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Operator-valued n-harmonic measure in the polydisc

by

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Abstract. An operator-valued multi-variable Poisson type integral is studied. In Section 2 we obtain a new equivalent condition for the existence of a so-called regular unitary dilation of an n-tuple $T = (T_1, \ldots, T_n)$ of commuting contractions. Our development in Section 2 also contains a new proof of the classical dilation result of S. Brehmer, B. Sz.-Nagy and I. Halperin. In Section 3 we turn to the boundary behavior of this operator-valued Poisson integral. The results obtained in this section improve upon an earlier result proved by R. E. Curto and F.-H. Vasilescu in [3].

0. Introduction. Let $\mathcal{H}$ be a (not necessarily separable) Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$. The space $\mathcal{L}(\mathcal{H})$ is normed by the *operator norm*, that is, $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$ for $T \in \mathcal{L}(\mathcal{H})$, and the operator inequality $T \geq 0$ in $\mathcal{L}(\mathcal{H})$ means that $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. A *contraction* is an operator $T \in \mathcal{L}(\mathcal{H})$ such that $\|T\| \leq 1$. For an n-tuple $T = (T_1, \ldots, T_n)$ of commuting operators in $\mathcal{L}(\mathcal{H})$ we denote the associated Brehmer quantities by

$$\Delta_T^{\varepsilon} = \sum_{0\leq\alpha\leq\varepsilon} (-1)^{|\alpha|} T^{*\alpha} T^{\alpha}, \tag{0.1}$$

where $T^* = (T_1^*, \ldots, T_n^*)$ and $\varepsilon \geq 0$ is a multi-index. Standard multi-index notation is used. For a multi-index $\alpha \in \mathbb{Z}^n$ we write

$$\alpha^+ = (\max(\alpha_1, 0), \ldots, \max(\alpha_n, 0)), \quad \alpha^- = -(\min(\alpha_1, 0), \ldots, \min(\alpha_n, 0)).$$

In particular, $\alpha = \alpha^+ - \alpha^-$ and $\alpha^+, \alpha^- \geq 0$.

In Theorem 2.1 we prove that with every n-tuple $T = (T_1, \ldots, T_n)$ of commuting contractions in $\mathcal{L}(\mathcal{H})$ satisfying the additional positivity condition (A) below, one can associate a positive $\mathcal{L}(\mathcal{H})$-valued operator measure ω_T on the unit n-torus $\mathbb{T}^n$ such that $\widehat{\omega}_T(\alpha) = T^{*\alpha^+} T^{\alpha^-}$ for $\alpha \in \mathbb{Z}^n$, where

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$\widehat{\omega}_T(\alpha) = \int e^{-i\alpha\cdot\theta}\, d\omega_T(e^{i\theta})$ is the αth Fourier coefficient of ω_T. For definitions of measure-theoretic concepts the reader is referred to Section 1. Let us note in passing that if such an operator measure ω_T exists, then it is uniquely determined. This is clear by uniqueness of Fourier coefficients. When the T_j's are strict contractions, that is, $\|T_j\| < 1$ for $1 \leq j \leq n$, the operator measure ω_T is given by the formula

$$d\omega_T(e^{i\theta}) = P(T; e^{i\theta})d\sigma(e^{i\theta}),$$

where $d\sigma$ is the normalized Lebesgue measure on $\mathbb{T}^n$ and $P(T; e^{i\theta})$ is the $\mathcal{L}(\mathcal{H})$-valued n-harmonic Poisson kernel defined by

$$P(T; e^{i\theta}) = \prod_{j=1}^{n}(I - e^{i\theta_j}T_j^*)^{-1} \cdot \varDelta_T^{(1,\dots,1)} \cdot \prod_{j=1}^{n}(I - e^{-i\theta_j}T_j)^{-1},$$

where $e^{i\theta} = (e^{i\theta_1}, \dots, e^{i\theta_n}) \in \mathbb{T}^n$ and $\varDelta_T^{\varepsilon}$ is given by (0.1). In Theorem 2.2 we turn to the converse of Theorem 2.1 and prove that if such a positive operator measure ω_T exists, then condition (B) below holds.

The development in Section 2 may be regarded as analogous to the existence theory of so-called regular unitary dilations developed by, most notably, S. Brehmer [2], B. Sz.-Nagy [14, 15] and I. Halperin [6, 7] in the early 1960's and more recently considered by R. E. Curto and F.-H. Vasilescu [3, 4]. A standard reference for this material is Section I.9 in the book [16] by B. Sz.-Nagy and C. Foiaş. In particular, our development in Section 2 contains a new equivalent condition for the existence of a regular unitary dilation of an n-tuple $T = (T_1, \dots, T_n)$ of commuting contractions in $\mathcal{L}(\mathcal{H})$. This condition reads as follows:

(A) There exists a sequence $r_k = (r_{k1}, \dots, r_{kn})$, $0 \leq r_{kj} < 1$, such that $r_k \to (1, \dots, 1)$ as $k \to \infty$ and $\varDelta^{(1,\dots,1)}_{(r_{k1}T_1,\dots,r_{kn}T_n)} \geq 0$ in $\mathcal{L}(\mathcal{H})$ for all k.

By the work of S. Brehmer, B. Sz.-Nagy and I. Halperin, an n-tuple T of commuting operators in $\mathcal{L}(\mathcal{H})$ has a regular unitary dilation if and only if the following condition holds:

(B) $\varDelta_T^{\varepsilon} \geq 0$ in $\mathcal{L}(\mathcal{H})$ for all $0 \leq \varepsilon \leq (1, \dots, 1)$.

Note that for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_k = \delta_{j,k}$, the inequality $\varDelta_T^{\varepsilon} \geq 0$ means that T_j is a contraction. It is straightforward to prove that (B) implies (A), see Proposition 2.1. The converse implication that (A) implies (B), valid when T is an n-tuple of commuting contractions, follows by Theorems 2.1 and 2.2 and our proof of this requires some more work. The condition using (A) seems formally weaker than (B). Note however that when T is a 2-tuple of contractions, (A) reduces to $\varDelta_T^{(1,1)} \geq 0$. Section 2 also contains a new proof of the Brehmer–Sz.-Nagy–Halperin result referred to above.

For an n-tuple $T=(T_1,\dots,T_n)$ of commuting strict contractions in $\mathcal{L}(\mathcal{H})$ and $\varphi\in C(\mathbb{T}^n)$ we consider the *$\mathcal{L}(\mathcal{H})$-valued n-harmonic Poisson integral* defined by the formula

$$P[\varphi](T)=\int_{\mathbb{T}^n} P(T;e^{i\theta})\varphi(e^{i\theta})\,d\sigma(e^{i\theta}),$$

where $P(T;e^{i\theta})$ is as above. Let now T be an n-tuple of commuting operators in $\mathcal{L}(\mathcal{H})$ satisfying (B). In Theorem 4.3 in [3] it is shown, under an extra structural assumption on T, that the limit $\lim_{r\to1-}P[\varphi](rT)$ exists in the strong operator topology of $\mathcal{L}(\mathcal{H})$. See also the last paragraph on page 793 in [3] where this result is announced. The purpose of Section 3 of this paper is to present some stronger results concerning this limit. In particular, we prove that

$$\lim_{\substack{r_j\to1\\0\le r_j<1}} P[\varphi](r_1T_1,\dots,r_nT_n)=\int_{\mathbb{T}^n}\varphi(e^{i\theta})\,d\omega_T(e^{i\theta})\quad\text{in }\mathcal{L}(\mathcal{H}).\tag{0.2}$$

We emphasize that in (0.2) the limit is taken in the operator norm of $\mathcal{L}(\mathcal{H})$, the n-tuple T of commuting operators in $\mathcal{L}(\mathcal{H})$ satisfies (B) (no extra structural assumption) and $\varphi\in C(\mathbb{T}^n)$ is arbitrary. In Theorems 3.1 and 3.2 we give more general results formulated in terms of operator measures of the type $d\omega_T$. We also observe some closely related summability results for the formal series $\sum\widehat{\varphi}(\alpha)T^{*\alpha^-}T^{\alpha^+}$, where

$$\widehat{\varphi}(\alpha)=\int_{\mathbb{T}^n} e^{-i\alpha\cdot\theta}\varphi(e^{i\theta})\,d\sigma(e^{i\theta})$$

denotes the αth Fourier coefficient of $\varphi\in C(\mathbb{T}^n)$. In particular, the formal series $\sum\widehat{\varphi}(\alpha)T^{*\alpha^-}T^{\alpha^+}$ is Abel summable to $\int\varphi\,d\omega_T$ in $\mathcal{L}(\mathcal{H})$.

A motivation for the study of these Poisson type integrals comes from the von Neumann inequality [11]. Indeed, by Proposition 1.1 and (1.1), we always have $\|\int\varphi\,d\omega_T\|\le\|\varphi\|_\infty$. This together with the identity $\int\varphi\,d\omega_T=\sum\widehat{\varphi}(\alpha)T^{*\alpha^-}T^{\alpha^+}$, valid for, say, $\varphi=\sum\widehat{\varphi}(\alpha)e^{i\alpha\cdot\theta}$ a trigonometric polynomial on $\mathbb{T}^n$, implies the von Neumann inequality

$$\Big\|\sum_{\alpha\in\mathbb{Z}^n}\widehat{\varphi}(\alpha)T^{*\alpha^-}T^{\alpha^+}\Big\|\le\|\varphi\|_\infty.$$

More generally, if we interpret the formal series $\sum\widehat{\varphi}(\alpha)T^{*\alpha^-}T^{\alpha^+}$ by means of Abel or Cesàro summation in $\mathcal{L}(\mathcal{H})$, then the same von Neumann inequality holds true for arbitrary $\varphi\in C(\mathbb{T}^n)$.

The $\mathcal{L}(\mathcal{H})$-valued n-harmonic Poisson kernel $P(T;e^{i\theta})$ seems to have first appeared in the paper [3] by R. E. Curto and F.-H. Vasilescu. We also remark that a similar $\mathcal{L}(\mathcal{H})$-valued $\mathcal{M}$-harmonic Poisson integral for the unit ball in $\mathbb{C}^n$ has been studied by F.-H. Vasilescu in [17]. However, the proof of the classical von Neumann inequality ($n = 1$) using the Poisson integral formula indicated in the previous paragraph is known to several people and dates back at least to the 1970's (private communication).

The proofs in this paper make use of measure theory including the F. Riesz representation theorem, some basic harmonic analysis and the operator-valued Poisson integral discussed above. This approach is different from that used by S. Brehmer, B. Sz.-Nagy and I. Halperin.

1. Measure theory. The purpose of this section is to review some facts about integration in Hilbert space. Let S be a set and $\mathfrak{S}$ a σ-algebra of subsets of S. By an $\mathcal{L}(\mathcal{H})$-valued *operator measure* on S we mean a finitely additive set function $\mu : \mathfrak{S} \to \mathcal{L}(\mathcal{H})$ such that the set functions $\mu_{x,y}$, $x, y \in \mathcal{H}$, defined by $\mu_{x,y}(E) = (\mu(E)x, y)$ for $E \in \mathfrak{S}$, are all complex measures on S. The *semi-variation* of μ, here denoted by $|\mu|$, is the set function defined by $|\mu|(E) = \sup_{\|x\|\le 1,\, \|y\|\le 1} |\mu_{x,y}|(E)$, where $|\mu_{x,y}|$ is the total variation of the complex measure $\mu_{x,y}$. An operator measure μ is said to be *positive* if $\mu(E) \ge 0$ in $\mathcal{L}(\mathcal{H})$ for every $E \in \mathfrak{S}$. A projection-valued operator measure μ with $\mu(S) = I$ is called a *spectral* measure (see §36 in [5]). In the literature, a positive operator measure μ with $\mu(S) = I$ is sometimes called a *quasi-spectral* measure or a *semi-spectral* measure.

The semi-variation of a positive operator measure is easily computed.

Proposition 1.1. *Let μ be a positive $\mathcal{L}(\mathcal{H})$-valued operator measure. Then $|\mu|(E) = \sup_{\|x\|\le 1}(\mu(E)x, x) = \|\mu(E)\|$ for every measurable set E.*

Proof. It suffices to verify that

$$\sup_{\|x\|\le 1, \|y\|\le 1} |\mu_{x,y}|(E) \le \sup_{\|x\|\le 1} (\mu(E)x, x).$$

Let $\{E_j\}$ be a finite partition of E into measurable sets and let $x, y \in \mathcal{H}$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\sum |\mu_{x,y}(E_j)| &= \sum |(\mu(E_j)x, y)| \\
&\le \sum (\mu(E_j)x, x)^{1/2} (\mu(E_j)y, y)^{1/2} \\
&\le \Big(\sum (\mu(E_j)x, x)\Big)^{1/2} \Big(\sum (\mu(E_j)y, y)\Big)^{1/2} \\
&= (\mu(E)x, x)^{1/2} (\mu(E)y, y)^{1/2}.
\end{aligned}$$

Thus, $|\mu_{x,y}|(E) = \sup_{\{E_j\}} \sum |\mu_{x,y}(E_j)| \le (\mu(E)x, x)^{1/2}(\mu(E)y, y)^{1/2}$. ∎

Next we recall the definition of the integral $\int f\, d\mu$, where f is a complex-valued measurable function and μ is an operator measure. Let $f : S \to \mathbb{C}$ be a measurable function such that the scalar integrals $\int f\, d\mu_{x,y}$, $x, y \in \mathcal{H}$, all exist in the usual Lebesgue sense. The function f is said to be *integrable* with respect to μ if $(x, y) \mapsto \int f\, d\mu_{x,y}$ is a bounded sesquilinear map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$.

It is straightforward to see that if this holds, then there exists an operator $T \in \mathcal{L}(\mathcal{H})$ (necessarily uniquely determined) such that $(Tx, y) = \int f\,d\mu_{x,y}$ for $x, y \in \mathcal{H}$. The integral $\int f\,d\mu$ is defined by $\int f\,d\mu = T$.

It is straightforward to see that if the integral $\int f\,d\mu$ exists, then there is an estimate

$$\Big\| \int_S f(s)\,d\mu(s) \Big\| \le \|f\|_\infty |\mu|(S), \tag{1.1}$$

where $\|f\|_\infty = \inf\{c > 0 : |\mu|(\{s \in S : |f(s)| > c\}) = 0\}$ is the essential supremum.

Next we observe that the integral $\int f\,d\mu$ exists if $f : S \to \mathbb{C}$ is a bounded measurable function and μ is an operator measure of finite total semi-variation, that is, $|\mu|(S) < \infty$. Indeed, it is clear that the integrals $\int f\,d\mu_{x,y}$, $x, y \in \mathcal{H}$, all exist and that we have the estimate $|\int f\,d\mu_{x,y}| \le \|f\|_\infty |\mu|(S) \|x\|\,\|y\|$.

Let us now specialize to the case when S is a locally compact Hausdorff space. In this case we take $\mathfrak{S}$ to be the σ-algebra of all Borel subsets of S. We require of μ that the $\mu_{x,y}$, $x, y \in \mathcal{H}$, are all regular complex Borel measures on S. Denote by $C_0(S)$ the space of all continuous functions on S vanishing at infinity. If μ is of finite total semi-variation, then (1.1) shows that the formula

$$\Lambda(f) = \int_S f(s)\,d\mu(s) \quad \text{for } f \in C_0(S) \tag{1.2}$$

defines a bounded linear map $\Lambda : C_0(S) \to \mathcal{L}(\mathcal{H})$ of norm less than or equal to $|\mu|(S)$. Since the $\mu_{x,y}$'s are all regular we have $\|\Lambda\| = |\mu|(S)$. Next we show that every bounded linear map $\Lambda : C_0(S) \to \mathcal{L}(\mathcal{H})$ is obtained in this way.

PROPOSITION 1.2. *Let S be a locally compact Hausdorff space and $\mathcal{H}$ a Hilbert space. Let $\Lambda : C_0(S) \to \mathcal{L}(\mathcal{H})$ be a bounded linear map. Then there exists an $\mathcal{L}(\mathcal{H})$-valued operator measure μ on S of finite total semi-variation such that* (1.2) *holds.*

Proof. For $x, y \in \mathcal{H}$, by the F. Riesz representation theorem (see Theorem 6.19 in [13]) there exists a unique complex regular Borel measure $\mu_{x,y}$ on S such that

$$(\Lambda(f)x, y) = \int_S f\,d\mu_{x,y} \quad \text{for } f \in C_0(S).$$

Clearly, the map $(x, y) \mapsto \mu_{x,y}$ is linear in x and conjugate linear in y. It is also clear that $|\mu_{x,y}|(S) \le \|\Lambda\|\,\|x\|\,\|y\|$. Let $E \subset S$ be a Borel set. It is straightforward to prove that there exists a unique operator $\mu(E) \in \mathcal{L}(\mathcal{H})$ such that $(\mu(E)x, y) = \mu_{x,y}(E)$ for $x, y \in \mathcal{H}$. It is clear that μ so defined is an $\mathcal{L}(\mathcal{H})$-valued operator measure of finite total semi-variation. By definition of the integral, $(\int f\,d\mu x, y) = \int f\,d\mu_{x,y}$. Thus, (1.2) holds. ∎

For easy reference we quote the following theorem of M. A. Naimark [9].

Theorem 1.1. *Let μ be a positive $\mathcal{L}(\mathcal{H})$-valued operator measure on S such that $\mu(S) = I$. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and an $\mathcal{L}(\mathcal{K})$-valued spectral measure E on S such that $\mu(\omega) = PE(\omega)|_{\mathcal{H}}$ for $\omega \in \mathfrak{S}$, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.*

For a proof of Theorem 1.1, apart from [9], we also refer to Section 7 in [14] or Section 6 in [8].

2. Construction of the n-harmonic measure. Let $\mathbb{D}^n$ be the unit polydisc and denote by $d\sigma$ the normalized Lebesgue measure on the unit n-torus $\mathbb{T}^n$. The *n-harmonic Poisson kernel* for $\mathbb{D}^n$ is the function defined by

$$P(z; e^{i\theta}) = \prod_{j=1}^{n} \frac{1 - |z_j|^2}{|e^{i\theta_j} - z_j|^2},$$

where $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$ and $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{T}^n$. Note that for $n = 1$ the function $P(z; e^{i\theta})$ is the usual Poisson kernel for the unit disc $\mathbb{D}$. For $r = (r_1, \ldots, r_n)$, $0 \leq r_j < 1$, we also write

$$P_r(e^{i\theta}) = P(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}; 1, \ldots, 1) = \sum_{\alpha \in \mathbb{Z}^n} r_1^{|\alpha_1|} \cdots r_n^{|\alpha_n|} e^{i\alpha\cdot\theta}.$$

An important feature is that the n-harmonic Poisson integral

$$\begin{aligned} u(z) &= P[\varphi](z) = P_r * \varphi(e^{i\theta}) \\ &= \int_{\mathbb{T}^n} P(z, e^{i\tau})\varphi(e^{i\tau})\, d\sigma(e^{i\tau}), \quad z_j = r_j e^{i\theta_j}, \end{aligned} \tag{2.1}$$

solves the n-harmonic Dirichlet problem

$$\begin{cases} \Delta_j u = 0 & \text{in } \mathbb{D}^n \ (1 \leq j \leq n), \\ u = \varphi & \text{on } \mathbb{T}^n, \end{cases}$$

where $\Delta_j = 4\bar{\partial}_j \partial_j$ is the Laplacian in the variable z_j. We note that if $\varphi \in C(\mathbb{T}^n)$, then $u = P[\varphi] \in C(\overline{\mathbb{D}}^n)$. We refer to the monograph [12] for more details.

Let $\mathcal{H}$ be a Hilbert space. For an n-tuple $T = (T_1, \ldots, T_n)$ of commuting operators in $\mathcal{L}(\mathcal{H})$ and a multi-index $\varepsilon \geq 0$ we consider the Brehmer quantity defined by

$$\Delta_T^{\varepsilon} = \sum_{0 \leq \alpha \leq \varepsilon} (-1)^{|\alpha|} T^{*\alpha} T^{\alpha}, \tag{2.2}$$

where $T^* = (T_1^*, \ldots, T_n^*)$.

PROPOSITION 2.1. *Let $0 \le r_j \le 1$ for $1 \le j \le n$. Let $T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^n$ be an n-tuple of commuting operators such that $\Delta_T^{\varepsilon} \ge 0$ in $\mathcal{L}(\mathcal{H})$ for $0 \le \varepsilon \le (1, \ldots, 1)$. Then $\Delta^{\varepsilon}_{(r_1T_1,\ldots,r_nT_n)} \ge 0$ in $\mathcal{L}(\mathcal{H})$ for $0 \le \varepsilon \le (1, \ldots, 1)$.*

Proof. It is clearly sufficient to prove that $\Delta^{\varepsilon}_{(r_1T_1,T_2,\ldots,T_n)} \ge 0$. Write $T = (T_1, T'')$ and $\alpha = (\alpha_1, \alpha'')$. In proving $\Delta^{\varepsilon}_{(r_1T_1,T'')} \ge 0$ we can clearly assume that $\varepsilon_1 = 1$. We have

$$\begin{aligned}\Delta^{\varepsilon}_{(r_1T_1,T'')} &= \sum_{0\le\alpha\le\varepsilon} (-1)^{|\alpha|}(r_1T_1)^{*\alpha_1}(T'')^{*\alpha''}(r_1T_1)^{\alpha_1}(T'')^{\alpha''}\\ &= \sum_{0\le\alpha''\le\varepsilon''} (-1)^{|\alpha''|}(T'')^{*\alpha''}(T'')^{\alpha''}\\ &\quad - \sum_{0\le\alpha''\le\varepsilon''} (-1)^{|\alpha''|}(r_1T_1)^{*}(T'')^{*\alpha''}(r_1T_1)(T'')^{\alpha''}\\ &= \Delta^{\varepsilon''}_{T''} - r_1^2T_1^*\Delta^{\varepsilon''}_{T''}T_1 \ge \Delta^{\varepsilon''}_{T''} - T_1^*\Delta^{\varepsilon''}_{T''}T_1 = \Delta^{\varepsilon}_{T} \ge 0 \quad \text{in } \mathcal{L}(\mathcal{H}),\end{aligned}$$

where we have used the fact that $\Delta^{\varepsilon''}_{T''}, \Delta^{\varepsilon}_{T} \ge 0$. ∎

Let $T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^n$ be an n-tuple of commuting operators such that the T_j's are strictly contractive, that is, $\|T_j\| < 1$ for $1 \le j \le n$. The $\mathcal{L}(\mathcal{H})$-*valued n-harmonic Poisson kernel* is defined by the formula

$$P(T; e^{i\theta}) = \prod_{j=1}^{n}(I - e^{i\theta_j}T_j^*)^{-1} \cdot \Delta_T^{(1,\ldots,1)} \cdot \prod_{j=1}^{n}(I - e^{-i\theta_j}T_j)^{-1}, \tag{2.3}$$

where $e^{i\theta} \in \mathbb{T}^n$ and Δ_T^{ε} is given by (2.2). Our first task is to compute the Fourier coefficients of $P(T;\cdot)$.

LEMMA 2.1. *Let $T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^n$ be an n-tuple of commuting strict contractions and let $P(T; e^{i\theta})$ be as in (2.3). Then*

$$P(T;\cdot)^\wedge(\alpha) = \int_{\mathbb{T}^n} P(T; e^{i\theta})e^{-i\alpha\cdot\theta}\, d\sigma(e^{i\theta}) = T^{*\alpha^+}T^{\alpha^-},$$

where

$$\alpha^+ = (\max(\alpha_1, 0), \ldots, \max(\alpha_n, 0)), \quad \alpha^- = -(\min(\alpha_1, 0), \ldots, \min(\alpha_n, 0)).$$

Proof. We compute

$$\begin{aligned}P(T; e^{i\theta}) &= \Big(\sum_{\alpha\ge0} e^{i\alpha\cdot\theta}T^{*\alpha}\Big)\Delta_T^{(1,\ldots,1)}\Big(\sum_{\alpha\ge0} e^{-i\alpha\cdot\theta}T^{\alpha}\Big)\\ &= \sum_{0\le\gamma\le(1,\ldots,1)} (-1)^{|\gamma|} \sum_{\alpha,\beta\ge0} e^{i(\alpha-\beta)\cdot\theta}T^{*(\alpha+\gamma)}T^{\beta+\gamma}.\end{aligned}$$

Thus,

$$P(T;\cdot)^\wedge(\delta) = \sum_{0\le\gamma\le(1,\ldots,1)} (-1)^{|\gamma|} \sum_{\substack{\delta=\alpha-\beta\\ \alpha,\beta\ge0}} T^{*(\alpha+\gamma)}T^{\beta+\gamma}. \tag{2.4}$$

We now note that the summation in the inner sum in (2.4) is over all pairs of multi-indices (α, β) of the form $\alpha = \delta^+ + \varepsilon$, $\beta = \delta^- + \varepsilon$, where $\varepsilon \geq 0$. Thus,

$$P(T;\cdot)^\wedge(\delta) = T^{*\delta^+}\Big(\sum_{0\leq\gamma\leq(1,\dots,1)} (-1)^{|\gamma|} \sum_{\varepsilon\geq 0} T^{*(\varepsilon+\gamma)} T^{\varepsilon+\gamma}\Big) T^{\delta^-}.$$

To complete the proof it now suffices to show that the double sum within parentheses equals I, and we do this by induction on $n \geq 1$. Let

$$s(T_1, \dots, T_n) = \sum_{0\leq\gamma\leq(1,\dots,1)} (-1)^{|\gamma|} \sum_{\varepsilon\geq 0} T^{*(\varepsilon+\gamma)} T^{\varepsilon+\gamma}.$$

Clearly, $s(T_1) = I$. For $n \geq 2$ we have

$$\begin{aligned} s(T_1, \dots, T_n) = &\sum_{\varepsilon_n=0}^{\infty} T_n^{*\varepsilon_n} s(T_1, \dots, T_{n-1}) T_n^{\varepsilon_n} \\ &- \sum_{\varepsilon_n=0}^{\infty} T_n^{*(\varepsilon_n+1)} s(T_1, \dots, T_{n-1}) T_n^{\varepsilon_n+1}. \end{aligned}$$

Thus, $s(T_1, \dots, T_{n-1}) = I$ implies $s(T_1, \dots, T_n) = I$. ■

We can now prove the main result of this section.

THEOREM 2.1. *Let $T = (T_1, \dots, T_n) \in \mathcal{L}(\mathcal{H})^n$ be an n-tuple of commuting contractions. Assume that there exists a sequence $r_k = (r_{k1}, \dots, r_{kn})$, $0 \leq r_{kj} < 1$, such that $r_k \to (1, \dots, 1)$ as $k \to \infty$ and $\Delta^{(1,\dots,1)}_{(r_{k1}T_1,\dots,r_{kn}T_n)} \geq 0$ in $\mathcal{L}(\mathcal{H})$ for all k. Then there exists a (unique) positive $\mathcal{L}(\mathcal{H})$-valued operator measure ω_T on $\mathbb{T}^n$ such that $\widehat{\omega}_T(\alpha) = T^{*\alpha^+} T^{\alpha^-}$ for $\alpha \in \mathbb{Z}^n$.*

Proof. Clearly, by uniqueness of Fourier coefficients, the operator measure ω_T is uniquely determined. We proceed to prove the existence of ω_T. Consider the linear map

$$\Lambda : \varphi \mapsto \sum_{\alpha\in\mathbb{Z}^n} \widehat{\varphi}(\alpha) T^{*\alpha^-} T^{\alpha^+} \in \mathcal{L}(\mathcal{H})$$

defined for trigonometric polynomials φ on $\mathbb{T}^n$, and with values in $\mathcal{L}(\mathcal{H})$. By assumption, the Poisson kernel $P(r_{k1}T_1, \dots, r_{kn}T_n; e^{i\theta})$ is defined and by Lemma 2.1 we have

$$\begin{aligned} \text{(2.5)} \quad &\sum_{\alpha\in\mathbb{Z}^n} \widehat{\varphi}(\alpha) r_{k1}^{|\alpha_1|} \cdots r_{kn}^{|\alpha_n|} T^{*\alpha^-} T^{\alpha^+} \\ &\qquad = \int_{\mathbb{T}^n} P(r_{k1}T_1, \dots, r_{kn}T_n; e^{i\theta}) \varphi(e^{i\theta})\, d\sigma(e^{i\theta}). \end{aligned}$$

Now Proposition 1.1 and (1.1) yield the estimate

$$\Big\| \sum_{\alpha\in\mathbb{Z}^n} \widehat{\varphi}(\alpha) r_{k1}^{|\alpha_1|}\cdots r_{kn}^{|\alpha_n|} T^{*\alpha^-}T^{\alpha^+}\Big\| \le \|\varphi\|_\infty.$$

Letting $k\to\infty$ we obtain $\|\varLambda(\varphi)\| \le \|\varphi\|_\infty$. By approximation, the map $\varLambda$ uniquely extends to a bounded linear map $\varLambda : C(\mathbb{T}^n)\to\mathcal{L}(\mathcal{H})$ of norm less than or equal to 1. By Proposition 1.2 there exists an $\mathcal{L}(\mathcal{H})$-valued operator measure ω_T on $\mathbb{T}^n$ such that $\varLambda(\varphi) = \int\varphi\, d\omega_T$. Clearly, $\widehat{\omega}_T(\alpha) = T^{*\alpha^+}T^{\alpha^-}$. Also, by (2.5) it is clear that $\varphi\ge 0$ implies $\varLambda(\varphi)\ge 0$ in $\mathcal{L}(\mathcal{H})$ for trigonometric polynomials φ. Since a general $0\le\varphi\in C(\mathbb{T}^n)$ can be uniformly approximated by non-negative trigonometric polynomials, this shows that $0\le\varphi\in C(\mathbb{T}^n)$ implies $\varLambda(\varphi)\ge 0$. Thus, the operator measure ω_T is positive. ∎

Remark 2.1. By Proposition 2.1, the positivity assumption in Theorem 2.1 holds if T satisfies the Brehmer condition $\varDelta_T^\varepsilon \ge 0$ in $\mathcal{L}(\mathcal{H})$ for $0\le\varepsilon\le(1,\ldots,1)$.

Remark 2.2. If an n-tuple T of commuting strict contractions in $\mathcal{L}(\mathcal{H})$ is such that there exists an $\mathcal{L}(\mathcal{H})$-valued operator measure ω_T on $\mathbb{T}^n$ with $\widehat{\omega}_T(\alpha) = T^{*\alpha^+}T^{\alpha^-}$ for $\alpha\in\mathbb{Z}^n$, then the relation between ω_T and $P(T;e^{i\theta})$ is given by

$$d\omega_T(e^{i\theta}) = P(T;e^{i\theta})\, d\sigma(e^{i\theta}). \tag{2.6}$$

Indeed, this is clear by uniqueness of Fourier coefficients.

Next we turn to the converse of Theorem 2.1.

Theorem 2.2. *Let $T=(T_1,\ldots,T_n)\in\mathcal{L}(\mathcal{H})^n$ be an n-tuple of commuting operators such that there exists a positive $\mathcal{L}(\mathcal{H})$-valued operator measure ω on $\mathbb{T}^n$ with $\widehat{\omega}(\alpha) = T^{*\alpha^+}T^{\alpha^-}$ for $\alpha\in\mathbb{Z}^n$. Then $\varDelta_T^\varepsilon\ge 0$ in $\mathcal{L}(\mathcal{H})$ for $0\le\varepsilon\le(1,\ldots,1)$.*

Proof. Assume first that the T_j's are all strict contractions. By Remark 2.2, formula (2.6) holds with $\omega_T = \omega$. Since $\omega\ge 0$, we have $\int P(T;e^{i\theta})\varphi(e^{i\theta})\, d\sigma(e^{i\theta}) \ge 0$ in $\mathcal{L}(\mathcal{H})$ for $0\le\varphi\in C(\mathbb{T}^n)$. Clearly, this implies that $P(T;e^{i\theta})\ge 0$. By (2.3) we have $\varDelta_T^{(1,\ldots,1)}\ge 0$ in $\mathcal{L}(\mathcal{H})$.

We now consider the general case. Since $T_j = \int e^{-i\theta_j}\, d\omega(e^{i\theta})$, the operator T_j is a contraction. We consider the convolution $P_r*\omega$, $r=(r_1,\ldots,r_n)$, $0\le r_j<1$, defined by $P_r*\omega(e^{i\theta}) = \int P_r(e^{i(\theta-\tau)})\, d\omega(e^{i\tau})$. An easy computation shows that

$$(P_r*\omega)^\wedge(\alpha) = \int_{\mathbb{T}^n} (P_r*\omega)(e^{i\theta})e^{-i\alpha\cdot\theta}\, d\sigma(e^{i\theta}) = r_1^{|\alpha_1|}\cdots r_n^{|\alpha_n|}T^{*\alpha^+}T^{\alpha^-}.$$

Thus, the sequence $(r_1T_1,\ldots,r_nT_n)^{*\alpha^+}(r_1T_1,\ldots,r_nT_n)^{\alpha^-}$, $\alpha\in\mathbb{Z}^n$, is the Fourier coefficient sequence of the positive $\mathcal{L}(\mathcal{H})$-valued operator measure

$(P_r * \omega)d\sigma$. By the first part of the proof we have $\Delta^{(1,\dots,1)}_{(r_1T_1,\dots,r_nT_n)} \geq 0$. Letting $r \to (1,\dots,1)$ we obtain $\Delta^{(1,\dots,1)}_T \geq 0$.

We now prove that $\Delta^{\varepsilon}_T \geq 0$ for an arbitrary multi-index ε with $0 \leq \varepsilon \leq (1,\dots,1)$. Let $1 \leq j_1 < \dots < j_m \leq n$ be those indices j for which $\varepsilon_j = 1$. Consider the linear map $\Lambda : C(\mathbb{T}^m) \to \mathcal{L}(\mathcal{H})$ defined by

$$\Lambda : \varphi \mapsto \int_{\mathbb{T}^n} \varphi(e^{i\theta_{j_1}},\dots,e^{i\theta_{j_m}})\, d\omega(e^{i\theta}) \in \mathcal{L}(\mathcal{H}).$$

Clearly, the map Λ is positive and is, by Proposition 1.2, induced by a positive $\mathcal{L}(\mathcal{H})$-valued operator measure λ on $\mathbb{T}^m$ in the sense that $\Lambda(\varphi) = \int \varphi\, d\lambda$. By construction we have

$$\widehat{\lambda}(\alpha) = \int_{\mathbb{T}^m} e^{-i\alpha\cdot\theta}\, d\lambda(e^{i\theta}) = \int_{\mathbb{T}^n} e^{-i(\alpha_1\theta_{j_1}+\dots+\alpha_m\theta_{j_m})}\, d\omega(e^{i\theta}) = T^{*\beta^+}T^{\beta^-},$$

where $\beta = (\beta_1,\dots,\beta_n)$ is defined by $\beta_j = 0$ for $j \neq j_1,\dots,j_m$ and $\beta_{j_k} = \alpha_k$ for $k = 1,\dots,m$. Thus, the sequence $(T_{j_1},\dots,T_{j_m})^{*\alpha^+}(T_{j_1},\dots,T_{j_m})^{\alpha^-}$, $\alpha \in \mathbb{Z}^m$, is the Fourier coefficient sequence of λ. By what we have proven above, $\Delta^{\varepsilon}_T = \Delta^{(1,\dots,1)}_{(T_{j_1},\dots,T_{j_m})} \geq 0$ in $\mathcal{L}(\mathcal{H})$. ∎

We close this section with some comments on the relation of our results in this section to the so-called regular unitary dilations of S. Brehmer, B. Sz.-Nagy and I. Halperin. Recall that a *unitary representation* $U : \mathbb{Z}^n \to \mathcal{L}(\mathcal{H})$ is a function such that $U(\alpha)$ is a unitary operator for every $\alpha \in \mathbb{Z}^n$ and

$$U(\alpha+\beta) = U(\alpha)U(\beta) \quad \text{for } \alpha,\beta \in \mathbb{Z}^n.$$

It is easy to see that a unitary representation $U : \mathbb{Z}^n \to \mathcal{L}(\mathcal{H})$ naturally corresponds to an n-tuple $U = (U_1,\dots,U_n)$ of commuting unitary operators in $\mathcal{L}(\mathcal{H})$ by means of the formula $U(\alpha) = U^\alpha = U_1^{\alpha_1}\dots U_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}^n$.

By Stone's theorem (see [1]) every unitary representation $U : \mathbb{Z}^n \to \mathcal{L}(\mathcal{H})$ has the form

$$U(\alpha) = \int_{\mathbb{T}^n} e^{i\alpha\cdot\theta}\, dE(e^{i\theta}) \quad \text{for } \alpha \in \mathbb{Z}^n, \tag{2.7}$$

where E is an $\mathcal{L}(\mathcal{H})$-valued spectral measure on $\mathbb{T}^n$. Conversely, given a spectral measure E it is easy to see that formula (2.7) defines a unitary representation U.

Let $T = (T_1,\dots,T_n)$ be an n-tuple of commuting operators in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{K}$ be a Hilbert space containing $\mathcal{H}$ as a closed subspace. A unitary representation $U : \mathbb{Z}^n \to \mathcal{L}(\mathcal{K})$ is said to be a *regular unitary dilation* of T if

$$PU(\alpha)|_{\mathcal{H}} = T^{*\alpha^-}T^{\alpha^+} \quad \text{for all } \alpha \in \mathbb{Z}^n,$$

where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

It is well known that a regular unitary dilation corresponds to a positive operator measure of the type studied in this section. We have the following proposition.

PROPOSITION 2.2. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators in $\mathcal{L}(\mathcal{H})$. Then the following two assertions are equivalent.*

(1) *The n-tuple T has a regular unitary dilation $U : \mathbb{Z}^n \to \mathcal{L}(\mathcal{K})$.*
(2) *There exists a positive $\mathcal{L}(\mathcal{H})$-valued operator measure ω on $\mathbb{T}^n$ such that $\widehat{\omega}(\alpha) = T^{*\alpha^+}T^{\alpha^-}$ for $\alpha \in \mathbb{Z}^n$.*

Furthermore, when the above holds, we have

$$\omega(\sigma) = PE(\sigma)|_{\mathcal{H}} \quad \text{for } \sigma \in \mathfrak{S}, \tag{2.8}$$

where E is the spectral measure for U, P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, and $\mathfrak{S}$ denotes the σ-algebra of Borel subsets of $\mathbb{T}^n$.

Proof. Assume first that T has a regular unitary dilation $U : \mathbb{Z}^n \to \mathcal{L}(\mathcal{K})$. By Stone's theorem there exists an $\mathcal{L}(\mathcal{K})$-valued spectral measure E on $\mathbb{T}^n$ such that (2.7) holds. By compression to $\mathcal{H}$ we see that $T^{*\alpha^-}T^{\alpha^+} = \int e^{i\alpha\cdot\theta}\, d\omega(e^{i\theta})$ for $\alpha \in \mathbb{Z}^n$, where ω is as in (2.8). Now, clearly, $\widehat{\omega}(\alpha) = T^{*\alpha^+}T^{\alpha^-}$ for $\alpha \in \mathbb{Z}^n$.

Assume next that assertion (2) holds. By Theorem 1.1 there exists a larger Hilbert space $\mathcal{K}$ and an $\mathcal{L}(\mathcal{K})$-valued spectral measure E on $\mathbb{T}^n$ such that (2.8) holds. Formula (2.7) gives us a unitary representation $U : \mathbb{Z}^n \to \mathcal{L}(\mathcal{K})$. A straightforward verification shows that U is a regular unitary dilation of T. ∎

We also remark that by Bochner's theorem (see [10]) assertion (2) in the above proposition is equivalent to positive definiteness of the sequence $T^{*\alpha^+}T^{\alpha^-}$, $\alpha \in \mathbb{Z}^n$.

3. Boundary behavior of the Poisson integral. In this section we are concerned with the boundary behavior of the $\mathcal{L}(\mathcal{H})$-valued n-harmonic Poisson integral discussed in the introduction. More generally, we also prove some statements involving operator measures of the type ω_T. First we need a lemma.

LEMMA 3.1. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators in $\mathcal{L}(\mathcal{H})$ such that $\Delta_T^{\varepsilon} \geq 0$ in $\mathcal{L}(\mathcal{H})$ for $0 \leq \varepsilon \leq (1, \ldots, 1)$. Then*

$$\int_{\mathbb{T}^n} P[\varphi](r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n})\, d\omega_T(e^{i\theta}) = \int_{\mathbb{T}^n} \varphi(e^{i\theta})\, d\omega_{(r_1T_1,\ldots,r_nT_n)}(e^{i\theta})$$

for every $\varphi \in C(\mathbb{T}^n)$ and $r = (r_1, \ldots, r_n)$, $0 \leq r_j \leq 1$.

Proof. Let φ be a trigonometric polynomial on $\mathbb{T}^n$ and note that

$$P[\varphi](r_1e^{i\theta_1},\dots,r_ne^{i\theta_n}) = P_r * \varphi(e^{i\theta}) = \sum_{\alpha\in\mathbb{Z}^n} r_1^{|\alpha_1|}\cdots r_n^{|\alpha_n|}\widehat{\varphi}(\alpha)e^{i\alpha\cdot\theta}.$$

We now have

$$\int P[\varphi](r_1e^{i\theta_1},\dots,r_ne^{i\theta_n})\,d\omega_T(e^{i\theta}) = \sum_{\alpha\in\mathbb{Z}^n} r_1^{|\alpha_1|}\cdots r_n^{|\alpha_n|}\widehat{\varphi}(\alpha)T^{*\alpha^-}T^{\alpha^+}.$$

Since also

$$\int \varphi(e^{i\theta})\,d\omega_{(r_1T_1,\dots,r_nT_n)}(e^{i\theta}) = \sum_{\alpha\in\mathbb{Z}^n} \widehat{\varphi}(\alpha)r_1^{|\alpha_1|}\cdots r_n^{|\alpha_n|}T^{*\alpha^-}T^{\alpha^+},$$

we have proved the lemma for φ a trigonometric polynomial. The general case now follows by approximation. ∎

Remark 3.1. Let $0\le r_j\le 1$ for $1\le j\le n$. Note that by the results of Section 2, existence of $d\omega_{(r_1T_1,\dots,r_nT_n)}$ follows from that of $d\omega_T$.

Theorem 3.1. *Let $T=(T_1,\dots,T_n)$ be an n-tuple of commuting operators in $\mathcal{L}(\mathcal{H})$ such that $\Delta_T^\varepsilon\ge 0$ in $\mathcal{L}(\mathcal{H})$ for $0\le\varepsilon\le(1,\dots,1)$. Let $\varphi\in C(\mathbb{T}^n)$ and let $r=(r_1,\dots,r_n)$, $0\le r_j\le 1$ for $1\le j\le n$. Then*

$$\Big\|\int_{\mathbb{T}^n}\varphi\,d\omega_{(r_1T_1,\dots,r_nT_n)} - \int_{\mathbb{T}^n}\varphi\,d\omega_T\Big\| \le \max_{e^{i\theta}\in\mathbb{T}^n}|P[\varphi](r_1e^{i\theta_1},\dots,r_ne^{i\theta_n})-\varphi(e^{i\theta})|.$$

In particular, $\lim_{r\to(1,\dots,1)}\int\varphi\,d\omega_{(r_1T_1,\dots,r_nT_n)} = \int\varphi\,d\omega_T$ *in* $\mathcal{L}(\mathcal{H})$, *where* $0\le r_j\le 1$.

Proof. By Lemma 3.1, Proposition 1.1 and (1.1) we have

$$\begin{aligned}\Big\|\int\varphi\,d\omega_{(r_1T_1,\dots,r_nT_n)} &- \int\varphi\,d\omega_T\Big\| \\ &= \Big\|\int (P[\varphi](r_1e^{i\theta_1},\dots,r_ne^{i\theta_n})-\varphi(e^{i\theta}))\,d\omega_T(e^{i\theta})\Big\| \\ &\le \max_{e^{i\theta}\in\mathbb{T}^n}|P[\varphi](r_1e^{i\theta_1},\dots,r_ne^{i\theta_n})-\varphi(e^{i\theta})|.\end{aligned}$$

The last assertion of the theorem is clear since $P[\varphi](r_1e^{i\theta_1},\dots,r_ne^{i\theta_n}) \to\varphi(e^{i\theta})$ uniformly in $e^{i\theta}\in\mathbb{T}^n$ as $r\to(1,\dots,1)$. ∎

We can formulate the last continuity assertion of Theorem 3.1 more generally as follows.

Theorem 3.2. *Denote by $\mathcal{P}$ the set of all n-tuples $T\in\mathcal{L}(\mathcal{H})^n$ of commuting operators such that $\Delta_T^\varepsilon\ge 0$ in $\mathcal{L}(\mathcal{H})$ for $0\le\varepsilon\le(1,\dots,1)$. Then the map*

$$C(\mathbb{T}^n)\times\mathcal{P}\ni(\varphi,T)\mapsto\int_{\mathbb{T}^n}\varphi\,d\omega_T\in\mathcal{L}(\mathcal{H})$$

is continuous.

Proof. Let $T_j \to T$ in $\mathcal{L}(\mathcal{H})^n$ and $\varphi_j \to \varphi$ in $C(\mathbb{T}^n)$. We have to prove that $\int \varphi_j \, d\omega_{T_j} \to \int \varphi \, d\omega_T$ in $\mathcal{L}(\mathcal{H})$. First observe that $\int P \, d\omega_{T_j} \to \int P \, d\omega_T$ whenever P is a trigonometric polynomial on $\mathbb{T}^n$. Let $\varepsilon > 0$. Let P be a trigonometric polynomial such that $\|\varphi - P\|_\infty < \varepsilon/4$. For j large enough we have

$$\Big\| \int \varphi_j \, d\omega_{T_j} - \int \varphi \, d\omega_T \Big\| \le \Big\| \int (\varphi_j - \varphi) \, d\omega_{T_j} \Big\| + \Big\| \int (\varphi - P) \, d\omega_{T_j} \Big\| + \Big\| \int P \, d\omega_{T_j} - \int P \, d\omega_T \Big\| + \Big\| \int (P - \varphi) \, d\omega_T \Big\| < \varepsilon.$$

Thus, $\int \varphi_j \, d\omega_{T_j} \to \int \varphi \, d\omega_T$. ■

Let us now restrict our attention to the special case of Theorem 3.1 when $r = (r_1, \ldots, r_n)$ is such that $0 \le r_j < 1$ for $1 \le j \le n$. The last assertion of Theorem 3.1 then becomes

$$\lim_{\substack{r_j \to 1 \\ 0 \le r_j < 1}} P[\varphi](r_1 T_1, \ldots, r_n T_n) = \int_{\mathbb{T}^n} \varphi \, d\omega_T \quad \text{in } \mathcal{L}(\mathcal{H}). \tag{3.1}$$

A computation shows that

$$\int P(r_1 T_1, \ldots, r_n T_n; e^{i\theta}) \varphi(e^{i\theta}) \, d\sigma(e^{i\theta}) = \sum_{\alpha \in \mathbb{Z}^n} r_1^{|\alpha_1|} \cdots r_n^{|\alpha_n|} \widehat{\varphi}(\alpha) T^{*\alpha^-} T^{\alpha^+} \quad \text{in } \mathcal{L}(\mathcal{H}).$$

Thus, by (3.1), the formal series $\sum \widehat{\varphi}(\alpha) T^{*\alpha^-} T^{\alpha^+}$ is Abel summable to $\int \varphi \, d\omega_T$ in $\mathcal{L}(\mathcal{H})$, that is,

$$\lim_{\substack{r_j \to 1 \\ 0 \le r_j < 1}} \sum_{\alpha \in \mathbb{Z}^n} r_1^{|\alpha_1|} \cdots r_n^{|\alpha_n|} \widehat{\varphi}(\alpha) T^{*\alpha^-} T^{\alpha^+} = \int_{\mathbb{T}^n} \varphi \, d\omega_T \quad \text{in } \mathcal{L}(\mathcal{H}).$$

In the preceding paragraph we essentially used the fact that $P_r * \varphi \to \varphi$ in $C(\mathbb{T}^n)$. We now consider the analogous situation when the Poisson kernel P_r is replaced by the Fejér kernel K_N, $N = (N_1, \ldots, N_n)$, $N_j \ge 0$, defined by

$$K_N(e^{i\theta}) = \prod_{j=1}^n K_{N_j}(e^{i\theta_j}) = \sum_{-N \le \alpha \le N} \prod_{j=1}^n \left(1 - \frac{|\alpha_j|}{N_j + 1}\right) \cdot e^{i\alpha\cdot\theta}, \qquad e^{i\theta} \in \mathbb{T}^n.$$

Since $K_N * \varphi \to \varphi$ in $C(\mathbb{T}^n)$ as $N_j \to \infty$, we have $\int K_N * \varphi \, d\omega_T \to \int \varphi \, d\omega_T$ in $\mathcal{L}(\mathcal{H})$. This last limit assertion can be reformulated as saying that the formal series $\sum \widehat{\varphi}(\alpha) T^{*\alpha^-} T^{\alpha^+}$ is Cesàro summable to $\int \varphi \, d\omega_T$ in $\mathcal{L}(\mathcal{H})$, that is,

$$\lim_{N_j \to \infty} \sum_{|\alpha_j| \le N_j} \left(1 - \frac{|\alpha_1|}{N_1 + 1}\right) \cdots \left(1 - \frac{|\alpha_n|}{N_n + 1}\right) \widehat{\varphi}(\alpha) T^{*\alpha^-} T^{\alpha^+} = \int_{\mathbb{T}^n} \varphi \, d\omega_T \quad \text{in } \mathcal{L}(\mathcal{H}).$$

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