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Spectra of the difference, sum and product of idempotents

by

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Abstract. We give a simple proof of the relation between the spectra of the difference and product of any two idempotents in a Banach algebra. We also give the relation between the spectra of their sum and product.

By an idempotent in a unital Banach algebra $\mathcal{A}$ we mean an element p in $\mathcal{A}$ such that $p^2 = p$. The problem of determination of the spectrum of the difference and sum of a pair of idempotents in a Banach algebra from their product arose from many sources (see [1] and [3]).

In [3] it is shown that for two self-adjoint idempotents P and Q on a Hilbert space, the spectrum $\sigma(PQ)$ of the product PQ lies in the interval $[0, 1]$ and that

$$\sigma(PQ) \setminus \{0,1\} = \{1 - \mu^2 : \mu \in \sigma(P - Q) \setminus \{-1, 0, 1\}\}.$$

In this note, we shall generalize this result to an arbitrary pair of idempotents in a unital Banach algebra $\mathcal{A}$. The following theorem is our main result.

THEOREM 1. *Let $p, q \in \mathcal{A}$ be two idempotents. Then*

$$\begin{aligned}\sigma(pq) \setminus \{0,1\} &= \{1 - \mu^2 : \mu \in \sigma(p - q) \setminus \{-1, 0, 1\}\} \\ &= \{(1 - \mu)^2 : \mu \in \sigma(p + q) \setminus \{0, 1, 2\}\}.\end{aligned}$$

For the proof we need two lemmas. The first one is well known [2, p. 66].

LEMMA 2. *Let $x, y \in \mathcal{A}$. If $xy = 0$, then $\sigma(x+y)\setminus\{0\} = \sigma(x)\cup\sigma(y)\setminus\{0\}$.*

Proof. Just note that for any non-zero scalar λ, we have $\lambda - (x + y) = \lambda^{-1}(\lambda - x)(\lambda - y)$. Hence the result is checked easily. ∎

LEMMA 3. *If $p = p^2$ and $q = q^2$ in $\mathcal{A}$, then*

$$\sigma((e - p)(e - q)) \setminus \{0, 1\} = \sigma(pq) \setminus \{0, 1\},$$

where e denotes the unit element of $\mathcal{A}$.

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Proof. First we apply Lemma 2 for $x = p$ and $y = (e-p)(e-q)$. Since $xy = 0$, we have

$$\sigma(p + (e-p)(e-q)) \setminus \{0\} = \sigma(p) \cup \sigma((e-p)(e-q)) \setminus \{0\}. \tag{1}$$

From (1) and the fact that $\sigma(p) \subseteq \{0,1\}$, we deduce that

$$\sigma(p + (e-p)(e-q)) \setminus \{0,1\} = \sigma((e-p)(e-q)) \setminus \{0,1\}. \tag{2}$$

Since $p + (e-p)(e-q) = e - q + pq$, we have

$$\sigma(p + (e-p)(e-q)) = \sigma(e - q + pq). \tag{3}$$

Applying Lemma 2 for $x = pq$ and $y = e - q$, we obtain

$$\sigma(e - q + pq) \setminus \{0\} = \sigma(e-q) \cup \sigma(pq) \setminus \{0\}; \tag{4}$$

but $\sigma(e-q) \subseteq \{0,1\}$, so that

$$\sigma(e - q + pq) \setminus \{0,1\} = \sigma(pq) \setminus \{0,1\}. \tag{5}$$

From (1), (3) and (5), we conclude that

$$\sigma((e-p)(e-q)) \setminus \{0,1\} = \sigma(pq) \setminus \{0,1\},$$

which completes the proof. ■

Proof of the theorem. Write

$$(e - (p+q))^2 = (e-p)(e-q) + qp = e - (p-q)^2. \tag{6}$$

Using Lemma 2 for $x = (e-p)(e-q)$ and $y = qp$, we get

$$\sigma((e - (p+q))^2) \setminus \{0\} = \sigma((e-p)(e-q)) \cup \sigma(qp) \setminus \{0\}.$$

From Lemma 3, (6) and Jacobson's theorem (see for instance [2, p. 33]), it follows that

$$\sigma((e - (p+q))^2) \setminus \{0,1\} = \sigma(e - (p-q)^2) \setminus \{0,1\} = \sigma(pq) \setminus \{0,1\}.$$

Now the result follows by applying the spectral mapping theorem. ■

An element $a \in \mathcal{A}$ is called *quadratic* if it satisfies some non-trivial quadratic equation $(a - \alpha e)(a - \beta e) = 0$, where $\alpha, \beta \in \mathbb{C}$. We write $a = a(\alpha, \beta)$. If $\alpha \neq \beta$, then it is immediate to verify that $p = (\alpha - \beta)^{-1}(a - \beta)$ is an idempotent.

COROLLARY 4. *Let $\alpha, \beta, \mu, \nu \in \mathbb{C}$ and let $a = a(\alpha, \beta)$, $b = b(\mu, \nu)$ be quadratic elements in $\mathcal{A}$. If $\alpha \neq \beta$ and $\mu \neq \nu$, then*

$$\begin{aligned} &\sigma((a-\beta)(b-\nu)) \setminus \{0, (\alpha-\beta)(\mu-\nu)\} \\ &= (\alpha-\beta)(\mu-\nu) - \left\{ \frac{\lambda^2}{(\alpha-\beta)(\mu-\nu)} : \lambda \in \sigma((\mu-\nu)a - (\alpha-\beta)b) \setminus \mathcal{S}_1 \right\} \\ &= \left\{ \frac{((\alpha-\beta)(\mu-\nu) - \lambda)^2}{(\alpha-\beta)(\mu-\nu)} : \lambda \in \sigma((\mu-\nu)(a-\beta)(\alpha-\beta)(b-\nu)) \setminus \mathcal{S}_2 \right\}, \end{aligned} \tag{7}$$

where

$$\mathcal{S}_1 = \{2\nu\alpha - \alpha\mu - \beta\nu, \nu\alpha - \beta\mu, \alpha\mu + \beta\nu - 2\beta\mu\},$$
$$\mathcal{S}_2 = \{0, (\alpha - \beta)(\mu - \nu), 2(\alpha - \beta)(\mu - \nu)\}.$$

Proof. This follows from the fact that

$$p = \frac{1}{\alpha - \beta}(a - \beta) \quad \text{and} \quad q = \frac{1}{\mu - \nu}(b - \nu)$$

are idempotents. ∎

In the case where $\alpha = \beta$ and $\mu = \nu$ we obtain two nilpotents of order 2.

PROPOSITION 5. *Let a and b in $\mathcal{A}$ with $a^2 = b^2 = 0$; then*

$$\begin{aligned}\sigma(ab) \setminus \{0\} &= \{\lambda^2 : \lambda \in \sigma(a + b) \setminus \{0\}\} \\ &= \{-\lambda^2 : \lambda \in \sigma(a - b) \setminus \{0\}\}.\end{aligned}$$

Proof. To see this we use Lemma 2 for $x = ab$ and $y = ba$, and the Jacobson theorem. We get $\sigma(x + y) \setminus \{0\} = \sigma(x) \setminus \{0\}$. On the other hand $(a+b)^2 = x+y$, hence $\sigma(ab) \setminus \{0\} = \sigma((a+b)^2) \setminus \{0\}$. By a similar argument we infer that $\sigma(ab) \setminus \{0\} = -\sigma((a - b)^2) \setminus \{0\}$. ∎

REMARK. In the case of a nilpotent and an idempotent, the following example shows that there is no *quadratic* relation similar to (7).

EXAMPLE. On the Hilbert space $\mathbb{C}^2$, let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q_\alpha = \begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix}, \quad \alpha \in \mathbb{C}.$$

References

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