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The topological entropy versus level sets for interval maps (part II)

by

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Abstract. Let $f: [a,b] \to [a,b]$ be a continuous function of the compact real interval such that (i) $\operatorname{card} f^{-1}(y) \geq 2$ for every $y \in [a,b]$; (ii) for some $m \in \{\infty, 2, 3, \ldots\}$ there is a countable set $L \subset [a,b]$ such that $\operatorname{card} f^{-1}(y) \geq m$ for every $y \in [a,b] \setminus L$. We show that the topological entropy of f is greater than or equal to $\log m$. This generalizes our previous result for $m = 2$.

0. Introduction. The aim of this paper is to demonstrate a relationship of two characteristics of an interval map: its topological entropy and cardinalities of level sets. Our main result states that for an interval map—as opposed to circle maps or some maps on higher dimensional manifolds [Ma]—the cardinalities of level sets strongly determine the value of entropy. Elaborating our approach from [B1] we focus on the case of m-preimages for fixed $m \in \mathbb{N} \setminus \{1\}$ or m equal to infinity. In particular, Theorem 4.3 shows that if we forbid an exceptional case of one-point level sets, the dependence between entropy and the cardinalities of level sets is rather regular. Based on that and several known (always) non-transitive counterexamples we conjecture that this should be the case for a wider variety of *transitive maps on compact topological manifolds.*

Let $[a,b]$ be a compact real interval. We denote by $C([a,b])$ the set of all continuous functions which map $[a,b]$ into itself. Any element of $C([a,b])$ is called an *interval map*. For $m \in \{\infty, 2, 3, 4, \ldots\}$ let $L(m,[a,b])$ be the subset of $C([a,b])$ maps satisfying

$$(1_m) \qquad \forall y \in [a,b]: \quad \operatorname{card} f^{-1}(y) \geq m.$$

From [B1] we know that the topological entropy of any $f \in L(2,[a,b])$ is greater than or equal to $\log 2$. In this paper we extend that result as follows.

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Let $L^\sharp(m,[a,b])$ be the subset of $L(2,[a,b])$ defined by

$$(1^\sharp_m) \qquad \forall f \in L^\sharp(m,[a,b])\ \exists L \subset [a,b],\ L \text{ countable } \forall y \in [a,b]\setminus L: \quad \operatorname{card} f^{-1}(y) \ge m.$$

We show the following statement.

Theorem 4.3. *Let $f \in L^\sharp(m,[a,b])$. Then the topological entropy of f is greater than or equal to* $\log m$. *In particular*, *this is true for any map from $L(m,[a,b]) \subset L^\sharp(m,[a,b])$.*

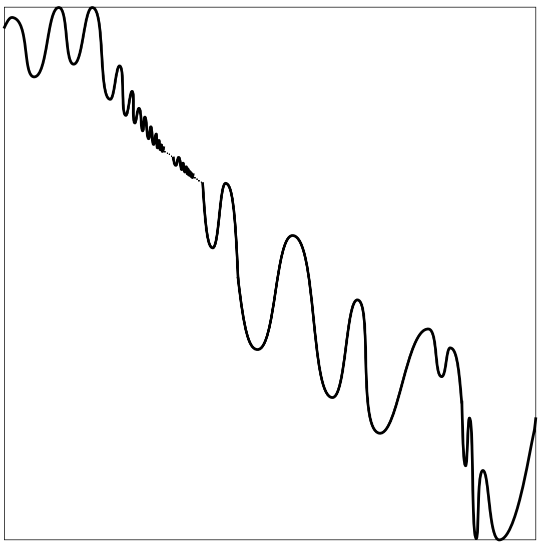

Fig. 1. $f \in L^\sharp(3,[a,b])$, $\operatorname{ent}(f) \ge \log 3$

Our main result is rather delicate. One can easily find an interval map of entropy zero that does not satisfy (1_∞) for exactly one point from $[a,b]$.

In this paper we use several times the following type of "horseshoe".

Definition 0.1. Let (X,ϱ) be a compact metric space, $f\colon X \to X$ be continuous and $S_0,S_1,\ldots,S_{m-1} \subset X$ be closed. We say that the sets $S_0,S_1,\ldots,S_{m-1}$ form an *m-horseshoe* if they are pairwise disjoint and

$$f(S_0)\cap f(S_1)\cap\cdots\cap f(S_{m-1}) \supset S_0\cup S_1\cup\cdots\cup S_{m-1}.$$

As an easy consequence of the definition of topological entropy we obtain the following [DGS].

Proposition 0.2. *Let (X,ϱ) be a compact metric space and $f\colon X\to X$ be continuous. If the sets $S_0,S_1,\ldots,S_{m-1}\subset X$ form an m-horseshoe then the topological entropy of f is greater than or equal to* $\log m$.

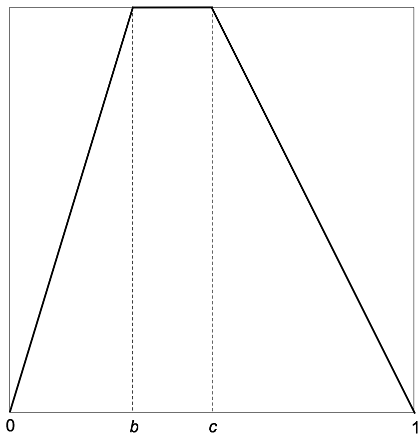

Fig 2. The sets $S_0 = [0, b]$, $S_1 = [c, 1]$ form a 2-horseshoe

The paper is organized as follows. In Section 1 we give some basic notation, definitions and known results (Theorems 1.1 and 1.3). Section 2 is devoted to the lemmas used throughout the paper. In Section 3 we analyze the properties of maps from $L^\sharp(m, [a, b])$, $m \in \{\infty, 2, \ldots\}$. In Section 4 we prove the key Lemma 4.1, Corollary 4.2 and Theorem 4.3.

Finally, Section 5 is devoted to an application of Theorem 4.3. We show that the entropy of a Besicovitch function (preserving the Lebesgue measure) is infinite.

1. Definitions and known results. By $\mathbb{N}$ we denote the set of positive integers. We will work with *topological dynamics* (X, T), where X is a compact metric space and T: $X \to X$ is a continuous map. (X, T) is *minimal* if $\{T^i(x)\colon\ i \in \mathbb{N}\}$ is dense in X for each $x \in X$. A subset M of X is T-*invariant* if $T(M) \subset M$, and *minimal* (in X) if M is closed, T-invariant and $(M, T|M)$ is minimal.

Let ϱ be a metric on the space X. We will use Bowen's definition of topological entropy [DGS]. A set $E \subset X$ is (n, ε)-*separated* (with respect to T) if, whenever $x, y \in E$, $x \neq y$ then $\max_{0 \le i \le n-1} \varrho(T^i(x), T^i(y)) > \varepsilon$. For a compact set $K \subset X$ we denote by $s(n, \varepsilon, K)$ the largest cardinality of any (n, ε)-separated subset of K. Put

$$\operatorname{ent}(T, K) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, K)$$

and $\operatorname{ent}(T) = \operatorname{ent}(T, X)$. The quantity $\operatorname{ent}(T)$ is called the *topological entropy* of T.

A topological dynamics (Y, S) is a *factor* of (X, T) if there is a continuous surjective factor map h: $X \to Y$ such that $h \circ T = S \circ h$.

THEOREM 1.1 ([Bo]). *If* (Y,S) *is a factor of* (X,T) *then*
$$\operatorname{ent}(S) \le \operatorname{ent}(T) \le \operatorname{ent}(S) + \sup_{y\in Y} \operatorname{ent}(T, h^{-1}(\{y\})).$$

As usual, the *ω-limit set* $\omega_T(x)$ of $x \in X$ consists of all the limit points of $\{T^i(x)\colon\ i \in \mathbb{N}\}$. A set $P \subset X$ is called a *periodic orbit* (of period n) if $P = \{x, T(x), \ldots, T^{n-1}(x)\}$ for some $x \in X$ and $n \in \mathbb{N}$ with $T^n(x) = x$. A normalized Borel measure μ on X is T-*invariant* if $\mu(T^{-1}(E)) = \mu(E)$ for each Borel set $E \subset X$.

Now we list several useful properties of minimal sets. As is well known they can be considered in any topological dynamics.

LEMMA 1.2. (i) *For each* $x \in X$, *the ω-limit set* $\omega_T(x)$ *contains some minimal set.*

(ii) *Any minimal set in* X *is either finite and then a periodic orbit of* T, *or infinite and then uncountable.*

(iii) *If* (X,T) *is minimal and a measure* μ *on* X *is* T-*invariant then either* X *is finite and then* μ *is atomic*, *or* X *is infinite and then* μ *is nonatomic. In any case* $\operatorname{supp}\mu = X$.

(iv) *Let* $M \subset X$ *be minimal in* X. *If* M *is infinite then for each countable closed set* $C \subset M$ *and* $x \in M$ *we have* $\lim_n n^{-1} \operatorname{card}\{0 \le i \le n-1\colon\ T^i(x) \in C\} = 0$.

Proof. See [BC] for (i)–(iii).

Let us prove (iv). Notice that by our assumption and (ii), M is uncountable. Suppose to the contrary there is an increasing sequence $\{k_n\}_{n=1}^{\infty}$ such that $\lim_n k_n^{-1} C(k_n, x) = a \in (0,1]$, where $C(n,x) = \operatorname{card}\{0 \le i \le n-1\colon\ T^i(x) \in C\}$. Then using the standard method [DGS, Prop. 2.7] we can find an atomic T-invariant measure μ for which $\mu(C) > 0$ and $\operatorname{supp}\mu \subsetneq M$, a contradiction with (ii) and (iii). ■

We will use the *symbolic dynamics* [DGS]. For $m \in \mathbb{N}$ consider the set $N_m = \{0, 1, \ldots, m-1\}$ as a space with the discrete topology, and denote by Ω_m the infinite product space $\prod_{i=0}^{\infty} X_i$, where $X_i = N_m$ for all i. The shift map $\sigma_m\colon \Omega_m \to \Omega_m$ (in what follows we write σ instead of σ_m) is defined by $(\sigma(\omega))_i = \omega_{i+1}$ for $i \in \mathbb{N} \cup \{0\}$. Obviously, each (Ω_m, σ) is a topological dynamics.

It is known [DGS, Prop. 16.11] that for $\Omega \subset \Omega_m$ closed,

$$\operatorname{ent}(\sigma, \Omega) = \lim_n \frac{1}{n} \log \operatorname{card} \Omega(n), \tag{2}$$

where $\Omega(n) = \{\omega(n) = (\omega_0, \ldots, \omega_{n-1})\colon \omega \in \Omega\}$.

The following remarkable result concerns the topological entropy of subshifts in (Ω_m, σ).

THEOREM 1.3 ([G]). *Let $m \in \mathbb{N}$. For any positive ε there is a minimal set Γ in Ω_m such that* $\mathrm{ent}(\sigma, \Gamma) > -\varepsilon + \log m$.

The following easy lemma is needed in the proof of Theorem 4.3. Put $\Omega_{j,k} = \{\omega \in \Omega_m\colon \omega_{2i+j} \neq k$ for each $i \in \mathbb{N} \cup \{0\}\}$ for $j \in N_2$, $k \in N_m$, and

$$\Omega(M(\infty)) = \bigcup_{(j,k)\in N_2\times N_m} \Omega_{j,k}. \tag{3}$$

LEMMA 1.4. *Let $\Omega = \Omega(M(\infty))$ be as in* (3).

(i) *The set Ω is closed σ-invariant in Ω_m and* $\mathrm{ent}(\sigma,\Omega) = \frac{1}{2}\log[m(m-1)]$.
(ii) *For each $k_0 \geq 2$,*
$$\frac{1}{2}\log[m(m-1)] \leq \frac{1}{k_0}\log(m-1) + \frac{k_0-1}{k_0}\log m.$$

Proof. (i) The closedness of Ω is clear. Since $\sigma(\Omega) \subset \Omega$, we can compute the entropy $\mathrm{ent}(\sigma, \Omega)$ using (2). Obviously, for each $n \in \mathbb{N}$ and $(j,k) \in N_2 \times N_m$ we have $\mathrm{card}\, \Omega_{j,k}(2n) = [m(m-1)]^n$, hence the conclusion follows. Property (ii) is clear. ∎

2. Lemmas. In what follows, by $[a,b]$, resp. (a,b) we always denote a compact, resp. open real interval. As usual, a map $f \in C([a,b])$ has a *strict local maximum*, resp. *minimum* at a point $x \in [a,b]$ if there is an $\varepsilon > 0$ such that for each $y \in [a,b] \cap ((x-\varepsilon, x+\varepsilon) \setminus \{x\})$ we have $f(y) < f(x)$, resp. $f(y) > f(x)$. In this case we say that $f(x)$ is a locally extremal value. We set $C_{\mathrm{extrem}}(f) = \{y \in [a,b]\colon y$ is a locally extremal value$\}$ and $C_{\mathrm{inter}}(f) = \{y \in [a,b]\colon f^{-1}(y)$ contains an interval$\}$.

The following lemma is well known.

LEMMA 2.1. *Let $f \in C([a,b])$. Then the set $C_{\mathrm{inter}}(f) \cup C_{\mathrm{extrem}}(f)$ is countable.*

Proof. Obviously $C_{\mathrm{inter}}(f)$ is countable. Moreover, there is a countable set of points in $[a,b]$ where a map f attains its strict local extreme [Br], hence also $C_{\mathrm{extrem}}(f)$ is countable. ∎

Let $\mathcal{J} = \{J_\alpha\}_\alpha$ and $\mathcal{K} = \{K_\beta\}_\beta$ be two systems of open subintervals of (a,b). Then $\mathcal{K}$ is said to be *finer* than $\mathcal{J}$ if every K_β is contained in some J_α. In what follows by a countable set we also mean a finite one.

LEMMA 2.2. (i) *For any $T \subset \mathbb{R}$ the set $\{x \in T\colon x$ is a one-sided limit point of $T\}$ is countable.*

(ii) *Let $\mathcal{J} = \{J_\alpha\}_\alpha$ be a system of open subintervals of (a,b) for which $(a,b) \setminus \bigcup_\alpha J_\alpha$ is countable. There is a countable system $\mathcal{K} = \{K_n\}_n$ of pairwise disjoint open subintervals of (a,b) that is finer than $\mathcal{J}$ and such that $(a,b) \setminus \bigcup_n K_n$ is countable.*

Proof. Conclusion (i) is clear.

Let us prove (ii). Since each J_α can be expressed as an increasing union of open intervals with rational endpoints, there is a countable system $\mathcal{L} = \{L_n\}_{n\in\mathbb{N}}$ of open intervals which is finer than $\mathcal{J}$ and such that $(a,b) \setminus \bigcup_{n\in\mathbb{N}} L_n = (a,b) \setminus \bigcup_\alpha J_\alpha$. To construct $\mathcal{K}$, in the first step we put $K_1 = L_1$. Suppose we have already defined open intervals $K_1, \dots, K_l$ in $m-1$ steps; then the new open intervals from $\mathcal{K}$ given by the mth step are the nonempty connected components of $L_m \setminus \overline{\bigcup_{i=1}^l K_i}$. Now the reader can verify that the resulting countable system $\mathcal{K}$ satisfies (ii). ∎

As usual, for $y \in [a,b]$ by a left neighbourhood of y in the relative topology we mean any set containing an interval $(y-\delta, y] \cap [a,b]$ with some δ positive; right and two-sided neighbourhoods are defined analogously.

DEFINITION. Let $f \in C([a,b])$. We say that $x \in [a,b]$ is *left regular* if for each two-sided neighbourhood $U(x)$ of x the set $f(U(x))$ is a left neighbourhood of $f(x)$; a *right regular*, resp. *regular point* is defined analogously. We denote the corresponding sets of regular points by $R_{\mathrm{lreg}}(f), R_{\mathrm{rreg}}(f), R_{\mathrm{reg}}(f)$ respectively. Obviously, $R_{\mathrm{reg}}(f) = R_{\mathrm{lreg}}(f) \cap R_{\mathrm{rreg}}(f)$.

For $f \in C([a,b])$ we define the following sets (see Lemma 2.1):

$$\begin{aligned}
C_{\mathrm{reg}}(f) &= \{y \in (a,b) \colon \operatorname{card}(f^{-1}(y) \cap R_{\mathrm{reg}}(f)) \geq m\},\\
C_{\mathrm{rreg}}(f) &= \{y \in (a,b) \colon y \notin C_{\mathrm{reg}} \cup C_{\mathrm{extrem}} \cup C_{\mathrm{inter}}\\
&\qquad\qquad \& \operatorname{card}(f^{-1}(y) \cap R_{\mathrm{rreg}}(f)) \geq m\},\\
C_{\mathrm{lreg}}(f) &= \{y \in (a,b) \colon y \notin C_{\mathrm{rreg}} \cup C_{\mathrm{reg}} \cup C_{\mathrm{extrem}} \cup C_{\mathrm{inter}}\\
&\qquad\qquad \& \operatorname{card}(f^{-1}(y) \cap R_{\mathrm{lreg}}(f)) \geq m\}.
\end{aligned}$$

For $y \in [a,b]$ we put $T(y) = \{(t_0, \dots, t_{m-1}) \colon t_0 < \dots < t_{m-1} \;\&\; t_i \in f^{-1}(y)\} \subset [a,b]^m$ and fix a map $\phi \colon C_{\mathrm{lreg}}(f) \cup C_{\mathrm{rreg}}(f) \cup C_{\mathrm{reg}}(f) \to [a,b]^m$ satisfying

$$(*) \quad \phi(y) = (t_0, \dots, t_{m-1}) \in T(y), \quad t_i \in f^{-1}(y) \cap R_j(f) \text{ if } y \in C_j(f),\quad j \in \{\mathrm{lreg}, \mathrm{rreg}, \mathrm{reg}\}.$$

The next lemma will be important when proving our main result. We use the notation $C = C_{\mathrm{lreg}}(f) \cup C_{\mathrm{rreg}}(f) \cup C_{\mathrm{reg}}(f)$, $N_m = \{0, 1, \dots, m-1\}$; for $t \in \mathbb{R}^m$ we put $\|t\| = \min_{0 \leq i \leq m-2} |t_{i+1} - t_i|$, and for a map f, $y \in \mathbb{R}$ and $\varepsilon > 0$,

$$J(\varepsilon, y) = \begin{cases} (y-\varepsilon, y), & y \in C_{\mathrm{lreg}}(f),\\ (y, y+\varepsilon), & y \in C_{\mathrm{rreg}}(f),\\ (y-\varepsilon, y+\varepsilon), & y \in C_{\mathrm{reg}}(f). \end{cases}$$

LEMMA 2.3. *Let $f \in C([a,b])$.*

(i) *If $y \in [a,b]$ and $y \notin C_{\mathrm{inter}}(f)$ then $f^{-1}(y) \subset R_{\mathrm{lreg}}(f) \cup R_{\mathrm{rreg}}(f)$.*

(ii) *For any $y' \in C$ there is an $\varepsilon(y') > 0$ such that $J(\varepsilon(y'), y') \subset (a,b)$ and*

$$\forall y \in J(\varepsilon(y'), y')\ \exists t \in T(y)\ \forall i \in N_m: \quad t_i \in (\phi(y')_i - \delta, \phi(y')_i + \delta),$$

where $\delta = \|\phi(y')\|/100$ (see $(\star)$).

In statements (iii)–(iv) *we assume that the set $(a,b) \setminus C$ is countable.*

(iii) *There exists a countable system $\{K_n\}_n$ of pairwise disjoint open subintervals of (a,b) that is finer than $\{J(\varepsilon(y), y)\}_{y \in C}$ (see* (ii)*) and such that $[a,b] \setminus \bigcup_n K_n$ is countable. Moreover, there exists a map $\Psi\colon \mathbb{N} \to C$ such that $K_n \subset J(\varepsilon(\Psi(n)), \Psi(n))$ for each $n \in \mathbb{N}$.*

(iv) *There is a map $\psi\colon \bigcup_n K_n \to [a,b]^m$ such that if $K_n \subset J(\varepsilon(y'), y')$ where $\Psi(n) = y'$ then for each $y \in K_n$ we have $\psi(y) = t \in T(y)$ and*

$$\forall i \in N_m: \quad t_i \in (\phi(y')_i - \delta, \phi(y')_i + \delta).$$

Proof. (i) The reader can easily verify that a point $x \in [a,b]$ is not (left, right) regular if and only if f is constant on some neighbourhood of x.

Let us prove (ii) when $y' \in C_{\mathrm{lreg}}(f)$ (the other cases are similar). Since $y' \in C_{\mathrm{lreg}}(f)$, for $\phi(y') \in T(y')$ and δ defined above the set $f((\phi(y')_i - \delta, \phi(y')_i + \delta))$ is a left neighbourhood of y' for each $i \in N_m$. Now we can choose $\varepsilon(y')$ sufficiently small to satisfy

$$J(\varepsilon(y'), y') \subset \bigcap_{i=0}^{m-1} f((\phi(y')_i - \delta, \phi(y')_i + \delta)),$$

which proves (ii) for $y' \in C_{\mathrm{lreg}}(f)$.

Let us show (iii). Notice that if $(a,b) \setminus C$ is countable then so is $A = (a,b) \setminus \bigcup_{y \in C} J(\varepsilon(y), y)$. Indeed, $A \subset (A \cap C) \cup ([a,b] \setminus C)$ and $A \cap C$ is countable by Lemma 2.2(i). Now (iii) is a consequence of Lemma 2.2(ii). The existence of Ψ comes from the fact that $\{K_n\}_n$ is finer than $\{J(\varepsilon(y), y)\}_{y \in C}$.

Property (iv) is an easy consequence of (ii) and (iii). ∎

3. Properties of maps from $L^\sharp(m, [a,b])$, $m \in \{\infty, 2, 3, \ldots\}$. In what follows for $f \in C([a,b])$ we use the notation

$$B_1(f) = \{x \in [a,b]\colon f(y) \ge f(x),\ \forall y \in [a,x]\ \&\ f(x) \ge f(y),\ \forall y \in [x,b]\},$$
$$B_2(f) = \{x \in [a,b]\colon f(y) \le f(x),\ \forall y \in [a,x]\ \&\ f(x) \le f(y),\ \forall y \in [x,b]\},$$

and $B(f) = B_1(f) \cup B_2(f)$. If there is no ambiguity we often write B_i, resp. B instead of $B_i(f)$, resp. $B(f)$.

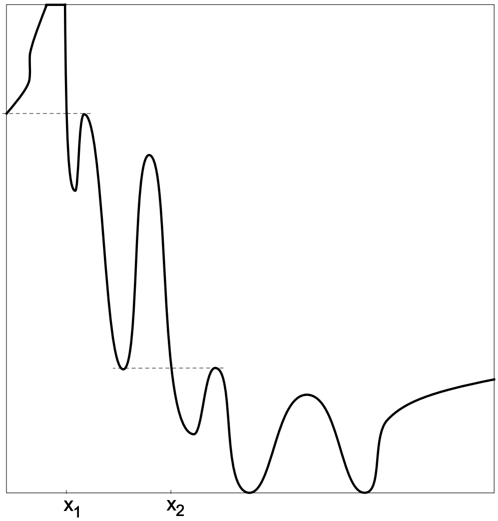

Fig 3. $f \in L(2, [a,b])$ and $x_1, x_2 \in B_1(f)$

For $f \in L(m, [a,b])$ and $y \in [a,b]$ we put $m_y = m_y(f) = \min f^{-1}(y)$ and $M_y = M_y(f) = \max f^{-1}(y)$. The closed sets $S_0 = S_0(f)$, $S_{m-1} = S_{m-1}(f)$, $S_{(0,m-1)} = S_{(0,m-1)}(f)$ are defined as

$$S_0 = \overline{\{m_y\colon y \in [a,b]\}},$$

(4)

$$S_{m-1} = \overline{\{M_y\colon y \in [a,b]\}}, \qquad S_{(0,m-1)} = S_0 \cap S_{m-1}.$$

Since for every $m \in \{\infty, 2, 3, 4, \ldots\}$

$$L(m, [a,b]) \subset L^\sharp(m, [a,b]) \subset L(2, [a,b])$$

we can apply the results developed in [B1] for maps from $L(2, [a,b])$.

LEMMA 3.1 ([B1]). *Let $f \in L(2, [a,b])$ and $S_{(0,m-1)} \neq \emptyset$.*

(i) *Either B_1 or B_2 is empty, hence $B \in \{B_1, B_2\}$.*
(ii) *$S_{(0,m-1)} \subset B \setminus \{a,b\}$.*
(iii) *The closed set B can be expressed as a union $(n \geq 1)$*

$$\{b_n\}_{n<\mathcal{K}} \cup \bigcup_{n<\mathcal{L}} [b_n^-, b_n^+],$$

where $b_n^- < b_n^+$ for each cardinal n, $1 \leq n < \mathcal{L}$; in the topology of $[a,b]$, the points a, b are not limit points of the set $\{b_n\}_{n<\mathcal{K}} \cup \bigcup_{n<\mathcal{L}} \{b_n^-, b_n^+\}$ and no point $b_m \in \{b_n\}_{n<\mathcal{K}}$ is a two-sided limit point of that set, hence $\mathcal{K}$, $\mathcal{L}$ are at most countable cardinals.
(iv) *If $\omega_f \subset B_1$ is an ω-limit set then either $\omega_f = \{p\}$ and $p \in \mathrm{Fix}(f)$, or ω_f is a periodic orbit of period 2.*
(v) *If $\omega_f \subset B_2$ is an ω-limit set then $\omega_f = \{p\}$ and $p \in \mathrm{Fix}(f)$.*

(vi) $a \in B_1$ ($a \in B_2$), *resp.* $b \in B_1$ ($b \in B_2$) *if and only if* $f(a) = b$ ($f(a) = a$), *resp.* $f(b) = a$ ($f(b) = b$).

(vii) $\operatorname{card}(B_1 \cap \operatorname{Fix}(f)) \le 1$.

LEMMA 3.2. *Let* $f \in L(2, [a, b])$.

(i) *If* $B_1(f) = B_2(f) = \emptyset$ *then* $S_{(0,m-1)} = \emptyset$.

(ii) *If* $B_1(f) \ne \emptyset \ne B_2(f)$ *then for some* $a \le a_1 < b_1 \le b$ *either* $f([a, a_1]) = \{a\}$, $f([b_1, b]) = \{a\}$, $B_2(f) = [a, a_1]$, $B_1(f) = [b_1, b]$, *or* $f([a, a_1]) = \{b\}$, $f([b_1, b]) = \{b\}$, $B_1(f) = [a, a_1]$, $B_2(f) = [b_1, b]$. *In any case* $S_{(0,m-1)} = \emptyset$.

Proof. Property (i) is a consequence of Lemma 3.1(ii). For (ii) see [B, Cor. L.2.1]. ∎

We have seen that for $f \in L^\sharp(m, [a, b])$ if $\omega_f(x) \subset B$ then $\omega_f(x)$ has a simple structure. In fact it is a periodic orbit and $\operatorname{card} \omega_f(x) \le 2$. However, the number of different ω-limit sets that are subsets of B can be infinite. Fortunately, for each such f one can consider a simplified version g of f (more precisely, a factor $([a, b], g)$ of $([a, b], f)$) which is in $L^\sharp(m, [a, b])$ again and has a very poor structure of ω-limit sets in $B(g)$. The precise statement is given in Lemma 3.3.

Now we introduce some useful notation. For intervals $J = [\alpha, \beta] \subset [a, b]$ and $K = [\gamma, \delta] \subset [a, b]$, where $a \le \alpha < \gamma \le \delta < \beta \le b$, the symbol $h(J, K)$ denotes a continuous nondecreasing piecewise affine map from $[a, b]$ onto $[a, b]$ that is constant on $[a, \alpha]$, K and $[\beta, b]$.

LEMMA 3.3. *Let* $f \in L^\sharp(m, [a, b])$ *and assume that* $\emptyset \ne B(f) \in \{B_1(f), B_2(f)\}$. *There is a map* $g \in L^\sharp(m, [a, b])$ *such that* $([a, b], g)$ *is a factor of* $([a, b], f)$ *and one of the following possibilities holds.*

(i) $B_2(g) = \emptyset$ *and if* $\omega_g(x) \subset B_1(g)$ *then either* $\omega_g(x) = \{a, b\}$ *or* $\omega_g(x) = \{p\}$ *for some* $p \in \operatorname{Fix}(g) \cap B_1(g)$.

(ii) $B_1(g) = \emptyset$ *and if* $\omega_g(x) \subset B_2(g)$ *then* $\omega_g(x) = \{p\}$ *for some* $p \in \operatorname{Fix}(g) \cap \{a, b\}$.

Proof. Without loss of generality we can assume that $B_1(f) \ne \emptyset$ and $B_2(f) = \emptyset$. We show that (i) holds in this case. Set

$$D = \{(x, f(x)) \in B_1 \times B_1 \colon f^2(x) = x < f(x)\} \cup \{(a, b)\}.$$

By Lemma 3.1(vii),(iv) there is nothing to prove if $D = \{(a, b)\}$. In this case we put $g = f$.

For $(u, v) \in D$ we can consider a uniquely determined factor $([a, b], f_u)$ of $([a, b], f)$ with a factor map $h(J, K)$, where $\alpha = u$, $\beta = v$ and $\gamma = \delta$. Now,

$$y := \max\{u \colon (u, v) \in D \;\&\; f_u \in L^\sharp(m, [a, b])\}$$

exists, since otherwise $f \notin L^\sharp(m,[a,b])$. Define

$$D_1 = \{x\colon x > y \,\&\, (x, f(x)) \in D\}.$$

If $D_1 = \emptyset$, we can put $g = f_y$. Otherwise $z := \min D_1$ exists, $z > y$ and for $(y, \widetilde{y}), (z, f(z)) \in D$ there is a factor $([a,b], g)$ of $([a,b], f)$ with a factor map $h(J,K)$, where $\alpha = y < \gamma = z < \delta = f(z) < \beta = \widetilde{y}$.

Summarizing, at least one of the following possibilities holds: (i) $g = f$, (ii) $g(a) = b$, $g(b) = a$, (iii) $g(c) = c$, where $c = h(J,K)(\gamma) \in (a,b)$. This implies that $B_2(g) = \emptyset$.

Obviously, $g \in L^\sharp(m,[a,b])$ and from Lemma 3.1(iv) and our choice of y and z satisfying $D_1 \cap (y,z) = \emptyset$ property (i) follows.

If $B_2(f) \neq \emptyset$ and $B_1(f) = \emptyset$ then the existence of g satisfying (ii) can be shown similarly. ■

For $g \in L^\sharp(m[a,b])$ consider the following four properties (A)–(D):

$$(\spadesuit)\begin{cases} \text{(A) } B_1(g) = B_2(g) = \emptyset; \\ \text{(B) } B_1(g) \neq \emptyset \neq B_2(g); \\ \text{(C) } g \text{ satisfies the conclusion of Lemma 3.3(i);} \\ \text{(D) } g \text{ satisfies the conclusion of Lemma 3.3(ii).} \end{cases}$$

We set

$$L^\star(m,[a,b]) = \{g \in L^\sharp(m,[a,b])\colon g \text{ has some of properties (A)–(D)}\}.$$

Lemma 3.4. *Let $g \in L^\star(m,[a,b])$. There is a positive integer $k_0 = k_0(g) \geq 2$ such that for any $x \in B(g)$ we have*

$$g^k(x) \in ([a,b] \setminus B(g)) \cup (\mathrm{Fix}(g) \cap B(g)) \cup \{a,b\} \quad \textit{for some } k < k_0.$$

Proof. The statement is true for g satisfying (A). For (B), use Lemma 3.2(ii). Now, suppose (C) holds. By Lemma 3.1(iii) the endpoints a, b are not limit points of $\{b_n\}_{n<\mathcal{K}} \cup \bigcup_{n<\mathcal{L}} \{b_n^-, b_n^+\}$. By the same lemma, if $\mathrm{Fix}(g) \cap B_1(g) \neq \emptyset$ then no point in this set is a two-sided limit point of $B_1(g)$. Since by our assumption $B_1(g)$ contains no other ω-limit set (a 2-cycle), there is a $k_0 \geq 2$ such that $B(g) \setminus ((\mathrm{Fix}(g) \cap B(g)) \cup \{a,b\})$ contains at most k_0 consecutive iterates of any point of $B(g)$. The case when g satisfies (D) can be verified similarly. ■

The next lemma uses the notation introduced in Section 2 before Lemma 2.3.

Lemma 3.5. *Let $g \in L^\star(m,[a,b])$. Then $(a,b) \setminus (C_{\mathrm{lreg}}(f) \cup C_{\mathrm{rreg}}(f) \cup C_{\mathrm{reg}}(f))$ is countable.*

Proof. Lemma 2.1 implies that it is sufficient to show

$$(a,b) \setminus (C_{\mathrm{lreg}}(f) \cup C_{\mathrm{rreg}}(f) \cup C_{\mathrm{reg}}(f)) \subset C_{\mathrm{inter}}(f) \cup C_{\mathrm{extrem}}(f) \cup L,$$

where L is the countable set given in $(1^\sharp_m)$ of the introduction. Take $y \in (a,b) \setminus (C_{\rm lreg}(f) \cup C_{\rm rreg}(f) \cup C_{\rm reg}(f))$ and suppose that $y \notin C_{\rm inter}(f) \cup L$. Then $\operatorname{card} f^{-1}(y) \geq m$ and Lemma 2.3(i) shows that $\operatorname{card} f^{-1}(y)$ is finite. Since $y \notin C_{\rm reg}(f)$ there exists an $x \in f^{-1}(y)$ which is not regular. Since x is an isolated point of $f^{-1}(y)$ we have $f(x) = y \in C_{\rm extrem}(f)$. ■

Let $g \in L^\star(m,[a,b])$. We define closed sets $S_0 = S_0(g)$, $S_1 = S_1(g)$, $\ldots$, $S_{m-1} = S_{m-1}(g)$ as follows: S_0, S_{m-1} are as in (4). By Lemma 3.5 we can use the pairwise disjoint countable system $\{K_n\}_n$ and the map $\psi\colon \bigcup_n K_n \to [a,b]^m$ from Lemma 2.3(iii),(iv). For each $i \in \{1,\ldots,m-2\}$ we put

$$S_i = S_i(g) = \overline{\Big\{\psi(y)_i \colon y \in \bigcup_n K_n\Big\}}.$$

Also we put

$$S = S(g) = \bigcap_{i=0}^{\infty} g^{-i}(S_0 \cup S_1 \cup \cdots \cup S_{m-1}). \tag{5}$$

The reader can verify that since $g \in L^\star(m,[a,b])$, by Lemmas 3.5 and 2.3 the sets S_i, $i \in N_m$, satisfy $[a,b] = \bigcap_{i=0}^{m-1} g(S_i) \supset \bigcup_{i=0}^{m-1} S_i$. There are a finite number of nontrivial intersections of elements of $\mathcal{H} = \{S_0, S_1, \ldots, S_{m-1}\}$, i.e., of sets

$$S_{(i(1),\ldots,i(q))} = \bigcap_{j=1}^{q} S_{i(j)}, \qquad 0 \leq i(1) < \cdots < i(q) \leq k-1 \;\&\; 2 \leq q \leq k.$$

We define the *kernel* of $\mathcal{H}$ by $\operatorname{Ker}\mathcal{H} = \bigcup_{i(1)\neq i(2)} S_{(i(1),i(2))}$, and the *center* of $\mathcal{H}$ by $\operatorname{Cen}\mathcal{H} = \bigcap_{i=0}^{m-1} S_i$. Clearly, both $\operatorname{Ker}\mathcal{H}$ and $\operatorname{Cen}\mathcal{H}$ are closed.

LEMMA 3.6. *Let $g \in L^\star(m,[a,b])$, $\mathcal{H} = \{S_0, S_1, \ldots, S_{m-1}\}$, and* $\operatorname{Ker}\mathcal{H}$ *be as above. Then $g(\operatorname{Ker}\mathcal{H})$ is countable.*

Proof. By our construction of $S_0, \ldots, S_{m-1}$, if $x \in S_{(i,j)}$ for $i \neq j$ then $g(x) \in [a,b] \setminus \bigcup_n K_n$, which is a countable set by Lemma 2.3(iii). ■

4. The proof of the main result. As before, for $g \in L^\star(m,[a,b])$ we consider the closed sets $S_0(g), \ldots, S_{m-1}(g)$ and also the set $S = S(g)$ given by (5). If $x \in S$ then by its *itinerary* with respect to $S_0, S_1, \ldots, S_{m-1}$ we mean any $\omega \in \Omega_m$ such that $g^i(x) \in S_{\omega_i}$ for $i \in \mathbb{N} \cup \{0\}$. For $M \subset S$ we denote by $\Omega(M)$ the least closed σ-invariant subset of Ω_m that contains all possible itineraries of points of M with respect to $S_0, S_1, \ldots, S_{m-1}$. In particular, if $M = \operatorname{Fix}(f) \cap \operatorname{Cen}\mathcal{H} \neq \emptyset$ then $\Omega(M) = \Omega_m$, hence $\operatorname{ent}(\sigma, \Omega(M)) = \log m$.

For $g \in L^\star(m,[a,b])$ we fix the value $k_0 = k_0(g) \geq 2$ given by Lemma 3.4. Here is the key lemma:

LEMMA 4.1. *Let* $g \in L^\star(m,[a,b])$. *If* $M \subset S$ *is minimal and* $M \neq \mathrm{Fix}(g) \cap \mathrm{Cen}\,\mathcal{H}$ *then*

$$\text{(6)} \qquad \mathrm{ent}(\sigma, \Omega(M)) \le \max\Big(\mathrm{ent}(g,M),\ \frac{1}{k_0}\log(m-1) + \frac{k_0-1}{k_0}\log m\Big).$$

Proof. Put $X = \{(x,\omega) \colon x \in M \ \&\ g^i(x) \in S_{\omega_i}$ for each $i \in \mathbb{N} \cup \{0\}\}$. The map $G = g \times \sigma$ defined by $G(x,\omega) = (g(x), \sigma(\omega))$ is continuous on the compact metric space X (with respect to the product metric). Moreover, the dynamical system (M,g), resp. $(\Omega(M),\sigma)$ is a factor of (X,G) given by the (factor map) projection $\Pi_1 \colon X \to M$, resp. $\Pi_2 \colon X \to \Omega(M)$. Using Theorem 1.1 we see that

$$\text{(7)} \qquad \mathrm{ent}(\sigma, \Omega(M)) \le \mathrm{ent}(G) \le \mathrm{ent}(g,M) + \sup_{x \in M} \mathrm{ent}(G, \Pi_1^{-1}(\{x\})).$$

Moreover, $\Lambda_x = \Pi_2(\Pi_1^{-1}(\{x\}))$ is a closed subset of Ω_m whenever $x \in M$. By (2) we have

$$\text{(8)} \qquad \mathrm{ent}(G, \Pi_1^{-1}(\{x\})) = \lim_n \frac{1}{n} \log \mathrm{card}\, \Lambda_x(n).$$

Concerning the relationship of the sets M, $\mathrm{Cen}\,\mathcal{H}$, $\mathrm{Ker}\,\mathcal{H}$ we consider several possibilities (see Lemma 1.2).

CASE I: *M is a cycle.* Then $\mathrm{ent}(g,M) = 0$ and to prove (6) we need to verify that

$$\mathrm{ent}(\sigma, \Omega(M)) \le \frac{1}{k_0}\log(m-1) + \frac{k_0-1}{k_0}\log m.$$

CASE I(a): $M \cap \mathrm{Cen}\,\mathcal{H} = \emptyset$. This is true if g satisfies (A) or (B) of (♠) (see Lemma 3.2). Our assumption implies that for each $x \in M$ and positive integer n we have $\mathrm{card}\,\Lambda_x(n) \le (m-1)^n$, hence (8) yields $\mathrm{ent}(G, \Pi_1^{-1}(\{x\})) \le \log(m-1)$. Now the property (6) is a consequence of (7) and of the inequality $\log(m-1) \le \frac{1}{k_0}\log(m-1) + \frac{k_0-1}{k_0}\log m$.

CASE I(b): $M \cap \mathrm{Cen}\,\mathcal{H} \neq \emptyset$. Then g satisfies (C) or (D) of (♠). Moreover, $\emptyset \neq \mathrm{Cen}\,\mathcal{H} \subset S_{(0,m-1)} \subset B \setminus \{a,b\}$ by Lemma 3.1(ii). Since $M \neq \mathrm{Fix}(g) \cap \mathrm{Cen}\,\mathcal{H}$, using Lemma 3.3 we obtain $M \setminus B \neq \emptyset$. By Lemma 3.4, for each $n \in \mathbb{N}$,

$$\mathrm{card}\,\Omega(M)(n) \le \mathrm{card}\,M \cdot (m-1)^{n/k_0}\ m^{n-n/k_0},$$

hence $\mathrm{ent}(\sigma, \Omega(M)) \le \frac{1}{k_0}\log(m-1) + \frac{k_0-1}{k_0}\log m$ by (2). Thus, (6) is true in this case.

CASE II: *M is infinite.* In this case we show that $\mathrm{ent}(G, \Pi_1^{-1}(\{x\})) = 0$ for each $x \in M$. Then from (7) we will obtain $\mathrm{ent}(\sigma, \Omega(M)) \le \mathrm{ent}(g,M)$, proving (6).

Fix $x \in M$, put $C = M \cap \operatorname{Ker}\mathcal{H}$ and $\widetilde{C} = M \cap g(\operatorname{Ker}\mathcal{H})$, and set, as in the proof of Lemma 1.2(iv), $C(n,x) = \operatorname{card}\{0 \le i \le n-1\colon g^i(x) \in C\}$ and $\widetilde{C}(n,x) = \operatorname{card}\{0 \le i \le n-1\colon g^i(x) \in \widetilde{C}\}$. Clearly $C(n,x) \le \widetilde{C}(n+1,x)$ for each n. If $s(n,\varepsilon) = s(n,\varepsilon,\Pi_1^{-1}(\{x\}))$ denotes the maximal cardinality of an (n,ε)-separated subset of $\Pi_1^{-1}(\{x\})$ (with respect to G), by the definition of $\operatorname{Ker}\mathcal{H}$ we have $s(n,\varepsilon) \le m^{C(n,x)}$ for any sufficiently small ε. It follows from Lemmas 3.6 and 1.2(iv) that

$$\limsup_{n\to\infty} \frac{1}{n}\log s(n,\varepsilon) \le \lim_{n\to\infty} \frac{1}{n}\log m^{C(n,x)} \le \lim_{n\to\infty}\frac{1}{n}\log m^{\widetilde{C}(n+1,x)} = 0,$$

hence $\operatorname{ent}(G,\Pi_1^{-1}(\{x\})) = 0$. ∎

Corollary 4.2. *Under the assumptions of Lemma* 4.1,

$$\operatorname{ent}(\sigma,\Omega(M)) \le \max\Big(\operatorname{ent}(g),\ \frac{1}{k_0}\log(m-1) + \frac{k_0-1}{k_0}\log m\Big).$$

Proof. By the definition, $\operatorname{ent}(g,M) \le \operatorname{ent}(g)$. Now apply Lemma 4.1. ∎

As before, we use the notation $N_m = \{0,1,\dots,m-1\}$.

Definition. Let $\Omega \subset \Omega_m$ and $j,k \in \mathbb{N}$, $j \le k$. We say that $\omega(k) \in \Omega(k)$ *contains* $\omega = (\omega_0,\dots,\omega_{j-1}) \in N_m^j$ if for some $l \in \{0,\dots,k-j\}$ and each $i \in \{0,\dots,j-1\}$,

$$\omega(k)_{l+i} = \omega_i.$$

Definition. Let $g \in L^\star(m,[a,b])$. We will say that $\omega = (\omega_0,\dots,\omega_{j-1}) \in N_m^j$ is a *j-itinerary* of $x \in [a,b]$ if $g^i(x) \in S_{\omega_i}(g)$ for $i \in \{0,\dots,j-1\}$. We say that a *j-itinerary of x does not exist* if $\{x,\dots,g^{j-1}(x)\} \not\subseteq S_0(g) \cup S_1(g) \cup \dots \cup S_{m-1}(g)$.

Combining Lemma 4.1 and Corollary 4.2 with the results of Sections 1 and 2 we now obtain the main result of this paper.

Theorem 4.3. *Let $f \in L^\sharp(m,[a,b])$. Then the topological entropy of f is greater than or equal to* $\log m$. *In particular, this is true for any map from* $L(m,[a,b]) \subset L^\sharp(m,[a,b])$.

Proof. Let $f \in L^\sharp(m,[a,b])$. There is nothing to prove if $\operatorname{Ker}\mathcal{H} = \emptyset$. In this case $S_0(f), S_1(f), \dots, S_{m-1}(f)\}$ form an m-horseshoe and so $\operatorname{ent}(f) \ge \log m$ by Proposition 0.2.

Now, suppose $\operatorname{Ker}\mathcal{H} \ne \emptyset$. By Lemmas 3.3 and 3.4, instead of f we can consider the map $g \in L^\star(m,[a,b])$ such that $\operatorname{ent}(f) \ge \operatorname{ent}(g)$. Obviously it is sufficient to prove $\operatorname{ent}(g) \ge \log m$.

In what follows all sets are taken with respect to g. The inequality $\operatorname{ent}(g) \ge \log m$ is clear if $\operatorname{Ker}\mathcal{H} = \emptyset$ since in this case the sets $S_0, S_1, \dots, S_{m-1}$ form an m-horseshoe.

Suppose to the contrary that $\operatorname{Ker}\mathcal{H} \neq \emptyset$ and $\operatorname{ent}(g) < \log m$. Let $k_0 \geq 2$ be as in Lemma 3.4. Using Theorem 1.3 we can consider a minimal set Γ in Ω_m such that

$$\operatorname{ent}(\sigma, \Gamma) > \max\left(\operatorname{ent}(g), \frac{1}{k_0}\log(m-1) + \frac{k_0-1}{k_0}\log m\right). \tag{9}$$

Lemma 1.2(i) shows that for each $x \in \operatorname{Ker}\mathcal{H}$ there is a minimal set $M(x)$ in $[a,b]$ such that $M(x) \subset \omega_g(x)$.

Put $B_S = \{x \in S \cap \operatorname{Ker}\mathcal{H}\colon M(x) \neq \operatorname{Fix}(g) \cap \operatorname{Cen}\mathcal{H}\}$ (see (5) for S). We deduce from Lemma 4.1 that (6) is true for $M(x)$ and $\operatorname{ent}(\sigma, \Omega(M(x)))$ when $x \in B_S$. Hence by the minimality of Γ, Lemma 1.4 and (9) (for $x = \infty$ see (3)),

$$\forall x \in B_S \cup \{\infty\}\colon \quad \Omega(M(x)) \cap \Gamma = \emptyset.$$

Since Γ is σ-invariant we even see that for each $x \in B_S \cup \{\infty\}$ there is $n(x) \in \mathbb{N}$ such that

$$\text{no } \gamma \in \Gamma(m) \text{ contains } \omega(n(x)) \tag{10}$$

whenever $m \geq n(x)$ and $\omega(n(x)) \in \Omega(M(x)(n(x))$.

Now we define an open cover $\{U(x)\}_{x\in\operatorname{Ker}\mathcal{H}}$ of $\operatorname{Ker}\mathcal{H}$ in three steps:

(i) If $x \in (\operatorname{Ker}\mathcal{H}) \setminus S$ and $g^{m(x)}(x) \notin S_0 \cup S_1 \cup \cdots \cup S_{m-1}$, choose $U(x)$ in such a way that $g^{m(x)}(U(x)) \cap (S_0 \cup S_1 \cup \cdots \cup S_{m-1}) = \emptyset$.

(ii) If $x \in B_S$ then we can consider $m(x) \in \mathbb{N}$ such that for any itinerary ω of x, $\omega(m(x))$ contains some element of $\Omega(M(x))(n(x))$; now, using the continuity of g, choose a neighbourhood $U(x)$ of x such that for any $y \in U(x)$ either the $m(x)$-itinerary of y does not exist or for any itinerary ω of y, $\omega(m(x))$ contains some element of $\Omega(M(x))(n(x))$.

(iii) Let $x \in S \cap \operatorname{Ker}\mathcal{H}$ be such that $M(x) = \operatorname{Fix}(g) \cap \operatorname{Cen}\mathcal{H} = \{p\}$. Since $\operatorname{Cen}\mathcal{H} \subset S_{(0,m-1)}$, from Lemmas 3.1(ii) and 3.3(i) we get $p \in B_1 \cap (a,b)$. We know that $\operatorname{card} g^{-1}(p) \geq 2$. Let $z \in g^{-1}(p) \setminus \{p\}$. Using the definition (4) of S_0, S_{m-1} the reader can verify that if $z < p$, resp. $z > p$ then for some small positive η we have $S_0 \cap (p-\eta, p) = \emptyset$, resp. $S_{m-1} \cap (p, p+\eta) = \emptyset$. Therefore we can consider $m(x) \in \mathbb{N}$ and $U(x)$ such that for any $y \in U(x)$ either $g^i(y) = p$ for some $i \leq m(x)$, or the $m(x)$-itinerary of y does not exist, or for any itinerary ω of y, $\omega(m(x))$ contains some element of $\Omega(M(\infty))(n(\infty))$.

Obviously we have found the pairs $U(x), m(x)$, where $\{U(x)\}_{x\in\operatorname{Ker}\mathcal{H}}$ is an open cover of the compact set $\operatorname{Ker}\mathcal{H}$; let $\{U(x_1), \ldots, U(x_k)\}$ be its finite subcover, and put

$$k^\star = \max\{m(x_1), \ldots, m(x_k)\}.$$

To finish the proof we define

$$R_i = S_i \setminus (\operatorname{Fix}(g) \cap \operatorname{Cen}\mathcal{H}), \qquad i \in N_m.$$

Since $\bigcap_{i=0}^{m-1} g(R_i) \supset \bigcup_{i=0}^{m-1} R_i$, for each $l \in \mathbb{N}$ and $\gamma \in \Gamma(l)$ there is $x = x(\gamma) \in \bigcup_{i=0}^{m-1} R_i$ such that for each $i \in N_l$ we have

$$g^i(x) \in R_{\gamma_i}, \qquad g^i(x) \notin \mathrm{Fix}(g) \cap \mathrm{Cen}\,\mathcal{H}. \tag{11}$$

It is clear that the sets $T_i = R_i \setminus \bigcup_{j=1}^{k} U(x_j)$, $i \in N_m$, are closed. Moreover,

$$\delta = \min\{\mathrm{dist}(T_i, T_j)\colon i \neq j\} > 0.$$

Suppose that for some $l > k^\star$, $\gamma \in \Gamma(l)$, $x(\gamma)$ and $i \in \{0, \ldots, l-1-k^\star\}$ we have $g^i(x(\gamma)) \in U(x_j)$. Then by definition of $\{U(x)\}_{x\in \mathrm{Ker}\,\mathcal{H}}$ either the $k^\star$-itinerary of $g^i(x(\gamma))$ does not exist, or γ contains some element of $\Omega(M(x_j))(n(x_j))$, which is impossible by (11) and (10). This implies that for any $l > k^\star$, $\gamma \in \Gamma(l)$ and $x(\gamma)$ we have

$$\{g^i(x(\gamma))\}_{i=0}^{l-1-k^\star} \subset T_0 \cup T_1 \cup \cdots \cup T_{m-1}.$$

Now, estimating the topological entropy of g we have, for some $\varepsilon < \delta$ and each $l > k^\star$,

$$s(l-1-k^\star, \varepsilon, [a,b]) \geq \mathrm{card}\,\Gamma(l)/m^{k^\star},$$

hence by (9) and (2), $\mathrm{ent}(g) \geq \mathrm{ent}(\sigma, \Gamma) > \mathrm{ent}(g)$—a contradiction. The proof of our theorem is finished. ∎

5. The topological entropy of a Besicovitch function. For the Lebesgue measure λ we define

$$C(\lambda) = \{f \in C([0,1])\colon \forall \text{ Borel } A \subset [0,1]\colon \lambda(A) = \lambda(f^{-1}(A))\}.$$

By a *Besicovitch function* we mean a function which has a unilateral derivative (finite or infinite) at no point. In [B2], [B3] we have constructed Besicovitch functions in $C(\lambda)$. Now we show that such maps have an infinite topological entropy. First, let us repeat the construction from [B2]. Also we correct an inaccuracy there (compare the definition of ϕ).

Construction. Let $k > 4$. Set

$$D = [0, 1/2] \setminus L, \quad \text{where} \quad L = \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{2^{m-1}} r_{m,p},$$

and the open intervals $r_{m,p} = (a_{m,p}, b_{m,p})$ are constructed as follows:

(α) $d_{1,1} = [0, 1/2]$, $r_{1,1} \subset d_{1,1}$, $\lambda(r_{1,1}) = 1/2k$, $b_{1,1}$ is the centre of $d_{1,1}$;

(β) if $d_{n,1}, \ldots, d_{n,2^{n-1}}$ are the intervals of $[0,1/2] \setminus \bigcup_{q=1}^{n-1} \bigcup_{p=1}^{2^{q-1}} r_{q,p}$ for $n > 1$ (from left to right), then $r_{n,p} \subset d_{n,p}$, $b_{n,p}$ is the centre of $d_{n,p}$ and $\lambda(r_{n,p}) = 1/2k^n$.

Obviously, $\lambda(L) = 1/2(k-2)$ and $\lambda(D) = (k-3)/2(k-2)$.

Let ϕ: $[0,1/2]\to[0,1]$ be a nondecreasing continuous function such that $\phi(0)=0$, $\phi(1/2)=1$, ϕ is constant on every interval $r_{m,p}$, and $\phi(r_{m,p})=\{(2p-1)/2^m\}$. Define a function p: $[0,1]\to[0,1]$ by

$$p(x)=\begin{cases}\phi(x), & x\in[0,1/2],\\ \phi(1-x), & x\in[1/2,1].\end{cases}$$

The function p and the interval $[0,1]$ form the well-known step triangle [P].

The above procedure will be called the construction of a step triangle with *base* $[0,1]$, *height* 1 and *parameter* k.

We have seen that the base $[0,1]$ lies below the vertex $(1/2,1)$—in such a case we say that the step triangle is *positively oriented.* The set $\{(x,p(x))\colon x\in[0,1/2]\}$, resp. $\{(x,p(x))\colon x\in[1/2,1]\}$ is the left, resp. right side of triangle. Further, put $u_y=\{(x,y)\colon x\in[0,1]\}$ and let $g(f)$ be the graph of the function f.

Now, we can construct a function f as follows:

(c_0) construct a positively oriented step triangle with base $[0,1]$, height 1 and parameter k; the sides of the step triangle define a function f_0;

(c_n) for $n>0$, construct step triangles (positively or negatively oriented) whose bases are intervals of the set $\bigcup_{p=1}^{2^{n-1}}u_{2p-1/2^n}\cap g(f_{n-1})$, height $1/2^n$ and parameter k; the constructed triangles are placed inside the bigger triangle, with bases on its sides; the union of sides of all triangles constructed so far defines a function f_n.

Finally, put $f=\lim_{n\to\infty}f_n$ (obviously $\varrho(f_{n-1},f_n)=1/2^n$).

Theorem 5.1 ([B2], [B3]). *$f\in C(\lambda)$ and f is a Besicovitch function.*

In order to illustrate how our Theorem 4.3 can be used we will prove that $\operatorname{ent}(f)=\infty$. Since $\operatorname{ent}(f^n)=n\operatorname{ent}(f)$ for each $n\in\mathbb{N}$, by Theorem 4.3 it is sufficient to show that

Theorem 5.2. *$f^2\in L^\sharp(\infty,[0,1])$.*

Proof. Since $f(0)=f(1)=0$ and $f(1/2)=1$ we have $f^2\in L(2,[0,1])$. Put $M=\{p/2^n\colon n\in\mathbb{N}\cup\{0\},\ p\in\{0,1,\ldots,2^n\}\}$ and suppose that $y\in[0,1]\setminus M$. We will show that $\operatorname{card} f^{-1}(y)=\infty$. Otherwise there would be the smallest step triangle T such that u_y has a nonempty intersection with its sides. Without loss of generality we can assume that this step triangle T has a positive orientation, it is of height $1/2^n$ and has its base in $u_{(2p-1)/2^n}$. Since $y\notin M$ there is a unique positive integer m such that for $L=\sum_{i=1}^m 1/2^{n+i}$ we have

$$y\in\left(\frac{2p-1}{2^n}+L-\frac{1}{2^{n+m}},\frac{2p-1}{2^n}+L\right).$$

Then from our construction it follows that u_y has a nonempty intersection with sides of a negatively oriented step triangle (placed inside T and with base on a side of T) of height $1/2^{n+m}$ and with base in $u_{(2p-1)/2^n+L}$. This is a contradiction.

Now, from $(f^2)^{-1} = f^{-1}(f^{-1})$ we obtain

$$\mathrm{card}(f^2)^{-1}(y) \begin{cases} \geq 2, & y \in M, \\ = \infty, & y \notin M. \end{cases}$$

Since M is countable the conclusion $f^2 \in L^\sharp(\infty, [0,1])$ follows. ∎

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