The Hausdorff operators on the real Hardy spaces $H^p(\mathbb{R})$

by

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Abstract. We prove that the Hausdorff operator generated by a function $\phi$ is bounded on the real Hardy space $H^p(\mathbb{R})$, $0 < p \leq 1$, if the Fourier transform $\hat{\phi}$ of $\phi$ satisfies certain smoothness conditions. As a special case, we obtain the boundedness of the Cesàro operator of order $\alpha$ on $H^p(\mathbb{R})$, $2/(2\alpha + 1) < p \leq 1$. Our proof is based on the atomic decomposition and molecular characterization of $H^p(\mathbb{R})$.

1. Introduction. Let $0 < p \leq 1$ and $H^p(\mathbb{R})$ be the real Hardy space, that is, the space of the boundary distributions $f(x) = \Re F(x)$ of the real parts $\Re F(z)$ of functions $F(z)$ in the Hardy space $H^p(\mathbb{R}_+^2) = \{F(z) : F$ is analytic in $\mathbb{R}_+^2$ and $\|F\|_{H^p(\mathbb{R}_+^2)} = \sup_{t>0} (\int_{-\infty}^{\infty} |F(x + it)|^p dx)^{1/p} < \infty\}$ on the upper half plane $\mathbb{R}_+^2 = \{z = x + it : t > 0\}$, with the norm $\|f\|_{H^p(\mathbb{R})} = \|F\|_{H^p(\mathbb{R}_+^2)}$. The Fourier transform of a function $f(x)$ in $\mathbb{R}$ is given by $[f(x)]^\wedge(\xi) = \hat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$.

Let $\phi \in L^1(\mathbb{R})$. For a function $f$ in $\mathbb{R}$, we define a function $\mathcal{H}_\phi f$ in $\mathbb{R}$ so that its Fourier transform $\hat{\mathcal{H}_\phi f}$ satisfies

$$\hat{\mathcal{H}_\phi f}(t) = \int_{\mathbb{R}} \hat{f}(t\xi)\phi(\xi) d\xi, \quad t \in \mathbb{R}.$$ 

The operator $\mathcal{H}_\phi$ is called the Hausdorff operator generated by $\phi$. For simplicity, we also write $\mathcal{H}f$ instead of $\mathcal{H}_\phi f$.

For $\alpha = 1, 2, \ldots$, the Cesàro operator $\mathcal{C}_\alpha$ of order $\alpha$ is given by

$$\mathcal{C}_\alpha f(t) = \begin{cases} \frac{\alpha}{t^\alpha} \int_0^t \hat{f}(\xi)(t - \xi)^{\alpha-1} d\xi & (t \neq 0), \\ \hat{f}(0) & (t = 0). \end{cases}$$

We note that $\mathcal{C}_\alpha = \mathcal{H}_\phi$ when $\phi(\xi) = \alpha(1 - \xi)^{\alpha-1}\chi_{(0,1)}(\xi)$, where $\chi_{(0,1)}$ is the characteristic function of the interval $(0, 1)$.

2000 Mathematics Subject Classification: Primary 47B38; Secondary 42B30.

Key words and phrases: Hausdorff operator, Cesàro operator, real Hardy space.
Giang and Móricz [4] proved the following result:

**Theorem A** (Giang and Móricz [4], Theorem 1). *The Cesàro operator $C_1$ of order 1 is a bounded operator on $H^1(\mathbb{R})$.*

Recently, Liflyand and Móricz [8] obtained the following generalization of this result to the Hausdorff operator by using the closed graph theorem and the fact that if $f \in L^1(\mathbb{R})$ satisfies $\hat{f}(t) = 0$ for $t < 0$, then $f \in H^1(\mathbb{R})$.

**Theorem B** (Liflyand and Móricz [8], Theorem 2). *If $\phi \in L^1(\mathbb{R})$, then the Hausdorff operator $\mathcal{H}_\phi$ is a bounded operator on $H^1(\mathbb{R})$.*

The purpose of this paper is to extend this result to $H^p(\mathbb{R})$ with index $p$ smaller than one under certain smoothness conditions on $\phi$ and to show that the extension gives the boundedness of the Cesàro operator $C_\alpha$ on $H^p(\mathbb{R})$, $1 \geq p > 2/(2\alpha + 1)$. The atomic decomposition and molecular characterization of $H^p(\mathbb{R})$ will play an essential role in our proof.

Historically, for the periodic case, Hardy [6] proved that if $\sum_{n=0}^{\infty} a_n \cos nx$ is the Fourier series of a function in $L^p(0, \pi)$, then so is $\sum_{n=0}^{\infty} (Ta)_n \cos nx$ for $1 \leq p < \infty$, where $(Ta)_0 = a_0$, $(Ta)_n = (a_1 + \ldots + a_n)/n$, $n = 1, 2, \ldots$, and the same is true for sine series. Kinukawa and Igarì [7] showed that if $\sum_{n=1}^{\infty} b_n \sin nx$ is a Fourier series, then the conjugate series $\sum_{n=1}^{\infty} (Tb)_n \cos nx$ is a Fourier series. Siskakis [9] obtained the same type of theorem in the Hardy space $H^1$ of the unit circle, that is, the operator $C_\alpha$ defined by $C_\alpha f(z) = \sum_{n=0}^{\infty} ((n+1)^{-1/2} \sum_{k=0}^{n} a_k) z^n$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, is bounded on $H^1$. For the real line case, Goldberg [5] investigated the properties of the operator $\mathcal{H}_\phi$ on the spaces $L^p(\mathbb{R})$ with $1 < p \leq 2$. Georgakis [3] studied the Fourier analytic properties of $\mathcal{H}_\phi$ on the space of complex bounded regular Borel measures on $\mathbb{R}$, and as a special case he showed that if $\phi \in L^1(\mathbb{R})$, then $\mathcal{H}_\phi$ is a bounded operator on $L^1(\mathbb{R})$. Giang and Móricz [4] and Liflyand and Móricz [8] followed as stated above.

**2. Results.** Let $0 < p \leq 1$ and $k$ be an integer, $k \geq 1/p - 1$. A real-valued function $a(x)$ is called a $(p, 2, k)$-atom if (i) $a(x)$ is supported in an interval $[c, c + h]$, (ii) $\|a\|_2 = (\int_{\mathbb{R}} |a(x)|^2 dx)^{1/2} \leq h^{1/2 - 1/p}$, and (iii) $\int_{\mathbb{R}} x^j a(x) dx = 0$ for $j = 0, 1, \ldots, k$. Then the atomic decomposition says that if $f \in H^p(\mathbb{R})$, then there exist a sequence $\{a_j\}$ of $(p, 2, k)$-atoms and a sequence $\{\lambda_j\}$ of real numbers with $\lambda_j a_j$ the series converging to $f$ in $H^p(\mathbb{R})$ and also in the sense of tempered distributions.

A real-valued function $M(x)$ is called a $(p, 2, b)$-molecule centered at $x_0$ if $M(x)$ satisfies the following conditions: (i) $N(M) = \|M\|_{L^1}^{1 - \theta} \|\cdot - x_0 \|^b M\|_{L^2}^\theta < \infty$, where $b > 1/p - 1/2$, $\theta = (1/p - 1/2)/b$, and (ii) $\int_{\mathbb{R}} x^j M(x) = 0,$
Let $0 < p \leq 1$ and $r$ be the smallest integer such that $r > 1/p - 1/2$. Suppose that $\phi \in L^1(\mathbb{R})$ satisfies the following:

(i) $\int_{\mathbb{R}} |\xi|^{-1/2} |\phi(\xi)| \, d\xi < \infty$,

(ii) $\phi \in C^{2r}(\mathbb{R})$ with $\sup_x |x|^r |\hat{\phi}^{(r)}(x)| < \infty$, $\sup_x |x|^r |\hat{\phi}^{(2r)}(x)| < \infty$.

Then, for a $(p, 2, r-1)$-atom $a$, $\mathcal{H}_\phi a$ is a $(p, 2, r)$-molecule centered at 0, and

$$N(\mathcal{H}_\phi a) \leq C,$$

where $C$ is independent of the atoms $a$.

Our theorem and its corollary are as follows:

**Theorem.** Let $0 < p \leq 1$. Suppose that $\phi \in L^1(\mathbb{R})$ satisfies the same conditions as in the Lemma. Then the Hausdorff operator $\mathcal{H}_\phi$ is a bounded operator on $H^p(\mathbb{R})$.

**Corollary.** Let $\alpha = 1, 2, \ldots$ If $2/(2\alpha + 1) < p \leq 1$, then the Cesàro operator $C_\alpha$ of order $\alpha$ is a bounded operator on $H^p(\mathbb{R})$.

Other typical summability kernels covered by the theorem are, for example, $e^{-\xi^2}$ and $e^{-|\xi|}$. The corresponding operators $\mathcal{H}_\phi$ are bounded on $H^p(\mathbb{R})$ for every $p$ with $0 < p \leq 1$ since the functions $e^{-\xi^2}$ and $e^{-|\xi|}$ satisfy the conditions of the Lemma for every $p$ with $0 < p \leq 1$.

The Lemma will be proved in the next section. A discussion on defining the values of the Hausdorff operator for $H^p(\mathbb{R})$ functions and the proof of the Theorem will be given in the fourth section by using the main estimate of the Lemma.

3. **Proof of the Lemma.** Let $0 < p \leq 1$ and $r$ be the smallest integer such that $r > 1/p - 1/2$. We begin with estimating the molecular norm $N(\mathcal{H}_\phi a)$ for a $(p, 2, r-1)$-atom $a$. By the Plancherel theorem and the identity $[xf(x)]^\wedge(t) = i(d/dt)\hat{f}(t)$, we have $\|\mathcal{H}a\|_2 = \|\hat{\mathcal{H}}a\|_2$, and

$$\|x^r\mathcal{H}a(x)\|_2 = \|[x^r\mathcal{H}a(x)]^\wedge(t)\|_2 = \|(d/dt)^r\hat{\mathcal{H}}a(t)\|_2,$$

which leads to

$$N(\mathcal{H}a) = \|\hat{\mathcal{H}}a\|_2^{1-\theta} \|(d/dt)^r\hat{\mathcal{H}}a(t)\|_2^\theta,$$
where

\[ \theta = (1/p - 1/2)/r = \frac{2 - p}{2pr}. \]

To estimate \( \| \hat{\mathcal{H}}a \|_2 \), we apply the generalized Minkowski inequality. We have

\[
\| \hat{\mathcal{H}}a \|_2 \leq \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\hat{a}(t\xi)|^2 \, dt \right\}^{1/2} |\phi(\xi)| \, d\xi
= \|\hat{a}\|_2 \int_{\mathbb{R}} |\xi|^{-1/2} |\phi(\xi)| \, d\xi = D_0 \|a\|_2,
\]

where \( D_0 = \int_{\mathbb{R}} |\xi|^{-1/2} |\phi(\xi)| \, d\xi \), which implies

\[ N(\mathcal{H}a) \leq D_0 \|a\|_2^{1-\theta} \| (d/dt)^r \hat{\mathcal{H}}a(t) \|_2^\theta. \]

It is enough to show that \( \| (d/dt)^r \hat{\mathcal{H}}a(t) \|_2 \leq C \|a\|_2^\delta \), where

\[ 1 - \theta + \delta \theta = 0, \quad \text{that is,} \quad \delta = 1 - 1/\theta = \frac{2 - (1 + 2r)p}{2 - p}, \]

and the constant \( C \) is independent of the atoms \( a \).

Noting that \( \hat{\mathcal{H}}a(t) = \int_{\mathbb{R}} a(x) \hat{\phi}(tx) \, dx \), we have

\[
\frac{d^r}{dt^r} \hat{\mathcal{H}}a(t) = \int_{\mathbb{R}} a(x) x^r \hat{\phi}(r)(tx) \, dx.
\]

Let \( \mu \) be a positive number, which will be chosen later, and write

\[
\| (d/dt)^r \hat{\mathcal{H}}a(t) \|_2^2 = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} a(x) x^r \hat{\phi}(r)(tx) \, dx \right\}^2 \, dt
= \left\{ \int_{|t| \leq \mu} + \int_{|t| > \mu} \right\} \left\{ \int_{\mathbb{R}} a(x) x^r \hat{\phi}(r)(tx) \, dx \right\}^2 \, dt
= I_\mu + J_\mu, \quad \text{say.}
\]

We first treat \( J_\mu \). It follows that

\[
J_\mu = \int_{|t| > \mu} \left| t^{-r} \int_{\mathbb{R}} a(x)(tx)^r \hat{\phi}(r)(tx) \, dx \right|^2 \, dt
\leq \int_{|t| > \mu} \left| t^{-2r} \, dt \left( \sup_x |x|^r \hat{\phi}(r)(x) \right) \right|^2 \left( \int_{\mathbb{R}} |a(x)| \, dx \right)^2.
\]

By Schwarz’s inequality, we have

\[
\int_{\mathbb{R}} |a(x)| \, dx \leq \|a\|_2 h^{1/2},
\]

which is bounded by \( \|a\|_2^{2(1-p)/(2-p)} \), since the condition \( \|a\|_2 \leq h^{1/2-1/p} \) implies \( h \leq \|a\|_2^{-2p/(2-p)} \), where the support interval of \( a \) is \([c, c+h]\). Thus,
we have

(1) \[ J_\mu \leq (2K^2/(2r - 1))\mu^{-2r+1}\|a\|_2^{4(1-p)/(2-p)}, \]

where \( K = \sup_x |x^r|\hat{\phi}(r)(x)|. \)

To estimate \( I_\mu \), we consider the inner integral \( \int_R a(x)x^r\hat{\phi}(r)(tx)dx \) of \( I_\mu \). Let \( g_t(x) = x^r\hat{\phi}(r)(tx) \). By the Taylor expansion of \( g_t \) at \( x = c \) and the vanishing moment property of atoms, we have

\[ \int_R a(x)g_t(x)dx = \frac{1}{r!}\int_R a(x)g_t^{(r)}(\tilde{x})(x - c)^r dx, \quad c < \tilde{x} < c + h, \]

where \([c, c + h]\) is the support interval of \( a \) and \( g_t^{(r)} \) is the \( r \)th derivative of \( g_t \) with respect to \( x \). Since

\[ g_t^{(r)}(\tilde{x}) = \sum_{j=0}^r \binom{r}{j} \frac{r!}{j!}(t\tilde{x})^j\hat{\phi}^{(r+j)}(t\tilde{x}), \]

it follows that

(2) \[ |g_t^{(r)}(\tilde{x})| \leq C \sum_{j=0}^r \sup_x |x|^j|\hat{\phi}^{(r+j)}(x)| \]

\[ \leq C(\sup_x |\hat{\phi}(r)(x)| + \sup_x |x^r|\hat{\phi}^{(2r)}(x)|) = K', \quad \text{say}, \]

where \( K' \) depends only on \( \phi \) and \( r \), that is, \( p \). (We shall refer to the last inequality (2) in the Remark below.) Thus, we have

\[ \left| \int_R a(x)x^r\hat{\phi}(r)(tx)dx \right| \leq K' \int_R |a(x)| \cdot |x - c|^r dx \]

\[ \leq C\|a\|_2h^{r+1/2} \leq C\|a\|_2^{2(1-p-pr)/(2-p)}, \]

where \( C \) depends only on \( \phi \) and \( p \). The last inequality follows from \( h \leq \|a\|_2^{-2p/(2-p)} \). We have

(3) \[ I_\mu \leq C\mu\|a\|_2^{4(1-p-pr)/(2-p)}. \]

Therefore,

\[ \|(d/dt)^r\hat{H}a(t)\|_2 \leq \frac{1}{\mu} + \frac{1}{\mu} \]

\[ \leq C(\mu^{1/2}\|a\|_2^{2(1-p-pr)/(2-p)} + \mu^{-r+1/2}\|a\|_2^{2(1-p)/(2-p)}) \]

\[ \leq C(\mu^{1/2}\|a\|_2^{2(1-p-pr)/(2-p)-\delta}) + \mu^{-r+1/2}\|a\|_2^{2(1-p)/(2-p)-\delta})\|a\|_2^{\delta}, \]
where \( C \) depends only on \( p \) and \( \phi \). It follows that
\[
\mu^{1/2} \|a\|^2 \frac{2(1-p-pr)/(2-p)-\delta}{2-p} = \left( \mu \|a\|^2 \frac{(1-1-p)/(2-p)-\delta}{2-p} \right)^{1/2},
\]
and
\[
\mu^{-r+1/2} \|a\|^2 \frac{2(1-p)/(2-p)-\delta}{-r+1/2} = \left( \mu \|a\|^2 \frac{(2(1-p)/(2-p)-\delta)/(-r+1/2)}{-r+1/2} \right)^{-r+1/2},
\]
and
\[
\frac{4(1-p-pr)}{2-p} - 2\delta = \frac{2(1-p)/(2-p) - \delta}{-r+1/2} = \frac{-2p}{2-p}.
\]
Therefore, choosing \( \mu = \|a\|^{2p/(2-p)} \), we get
\[
\|(d/dt)^{r} \hat{H}a(t)\|_2 \leq C \|a\|^\delta,
\]
which leads to the desired inequality
\[
N(\hat{H}a) \leq C,
\]
where \( C \) depends only on \( p \) and \( \phi \).

We turn to the moment condition. It follows that
\[
\int x^j \hat{H}a(x) \, dx = [x^j \hat{H}a(x)]^\wedge(0) = i^j \frac{d^j \hat{H}a}{dt^j}(0)
\]
\[
= i^j \int \hat{a}^{(j)}(t\xi) \xi^j \phi(\xi) \, d\xi \bigg|_{t=0}.
\]
It follows from the moment condition for a \((p, 2, r-1)\)-atom \( a \) that \( \hat{a}^{(j)}(0) = (-i)^j \int a(x) \, dx = 0, j = 0, 1, \ldots, r-1. Since L \leq r-1, the moment condition for \( \hat{H}a \) follows. The proof of the Lemma is complete.

**Remark.** Let \( r = 1, 2, \ldots \) The following inequalities hold:
\[
\sup_x |x|^j |f^{(j)}(x)| \leq C(\sup_x |f(x)| + \sup_x |x|^r |f^{(r)}(x)|), \quad j = 1, \ldots, r-1,
\]
where \( C \) depends only on \( r \). These inequalities yield (2) by taking \( f = \hat{\phi}^{(r)}. \)
We get (5) by following the proof of the known inequalities (cf. [1, Ch. 2, Theorem 5.6]):
\[
u^j \sup_x |f^{(j)}(x)| \leq C(\sup_x |f(x)| + u^r \sup_x |f^{(r)}(x)|), \quad j = 1, \ldots, r-1,
\]
where \( u > 0 \) is arbitrary and \( C \) depends only on \( r \). For the reader’s convenience, we give a proof of (5). Let \( 1 = \lambda_1 < \ldots < \lambda_{r-1} = 2. \)

It follows from the Taylor formula that
\[
f(x + \lambda_l x) = \sum_{j=0}^{r-1} \frac{\lambda_l x)^j}{j!} f^{(j)}(x) + \lambda_l x \int_0^{\lambda_l x} \frac{(\lambda_l x - t)^{r-1}}{(r-1)!} f^{(r)}(x + t) \, dt
\]
for \( l = 1, \ldots, r-1. \) We denote the remainder term by \( R_r(x, l) \). We have a
linear system

\[ \sum_{j=1}^{r-1} \lambda_l \frac{x^j}{j!} f^{(j)}(x) = f(x + \lambda_l x) - f(x) - R_r(x, l), \quad l = 1, \ldots, r - 1. \]

Since the Vandermonde determinant is nonzero, this system can be solved for \( x^j f^{(j)}(x)/j! \). Further, the remainder term \( R_r(x, l) \) is estimated as follows:

\[ |R_r(x, l)| \leq C(\sup_x |x|^r |f^{(r)}(x)|) \int_0^1 (\lambda_l - u)^{r-1} (1 + u)^r \, du \]

for \( l = 1, \ldots, r - 1 \), where \( C \) depends only on \( r \), which leads to (5).

5. Proof of the Theorem. We first discuss defining the value \( \mathcal{H}_\phi f \) for \( f \in H^p(\mathbb{R}), 0 < p < 1 \). We use the fact that a function of the Lipschitz space \( \Lambda_1/p-1(\mathbb{R}) \) defines a continuous linear functional on \( H^p(\mathbb{R}) \) (cf. [2, III.5]). Let \( 0 < p < 1 \) and \( r \) be the smallest integer such that \( r > 1/p - 1/2 \). Suppose that \( \phi \) satisfies the same conditions as in the Lemma.

We put \( (\widehat{\phi})_t(x) = \widehat{\phi}(xt) \). Then we have \( |(d/dx)^j(\widehat{\phi})_t(x)| \leq A_j |t|^j, j = 0, 1, \ldots, r \), where \( A_j = \sup_x |\widehat{\phi}^{(j)}(x)| \). The constants \( A_j \) are finite, which follows from (6). Hence

\[ \| (\widehat{\phi})_t \|_{\Lambda_1/p-1(\mathbb{R})} \leq C(1 + |t|^r). \]

This implies \( (\widehat{\phi})_t \in \Lambda_1/p-1(\mathbb{R}) \), that is, for every \( t \in \mathbb{R} \) the function \( (\widehat{\phi})_t \) defines a continuous linear functional of \( H^p(\mathbb{R}) \) and

\[ \| (f, (\widehat{\phi})_t) \| \leq C(1 + |t|^r) \| f \|_{H^p(\mathbb{R})} \quad \text{for} \quad f \in H^p(\mathbb{R}), \]

where \( \langle \cdot, \cdot \rangle \) is the duality paring between \( H^p(\mathbb{R}) \) and \( \Lambda_1/p-1(\mathbb{R}) \), and \( C \) is independent of \( f \) and \( t \). We define \( \mathcal{H}_\phi f \) for \( f \in H^p(\mathbb{R}) \) as the inverse Fourier transform of the tempered function \( \langle f, (\widehat{\phi})_t \rangle \) with respect to the variable \( t \), that is, \( \mathcal{H}_\phi f(t) = \langle f, (\widehat{\phi})_t \rangle \). This coincides with the original definition when \( f, \phi \in L^1(\mathbb{R}) \) since

\[ \mathcal{H}_\phi f(t) = \int_{\mathbb{R}} \hat{f}(t\xi) \phi(\xi) \, d\xi = \int_{\mathbb{R}} f(x) \widehat{\phi}(tx) \, dx. \]

We turn to the proof of the Theorem. Let \( 0 < p \leq 1 \) and \( r \) be the smallest integer such that \( r > 1/p - 1/2 \). Let \( f \in H^p(\mathbb{R}) \). We have an atomic decomposition \( f = \sum_{j=0}^\infty \lambda_j a_j \), where \( \sum_{j=0}^\infty |\lambda_j|^p \leq C \| f \|_{H^p(\mathbb{R})}^p \) and \( a_j \) is a \((p, 2, r - 1)\)-atom. By the Lemma, we have

\[ \sum_{j=0}^\infty N(\lambda_j \mathcal{H}_\phi a_j)^p = \sum_{j=0}^\infty |\lambda_j|^p N(\mathcal{H}_\phi a_j)^p \leq C \sum_{j=0}^\infty |\lambda_j|^p \leq C \| f \|_{H^p(\mathbb{R})}^p. \]
Thus, the series \( \sum_{j=0}^{\infty} \lambda_j \mathcal{H}_\phi a_j \) converges to a tempered distribution \( g \) in \( S' \) and \( \|g\|_{H^p(\mathbb{R})} \leq C \|f\|_{H^p(\mathbb{R})} \), where \( S \) is the Schwartz space. It is enough to show that \( g = \mathcal{H}_\phi f \) in \( S' \). Let \( \psi \in S \). It follows that

\[
(g, \hat{\psi}) = \sum_{j=0}^{\infty} \lambda_j \langle \mathcal{H}_\phi a_j, \hat{\psi} \rangle = \sum_{j=0}^{\infty} \lambda_j \langle \hat{\mathcal{H}_\phi a_j}, \psi \rangle = \sum_{j=0}^{\infty} \lambda_j \int_{\mathbb{R}} \mathcal{H}_\phi a_j(t) \psi(t) \, dt = \sum_{j=0}^{\infty} \lambda_j \int_{\mathbb{R}} (a_j, (\hat{\phi})_t) \psi(t) \, dt,
\]

where \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( S' \) and \( S \). By the fact that \( |\langle a_j, (\hat{\phi})_t \rangle| \leq C \|\hat{\phi}\|_{L_{1/p-1}(\mathbb{R})} \) and by (7), we can change the order of the sum and integral in the last term. It follows that

\[
(g, \hat{\psi}) = \int_{\mathbb{R}} \int_{\mathbb{R}} \lambda_j (a_j, (\hat{\phi})_t) \psi(t) \, dt = \int_{\mathbb{R}} (f, (\hat{\phi})_t) \psi(t) \, dt = \langle \mathcal{H}_\phi f, \psi \rangle = \langle \mathcal{H}_\phi f, \hat{\psi} \rangle.
\]

Therefore, we have \( g = \mathcal{H}_\phi f \) in \( S' \), which completes the proof of the Theorem.

Finally, we prove the Corollary. We put \( \phi_\alpha(x) = \alpha(1 - \xi)\alpha-1\chi_{(0,1)}(\xi) \). Then \( C_\alpha = \mathcal{H}_{\phi_\alpha} \). Trivially, \( \phi_\alpha \in L^1(\mathbb{R}) \) and \( \int_{\mathbb{R}} |\xi|^{-1/2} |\phi_\alpha(\xi)| \, d\xi < \infty \). We check condition (ii) of the Lemma for \( \hat{\phi_\alpha} \). We have

\[
\hat{\phi_\alpha}(x) = \alpha!(-ix)^{-\alpha} \left( e^{-ix} - \sum_{j=0}^{\alpha-1} \frac{(-ix)^j}{j!} \right).
\]

We easily see that \( \hat{\phi_\alpha} \in C^\infty(\mathbb{R}) \) and

\[
\sup_{x} \left| x^{\alpha} |\hat{\phi_\alpha}(x)| \right| < \infty, \quad \sup_{x} \left| x^{\alpha} |\hat{\phi_\alpha}^{(2\alpha)}(x)| \right| < \infty
\]

for \( \alpha = 1, 2, \ldots \). Therefore, the Theorem yields the Corollary.


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*Received December 20, 2000
Revised version May 21, 2001*