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# Common zero sets of equivalent singular inner functions

by

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**Abstract.** Let $\mu$ and $\lambda$ be bounded positive singular measures on the unit circle such that $\mu \perp \lambda$. It is proved that there exist positive measures $\mu_0$ and $\lambda_0$ such that $\mu_0 \sim \mu$, $\lambda_0 \sim \lambda$, and $\{|\psi_{\mu_0}| < 1\} \cap \{|\psi_{\lambda_0}| < 1\} = \emptyset$, where $\psi_\mu$ is the associated singular inner function of $\mu$. Let $\mathcal{Z}(\mu) = \bigcap_{\{\nu;\, \nu\sim\mu\}} Z(\psi_\nu)$ be the common zeros of equivalent singular inner functions of $\psi_\mu$. Then $\mathcal{Z}(\mu) \neq \emptyset$ and $\mathcal{Z}(\mu) \cap \mathcal{Z}(\lambda) = \emptyset$. It follows that $\mu \ll \lambda$ if and only if $\mathcal{Z}(\mu) \subset \mathcal{Z}(\lambda)$. Hence $\mathcal{Z}(\mu)$ is the set in the maximal ideal space of $H^\infty$ which relates naturally to the set of measures equivalent to $\mu$. Some basic properties of $\mathcal{Z}(\mu)$ are given.

**1. Introduction.** Let $H^\infty$ be the Banach algebra of bounded analytic functions on the open unit disk $D$. We denote by $\mathcal{M} = M(H^\infty)$ the maximal ideal space of $H^\infty$, the space of non-zero multiplicative linear functionals of $H^\infty$ with the weak* topology. We think of $D$ as an open subset of $\mathcal{M}$. Identifying a function in $H^\infty$ with its Gelfand transform, we regard $H^\infty$ as a closed subalgebra of $C(\mathcal{M})$, the space of continuous functions on $\mathcal{M}$. Identifying a function in $H^\infty$ with its boundary function, we also view $H^\infty$ as an (essential) supremum norm closed subalgebra of $L^\infty$, the usual Lebesgue space on the unit circle $\partial D$. We may consider the maximal ideal space $M(L^\infty)$ of $L^\infty$ to be a subset of $\mathcal{M}$, and it is known that $M(L^\infty)$ is the Shilov boundary of $H^\infty$ (see [10]). For a point $x \in \mathcal{M}$, there exists a probability measure $\mu_x$ on $M(L^\infty)$ such that $f(x) = \int_{M(L^\infty)} f\, d\mu_x$ for every $f \in H^\infty$. We denote by $\operatorname{supp}\mu_x$ the closed support set of $\mu_x$. A function $f$ in $H^\infty$ is called *inner* if $|f| = 1$ on $M(L^\infty)$. For a function $f$ in $H^\infty$, we use the following notations:

$$\{|f| < 1\} = \{x \in \mathcal{M} \setminus D;\ |f(x)| < 1\}, \qquad Z(f) = \{x \in \mathcal{M} \setminus D;\ f(x) = 0\}.$$

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Note that these are subsets of $\mathcal{M}\backslash D$. For $\zeta\in\partial D$, let $\mathcal{M}_\zeta=\{x\in\mathcal{M};\ z(x)=\zeta\}$, where $z$ is the identity function on $D$. For a subset $E$ of $\mathcal{M}$, we denote by $\overline{E}$ its weak* closure in $\mathcal{M}$.

For a sequence $\{z_n\}_n$ in $D$ with $\sum_{n=1}^{\infty}(1-|z_n|)<\infty$, there is the associated Blaschke product

$$b(z)=\prod_{n=1}^{\infty}\frac{-\overline{z}_n}{|z_n|}\,\frac{z-z_n}{1-\overline{z}_nz},\qquad z\in D.$$

Blaschke products are typical inner functions. Moreover if for every bounded sequence $\{a_n\}_n$ of complex numbers there exists $f\in H^\infty$ such that $f(z_n)=a_n$ for every $n$, then both $\{z_n\}_n$ and the associated Blaschke product $b$ are called *interpolating*. In this case, we have $Z(b)=\overline{\{z_n\}_n}\setminus\{z_n\}_n$ (see [10, p. 205]). We denote by $S(b)$ the set of cluster points of $\{z_n\}_n$ in the closed unit disk.

For $x,y\in\mathcal{M}$, let

$$\begin{aligned}\varrho(x,y)&=\sup\{|f(y)|;\ f\in H^\infty,\ f(x)=0,\ \|f\|_\infty=1\},\\ P(x)&=\{w\in\mathcal{M};\ \varrho(x,w)<1\}.\end{aligned}$$

The set $P(x)$ is called the *Gleason part* containing $x$. When $P(x)=\{x\}$, both $x$ and $P(x)$ are called *trivial*. We denote by $G$ the set of non-trivial points in $\mathcal{M}$. In [11], Hoffman proved that $G\setminus D$ is the set of points $x$ in $\mathcal{M}\setminus D$ such that $b(x)=0$ for some interpolating Blaschke product $b$, and $G$ is open in $\mathcal{M}$. See [11] for the study of the structure of $\mathcal{M}$ and $G$.

Let $M_{\rm s}^+$ be the set of bounded positive (non-zero) measures on $\partial D$ singular with respect to the Lebesgue measure on $\partial D$. For $\mu\in M_{\rm s}^+$, we denote by $\operatorname{supp}\mu$ the closed support set of $\mu$ and by $\|\mu\|$ the total variation norm of $\mu$. We also denote by $M_{\rm s,d}^+$ and $M_{\rm s,c}^+$ the sets of discrete and continuous measures in $M_{\rm s}^+$, respectively. For $\zeta\in\partial D$, let $\delta_\zeta$ be the unit point mass at $\zeta$. For $\mu,\lambda\in M_{\rm s}^+$, we write $\mu\ll\lambda$ if $\mu$ is absolutely continuous with respect to $\lambda$, and $\mu\perp\lambda$ if $\mu$ and $\lambda$ are mutually singular; moreover, $\mu\wedge\lambda$ is the lower bound of $\mu$ and $\lambda$. For $\mu,\nu\in M_{\rm s}^+$, we write $\mu\sim\nu$ if $\mu$ and $\nu$ are equivalent, that is, $\mu\ll\nu$ and $\nu\ll\mu$. For each $\mu\in M_{\rm s}^+$, let

$$\psi_\mu(z)=\exp\Big(-\int_{\partial D}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(e^{i\theta})\Big),\qquad z\in D.$$

Then $\psi_\mu$ is called a *singular inner function*; it may be extended continuously on $\partial D\backslash\operatorname{supp}\mu$ and $|\psi_\mu|=1$ on $\mathcal{M}_\zeta$ for $\zeta\notin\operatorname{supp}\mu$ (see [5, 10]). When $\mu\sim\nu$, we say that $\psi_\mu$ and $\psi_\nu$ are *equivalent singular inner functions*. We have

$$|\psi_\mu(z)|=\exp\Big(-\int_{\partial D}P_z(e^{i\theta})\,d\mu(e^{i\theta})\Big),\qquad z\in D,$$

where $P_z$ is the Poisson kernel. We put

$$\mathcal{Z}(\mu) = \bigcap_{\{\nu \in M_s^+;\, \nu \sim \mu\}} Z(\psi_\nu), \qquad \mathcal{W}(\mu) = \bigcap_{\{\nu \in M_s^+;\, \nu \sim \mu\}} \{|\psi_\nu| < 1\}.$$

Then $\mathcal{Z}(\mu) \subset \mathcal{W}(\mu)$. In [13], the author proved that if $\mu, \lambda \in M_{\mathrm{s,d}}^+$ and $\mu \perp \lambda$, then $\mathcal{W}(\mu) \cap \mathcal{W}(\lambda) = \emptyset$.

The purpose of this paper is to study $\mathcal{Z}(\mu)$ and $\mathcal{W}(\mu)$ for $\mu \in M_s^+$. The motivation for this study comes from [12] and [14]. In [12], the author studied certain properties of Blaschke products, and in [14] similar properties for singular inner functions. In Section 2, we prove that if $\mu, \lambda \in M_s^+$ with $\mu \perp \lambda$, then there are $\mu_0, \lambda_0 \in M_s^+$ such that $\mu_0 \sim \mu$, $\lambda_0 \sim \lambda$, and $\{|\psi_{\mu_0}| < 1\} \cap \{|\psi_{\lambda_0}| < 1\} = \emptyset$. Then we get $\mathcal{W}(\mu) \cap \mathcal{W}(\lambda) = \emptyset$ and $\mathcal{Z}(\mu) \cap \mathcal{Z}(\lambda) = \emptyset$. Hence $\mathcal{Z}(\mu)$ is the set in $\mathcal{M} \setminus D$ related to the class of measures equivalent to $\mu$. From the point of view of the study of measures on $\partial D$, the set $\mathcal{Z}(\mu)$ is interesting and important. In Section 3, we prove that

$$\mathcal{W}(\mu) = \mathcal{Z}(\mu) \cup \bigcup_{\{\zeta \in \partial D;\, \mu(\{\zeta\}) \neq 0\}} \{|\psi_{\delta_\zeta}| < 1\}.$$

Hence if $\mu \in M_{\mathrm{s,c}}^+$, then $\mathcal{W}(\mu) = \mathcal{Z}(\mu)$. Moreover, we prove that for $\zeta \in \partial D$, if $\mu(\{\zeta\}) = 0$ then

$$\mathcal{Z}(\mu) \subset \overline{\bigcup_{\{\xi \in \partial D;\, \xi \neq \zeta\}} \mathcal{M}_\xi}.$$

In Section 4, we prove that for $\zeta \in \partial D$ there exists a Blaschke product $b$ such that $S(b) = \{\zeta\}$ and $\mathcal{Z}(\mu) \cap \overline{\{|b| < 1\}} = \emptyset$ for every $\mu \in M_s^+$. Also we show that for every Blaschke product $b$ with $S(b) = \partial D$ there exists $\mu \in M_{\mathrm{s,c}}^+$ such that $Z(b) \cap \mathcal{Z}(\mu) \neq \emptyset$.

By [4, p. 162], $Z(\psi_\mu)$ contains a trivial point for every $\mu \in M_s^+$. Hence $\mathcal{Z}(\mu)$ contains trivial points too. Let $\operatorname{int} \mathcal{Z}(\mu)$ denote the interior of $\mathcal{Z}(\mu)$ in $\mathcal{M} \setminus D$. If $\mu \in M_{\mathrm{s,c}}^+$, we have $Z(b) \not\subset \mathcal{Z}(\mu)$ for every interpolating Blaschke product $b$. This implies that $\operatorname{int} \mathcal{Z}(\mu) = \emptyset$. Note that $\operatorname{int} Z(\psi_\mu) \neq \emptyset$. Since the set $G$ of non-trivial points is open, one can ask whether $\mathcal{Z}(\mu) \cap G = \emptyset$ or not. To answer this, in Section 5 we study interpolating Blaschke products. For a non-empty closed subset $K$ of $\partial D$ which has Lebesgue measure zero, we construct an interpolating Blaschke product $b_K$ with certain properties. In Section 6, we prove that $Z(b_K) \cap \mathcal{Z}(\mu) \neq \emptyset$ for every $\mu \in M_s^+$ with $\operatorname{supp} \mu \subset K$. Hence $\mathcal{Z}(\mu)$ contains non-trivial points for every $\mu \in M_s^+$.

Let $\mu \in M_s^+$. We denote by $M(L^\infty(\mu))$ the maximal ideal space of the Banach algebra $L^\infty(\mu)$. In Section 6, we establish the existence of a natural map $\Phi_\mu$ from $M(L^\infty(\mu))$ to the family of closed subsets of $\mathcal{Z}(\mu)$ such that

$$\mathcal{Z}(\mu) = \bigcup_{x \in M(L^\infty(\mu))} \Phi_\mu(x)$$

and $\Phi_\mu(x) \cap \Phi_\mu(y) = \emptyset$ if $x \neq y$. Hence we may think of $\{\Phi_\mu(x); x \in M(L^\infty(\mu))\}$ as an atomic decomposition of the measure $\mu$ in $\mathcal{M} \setminus D$ in some sense. Also we prove that every $\Phi_\mu(x)$ contains non-trivial points.

**2. Mutually singular measures.** For a subset $E$ of $D \cup \partial D$, we denote by $\operatorname{cl} E$ the closure of $E$ in the complex plane. In this section, we prove that $\mathcal{W}(\mu) \cap \mathcal{W}(\lambda) = \emptyset$ if $\mu, \lambda \in M_{\mathrm{s}}^+$ and $\mu \perp \lambda$. First, we prove the following theorem.

THEOREM 2.1. *Let $\mu, \lambda \in M_s^+$ and $\mu \perp \lambda$. Then there exist $\mu_0, \lambda_0 \in M_{\mathrm{s}}^+$ such that $\mu_0 \sim \mu, \lambda_0 \sim \lambda$, and $\{|\psi_{\mu_0}| < 1\} \cap \{|\psi_{\lambda_0}| < 1\} = \emptyset$.*

*Proof.* Since $\mu \perp \lambda$, there exists a measurable subset $A \subset \partial D$ such that $\mu(A) = \|\mu\|$ and $\lambda(\partial D \setminus A) = \|\lambda\|$. By the regularity of the measures, there exist sequences $\{\mu_n\}_n$ and $\{\lambda_n\}_n$ of measures in $M_{\mathrm{s}}^+$ such that $\operatorname{supp} \mu_n \subset A$, $\operatorname{supp} \lambda_n \subset \partial D \setminus A$, and

$$\mu = \sum_{n=1}^\infty \mu_n, \qquad \lambda = \sum_{n=1}^\infty \lambda_n. \tag{2.1}$$

Then

$$\operatorname{supp} \mu_n \cap \operatorname{supp} \lambda_k = \emptyset \quad \text{ for all } n, k. \tag{2.2}$$

Let $\{\delta_n\}_n$ be a sequence of numbers such that

$$0 < \delta_n < 1, \qquad \prod_{n=1}^\infty \delta_n > 0. \tag{2.3}$$

For each $0 < s < 1$, let

$$U_{\mu_n}(s) = \{z \in D; |\psi_{\mu_n}(z)| < s\}, \qquad U_{\lambda_n}(s) = \{z \in D; |\psi_{\lambda_n}(z)| < s\}. \tag{2.4}$$

Then $U_{\mu_n}(s_1) \subset U_{\mu_n}(s_2)$ if $s_1 < s_2$, and

$$\bigcap_{0<s<1} \operatorname{cl} U_{\mu_n}(s) = \operatorname{supp} \mu_n, \qquad \bigcap_{0<s<1} \operatorname{cl} U_{\lambda_n}(s) = \operatorname{supp} \lambda_n.$$

Hence by (2.2), we have

$$\sup_{z \in U_{\mu_k}(s)} |\psi_{\lambda_n}(z)| \to 1, \qquad \sup_{z \in U_{\lambda_k}(s)} |\psi_{\mu_n}(z)| \to 1 \quad \text{ as } s \to 0 \text{ for all } n, k.$$

Then by induction, we may take $\{s_n\}_n$ and $\{t_n\}_n$ such that

$$U_{\mu_n}(s_n) \cap U_{\lambda_k}(t_k) = \emptyset \quad \text{for all } n, k, \tag{2.5}$$

$$\left|\prod_{j=1}^{n}\psi_{\lambda_j}\right| \geq \delta_n \quad \text{on } \bigcup_{k=n}^{\infty} U_{\mu_k}(s_k),$$

(2.6)

$$\left|\prod_{j=1}^{n}\psi_{\mu_j}\right| \geq \delta_n \quad \text{on } \bigcup_{k=n}^{\infty} U_{\lambda_k}(t_k).$$

Next, let $\{a_n\}_n$ and $\{b_n\}_n$ be sequences of numbers satisfying

$$0 < a_n < 1, \quad 0 < b_n < 1, \tag{2.7}$$

$$s_n^{a_n} \geq \delta_n, \quad t_n^{b_n} \geq \delta_n \quad \text{for every } n. \tag{2.8}$$

Let

$$\mu_0 = \sum_{n=1}^{\infty} a_n\mu_n, \quad \lambda_0 = \sum_{n=1}^{\infty} b_n\lambda_n. \tag{2.9}$$

Then by (2.1) and (2,7), $\mu_0, \lambda_0 \in M_{\mathrm{s}}^+$, $\mu_0 \sim \mu$, and $\lambda_0 \sim \lambda$.

For $z \in D \setminus \bigcup_{j=1}^{\infty} U_{\mu_j}(s_j)$, we have

$$\begin{aligned}
|\psi_{\mu_0}(z)| &= \prod_{j=1}^{k} |\psi_{\mu_j}(z)|^{a_j} \prod_{j=k+1}^{\infty} |\psi_{\mu_j}(z)|^{a_j} && \text{by (2.9)} \\
&\geq \prod_{j=1}^{k} |\psi_{\mu_j}(z)| \prod_{j=k+1}^{\infty} s_j^{a_j} && \text{by (2.4)} \\
&\geq \prod_{j=1}^{k} |\psi_{\mu_j}(z)| \prod_{j=k+1}^{\infty} \delta_j && \text{by (2.8).}
\end{aligned}$$

Hence

$$|\psi_{\mu_0}| \geq \prod_{j=1}^{k} |\psi_{\mu_j}| \prod_{j=k+1}^{\infty} \delta_j \quad \text{on } D \setminus \bigcup_{j=1}^{\infty} U_{\mu_j}(s_j) \text{ for every } k. \tag{2.10}$$

Similarly,

$$|\psi_{\lambda_0}| \geq \prod_{j=1}^{k} |\psi_{\lambda_j}| \prod_{j=k+1}^{\infty} \delta_j \quad \text{on } D \setminus \bigcup_{j=1}^{\infty} U_{\lambda_j}(t_j) \text{ for every } k. \tag{2.11}$$

Now suppose that $\{|\psi_{\mu_0}| < 1\} \cap \{|\psi_{\lambda_0}| < 1\} \neq \emptyset$. Then by the corona theorem [3], there exist $0 < \delta < 1$ and a sequence $\{z_n\}_n$ in $D$ such that $|z_n| \to 1$ and

$$|\psi_{\mu_0}(z_n)| < \delta, \quad |\psi_{\lambda_0}(z_n)| < \delta \quad \text{for every } n. \tag{2.12}$$

By (2.3), there exists a positive integer $k_0$ such that

$$\prod_{j=k_0+1}^{\infty} \delta_j > \delta^{1/2}. \tag{2.13}$$

Considering a subsequence of $\{z_n\}_n$, we may further assume that either

$$z_n \in \Big(D \setminus \bigcup_{j=1}^{\infty} U_{\mu_j}(s_j)\Big) \cap \Big(D \setminus \bigcup_{j=1}^{\infty} U_{\lambda_j}(t_j)\Big) \quad \text{for every } n, \tag{2.14}$$

$$z_n \in \bigcup_{j=1}^{\infty} U_{\mu_j}(s_j) \quad \text{for every } n, \tag{2.15}$$

or

$$z_n \in \bigcup_{j=1}^{\infty} U_{\lambda_j}(t_j) \quad \text{for every } n. \tag{2.16}$$

For each case we shall obtain a contradiction.

First, suppose that (2.14) holds. By (2.10), (2.12), and (2.13),

$$\delta > \prod_{j=1}^{k_0} |\psi_{\mu_j}(z_n)| \prod_{j=k_0+1}^{\infty} \delta_j > \delta^{1/2} \prod_{j=1}^{k_0} |\psi_{\mu_j}(z_n)| \quad \text{for every } n.$$

Then

$$\prod_{j=1}^{k_0} |\psi_{\mu_j}(z_n)| \le \delta^{1/2} < 1 \quad \text{for every } n.$$

Similarly,

$$\prod_{j=1}^{k_0} |\psi_{\lambda_j}(z_n)| \le \delta^{1/2} < 1 \quad \text{for every } n.$$

Hence

$$\mathrm{cl}\,\{z_n\}_n \setminus \{z_n\}_n \subset \Big(\bigcup_{j=1}^{k_0} \operatorname{supp} \mu_j\Big) \cap \Big(\bigcup_{j=1}^{k_0} \operatorname{supp} \lambda_j\Big).$$

But this contradicts (2.2). Therefore (2.14) does not occur.

Next, suppose that (2.15) holds. Then by (2.5),

$$\{z_n\}_n \subset D \setminus \bigcup_{j=1}^{\infty} U_{\lambda_j}(t_j). \tag{2.17}$$

Taking a subsequence of $\{z_n\}_n$, we may further assume that either

$$\{z_n\}_n \subset \bigcup_{j=1}^{m} U_{\mu_j}(s_j) \quad \text{for some } m \ge k_0, \tag{2.18}$$

or

$$\{z_n\}_n \cap \bigcup_{j=1}^{m} U_{\mu_j}(s_j) \quad \text{is a finite set for every } m. \tag{2.19}$$

Suppose that (2.18) holds. Then by (2.4), we have

$$\prod_{j=1}^{m} |\psi_{\mu_j}(z_n)| < \max_{1\le j\le m} s_j < 1 \quad \text{for every } n.$$

Hence

$$\operatorname{cl}\{z_n\}_n \setminus \{z_n\}_n \subset \bigcup_{j=1}^{m} \operatorname{supp}\mu_j. \tag{2.20}$$

By (2.11), (2.12), (2.13), and (2.17),

$$\delta > |\psi_{\lambda_0}(z_n)| \ge \prod_{j=1}^{k_0} |\psi_{\lambda_j}(z_n)| \prod_{j=k_0+1}^{\infty} \delta_j > \delta^{1/2} \prod_{j=1}^{k_0} |\psi_{\lambda_j}(z_n)|.$$

Thus we have

$$\prod_{j=1}^{k_0} |\psi_{\lambda_j}(z_n)| < \delta^{1/2} < 1 \quad \text{for every } n.$$

Therefore

$$\operatorname{cl}\{z_n\}_n \setminus \{z_n\}_n \subset \bigcup_{j=1}^{k_0} \operatorname{supp}\lambda_j. \tag{2.21}$$

Hence (2.20) and (2.21) contradict (2.2).

Next, suppose that (2.19) holds. Then for each $k$, we have

$$\begin{aligned}\liminf_{n\to\infty} |\psi_{\lambda_0}(z_n)| &\ge \liminf_{n\to\infty} \prod_{j=1}^{k} |\psi_{\lambda_j}(z_n)| \prod_{j=k+1}^{\infty} \delta_j \quad \text{by (2.11) and (2.17)}\\ &\ge \prod_{j=k}^{\infty} \delta_j \quad \text{by (2.6), (2.15), and (2.19).}\end{aligned}$$

Thus by (2.3), we have $|\psi_{\lambda_0}(z_n)| \to 1$ as $n \to \infty$. This contradicts (2.12). Therefore (2.15) does not occur.

Similarly, we may prove that (2.16) does not occur. Thus we get our assertion. ■

As an application of Theorem 2.1, we have the following.

THEOREM 2.2. *Let $\mu, \lambda \in M_{\rm s}^{+}$ be such that $\mu \perp \lambda$. Then $\mathcal{W}(\mu) \cap \mathcal{W}(\lambda) = \emptyset$, and consequently, $\mathcal{Z}(\mu) \cap \mathcal{Z}(\lambda) = \emptyset$.*

This theorem says that the singularity of measures on $\partial D$ may be represented in the maximal ideal space $\mathcal{M}$ of $H^\infty$ as disjoint closed subsets. So to study the behavior of singular inner functions, it is important to study the sets $\mathcal{Z}(\mu)$.

**3.** $\mathcal{Z}(\mu)$ **and** $\mathcal{W}(\mu)$**.** Recall that for $\mu \in M_{\mathrm{s}}^+$,

$$\mathcal{Z}(\mu) = \bigcap_{\{\nu\in M_{\mathrm{s}}^+;\,\nu\sim\mu\}} Z(\psi_\nu), \qquad \mathcal{W}(\mu) = \bigcap_{\{\nu\in M_{\mathrm{s}}^+;\,\nu\sim\mu\}} \{|\psi_\nu| < 1\}.$$

Thus $\mathcal{Z}(\mu) \subset \mathcal{W}(\mu)$ and $\mathcal{W}(\mu)$ is a subset of $\mathcal{M} \setminus (D \cup M(L^\infty))$. In this section, we study the properties of $\mathcal{Z}(\mu)$ and $\mathcal{W}(\mu)$. We note that if $\mu, \lambda \in M_{\mathrm{s}}^+$ and $\mu \sim \lambda$, then $\mathcal{Z}(\mu) = \mathcal{Z}(\lambda)$ and $\mathcal{W}(\mu) = \mathcal{W}(\lambda)$.

First, we prove the following.

THEOREM 3.1. *Let $\mu \in M_{\mathrm{s}}^+$ and $\zeta \in \operatorname{supp}\mu$. Then $\mathcal{Z}(\mu) \cap \mathcal{M}_\zeta \neq \emptyset$, and consequently, $\mathcal{Z}(\mu) \neq \emptyset$.*

To prove this, we use the following lemma.

LEMMA 3.2. *Let $\mu \in M_{\mathrm{s}}^+$ and $E$ be a closed subset of $\mathcal{M}$ such that $\mathcal{Z}(\mu) \cap E = \emptyset$. Then there exists $\nu \in M_{\mathrm{s}}^+$ such that $\nu \sim \mu$ and $Z(\psi_\nu) \cap E = \emptyset$.*

*Proof.* By our assumption, there exist $\nu_1, \ldots, \nu_n \in M_{\mathrm{s}}^+$ such that $\nu_j \sim \mu$ and

$$\sum_{j=1}^{n} |\psi_{\nu_j}| > 0 \quad \text{on } E. \tag{3.1}$$

Let $\nu$ be the lower bound of $\{\nu_j\}_{j=1}^n$, that is, $\nu = \bigwedge_{j=1}^n \nu_j$. Then $\nu \neq 0$ and $\nu \sim \mu$. Since $\nu \le \nu_j$, we have $|\psi_{\nu_j}| \le |\psi_\nu|$ on $\mathcal{M}$. Hence by (3.1), $0 < |\psi_\nu|$ on $E$. ∎

*Proof of Proposition 3.1.* Let $\nu \in M_{\mathrm{s}}^+$ and $\nu \sim \mu$. Since $\zeta \in \operatorname{supp}\nu$, it follows that $Z(\psi_\nu) \cap \mathcal{M}_\zeta \neq \emptyset$ (see [5, p. 76]). By Lemma 3.2, we have $\mathcal{Z}(\mu) \cap \mathcal{M}_\zeta \neq \emptyset$. ∎

The following lemma lists elementary properties of $\mathcal{Z}(\mu)$ and $\mathcal{W}(\mu)$.

LEMMA 3.3. *Let $\mu_1, \mu_2 \in M_{\mathrm{s}}^+$.*

- (i) *If $\mu_1 \perp \mu_2$, then $\mathcal{Z}(\mu_1 + \mu_2) = \mathcal{Z}(\mu_1) \cup \mathcal{Z}(\mu_1)$ and $\mathcal{W}(\mu_1 + \mu_2) = \mathcal{W}(\mu_1) \cup \mathcal{W}(\mu_1)$.*
- (ii) *If $\mu_1 \ll \mu_2$, then $\mathcal{Z}(\mu_1) \subset \mathcal{Z}(\mu_2)$ and $\mathcal{W}(\mu_1) \subset \mathcal{W}(\mu_2)$.*
- (iii) *$\mathcal{Z}(\mu_1 + \mu_2) = \mathcal{Z}(\mu_1) \cup \mathcal{Z}(\mu_2)$ and $\mathcal{W}(\mu_1 + \mu_2) = \mathcal{W}(\mu_1) \cup \mathcal{W}(\mu_2)$.*
- (iv) *If $\mu_1 \wedge \mu_2 \neq 0$, then $\mathcal{Z}(\mu_1 \wedge \mu_2) = \mathcal{Z}(\mu_1) \cap \mathcal{Z}(\mu_2)$ and $\mathcal{W}(\mu_1 \wedge \mu_2) = \mathcal{W}(\mu_1) \cap \mathcal{W}(\mu_2)$.*

*Proof.* We only prove the properties of $\mathcal{Z}(\mu)$; those of $\mathcal{W}(\mu)$ are established similarly.

(i) Suppose that $\mu_1 \perp \mu_2$. Let $\nu \in M_{\mathrm{s}}^+$. Then $\nu \sim \mu_1 + \mu_2$ if and only if $\nu = \nu_1 + \nu_2$ for some $\nu_1, \nu_2 \in M_{\mathrm{s}}^+$ with $\nu_1 \sim \mu_1$ and $\nu_2 \sim \mu_2$. Since $\psi_{\nu_1+\nu_2} = \psi_{\nu_1}\psi_{\nu_2}$, we have $Z(\psi_{\nu_1+\nu_2}) = Z(\psi_{\nu_1}) \cup Z(\psi_{\nu_2})$. Then by Theorem 2.2, $\mathcal{Z}(\mu_1 + \mu_2) = \mathcal{Z}(\mu_1) \cup \mathcal{Z}(\mu_2)$.

(ii) Suppose that $\mu_1 \ll \mu_2$. Then $\mu_2 = \nu_1 + \nu_2$, where $\nu_1 \sim \mu_1$ and $\nu_1 \perp \nu_2$. Hence by (i), $\mathcal{Z}(\mu_1) = \mathcal{Z}(\nu_1) \subset \mathcal{Z}(\nu_1) \cup \mathcal{Z}(\nu_2) = \mathcal{Z}(\mu_2)$.

(iii) By (ii), we have $\mathcal{Z}(\mu_1) \cup \mathcal{Z}(\mu_2) \subset \mathcal{Z}(\mu_1 + \mu_2)$. To prove the reverse inclusion, write $\mu_1 + \mu_2 = \nu_1 + \nu_2$, where $\nu_1, \nu_2 \in M_{\mathrm{s}}^+$ are such that $\nu_1 \sim \mu_1$, $\nu_1 \perp \nu_2$, and $\nu_2 \ll \mu_2$. Then by (i) and (ii),

$$\mathcal{Z}(\mu_1 + \mu_2) = \mathcal{Z}(\nu_1) \cup \mathcal{Z}(\nu_2) = \mathcal{Z}(\mu_1) \cup \mathcal{Z}(\nu_2) \subset \mathcal{Z}(\mu_1) \cup \mathcal{Z}(\mu_2).$$

(iv) By (ii), $\mathcal{Z}(\mu_1 \wedge \mu_2) \subset \mathcal{Z}(\mu_1) \cap \mathcal{Z}(\mu_2)$. Write $\mu_1 = \nu_1 + \nu_2$, where $\nu_1 \sim \mu_1 \wedge \mu_2$ and $\nu_2 \perp \mu_2$. Then by (i),

$$\mathcal{Z}(\mu_1) = \mathcal{Z}(\nu_1) \cup \mathcal{Z}(\nu_2) = \mathcal{Z}(\mu_1 \wedge \mu_2) \cup \mathcal{Z}(\nu_2).$$

By Theorem 2.2, $\mathcal{Z}(\nu_2) \cap \mathcal{Z}(\mu_2) = \emptyset$. By (ii), $\mathcal{Z}(\mu_1 \wedge \mu_2) \subset \mathcal{Z}(\mu_2)$. Hence

$$\mathcal{Z}(\mu_1) \cap \mathcal{Z}(\mu_2) = \mathcal{Z}(\mu_1 \wedge \mu_2) \cap \mathcal{Z}(\mu_2) = \mathcal{Z}(\mu_1 \wedge \mu_2). \ \blacksquare$$

PROPOSITION 3.4. *Let $\mu_1, \mu_2 \in M_{\mathrm{s}}^+$. Then $\mu_1 \ll \mu_2$ if and only if $\mathcal{Z}(\mu_1) \subset \mathcal{Z}(\mu_2)$.*

*Proof.* The "only if" part follows from Lemma 3.3(ii). Suppose that $\mu_1 \not\ll \mu_2$. Write $\mu_1 = \nu_1 + \nu_2$, where $\nu_1 \perp \mu_2$ and $\nu_2 \ll \mu_2$. Then $\nu_1 \neq 0$. By Proposition 3.1, we have $\mathcal{Z}(\nu_1) \neq \emptyset$. Since $\nu_1 \ll \mu_1$, Lemma 3.3(ii) yields $\mathcal{Z}(\nu_1) \subset \mathcal{Z}(\mu_1)$. Since $\nu_1 \perp \mu_2$, by Theorem 2.2 we have $\mathcal{Z}(\nu_1) \cap \mathcal{Z}(\mu_2) = \emptyset$. Thus we get $\mathcal{Z}(\mu_1) \not\subset \mathcal{Z}(\mu_2)$. $\blacksquare$

The following shows a relation between $\mathcal{W}(\mu)$ and $\mathcal{Z}(\mu)$.

THEOREM 3.5. *Let $\mu \in M_{\mathrm{s}}^+$. Then*

$$\mathcal{W}(\mu) = \mathcal{Z}(\mu) \cup \bigcup_{\{\zeta \in \partial D;\, \mu(\{\zeta\}) \neq 0\}} \{|\psi_{\delta_\zeta}| < 1\}.$$

*Proof.* The $\supset$ inclusion follows from the definition of $\mathcal{W}(\mu)$. To prove the reverse inclusion, let

$$x \in \mathcal{W}(\mu) \setminus \bigcup_{\{\zeta \in \partial D;\, \mu(\{\zeta\}) \neq 0\}} \{|\psi_{\delta_\zeta}| < 1\}. \tag{3.2}$$

It is sufficient to prove that $x \in \mathcal{Z}(\mu)$. Suppose not. Then there exists $\nu \in M_{\mathrm{s}}^+$ such that $\nu \sim \mu$ and $\psi_\nu(x) \neq 0$. We may assume that $x \in \mathcal{M}_1$.

First, suppose that $\mu(\{1\}) = 0$. Let $I_0 = \partial D$ and $I_n = \{e^{i\theta};\ -1/n \leq \theta \leq 1/n\}$ for every positive integer $n$. Set $\nu_n = \nu_{|(I_{n-1} \setminus I_n)}$. Then $\nu = \sum_{n=1}^{\infty} \nu_n$.

Let

$$\nu_0 = \sum_{n=1}^{\infty} \nu_n / n.$$

Then $\nu_0 \sim \nu \sim \mu$ and

$$k\nu_0 \le \nu + \sum_{n=1}^{k} k\nu_n \quad \text{for all } k. \tag{3.3}$$

Since $\operatorname{supp} \nu_n \subset \operatorname{cl}(I_{n-1} \setminus I_n)$, it follows that $1 \notin \operatorname{supp} \nu_n$. Hence $|\psi_{\nu_n}| = 1$ on $\mathcal{M}_1$ for every $n$. Since $x \in \mathcal{M}_1$, by (3.3),

$$|\psi_\nu(x)| = |\psi_\nu(x)| \prod_{n=1}^{k} |\psi_{\nu_n}(x)|^k \le |\psi_{\nu_0}(x)|^k \quad \text{for all } k.$$

Since $\psi_\nu(x) \ne 0$, we have $|\psi_{\nu_0}(x)| = 1$, so that $x \notin \mathcal{W}(\mu)$. This contradicts (3.2). Thus if $\mu(\{1\}) = 0$, then $x \in \mathcal{Z}(\mu)$.

Next, suppose that $\mu(\{1\}) = c > 0$. Write $\mu = c\delta_1 + \mu_1$, where $\mu_1 \perp \delta_1$. Then by Lemma 3.3(i), $\mathcal{W}(\mu) = \{|\psi_{\delta_1}| < 1\} \cup \mathcal{W}(\mu_1)$, so that we may rewrite condition (3.2) as

$$x \in \mathcal{W}(\mu_1) \setminus \bigcup_{\{\zeta \in \partial D;\, \mu_1(\{\zeta\}) \ne 0\}} \{|\psi_{\delta_\zeta}| < 1\}.$$

By the previous paragraph, $x \in \mathcal{Z}(\mu_1)$. By Lemma 3.3(ii), $\mathcal{Z}(\mu_1) \subset \mathcal{Z}(\mu)$. Hence $x \in \mathcal{Z}(\mu)$. ∎

COROLLARY 3.6. *Let $\mu \in M_s^+$ and $\zeta \in \partial D$. If $\mu(\{\zeta\}) = 0$, then $\mathcal{Z}(\mu) \cap \mathcal{M}_\zeta = \mathcal{W}(\mu) \cap \mathcal{M}_\zeta$.*

PROPOSITION 3.7. *Let $\mu \in M_s^+$ and $E$ be a closed subset of $\partial D$. Let $A$ be an $F_\sigma$-subset of $\mathcal{M}$ such that $A \cap \overline{\bigcup_{\xi \in \partial D \setminus E} \mathcal{M}_\xi} = \emptyset$. If $\mu(E) = 0$, then there exists $\nu \in M_s^+$ such that $\nu \sim \mu$ and $|\psi_\nu| = 1$ on $A$.*

*Proof.* By our assumption, $A = \bigcup_{j=1}^{\infty} A_j$, where $A_j$ is a closed set. Then there is a sequence $\{U_j\}_j$ of open subsets of $\mathcal{M}$ such that

$$A_j \subset U_j, \quad \overline{U}_j \cap \overline{\bigcup_{\xi \in \partial D \setminus E} \mathcal{M}_\xi} = \emptyset \quad \text{for every } j. \tag{3.4}$$

Let $I_0 = \partial D$ and $\{I_n\}_n$ be a sequence of open subsets of $\partial D$ such that $I_n \subset I_{n-1}$ and $\bigcap_{n=1}^{\infty} I_n = E$. Set $\mu_n = \mu_{|(I_{n-1} \setminus I_n)}$. Since $\mu(E) = 0$, we have $\mu = \sum_{n=1}^{\infty} \mu_n$. Since $E \cap \operatorname{supp} \mu_n = \emptyset$, it follows that $|\psi_{\mu_n}| = 1$ on $\bigcup_{\zeta \in E} \mathcal{M}_\zeta$. Then by (3.4), $\overline{U}_j \setminus D \subset \bigcup_{\zeta \in E} \mathcal{M}_\zeta$. Hence for every $n$ and $j$,

$$|\psi_{\mu_n}(z)| \to 1 \quad \text{as } |z| \to 1 \text{ and } z \in U_j \cap D. \tag{3.5}$$

Let $\{\varepsilon_n\}_n$ be a sequence of positive numbers such that

$$\prod_{n=1}^{\infty} \varepsilon_n > 0, \quad 0 < \varepsilon_n < 1 \quad \text{for every } n. \tag{3.6}$$

Then by (3.5), there exists a sequence $\{a_n\}_n$ of positive numbers such that $0 < a_n < 1$ and

$$|\psi_{\mu_n}(z)|^{a_n} \geq \varepsilon_n \quad \text{on } U_j \cap D \text{ for } 1 \leq j \leq n. \tag{3.7}$$

Let

$$\nu = \sum_{n=1}^{\infty} a_n \mu_n.$$

Then $\nu \in M_{\mathrm{s}}^+$, $\nu \sim \mu$, and for any positive integers $j$ and $m$, we have

$$\begin{aligned} \liminf_{|z|\to 1,\, z \in U_j \cap D} |\psi_\nu(z)| &= \liminf_{|z|\to 1,\, z \in U_j \cap D} \prod_{n=1}^{\infty} |\psi_{\mu_n}(z)|^{a_n} \\ &= \liminf_{|z|\to 1,\, z \in U_j \cap D} \prod_{n=m}^{\infty} |\psi_{\mu_n}(z)|^{a_n} \quad \text{by (3.5)} \\ &\geq \prod_{n=m}^{\infty} \varepsilon_n \quad \text{by (3.7)}. \end{aligned}$$

Hence by (3.6),

$$\liminf_{|z|\to 1,\, z \in U_j \cap D} |\psi_\nu(z)| = 1 \quad \text{for every } j.$$

By the corona theorem and (3.4), $A_j \subset \overline{U_j \cap D}$. Therefore $|\psi_\nu| = 1$ on $A_j$ for every $j$. Thus $|\psi_\nu| = 1$ on $A$. ∎

Corollary 3.8. *Let $\mu \in M_{\mathrm{s}}^+$ and $E$ be a closed subset of $\partial D$. If $\mu(E) = 0$, then*

$$\mathcal{Z}(\mu) \subset \mathcal{W}(\mu) \subset \overline{\bigcup_{\xi \in \partial D \setminus E} \mathcal{M}_\xi}.$$

This follows from Proposition 3.7.

Corollary 3.9. *Let $\mu \in M_{\mathrm{s}}^+$. Then $\mathcal{W}(\mu) = \mathcal{Z}(\mu)$ if and only if $\mu \in M_{\mathrm{s,c}}^+$.*

*Proof.* Suppose that $\mu(\{\zeta\}) > 0$ for some $\zeta \in \partial D$. Write $\mu = a\delta_\zeta + \mu_1$, where $\mu_1(\{\zeta\}) = 0$. Then by Lemma 3.3,

$$\mathcal{W}(\mu) = \{|\psi_{\delta_\zeta}| < 1\} \cup \mathcal{W}(\mu_1), \quad \mathcal{Z}(\mu) = Z(\psi_{\delta_\zeta}) \cup \mathcal{Z}(\mu_1).$$

Since $\{|\psi_{\delta_\zeta}| < 1\} \cap \overline{\bigcup_{\{\xi \in \partial D;\, \xi \neq \zeta\}} \mathcal{M}_\xi} = \emptyset$, by Corollary 3.8 we have

$$\mathcal{W}(\mu) \cap \{|\psi_{\delta_\zeta}| < 1\} = \{|\psi_{\delta_\zeta}| < 1\}, \quad \mathcal{Z}(\mu) \cap \{|\psi_{\delta_\zeta}| < 1\} = Z(\psi_{\delta_\zeta}).$$

Therefore $\mathcal{W}(\mu) \neq \mathcal{Z}(\mu)$

The converse follows from Theorem 3.5. ∎

Corollary 3.10. *Let $\mu \in M^+_{\mathrm{s,c}}$ be such that $x \in \mathcal{Z}(\mu)$. Let $y \in \mathcal{M} \setminus D$ and* $\operatorname{supp} \mu_x \subset \operatorname{supp} \mu_y$. *Then $y \in \mathcal{Z}(\mu)$.*

*Proof.* Let $\nu \in M^+_{\mathrm{s}}$ and $\nu \sim \mu$. Since $\psi_\nu(x) = 0$, we have $|\psi_\nu(y)| < 1$. Hence $y \in \mathcal{W}(\mu)$. By Corollary 3.9, $y \in \mathcal{Z}(\mu)$. ∎

**4. Blaschke products and singular inner functions.** Let $b$ be a Blaschke product with zeros $\{z_n\}_n$. Recall that $S(b)$ is the set of cluster points of $\{z_n\}_n$ in $\partial D$. Then $S(b)$ is the set of points in $\partial D$ to which $b$ may not be extended continuously. Moreover, we have

$$\{|b| < 1\} \cap \overline{\bigcup_{\xi \in \partial D \setminus S(b)} \mathcal{M}_\xi} = \emptyset. \tag{4.1}$$

There exists a sequence $\{p_n\}_n$ of positive integers such that $p_n \to \infty$ as $n \to \infty$ and

$$b_1(z) = \prod_{n=1}^{\infty} \left( \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z} \right)^{p_n}, \quad z \in D,$$

is a Blaschke product. Then $S(b_1) = S(b)$ and

$$\overline{\{|b| < 1\}} \subset Z(b_1) \subset \{|b_1| < 1\}.$$

Hence by (4.1),

$$\overline{\{|b| < 1\}} \cap \overline{\bigcup_{\xi \in \partial D \setminus S(b)} \mathcal{M}_\xi} = \emptyset. \tag{4.2}$$

Moreover, if

$$\lim_{k \to \infty} \prod_{n\,:\,n \neq k} \left| \frac{z_n - z_k}{1 - \overline{z}_k z_n} \right| = 1,$$

then both $b$ and $\{z_n\}_n$ are called *sparse* (or *thin*).

Suppose that $b$ is sparse. Then

$$\{|b| < 1\} = \bigcup_{x \in Z(b)} P(x) \tag{4.3}$$

(see [7, 9]). For every sequence $\{z_n\}_n$ in $D$ with $|z_n| \to 1$ as $n \to \infty$, there exists a sparse subsequence of $\{z_n\}_n$ (see [6]).

Lemma 4.1. *Let $b$ be a sparse Blaschke product. Let $\varphi$ be an inner function such that $|\varphi| = 1$ on $Z(b)$. Then $|\varphi| = 1$ on $\overline{\{|b| < 1\}}$.*

*Proof.* Let $y \in \{|b| < 1\}$. Then by (4.3), $y \in P(x)$ for some $x \in Z(b)$. By [4, p. 143], $\operatorname{supp} \mu_y = \operatorname{supp} \mu_x$. Since $|\varphi(x)| = 1$, we have $\varphi = \varphi(x)$ on $\operatorname{supp} \mu_y$. Hence $\varphi(y) = \int_{M(L^\infty)} \varphi \, d\mu_y = \varphi(x)$. Thus $|\varphi(y)| = 1$. ∎

First, we prove the following.

PROPOSITION 4.2. *Let $\mu \in M_s^+$. Then there is a sparse Blaschke product $b$ such that $S(b) = \operatorname{supp}\mu$ and $\mathcal{Z}(\mu) \cap \overline{\{|b| < 1\}} = \emptyset$.*

*Proof.* Since $|\psi_\mu| = 1$ on $M(L^\infty)$, by the corona theorem there exists a sequence $\{z_n\}_n$ in $D$ such that $|\psi_\mu(z_n)| \to 1$ as $n \to \infty$ and $\operatorname{cl}\{z_n\}_n \setminus \{z_n\}_n = \operatorname{supp}\mu$. Considering a subsequence, we may assume that $\{z_n\}_n$ is sparse. Let $b$ be the associated Blaschke product. Then $S(b) = \operatorname{supp}\mu$ and $|\psi_\mu| = 1$ on $Z(b)$. By Lemma 4.1, $Z(\psi_\mu) \cap \overline{\{|b| < 1\}} = \emptyset$. Thus $\mathcal{Z}(\mu) \cap \overline{\{|b| < 1\}} = \emptyset$. ∎

COROLLARY 4.3. *Let $b$ be a Blaschke product. If $\mu \in M_s^+$ and $\mu(S(b)) = 0$, then $\mathcal{Z}(\mu) \cap \overline{\{|b| < 1\}} = \emptyset$.*

*Proof.* By (4.2), $\overline{\{|b| < 1\}} \cap \overline{\bigcup_{\xi \in \partial D \setminus S(b)} \mathcal{M}_\xi} = \emptyset$; now apply Corollary 3.8. ∎

COROLLARY 4.4. *Let $\mu \in M_{s,c}^+$. Then $Z(b) \not\subset \mathcal{Z}(\mu)$ for every Blaschke product $b$.*

*Proof.* Let $\{z_n\}_n$ be the zeros of $b$ in $D$. Then there is a subsequence $\{z_{n_j}\}_j$ such that $z_{n_j} \to \zeta$ for some $\zeta \in \partial D$. Let $b_1$ be the Blaschke product with zeros $\{z_{n_j}\}_j$. Then $S(b_1) = \{\zeta\}$. Hence by Corollary 4.3, $\mathcal{Z}(\mu) \cap Z(b_1) = \emptyset$. Since $Z(b_1) \subset Z(b)$, we obtain our assertion. ∎

COROLLARY 4.5. *Let $\mu \in M_{s,c}^+$. Then* $\operatorname{int}\mathcal{Z}(\mu) = \emptyset$.

*Proof.* Suppose that $\operatorname{int}\mathcal{Z}(\mu) \neq \emptyset$. Then there is an interpolating Blaschke product $b$ such that $Z(b) \subset \operatorname{int}\mathcal{Z}(\mu)$. But by Corollary 4.4, $Z(b) \not\subset \mathcal{Z}(\mu)$. This is a contradiction. ∎

We have $\mathcal{W}(\mu) \cap M(L^\infty) = \emptyset$ for every $\mu \in M_s^+$. Then by Corollary 3.8, for each $\zeta \in \partial D$ we have

$$\mathcal{M}_\zeta \cap \overline{\bigcup_{\{\mu \in M_s^+;\, \mu(\{\zeta\})=0\}} \mathcal{W}(\mu)} \subset \mathcal{M}_\zeta \cap \overline{\bigcup_{\{\xi \in \partial D;\, \xi \neq \zeta\}} \mathcal{M}_\xi}.$$

Moreover we have the following.

PROPOSITION 4.6. *Let $\zeta \in \partial D$. Then*

$$\mathcal{M}_\zeta \cap \overline{\bigcup_{\{\mu \in M_s^+;\, \mu(\{\zeta\})=0\}} \mathcal{W}(\mu)} \subsetneq \mathcal{M}_\zeta \cap \overline{\bigcup_{\{\xi \in \partial D;\, \xi \neq \zeta\}} \mathcal{M}_\xi}.$$

To prove this, we need a lemma.

LEMMA 4.7. *Let $\zeta \in \partial D$. Then there exists a sparse Blaschke product $b$ satisfying the following conditions.*

(i) $S(b) = \{\zeta\}$.
(ii) *Let $\mu \in M_s^+$. Then there exists $\nu \in M_s^+$ such that $\nu \sim \mu$ and $|\psi_\nu| = 1$ on $\overline{\{|b| < 1\}}$.*

*Proof.* There exists a sequence $\{z_n\}_n$ in $D$ such that $|\psi_{\delta_\zeta}(z_n)| \to 1$ and $z_n \to \zeta$ as $n \to \infty$. Considering a subsequence, we may assume that $\{z_n\}_n$ is sparse. Let $b$ be the Blaschke product with zeros $\{z_n\}_n$. Then $S(b) = \{\zeta\}$, and by (4.2),

$$\overline{\{|b|<1\}} \cap \overline{\bigcup_{\{\xi\in\partial D;\, \xi\neq\zeta\}} \mathcal{M}_\xi} = \emptyset.$$

Let $\mu \in M_s^+$. Write $\mu = a\delta_\zeta + \mu_1$, where $\mu_1(\{\zeta\}) = 0$. Then by Proposition 3.7, there exists $\nu_1 \in M_s^+$ such that $\nu_1 \sim \mu_1$ and $|\psi_{\nu_1}| = 1$ on $\overline{\{|b|<1\}}$. Since $Z(b) = \overline{\{z_n\}_n} \setminus \{z_n\}_n$, it follows that $|\psi_{\delta_\zeta}| = 1$ on $Z(b)$. By Lemma 4.1, $|\psi_{\delta_\zeta}| = 1$ on $\overline{\{|b|<1\}}$. Put $\nu = a\delta_\zeta + \nu_1$. Then $\nu \sim \mu$ and $|\psi_\nu| = |\psi_{\delta_\zeta}|^a |\psi_{\nu_1}| = 1$ on $\overline{\{|b|<1\}}$. ∎

*Proof of Proposition 4.6.* We may assume that $\zeta = 1$. Let $\{J_n\}_n$ be a sequence of open subarcs of $\partial D$ such that $J_n \subsetneq J_{n-1}$ and $\bigcap_{n=1}^\infty J_n = \{1\}$. Then there is a sequence $\{\xi_n\}_n$ such that $\xi_n$ is an interior point of $J_n \setminus J_{n-1}$ and $\xi_n \to 1$ as $n \to \infty$. We may assume that $\xi_n \neq \xi_k$ for $n \neq k$. Let $\mu \in M_s^+$ and $\mu(\{1\}) = 0$. Put $\mu_n = \mu_{|(J_{n-1}\setminus J_n)}$. Then $\mu = \sum_{n=1}^\infty \mu_n$. For each $n$, by Lemma 4.7 there exist a sparse Blaschke product $q_n$ and $\nu_n \in M_s^+$ such that $S(q_n) = \{\xi_n\}$, $\nu_n \sim \mu_n$, $\|\nu_n\| = \|\mu_n\|$, and $|\psi_{\nu_n}| = 1$ on $Z(q_n)$. Let $\nu = \sum_{n=1}^\infty \nu_n$. Then $\nu \in M_s^+$ and $\nu \sim \mu$. Since $\xi_n \notin \operatorname{supp}(\nu - \nu_n)$, we have $|\psi_{\nu-\nu_n}| = 1$ on $\mathcal{M}_{\xi_n}$. Since $S(q_n) = \{\xi_n\}$, it follows that $Z(q_n) \subset \mathcal{M}_{\xi_n}$. Hence

$$|\psi_\nu| = |\psi_{\nu-\nu_n}|\,|\psi_{\nu_n}| = 1 \quad \text{on } Z(q_n). \tag{4.4}$$

Let $\{w_{n,k}\}_k$ be the zeros of $q_n$. Then $w_{n,k} \to \xi_n$ as $k \to \infty$. Since $\xi_n \neq \xi_k$ for $n \neq k$, there is a sequence $\{N_n\}_n$ of positive integers such that $\{w_{n,k};\, k \geq N_n,\, n = 1,2,\ldots\}$ is a sparse sequence (see [8, Lemma 1.5]). Since $\xi_n \to 1$, taking $N_n$ sufficiently large, we may assume that $\mathrm{cl}\{w_{n,k};\, k \geq N_n\} \setminus \{w_{n,k};\, k \geq N_n\} = \{1\} \cup \{\xi_n\}_n$. Let $b$ be the associated sparse Blaschke product. Then $\bigcup_{n=1}^\infty Z(q_n) \subset Z(b)$ and $Z(b) \setminus \bigcup_{n=1}^\infty Z(q_n) \subset \mathcal{M}_1$. Hence by (4.4), $\{|\psi_\nu| < 1\} \cap Z(b) \subset \mathcal{M}_1$.

For each positive integer $j$, let $b_j$ be a subproduct of $b$ with zeros

$$\{w_{n,k};\, |\psi_\nu(w_{n,k})| < 1 - 1/j,\, k \geq N_n,\, n = 1,2,\ldots\}.$$

Then $Z(b_j) \subset \{|\psi_\nu| < 1\} \cap Z(b) \subset \mathcal{M}_1$. Hence

$$Z(b_j) \cap \overline{\bigcup_{\{\xi\in\partial D;\, \xi\neq 1\}} \mathcal{M}_\xi} = \emptyset.$$

We also have

$$\bigcup_{j=1}^\infty Z(b_j) = \{|\psi_\nu| < 1\} \cap Z(b).$$

Therefore by Proposition 3.7 (considering $E = \{1\}$), there exists $\lambda \in M_{\mathrm{s}}^{+}$ such that $\lambda \sim \mu$ and

$$(4.5) \qquad |\psi_\lambda| = 1 \quad \text{on } \{|\psi_\nu| < 1\} \cap Z(b).$$

Let $\sigma = \nu \wedge \lambda$. Then $\sigma \sim \mu$ and $|\psi_\sigma| \geq \max\{|\psi_\nu|, |\psi_\lambda|\}$. Hence by (4.5), $|\psi_\sigma| = 1$ on $Z(b)$. By Lemma 4.1, $\{|\psi_\sigma| < 1\} \cap \{|b| < 1\} = \emptyset$. Thus $\mathcal{W}(\mu) \cap \{|b| < 1\} = \emptyset$, so that

$$\{|b| < 1\} \cap \overline{\bigcup_{\{\mu \in M_{\mathrm{s}}^{+};\, \mu(\{1\})=0\}} \mathcal{W}(\mu)} = \emptyset.$$

Since $\{|b| < 1\} \cap \mathcal{M}_{\xi_n} \neq \emptyset$, it is not difficult to see that

$$\{|b| < 1\} \cap \mathcal{M}_1 \cap \overline{\bigcup_{\{\xi \in \partial D;\, \xi \neq 1\}} \mathcal{M}_\xi} \neq \emptyset.$$

Thus we get our assertion. ■

By Lemma 4.7, we have the following.

PROPOSITION 4.8. *Let $\zeta \in \partial D$. Then there exists a Blaschke product $b$ such that $S(b) = \{\zeta\}$ and $\mathcal{Z}(\mu) \cap \overline{\{|b| < 1\}} = \emptyset$ for every $\mu \in M_{\mathrm{s}}^{+}$.*

One may ask whether there is a Blaschke product $b$ such that $S(b) = \partial D$ and $\mathcal{Z}(\mu) \cap \overline{\{|b| < 1\}} = \emptyset$ for every $\mu \in M_{\mathrm{s}}^{+}$. The following says that the answer is "no".

THEOREM 4.9. *Let $b$ be a Blaschke product such that $S(b) = \partial D$. Then*

(i) *$\mathcal{Z}(\delta_\zeta) \cap Z(b) \neq \emptyset$ for some $\zeta \in \partial D$.*
(ii) *$\mathcal{Z}(\mu) \cap Z(b) \neq \emptyset$ for some $\mu \in M_{\mathrm{s,c}}^{+}$.*

*Proof.* Let

$$(4.6) \qquad \Gamma(e^{i\theta}) = \left\{ z \in D;\ \frac{|e^{i\theta} - z|}{1 - |z|} < 2 \right\}.$$

Then

$$(4.7) \qquad \lim_{|z| \to 1,\, z \in \Gamma(e^{i\theta})} \psi_{\delta_{e^{i\theta}}}(z) = 0$$

(see [5, p. 76]). Let $b$ be a Blaschke product such that $S(b) = \partial D$. Let $\{z_n\}_n$ be the zeros of $b$. Write

$$z_n = r_n e^{i\theta_n}.$$

By induction, we shall choose a subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$. Put $n_1 = 1$. Since $S(b) = \partial D$, $\{e^{i\theta_n}\}_n$ is dense in $\partial D$. Then by (4.6), there exists a positive integer $n_2$ such that

$$z_{n_1} \in \Gamma(e^{i\theta_{n_2}}), \quad \theta_{n_1} < \theta_{n_2}, \quad \theta_{n_2} - \theta_{n_1} < 1/2.$$

Then $z_{n_2} \in \Gamma(e^{i\theta_{n_2}})$, so that there exists $n_3$ such that

$$z_{n_1}, z_{n_2} \in \Gamma(e^{i\theta_{n_3}}), \quad \theta_{n_2} < \theta_{n_3}, \quad \theta_{n_3} - \theta_{n_2} < 1/2^2.$$

Continuing, we get $\{z_{n_j}\}_j$ satisfying

$$\text{(4.8)} \quad z_{n_k} \in \Gamma(e^{i\theta_{n_j}}) \quad \text{for } 1 \le k \le j, \quad \theta_{n_j} < \theta_{n_{j+1}} \quad \theta_{n_{j+1}} - \theta_{n_j} < 1/2^{j+1}.$$

Thus $\theta_{n_j} \to \theta_0$ as $j \to \infty$ for some $\theta_0$. By (4.8), $z_{n_k} \in \operatorname{cl} \Gamma(e^{i\theta_0})$ for every $k$. Then by (4.7), $\psi_{\delta_{e^{i\theta_0}}}(z_{n_k}) \to 0$ as $k \to \infty$, so that $Z(\psi_{\delta_{e^{i\theta_0}}}) \cap Z(b) \ne \emptyset$. Therefore we get $\mathcal{Z}(\delta_{e^{i\theta_0}}) \cap Z(b) \ne \emptyset$.

To prove (ii), we need to work more. In the proof of (i), we choose one point in each step. In the proof of (ii), we choose two points. Let

$$\Lambda_k = \{(\alpha_1, \ldots, \alpha_k); \alpha_j = 0 \text{ or } 1\}, \quad \Lambda_\infty = \{(\alpha_1, \alpha_2, \ldots); \alpha_j = 0 \text{ or } 1\}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \Lambda_k$, put $|\alpha| = k$ and $\alpha^j = (\alpha_1, \ldots, \alpha_j)$ for $j \le k$. By induction, we shall choose a sequence $\{n_\alpha; \alpha \in \Lambda_k\}$, $k = 1, 2, \ldots,$ of finite sets of positive integers. Take positive integers $n_0$ and $n_1$ such that $\theta_{n_0} < \theta_{n_1}$. We have

$$z_{n_0} \in \Gamma(e^{i\theta_{n_0}}), \quad z_{n_1} \in \Gamma(e^{i\theta_{n_1}}).$$

Then take $n_{(l,m)}$ for $l, m = 0, 1$ such that

$$\begin{gathered} z_{n_l} \in \Gamma(e^{i\theta_{n_{(l,m)}}}) \quad \text{for } l, m = 0, 1, \\ 0 < |\theta_{n_{(l,m)}} - \theta_{n_l}| < |\theta_{n_1} - \theta_{n_0}|/4 \quad \text{for } l, m = 0, 1, \\ \theta_{n_{(l,m)}} \ne \theta_{n_{(t,s)}} \quad \text{if } (l, m) \ne (t, s). \end{gathered}$$

Assume that $\{n_\alpha; \alpha \in \Lambda_j\}$, $1 \le j \le k$, are chosen so that $z_{n_{\alpha^j}} \in \Gamma(e^{i\theta_{n_\alpha}})$ for $1 \le j \le |\alpha|$ and $\theta_{n_\alpha} \ne \theta_{n_\beta}$ for $\alpha, \beta \in \bigcup_{j=1}^k \Lambda_j, \alpha \ne \beta$. Let $\alpha \in \Lambda_k$. Take $n_{(\alpha,0)}$ and $n_{(\alpha,1)}$ such that

$$\text{(4.9)} \qquad z_{n_{\alpha^j}} \in \Gamma(e^{i\theta_{n_{(\alpha,l)}}}) \quad \text{for } 1 \le j \le k \text{ and } l = 0, 1,$$

$$\text{(4.10)} \quad \begin{aligned} 0 &< |\theta_{n_{(\alpha,l)}} - \theta_{n_\alpha}| \\ &< \frac{1}{4} \min\Big\{ |\theta_{n_\lambda} - \theta_{n_\gamma}|; \lambda, \gamma \in \bigcup_{j=1}^{k} \Lambda_j, \lambda \ne \gamma \Big\} \quad \text{for } l = 0, 1. \end{aligned}$$

This finishes our induction.

Let $\alpha = (\alpha_1, \alpha_2, \ldots) \in \Lambda_\infty$. Put $\alpha^k = (\alpha_1, \ldots, \alpha_k) \in \Lambda_k$. Then by (4.10),

$$|\theta_{n_{\alpha^k}} - \theta_{n_{\alpha^j}}| < \left(\frac{1}{4}\right)^{k-1} \left( \sum_{l=1}^{j-k} \left(\frac{1}{4}\right)^l \right) |\theta_{n_1} - \theta_{n_0}| \quad \text{for } j > k.$$

Hence $\{\theta_{n_{\alpha^k}}\}_k$ converges to some point, say $\theta_\alpha$. By (4.9),

$$\text{(4.11)} \qquad z_{n_{\alpha^j}} \in \Gamma(e^{i\theta_\alpha}) \quad \text{for every } j.$$

Let $\beta \in \Lambda_\infty$ and $\alpha \neq \beta$. Then we may assume that

$$\alpha = (\alpha_1, \ldots, \alpha_k, 0, \alpha_{k+2}, \ldots), \qquad \beta = (\alpha_1, \ldots, \alpha_k, 1, \beta_{k+2}, \ldots).$$

By (4.10), we have

$$|\theta_{n_{\alpha^j}} - \theta_{n_{(\alpha_1,\ldots,\alpha_k,0)}}| < \sum_{l=1}^{j-k-1} \left(\frac{1}{4}\right)^l |\theta_{n_{(\alpha_1,\ldots,\alpha_k,0)}} - \theta_{n_{(\alpha_1,\ldots,\alpha_k,1)}}| \quad \text{for } j \geq k+2.$$

Hence

$$|\theta_\alpha - \theta_{n_{(\alpha_1,\ldots,\alpha_k,0)}}| < \frac{1}{3}\, |\theta_{n_{(\alpha_1,\ldots,\alpha_k,0)}} - \theta_{n_{(\alpha_1,\ldots,\alpha_k,1)}}|.$$

Similarly,

$$|\theta_\beta - \theta_{n_{(\alpha_1,\ldots,\alpha_k,1)}}| < \frac{1}{3}\, |\theta_{n_{(\alpha_1,\ldots,\alpha_k,0)}} - \theta_{n_{(\alpha_1,\ldots,\alpha_k,1)}}|.$$

Thus we get $\theta_\alpha \neq \theta_\beta$. By our construction, $\{\theta_\alpha;\ \alpha \in \Lambda_\infty\}$ is the set of cluster points of $\bigcup_{k=1}^\infty \{\theta_{n_\alpha};\ \alpha \in \Lambda_k\}$. Hence $\{\theta_\alpha;\ \alpha \in \Lambda_\infty\}$ is a perfect set. Then there exists $\mu \in M^+_{\mathrm{s,c}}$ such that $\operatorname{supp} \mu \subset \{\theta_\alpha;\ \alpha \in \Lambda_\infty\}$. By [5, p. 76],

$$\lim_{|z| \to 1,\, z \in \Gamma(\theta_\alpha)} \psi_\mu(z) = 0$$

for some $\alpha \in \Lambda_\infty$. Therefore by (4.11), we have $Z(\psi_\mu) \cap Z(b) \neq \emptyset$. By Lemma 3.2, we obtain $\mathcal{Z}(\mu) \cap Z(b) \neq \emptyset$. ∎

Here we have the following problem.

PROBLEM 4.10. Does there exist an interpolating Blaschke product $b_0$ such that $S(b_0) = \partial D$ and $\mathcal{Z}(\mu) \cap Z(b_0) \neq \emptyset$ for every $\mu \in M^+_{\mathrm{s}}$?

**5. Construction of interpolating Blaschke products.** For a measurable subset $E$ of $\partial D$, we denote by $|E|$ the Lebesgue measure of $E$. In this section, for a given closed subset $K$ of $\partial D$ with $|K| = 0$, we construct a special interpolating Blaschke product $b_K$ associated with $K$. In Section 6, we shall prove that $Z(b_K) \cap \mathcal{Z}(\mu) \neq \emptyset$ for every $\mu \in M^+_{\mathrm{s}}$ with $\operatorname{supp} \mu \subset K$.

THEOREM 5.1. *Let $K$ be a closed subset of $\partial D$ with $|K| = 0$. Then there exists a sequence $\{J_{n,j}\}_{j=1}^{N_n}$, $n = 1, 2, \ldots,$ of open arcs such that for every $n$ and $k$,*

(i) $$K \subset \bigcup_{j=1}^{N_n} J_{n,j} \subset \bigcup_{j=1}^{N_{n-1}} J_{n-1,j},$$

(ii) $$\sum_j \{|J_{n,j}|;\ J_{n,j} \subset J_{n-1,k}\} \leq |J_{n-1,k}|/2.$$

*Let* $e^{i\theta_{n,j}}$ *be the center of the arc* $J_{n,j}$ *and*

$$z_{n,j} = \left(1 - \frac{|J_{n,j}|}{2\pi}\right) e^{i\theta_{n,j}}.$$

*Then* $\{z_{n,j};\ 1 \le j \le N_n,\ n = 1, 2, \ldots\}$ *is an interpolating sequence and the set of cluster points of* $\{z_{n,j};\ 1 \le j \le N_n,\ n = 1, 2, \ldots\}$ *in the closed unit disk* $\overline{D}$ *coincides with* $K$.

Let $b_K$ be the Blaschke product with zeros $\{z_{n,j};\ 1 \le j \le N_n, n = 1, 2, \ldots\}$. We call $b_K$ the interpolating Blaschke product *associated with* $K$.

*Proof of Theorem 5.1.* Let $K$ be a non-empty closed subset of $\partial D$ and $|K| = 0$. Then $K$ is totally disconnected. For an open arc $V$ of $\partial D$ such that $V \cap K$ is a non-empty closed set, there are finitely many disjoint open arcs $\{V_j\}_{j=1}^{k}$ of $\partial D$ such that $V_j \cap K$ are non-empty closed sets and

$$V \cap K \subset \bigcup_{j=1}^{k} V_j \subset V, \qquad \sum_{j=1}^{k} |V_j| \le |V|/2.$$

Now using the above fact inductively, we shall choose a family $\{J_{n,j}\}_{j=1}^{N_n}$ of open arcs for each $n$. Let $J_0 = \partial D$. Put $V = J_0$ in the above; then there are finitely many disjoint open arcs $\{J_{1,j}\}_{j=1}^{N_1}$ of $\partial D$ such that $J_{1,j} \cap K$ are non-empty closed sets and

$$J_0 \cap K \subset \bigcup_{j=1}^{N_1} J_{1,j} \subset J_0, \qquad \sum_{j=1}^{N_1} |J_{1,j}| \le |J_0|/2.$$

We proceed to the next step. For each $J_{1,j}, 1 \le j \le N_1$, there are finitely many disjoint open arcs $\{J_{1,j,l}\}_{l=1}^{m_j}$ of $\partial D$ such that $J_{1,j,l} \cap K$ are non-empty closed sets and

$$J_{1,j} \cap K \subset \bigcup_{l=1}^{m_j} J_{1,j,l} \subset J_{1,j}, \qquad \sum_{l=1}^{m_j} |J_{1,j,l}| \le |J_{1,j}|/2.$$

Let $N_2 = \sum_{j=1}^{N_1} m_j$ and

$$\{J_{2,j}\}_{j=1}^{N_2} = \{J_{1,j,l};\ 1 \le j \le N_1,\ 1 \le l \le m_j\}.$$

We have

$$K \subset \bigcup_{j=1}^{N_2} J_{2,j}.$$

Continuing this process, at the $n$th step we have a finite family $\{J_{n,j}\}_{j=1}^{N_n}$ of disjoint open arcs of $\partial D$ such that for $1 \le k \le N_{n-1}$,

$$J_{n-1,k} \cap K \subset \bigcup_{j} \{J_{n,j};\ J_{n,j} \subset J_{n-1,k}\} \subset J_{n-1,k}, \tag{5.1}$$

$$J_{n,j} \cap K \text{ is non-empty closed for every } j \text{ with } 1 \le j \le N_n,$$

(5.2)
$$K \subset \bigcup_{j=1}^{N_n} J_{n,j} \subset \bigcup_{j=1}^{N_{n-1}} J_{n-1,j},$$

(5.3)
$$\sum_j \{|J_{n,j}|;\ J_{n,j} \subset J_{n-1,k}\} \le |J_{n-1,k}|/2.$$

Thus we get the first half of our assertion.

By the above, we have

(5.4)
$$\bigcap_{n=1}^{\infty} \bigcup_{j=1}^{N_n} J_{n,j} = K.$$

Let $1 \le j \le N_n$. For $l > n$, we have

$$\begin{aligned} \sum_t \{|J_{l,t}|;\ J_{l,t} \subset J_{n,j}\} &= \sum_k \sum_t \{|J_{l,t}|;\ J_{l,t} \subset J_{l-1,k} \subset J_{n,j}\} && \text{by (5.1)} \\ &= \frac{1}{2} \sum_k \{|J_{l-1,k}|;\ J_{l-1,k} \subset J_{n,j}\} && \text{by (5.3).} \end{aligned}$$

Hence

$$\sum_t \{|J_{l,t}|;\ J_{l,t} \subset J_{n,j}\} \le \left(\frac{1}{2}\right)^{l-n} |J_{n,j}|,$$

so that

(5.5)
$$\sum_{l=n}^{\infty} \sum_t \{|J_{l,t}|;\ J_{l,t} \subset J_{n,j}\} \le 2|J_{n,j}|.$$

For $n \ge 1$ and $1 \le j \le N_n$, let $e^{i\theta_{n,j}}$ be the center of the arc $J_{n,j}$,

$$z_{n,j} = \left(1 - \frac{|J_{n,j}|}{2\pi}\right) e^{i\theta_{n,j}},$$

and

(5.6)
$$R(z_{n,j}) = \{re^{i\theta};\ e^{i\theta} \in J_{n,j},\ 1 - |J_{n,j}|/2\pi \le r < 1\}.$$

Then $z_{n,j} \in R(z_{n,j})$ and $1 - |z_{n,j}| = |J_{n,j}|/2\pi$. By (5.4), $K$ is the set of cluster points of $\{z_{n,j};\ 1 \le j \le N_n,\ n = 1, 2, \ldots\}$ in $\overline{D}$.

We prove that $\{z_{n,j};\ 1 \le j \le N_n,\ n = 1, 2, \ldots\}$ is an interpolating sequence. It is not difficult to see that $\{z_{n,j}\}_{n,j}$ is $\varrho$-separated, that is,

$$\inf\{\varrho(z_{n,j}, z_{k,l});\ (n,j) \ne (k,l)\} > 0;$$

I leave the proof to the reader. To prove that $\{z_{n,j};\ 1 \le j \le N_n,\ n = 1, 2, \ldots\}$

is interpolating, it is sufficient to show that

$$\sigma = \sum_{n=1}^{\infty}\sum_{j=1}^{N_n}(1-|z_{n,j}|)\delta_{z_{n,j}} = \sum_{n=1}^{\infty}\sum_{j=1}^{N_n}|J_{n,j}|\delta_{z_{n,j}}/2\pi$$

is a Carleson measure (see [2] and also [5, pp. 286–287]). Let

$$\Omega = \{re^{i\theta};\ 1-\varepsilon \le r < 1,\ \theta_0 \le \theta \le \theta_0 + 2\pi\varepsilon\}, \quad \text{where } 0<\varepsilon<1, \tag{5.7}$$

be an arbitrary Carleson square. We need to show that there is an absolute constant $C$, independent of $\varepsilon$, such that

$$\sum_{n,j}\{|J_{n,j}|;\ z_{n,j}\in\Omega\} \le C\varepsilon. \tag{5.8}$$

By our construction, there exists a sequence $\{z_{n_k,j_k}\}_{k=1}^{\infty}$ (maybe a finite set) satisfying

$$z_{n_k,j_k}\in\Omega \quad \text{for every } k, \tag{5.9}$$

$$R(z_{n_k,j_k})\cap R(z_{n_i,j_l}) = \emptyset \quad \text{for every } k\ne l, \tag{5.10}$$

$$\text{if } z_{n,j}\in\Omega, \text{ there exists } k \text{ such that } R(z_{n,j})\subset R(z_{n_k,j_k}). \tag{5.11}$$

Then

$$\begin{aligned}\sum_{n,j}\{|J_{n,j}|;\ z_{n,j}\in\Omega\} &= \sum_{k=1}^{\infty}\Big(\sum_{n,j}\{|J_{n,j}|;\ R(z_{n,j})\subset R(z_{n_k,j_k})\}\Big) \quad \text{by (5.11)}\\ &= \sum_{k=1}^{\infty}\Big(\sum_{n,j}\{|J_{n,j}|;\ J_{n,j}\subset J_{n_k,j_k}\}\Big) \quad \text{by (5.6)}\\ &\le 2\sum_{k=1}^{\infty}|J_{n_k,j_k}| \quad \text{by (5.5).}\end{aligned}$$

By (5.6) and (5.10), $J_{n_k,j_k}\cap J_{n_m,j_m} = \emptyset$ if $k\ne m$. By (5.9),

$$J_{n_k,j_k}\cap\{e^{i\theta};\ \theta_0\le\theta\le\theta_0+2\pi\varepsilon\}\ne\emptyset$$

and

$$|\{e^{i\theta};\ \theta_0\le\theta\le\theta_0+2\pi\varepsilon\}| \ge |J_{n_k,j_k}|.$$

Hence by (5.7),

$$\sum_{k=1}^{\infty}|J_{n_k,j_k}| \le 6\pi\varepsilon.$$

Thus we get (5.8), so that $\{z_{n,j};\ 1\le j\le N_n,\ n=1,2,\dots\}$ is interpolating. This completes the proof. ∎

**6. Properties of $\mathcal{Z}(\mu)$.** First we prove the following theorem.

THEOREM 6.1. *Let $K$ be a non-empty closed subset of $\partial D$ with $|K| = 0$, and $\mu \in M_s^+$ be such that* $\operatorname{supp}\mu \subset K$. *Then $Z(b_K) \cap \mathcal{Z}(\mu) \neq \emptyset$, where $b_K$ is the interpolating Blaschke product associated with $K$.*

Let $K$ be a non-empty closed subset of $\partial D$ with $|K| = 0$. Generally, there are uncountably many measures $\{\mu_\alpha\}_{\alpha\in\Lambda}$ in $M_s^+$ such that $\operatorname{supp}\mu_\alpha \subset K$ and $\mu_\alpha \perp \mu_\beta$ if $\alpha \neq \beta$. By Theorems 3.1 and 6.1, $\{Z(b_K) \cap \mathcal{Z}(\mu_\alpha)\}_\alpha$ is a family of non-empty mutually disjoint subsets in $Z(b_K)$. So $b_K$ is a very convenient interpolating Blaschke product to study the properties of $\psi_\mu$ with $\operatorname{supp}\mu \subset K$.

*Proof of Theorem 6.1.* Let $\nu \in M_s^+$ and $\nu \sim \mu$. We show that

$$Z(b_K) \cap Z(\psi_\nu) \neq \emptyset. \tag{6.1}$$

Let $\{J_{n,j};\ 1 \le j \le N_n,\ n = 1, 2, \ldots\}$ and $\{z_{n,j};\ 1 \le j \le N_n,\ n = 1, 2, \ldots\}$ be families given in Theorem 5.1. First, we prove that

$$\limsup_{n\to\infty} \max_{1\le j\le N_n} \frac{\nu(J_{n,j})}{|J_{n,j}|} = \infty. \tag{6.2}$$

Suppose not. Then there exists a positive constant $C$ such that

$$\max_{1\le j\le N_n} \frac{\nu(J_{n,j})}{|J_{n,j}|} \le C \quad \text{for every } n. \tag{6.3}$$

Then for each $n$, we have

$$\begin{aligned}
\nu(K) &\le \sum_{j=1}^{N_n} \nu(J_{n,j}) && \text{by Theorem 5.1(i)}\\
&\le C \sum_{j=1}^{N_n} |J_{n,j}| && \text{by (6.3)}\\
&\le \frac{C}{2} \sum_{j=1}^{N_{n-1}} |J_{n-1,j}| && \text{by Theorem 5.1(ii)}\\
&\le \frac{2\pi C}{2^n}.
\end{aligned}$$

Hence $\nu(K) = 0$, contrary to our assumption, so that (6.2) holds.

By (6.2), there exist $\{n_k\}_k$ and $\{j_k\}_k$ such that $1 \le j_k \le N_{n_k}$ and

$$\frac{\nu(J_{n_k,j_k})}{|J_{n_k,j_k}|} \to \infty \quad \text{as } k \to \infty. \tag{6.4}$$

By Theorem 5.1,

$$|J_{n_k,j_k}| = 2\pi(1 - |z_{n_k,j_k}|). \tag{6.5}$$

Let $e^{it} \in J_{n_k,j_k}$. Then

$$\begin{aligned}|e^{it} - z_{n_k,j_k}| &\le \big||z_{n_k,j_k}| - e^{i\pi(1-|z_{n_k,j_k}|)}\big| \\ &\le (1 - |z_{n_k,j_k}|) + |1 - e^{i\pi(1-|z_{n_k,j_k}|)}| \\ &\le (1+\pi)(1 - |z_{n_k,j_k}|).\end{aligned}$$

Then

$$|P_{z_{n_k,j_k}}(e^{it})| = \frac{1 - |z_{n_k,j_k}|^2}{|e^{it} - z_{n_k,j_k}|^2} \ge \frac{1}{(1+\pi)^2(1 - |z_{n_k,j_k}|)}.$$

Hence by (6.5),

$$|P_{z_{n_k,j_k}}| \ge \frac{2\pi}{(1+\pi)^2|J_{n_k,j_k}|} \quad \text{on } J_{n_k,j_k}.$$

Consequently, we have

$$\begin{aligned}-\log|\psi_\nu(z_{n_k,j_k})| &= \int_0^{2\pi} P_{z_{n_k,j_k}}(e^{i\theta})\,d\nu(\theta) \ge \int_{J_{n_k,j_k}} P_{z_{n_k,j_k}}(e^{i\theta})\,d\nu(\theta) \\ &\ge \frac{2\pi\nu(J_{n_k,j_k})}{(1+\pi)^2|J_{n_k,j_k}|}.\end{aligned}$$

Therefore by (6.4), $\psi_\nu(z_{n_k,j_k}) \to 0$ as $k \to \infty$. Since $b_K$ is the Blaschke product with zeros $\{z_{n,j};\ 1 \le j \le N_n,\ n = 1,2,\ldots\}$, we obtain $Z(b_K) \cap Z(\psi_\nu) \ne \emptyset$. Then Lemma 3.2 yields the assertion. ■

Corollary 6.2. *Let $\mu \in M_{\mathrm{s}}^+$. Then $\mathcal{Z}(\mu)$ contains non-trivial points.*

*Proof.* Since $\mu$ is a singular measure, there exists a closed subset $K$ of $\partial D$ such that $|K| = 0$ and $\mu(K) > 0$. By Lemma 3.3(ii), $\mathcal{Z}(\mu_{|K}) \subset \mathcal{Z}(\mu)$, and by Theorem 6.1, $\emptyset \ne Z(b_K) \cap \mathcal{Z}(\mu_{|K}) \subset Z(b_K) \cap \mathcal{Z}(\mu)$. Since $b_K$ is interpolating, we have $Z(b_K) \subset G$. ■

Let $\mu \in M_{\mathrm{s}}^+$. We denote by $M(L^\infty(\mu))$ the maximal ideal space of the Banach algebra $L^\infty(\mu)$. Then $M(L^\infty(\mu))$ is a totally disconnected space. For $f \in L^\infty(\mu)$, let $\widehat{f}$ be the Gelfand transform of $f$. For a measurable subset $S$ of $\operatorname{supp}\mu$, there exists an open and closed subset $\widehat{S}$ of $M(L^\infty(\mu))$ such that $\widehat{\chi_S} = \chi_{\widehat{S}}$. Then the family $\{\chi_{\widehat{S}}\}_S$ coincides with the set of idempotents in $C(M(L^\infty(\mu)))$, the space of continuous functions on $M(L^\infty(\mu))$. We have $\widehat{S^{\mathrm{c}}} = (\widehat{S})^{\mathrm{c}}$. For each $x \in M(L^\infty(\mu))$, let

$$\Phi_\mu(x) = \bigcap_{\{S;\, x \in \widehat{S}\}} \mathcal{Z}(\mu_{|S}). \tag{6.6}$$

The set $\Phi_\mu(x)$ is a closed subset in $\mathcal{M}$ associated with the point $x \in M(L^\infty(\mu))$. It is interesting to study $\Phi_\mu(x)$ from the point of view of measures on $\partial D$.

We have the following.

THEOREM 6.3. *Let* $\mu \in M_{\mathrm{s}}^{+}$.

(i) $\emptyset \neq \Phi_\mu(x) \subset \mathcal{Z}(\mu)$ *for* $x \in M(L^\infty(\mu))$.
(ii) $\Phi_\mu(x) \cap \Phi_\mu(y) = \emptyset$ *if* $x, y \in M(L^\infty(\mu))$ *and* $x \neq y$.
(iii) $\mathcal{Z}(\mu) = \bigcup_{x \in M(L^\infty(\mu))} \Phi_\mu(x)$.

*Proof.* First, assume that $\mu = \delta_\zeta$ for some $\zeta \in \partial D$. Then $M(L^\infty(\mu))$ is a one-point set, say $\{x\}$, and it is easy to see that $\Phi_\mu(x) = Z(\psi_{\delta_\zeta}) = \mathcal{Z}(\delta_\zeta)$. Hence we obtain the assertion.

Next suppose that $\mu$ is not a point mass. Then $M(L^\infty(\mu))$ contains more than one point. Let $S$ be a measurable subset of $\operatorname{supp}\mu$. Then $\mu = \mu_{|S} + \mu_{|S^c}$ and $\mu_{|S} \perp \mu_{|S^c}$. Hence by Theorem 2.2, $\mathcal{Z}(\mu_{|S}) \cap \mathcal{Z}(\mu_{|S^c}) = \emptyset$ and $\mathcal{Z}(\mu_{|S}) \subset \mathcal{Z}(\mu)$. By Lemma 3.3, $\mathcal{Z}(\mu) = \mathcal{Z}(\mu_{|S}) \cup \mathcal{Z}(\mu_{|S^c})$. Thus if $\mu_{|S} \neq 0$, then $\mathcal{Z}(\mu_{|S})$ is a non-empty open and closed subset of $\mathcal{Z}(\mu)$.

Let $x \in M(L^\infty(\mu))$. Suppose that $\Phi_\mu(x) = \emptyset$. Then there exist $S_1, \ldots, S_n$ such that $x \in \widehat{S}_j$ for every $j$ and $\bigcap_{j=1}^n \mathcal{Z}(\mu_{|S_j}) = \emptyset$. Set $S = \bigcap_{j=1}^n S_j$. Then $x \in \widehat{S}$, so that $\mu_{|S} \neq 0$. Hence by Proposition 3.1, $\mathcal{Z}(\mu_{|S}) \neq \emptyset$. By Lemma 3.3, $\mathcal{Z}(\mu_{|S}) \subset \bigcap_{j=1}^n \mathcal{Z}(\mu_{|S_j})$. This is a contradiction. Thus we get (i).

Let $x, y \in M(L^\infty(\mu))$ and $x \neq y$. Then there exists $S$ such that $x \in \widehat{S}$ and $y \notin \widehat{S}$. We have $y \in \widehat{S^c}$, and hence by Theorem 2.2,

$$\Phi_\mu(x) \cap \Phi_\mu(y) \subset \mathcal{Z}(\mu_{|S}) \cap \mathcal{Z}(\mu_{|S^c}) = \emptyset.$$

Thus (ii) holds.

Suppose (iii) does not hold. Then there is $\zeta \in \mathcal{Z}(\mu)$ such that $\zeta \notin \Phi_\mu(x)$ for every $x \in M(L^\infty(\mu))$. Hence for each $x \in M(L^\infty)$, there exists a measurable subset $S_x$ of $\operatorname{supp}\mu$ such that $x \in \widehat{S}_x$ and $\zeta \notin \mathcal{Z}(\mu_{|S_x})$. Since $\widehat{S}_x$ is an open subset of $M(L^\infty(\mu))$, there exist $S_{x_1}, \ldots, S_{x_n}$ such that

$$M(L^\infty(\mu)) = \bigcup_{j=1}^n \widehat{S}_{x_j}.$$

Put $S = \bigcup_{j=1}^n S_{x_j}$. Then $\widehat{S} = \bigcup_{j=1}^n \widehat{S}_{x_j} = M(L^\infty(\mu))$, so that $\mu_{|S} = \mu$. By Lemma 3.3,

$$\mathcal{Z}(\mu) = \bigcup_{j=1}^n \mathcal{Z}(\mu_{|S_{x_j}}).$$

Hence $\zeta \in \mathcal{Z}(\mu_{|S_{x_j}})$ for some $j$. This is a contradiction. ∎

We have the following problem.

PROBLEM 6.4. Let $\mu \in M_{\mathrm{s}}^{+}$. Is $\Phi_\mu(x)$ a connected set for every $x \in M(L^\infty(\mu))$?

We give some results on the sets $\Phi_\mu(x)$.

Proposition 6.5. *Let* $\mu \in M_{\mathrm{s}}^{+}$ *and* $x \in M(L^\infty(\mu))$.

(i) *If* $\zeta \in \Phi_\mu(x)$, *then* $P(\zeta) \subset \Phi_\mu(x)$.
(ii) $\Phi_\mu(x)$ *contains trivial points.*
(iii) *If* $\mu \in M_{\mathrm{s,c}}^{+}$, $\zeta \in \Phi_\mu(x)$, $\operatorname{supp}\mu_\zeta \subset \operatorname{supp}\mu_\xi$, *and* $\xi \in \mathcal{M} \setminus D$, *then* $\xi \in \Phi_\mu(x)$.

*Proof.* Let $\zeta \in \Phi_\mu(x)$. Then $\psi_\nu(\zeta) = 0$ for every $\nu \in M_{\mathrm{s}}^{+}$ with $\nu \sim \mu$. Since $\psi_\nu$ is a singular inner function, we have $P(\zeta) \subset Z(\psi_\nu)$. Hence $P(\zeta) \subset \mathcal{Z}(\mu)$. Thus we get (i).

(ii) follows from (i) and Budde's theorem [1], and (iii) from Corollary 3.10 and (6.6). ■

One may ask whether each $\Phi_\mu(x)$ contains non-trivial points. Here is the answer.

Theorem 6.6. *Let* $\mu \in M_{\mathrm{s}}^{+}$ *and* $x \in M(L^\infty(\mu))$. *Then* $\Phi_\mu(x)$ *contains non-trivial points.*

*Proof.* Let $\mu \in M_{\mathrm{s}}^{+}$. By the regularity of $\mu$, there is a sequence $\{K_n\}_n$ (maybe finite) of non-empty closed subsets satisfying

$$|K_n| = 0 \quad \text{for every } n, \tag{6.7}$$

$$K_n \cap K_m = \emptyset \quad \text{if } n \neq m, \tag{6.8}$$

$$\mu = \sum_{n=1}^{\infty} \mu_{|K_n}. \tag{6.9}$$

For each $n$, there exists an interpolating Blaschke product $b_{K_n}$ associated with $K_n$. Let $\{w_{n,j}\}_j$ be the zeros of $b_{K_n}$ in $D$. Then by Theorem 5.1, $K_n$ is the set of cluster points of $\{w_{n,j}\}_j$ in $\overline{D}$. Then by (6.8), we have $\{|b_{K_n}| < 1\} \cap \{|b_{K_m}| < 1\} = \emptyset$ if $n \neq m$. By the proof of [8, Lemma 1.5], there is a sequence $\{k_j\}_j$ of positive integers such that $\{w_{n,j};\, j \geq k_n, n = 1, 2, \ldots\}$ is an interpolating sequence.

Let

$$b'_{K_n}(z) = \prod_{j=k_n}^{\infty} \frac{-\overline{w}_{n,j}}{|w_{n,j}|}\,\frac{z - w_{n,j}}{1 - \overline{w}_{n,j} z}, \qquad b_\mu(z) = \prod_{n=1}^{\infty} b'_{K_n}(z), \qquad z \in D.$$

Then $b_\mu$ is an interpolating Blaschke product and

$$Z(b'_{K_n}) = Z(b_{K_n}), \qquad \bigcup_{n=1}^{\infty} Z(b_{K_n}) \subset Z(b_\mu). \tag{6.10}$$

Let $S$ be a measurable subset of $\operatorname{supp}\mu$ such that $x \in \widehat{S}$. Since $\mu_{|S} \neq 0$, by (6.9) there exists a positive integer $n$ such that $\mu_{|K_n \cap S} \neq 0$. By (6.7) and Theorem 6.1, $Z(b_{K_n}) \cap \mathcal{Z}(\mu_{|K_n \cap S}) \neq \emptyset$. Then by (6.10), $Z(b_\mu) \cap \mathcal{Z}(\mu_{|K_n \cap S}) \neq \emptyset$. Hence by Lemma 3.3, we have $Z(b_\mu) \cap \mathcal{Z}(\mu_{|S}) \neq \emptyset$. In the same way as in the proof of Theorem 6.3(i), we have $Z(b_\mu) \cap \Phi_\mu(x) \neq \emptyset$. Since $b_\mu$ is interpolating, $\Phi_\mu(x)$ contains non-trivial points. ∎

PROBLEM 6.7. Let $\mu \in M_s^+$ and $x \in M(L^\infty(\mu))$. Does $\Phi_\mu(x)$ contain sparse points?

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