{\mathrm{a}}^2(U_j)$. Also, let $M_- = \bigoplus_{j=1}^p M_j$. Since $\sigma_{\mathrm{p}}(M_j) = \emptyset$ for each j, for $1 \le j, k \le p$ there is no non-zero finite rank operator F such that $M_j F = \lambda F M_k$. Consequently, M_- does not λ-commute with any non-zero finite rank operator.

Let $V_1, \ldots, V_l$ be the components of $P_+(T_3)$. For every $1 \le j \le l$, let N_j be the adjoint of the Bergman operator on $L_{\mathrm{a}}^2(V_j)$ and $N_+ = \bigoplus_{j=1}^p N_j$. As before, $\sigma_{\mathrm{c}}(N_j) = \emptyset$, and, for $1 \le j, k \le l$, there is no non-zero finite rank operator F such that $N_j F = \lambda F N_k$. Consequently, N_+ does not λ-commute with any non-zero finite rank operator.

For $1 \le j \le m$, let Δ_j be a disc included in D_j, R_j the operator of multiplication by z on $L^2(\Delta_j)$ (with respect to the area measure), and $R = \bigoplus_{j=1}^m R_j$. The same argument used for M_- implies that R does not λ-commute with any non-zero finite rank operator.

The Similarity Orbit Theorem (Theorem 9.1 in [AFHV]) implies that there is a sequence of operators similar to $T_4 = M_- \oplus M_+ \oplus R \oplus C$ converging to T_3 and since T_3 is in the interior of $\mathcal{C}_1(\lambda)$, so is T_4. Thus there is a non-zero finite rank operator F such that $T_4 F = \lambda F T_4$. Let $F = (F_{ij})_{i,j=1}^4$ be the decomposition of F with respect to the same subspaces used in the definition of T_4. By construction, $F_{ii} = 0$ for every $1 \le i \le 4$. Also, since $M_- F_{1j} =$

$\lambda F_{1j}X_j$ (X_j stands for the jth diagonal entry of T_4) and $\sigma_{\mathrm{p}}(M_-)=\emptyset$, we have $F_{1j}=0$ for $1\le j\le 4$. A similar argument gives $F_{3j}=0$ for $1\le j\le 4$. Because $X_iF_{i2}=\lambda F_{i2}M_+$ and $\sigma_{\mathrm{c}}(M_+)=\emptyset$, we infer that $F_{i2}=0$ for $1\le i\le 4$. With a similar argument, the same is true for F_{i4}. Therefore the only possible non-zero entries of F are F_{21}, F_{24} and F_{41}.

But $M_+F_{21}=\lambda F_{21}M_-$, $\sigma(M_+)\subset P_+(T)$, $\sigma(M_-)\subset P_-(T)$ and $\lambda P_-(T)\cap P_+(T)=\emptyset$ imply that $F_{21}=0$. Also, $M_+F_{24}=\lambda F_{24}C$ and, by the construction of C, $\sigma(M_+)\cap\sigma(\lambda C)=\emptyset$; this gives $F_{24}=0$. Finally, a similar argument implies that $F_{41}=0$, and hence $F=0$, which is a contradiction. ∎

For $\lambda=0$ the interior is easier to describe.

13. THEOREM. *For an operator T the following are equivalent*:

(a) $T\in\operatorname{int}\mathcal{C}_1(0)$;
(b) $T\in\operatorname{int}\mathcal{C}_2(0)$;
(c) $T\in\operatorname{int}\mathcal{C}_3(0)$;
(d) $0\in P_+(T)$.

Proof. It is clear that (a)⇒(b)⇒(c).

(c)⇒(d). For every operator A in $\mathcal{C}_3(0)$ we have $0\in\sigma_{\mathrm{p}}(A)$. Suppose that $T\in\operatorname{int}\mathcal{C}_3(0)$ and $0\notin P_+(T)$. If $0\in\sigma_{\mathrm{lre}}(T)\cup\sigma_{\mathrm{p}}^0(T)$ or 0 belongs to a component of the semi-Fredholm domain with index 0 that is included in $\sigma(T)$, then, by using Apostol–Morrel simple models [AM], for every $\varepsilon>0$ we can find an operator S_ε such that $\|T-S_\varepsilon\|<\varepsilon$ and $0\notin\sigma(S_\varepsilon)$. This contradicts the fact that $T\in\operatorname{int}\mathcal{C}_3(0)$. If $0\in P_-(T)$, then, by again using the Apostol–Morrel simple models, for every $\varepsilon>0$ we can find an operator S_ε such that $\|T-S_\varepsilon\|<\varepsilon$ and $0\notin\sigma_{\mathrm{p}}(S_\varepsilon)$. As in the previous case this leads to a contradiction.

(d)⇒(a). It is easy to see that the set of all operators T such that $0\in P_+(T)$ is open. For such an operator, if x is an eigenvector corresponding to 0 and F is the orthogonal projection onto the one-dimensional subspace generated by x, then $TF=0=0FT$. Hence $T\in\mathcal{C}_1(0)$. ∎

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University of Tennessee
Knoxville, TN 37996-1300, U.S.A.
E-mail: conway@math.utk.edu

State University of New York
Brockport, NY 14420, U.S.A.
E-mail: gprajitu@brockport.edu

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