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The Hausdorff operators on the real Hardy spaces $H^p(\mathbb{R})$

by

YUICHI KANJIN (Kanazawa)

Abstract. We prove that the Hausdorff operator generated by a function ϕ is bounded on the real Hardy space $H^p(\mathbb{R})$, $0 < p \le 1$, if the Fourier transform $\widehat{\phi}$ of ϕ satisfies certain smoothness conditions. As a special case, we obtain the boundedness of the Cesàro operator of order α on $H^p(\mathbb{R})$, $2/(2\alpha+1) < p \le 1$. Our proof is based on the atomic decomposition and molecular characterization of $H^p(\mathbb{R})$.

1. Introduction. Let $0 < p \le 1$ and $H^p(\mathbb{R})$ be the real Hardy space, that is, the space of the boundary distributions $f(x) = \Re F(x)$ of the real parts $\Re F(z)$ of functions $F(z)$ in the Hardy space $H^p(\mathbb{R}^2_+) = \{F(z) : F$ is analytic in $\mathbb{R}^2_+$ and $\|F\|_{H^p(\mathbb{R}^2_+)} = \sup_{t>0}(\int_{-\infty}^{\infty} |F(x+it)|^p dx)^{1/p} < \infty\}$ on the upper half plane $\mathbb{R}^2_+ = \{z = x + it : t > 0\}$, with the norm $\|f\|_{H^p(\mathbb{R})} = \|F\|_{H^p(\mathbb{R}^2_+)}$. The Fourier transform of a function $f(x)$ in $\mathbb{R}$ is given by $[f(x)]^\wedge(\xi) = \widehat{f}(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$.

Let $\phi \in L^1(\mathbb{R})$. For a function f in $\mathbb{R}$, we define a function $\mathcal{H}_\phi f$ in $\mathbb{R}$ so that its Fourier transform $\widehat{\mathcal{H}_\phi f}$ satisfies

$$\widehat{\mathcal{H}_\phi f}(t) = \int_{\mathbb{R}} \widehat{f}(t\xi)\phi(\xi)\,d\xi, \qquad t \in \mathbb{R}.$$

The operator $\mathcal{H}_\phi$ is called the *Hausdorff operator* generated by ϕ. For simplicity, we also write $\mathcal{H}f$ instead of $\mathcal{H}_\phi f$.

For $\alpha = 1, 2, \ldots$, the *Cesàro operator* $\mathcal{C}_\alpha$ of order α is given by

$$\widehat{\mathcal{C}_\alpha f}(t) = \begin{cases} \dfrac{\alpha}{t^\alpha}\displaystyle\int_0^t \widehat{f}(\xi)(t-\xi)^{\alpha-1}\,d\xi & (t \ne 0), \\ \widehat{f}(0) & (t = 0). \end{cases}$$

We note that $\mathcal{C}_\alpha = \mathcal{H}_\phi$ when $\phi(\xi) = \alpha(1-\xi)^{\alpha-1}\chi_{(0,1)}(\xi)$, where $\chi_{(0,1)}$ is the characteristic function of the interval $(0,1)$.

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Giang and Móricz [4] proved the following result:

THEOREM A (Giang and Móricz [4], Theorem 1). *The Cesàro operator* $\mathcal{C}_1$ *of order* 1 *is a bounded operator on* $H^1(\mathbb{R})$.

Recently, Liflyand and Móricz [8] obtained the following generalization of this result to the Hausdorff operator by using the closed graph theorem and the fact that if $f \in L^1(\mathbb{R})$ satisfies $\widehat{f}(t) = 0$ for $t < 0$, then $f \in H^1(\mathbb{R})$.

THEOREM B (Liflyand and Móricz [8], Theorem 2). *If* $\phi \in L^1(\mathbb{R})$, *then the Hausdorff operator* $\mathcal{H}_\phi$ *is a bounded operator on* $H^1(\mathbb{R})$.

The purpose of this paper is to extend this result to $H^p(\mathbb{R})$ with index p smaller than one under certain smoothness conditions on $\widehat{\phi}$ and to show that the extension gives the boundedness of the Cesàro operator $\mathcal{C}_\alpha$ on $H^p(\mathbb{R})$, $1 \geq p > 2/(2\alpha+1)$. The atomic decomposition and molecular characterization of $H^p(\mathbb{R})$ will play an essential role in our proof.

Historically, for the periodic case, Hardy [6] proved that if $\sum_{n=0}^{\infty} a_n \cos nx$ is the Fourier series of a function in $L^p(0,\pi)$, then so is $\sum_{n=0}^{\infty} (Ta)_n \cos nx$ for $1 \leq p < \infty$, where $(Ta)_0 = a_0$, $(Ta)_n = (a_1 + \ldots + a_n)/n$, $n = 1, 2, \ldots$, and the same is true for sine series. Kinukawa and Igari [7] showed that if $\sum_{n=1}^{\infty} b_n \sin nx$ is a Fourier series, then the conjugate series $\sum_{n=1}^{\infty} (Tb)_n \cos nx$ is a Fourier series. Siskakis [9] obtained the same type of theorem in the Hardy space H^1 of the unit disc, that is, the operator $\mathcal{C}_*$ defined by $\mathcal{C}_* f(z) = \sum_{n=0}^{\infty} \{(n+1)^{-1} \sum_{k=0}^{n} a_k\} z^n$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, is bounded on H^1. For the real line case, Goldberg [5] investigated the properties of the operator $\mathcal{H}_\phi$ on the spaces $L^p(\mathbb{R})$ with $1 < p \leq 2$. Georgakis [3] studied the Fourier analytic properties of $\mathcal{H}_\phi$ on the space of complex bounded regular Borel measures on $\mathbb{R}$, and as a special case he showed that if $\phi \in L^1(\mathbb{R})$, then $\mathcal{H}_\phi$ is a bounded operator on $L^1(\mathbb{R})$. Giang and Móricz [4] and Liflyand and Móricz [8] followed as stated above.

2. Results. Let $0 < p \leq 1$ and k be an integer, $k \geq 1/p - 1$. A real-valued function $a(x)$ is called a $(p,2,k)$-*atom* if (i) $a(x)$ is supported in an interval $[c, c+h]$, (ii) $\|a\|_2$ $(= \{\int_{\mathbb{R}} |a(x)|^2\,dx\}^{1/2}) \leq h^{1/2-1/p}$, and (iii) $\int_{\mathbb{R}} x^j a(x)\,dx = 0$ for $j = 0, 1, \ldots, k$. Then the atomic decomposition says that if $f \in H^p(\mathbb{R})$, then there exist a sequence $\{a_j\}$ of $(p,2,k)$-atoms and a sequence $\{\lambda_j\}$ of real numbers with $\sum_j |\lambda_j|^p \leq C\|f\|_{H^p(\mathbb{R})}^p$ such that $f = \sum_j \lambda_j a_j$, the series converging to f in $H^p(\mathbb{R})$ and also in the sense of tempered distributions.

A real-valued function $M(x)$ is called a $(p,2,b)$-*molecule* centered at x_0 if $M(x)$ satisfies the following conditions: (i) $N(M) = \|M\|_2^{1-\theta} \||\cdot - x_0|^b M\|_2^{\theta}$ $< \infty$, where $b > 1/p - 1/2$, $\theta = (1/p - 1/2)/b$, and (ii) $\int_{\mathbb{R}} x^j M(x) = 0$,

$j = 0, 1, \ldots, [1/p - 1]$, where $[1/p - 1]$ is the greatest integer not exceeding $1/p - 1$. We call $N(M)$ the molecular norm of $M(x)$. The molecular characterization asserts that if $f = \sum_j M_j$ with $(p, 2, b)$-molecules M_j as tempered distributions, and $\sum_j N(M_j)^p < \infty$, then $f \in H^p(\mathbb{R})$ and $\|f\|_{H^p(\mathbb{R})}^p \le C \sum_j N(M_j)^p$. For the atomic decomposition and molecular characterization, we may refer to [2, Chapter III].

The following lemma gives the main estimate:

Lemma. *Let $0 < p \le 1$ and r be the smallest integer such that $r > 1/p - 1/2$. Suppose that $\phi \in L^1(\mathbb{R})$ satisfies the following*:

(i) $\int_{\mathbb{R}} |\xi|^{-1/2} |\phi(\xi)|\, d\xi < \infty$,

(ii) $\widehat{\phi} \in C^{2r}(\mathbb{R})$ *with* $\sup_x |x|^r |\widehat{\phi}^{(r)}(x)| < \infty$, $\sup_x |x|^r |\widehat{\phi}^{(2r)}(x)| < \infty$.

Then, for a $(p, 2, r-1)$-atom a, $\mathcal{H}_\phi a$ is a $(p, 2, r)$-molecule centered at 0, *and*

$$N(\mathcal{H}_\phi a) \le C,$$

where C is independent of the atoms a.

Our theorem and its corollary are as follows:

Theorem. *Let $0 < p \le 1$. Suppose that $\phi \in L^1(\mathbb{R})$ satisfies the same conditions as in the Lemma. Then the Hausdorff operator $\mathcal{H}_\phi$ is a bounded operator on $H^p(\mathbb{R})$.*

Corollary. *Let $\alpha = 1, 2, \ldots$ If $2/(2\alpha + 1) < p \le 1$, then the Cesàro operator $\mathcal{C}_\alpha$ of order α is a bounded operator on $H^p(\mathbb{R})$.*

Other typical summability kernels covered by the theorem are, for example, $e^{-\xi^2}$ and $e^{-|\xi|}$. The corresponding operators $\mathcal{H}_\phi$ are bounded on $H^p(\mathbb{R})$ for every p with $0 < p \le 1$ since the functions $e^{-\xi^2}$ and $e^{-|\xi|}$ satisfy the conditions of the Lemma for every p with $0 < p \le 1$.

The Lemma will be proved in the next section. A discussion on defining the values of the Hausdorff operator for $H^p(\mathbb{R})$ functions and the proof of the Theorem will be given in the fourth section by using the main estimate of the Lemma.

3. Proof of the Lemma. Let $0 < p \le 1$ and r be the smallest integer such that $r > 1/p - 1/2$. We begin with estimating the molecular norm $N(\mathcal{H}_\phi a)$ for a $(p, 2, r-1)$-atom a. By the Plancherel theorem and the identity $[xf(x)]^\wedge(t) = i(d/dt)\widehat{f}(t)$, we have $\|\mathcal{H}a\|_2 = \|\widehat{\mathcal{H}a}\|_2$, and

$$\|x^r \mathcal{H}a(x)\|_2 = \|[x^r \mathcal{H}a(x)]^\wedge(t)\|_2 = \|(d/dt)^r \widehat{\mathcal{H}a}(t)\|_2,$$

which leads to

$$N(\mathcal{H}a) = \|\widehat{\mathcal{H}a}\|_2^{1-\theta} \|(d/dt)^r \widehat{\mathcal{H}a}(t)\|_2^{\theta},$$

where

$$\theta = (1/p - 1/2)/r = \frac{2-p}{2pr}.$$

To estimate $\|\widehat{\mathcal{H}a}\|_2$, we apply the generalized Minkowski inequality. We have

$$\begin{aligned}\|\widehat{\mathcal{H}a}\|_2 &\le \int_{\mathbb{R}} \Big\{ \int_{\mathbb{R}} |\widehat{a}(t\xi)|^2\,dt \Big\}^{1/2} |\phi(\xi)|\,d\xi \\ &= \|\widehat{a}\|_2 \int_{\mathbb{R}} |\xi|^{-1/2}|\phi(\xi)|\,d\xi = D_0\|a\|_2,\end{aligned}$$

where $D_0 = \int_{\mathbb{R}} |\xi|^{-1/2}|\phi(\xi)|\,d\xi$, which implies

$$N(\mathcal{H}a) \le D_0\|a\|_2^{1-\theta}\|(d/dt)^r\widehat{\mathcal{H}a}(t)\|_2^{\theta}.$$

It is enough to show that $\|(d/dt)^r\widehat{\mathcal{H}a}(t)\|_2 \le C\|a\|_2^{\delta}$, where

$$1-\theta+\delta\theta = 0, \quad \text{that is,} \quad \delta = 1 - 1/\theta = \frac{2-(1+2r)p}{2-p},$$

and the constant C is independent of the atoms a.

Noting that $\widehat{\mathcal{H}a}(t) = \int_{\mathbb{R}} a(x)\widehat{\phi}(tx)\,dx$, we have

$$\frac{d^r}{dt^r}\widehat{\mathcal{H}a}(t) = \int_{\mathbb{R}} a(x)x^r\widehat{\phi}^{(r)}(tx)\,dx.$$

Let μ be a positive number, which will be chosen later, and write

$$\begin{aligned}\|(d/dt)^r\widehat{\mathcal{H}a}(t)\|_2^2 &= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} a(x)x^r\widehat{\phi}^{(r)}(tx)\,dx\Big|^2 dt \\ &= \Big\{ \int_{|t|\le\mu} + \int_{|t|>\mu} \Big\} \Big| \int_{\mathbb{R}} a(x)x^r\widehat{\phi}^{(r)}(tx)\,dx\Big|^2 dt \\ &= I_\mu + J_\mu, \qquad \text{say.}\end{aligned}$$

We first treat J_μ. It follows that

$$\begin{aligned}J_\mu &= \int_{|t|>\mu} \Big| t^{-r}\int_{\mathbb{R}} a(x)(tx)^r\widehat{\phi}^{(r)}(tx)\,dx\Big|^2 dt \\ &\le \int_{|t|>\mu} |t|^{-2r}\,dt\,(\sup_x |x|^r|\widehat{\phi}^{(r)}(x)|)^2\Big(\int_{\mathbb{R}} |a(x)|\,dx\Big)^2.\end{aligned}$$

By Schwarz's inequality, we have

$$\int_{\mathbb{R}} |a(x)|\,dx \le \|a\|_2 h^{1/2},$$

which is bounded by $\|a\|_2^{2(1-p)/(2-p)}$, since the condition $\|a\|_2 \le h^{1/2-1/p}$ implies $h \le \|a\|_2^{-2p/(2-p)}$, where the support interval of a is $[c, c+h]$. Thus, we have

$$J_\mu \le (2K^2/(2r-1))\mu^{-2r+1}\|a\|_2^{4(1-p)/(2-p)}, \tag{1}$$

where $K = \sup_x |x|^r|\widehat{\phi}^{(r)}(x)|$.

To estimate I_μ, we consider the inner integral $\int_{\mathbb{R}} a(x)x^r\widehat{\phi}^{(r)}(tx)\,dx$ of I_μ. Let $g_t(x) = x^r\widehat{\phi}^{(r)}(tx)$. By the Taylor expansion of g_t at $x = c$ and the vanishing moment property of atoms, we have

$$\int_{\mathbb{R}} a(x)g_t(x)\,dx = \frac{1}{r!}\int_{\mathbb{R}} a(x)g_t^{(r)}(\widetilde{x})(x-c)^r\,dx, \qquad c < \widetilde{x} < c+h,$$

where $[c, c+h]$ is the support interval of a and $g_t^{(r)}$ is the rth derivative of g_t with respect to x. Since

$$g_t^{(r)}(\widetilde{x}) = \sum_{j=0}^{r}\binom{r}{j}\frac{r!}{j!}(t\widetilde{x})^j\widehat{\phi}^{(r+j)}(t\widetilde{x}),$$

it follows that

$$\begin{aligned} |g_t^{(r)}(\widetilde{x})| &\le C\sum_{j=0}^{r}\sup_x |x|^j|\widehat{\phi}^{(r+j)}(x)| \\ &\le C(\sup_x |\widehat{\phi}^{(r)}(x)| + \sup_x |x|^r|\widehat{\phi}^{(2r)}(x)|) = K', \qquad \text{say}, \end{aligned} \tag{2}$$

where K' depends only on ϕ and r, that is, p. (We shall refer to the last inequality (2) in the Remark below.) Thus, we have

$$\begin{aligned} \Big|\int_{\mathbb{R}} a(x)x^r\widehat{\phi}^{(r)}(tx)\,dx\Big| &\le K'\int_{\mathbb{R}} |a(x)|\cdot|x-c|^r\,dx \\ &\le C\|a\|_2h^{r+1/2} \le C\|a\|_2^{2(1-p-pr)/(2-p)}, \end{aligned}$$

where C depends only on ϕ and p. The last inequality follows from $h \le \|a\|_2^{-2p/(2-p)}$. We have

$$I_\mu \le C\mu\|a\|_2^{4(1-p-pr)/(2-p)}. \tag{3}$$

Therefore,

$$\begin{aligned} \|(d/dt)^r\widehat{\mathcal{H}a}(t)\|_2 &\le I_\mu^{1/2} + J_\mu^{1/2} \\ &\le C(\mu^{1/2}\|a\|_2^{2(1-p-pr)/(2-p)} + \mu^{-r+1/2}\|a\|_2^{2(1-p)/(2-p)}) \\ &\le C(\mu^{1/2}\|a\|_2^{2(1-p-pr)/(2-p)-\delta} \\ &\qquad + \mu^{-r+1/2}\|a\|_2^{2(1-p)/(2-p)-\delta})\|a\|_2^{\delta}, \end{aligned}$$

where C depends only on p and ϕ. It follows that

$$\mu^{1/2}\|a\|_2^{2(1-p-pr)/(2-p)-\delta} = (\mu\|a\|_2^{4(1-p-pr)/(2-p)-2\delta})^{1/2},$$
$$\mu^{-r+1/2}\|a\|_2^{2(1-p)/(2-p)-\delta} = (\mu\|a\|_2^{(2(1-p)/(2-p)-\delta)/(-r+1/2)})^{-r+1/2},$$

and

$$\frac{4(1-p-pr)}{2-p} - 2\delta = \frac{2(1-p)/(2-p)-\delta}{-r+1/2} = \frac{-2p}{2-p}.$$

Therefore, choosing $\mu = \|a\|_2^{2p/(2-p)}$, we get

$$\|(d/dt)^r \widehat{\mathcal{H}a}(t)\|_2 \le C\|a\|_2^{\delta}, \tag{4}$$

which leads to the desired inequality

$$N(\mathcal{H}a) \le C,$$

where C depends only on p and ϕ.

We turn to the moment condition. It follows that

$$\int_{\mathbb{R}} x^j \mathcal{H}a(x)\,dx = [x^j\mathcal{H}a(x)]^\wedge(0) = i^j \frac{d^j\widehat{\mathcal{H}a}}{dt^j}(0)$$
$$= i^j \int_{\mathbb{R}} \widehat{a}^{(j)}(t\xi)\xi^j\phi(\xi)\,d\xi\Big|_{t=0}.$$

It follows from the moment condition for a $(p,2,r-1)$-atom a that $\widehat{a}^{(j)}(0) = (-i)^j\int_{\mathbb{R}} x^j a(x)\,dx = 0, j = 0,1,\ldots,r-1$. Since $L \le r-1$, the moment condition for $\mathcal{H}a$ follows. The proof of the Lemma is complete.

Remark. Let $r = 1,2,\ldots$ The following inequalities hold:

$$\sup_x |x|^j|f^{(j)}(x)| \le C(\sup_x|f(x)| + \sup_x|x|^r|f^{(r)}(x)|), \quad j = 1,\ldots,r-1, \tag{5}$$

where C depends only on r. These inequalities yield (2) by taking $f = \widehat{\phi}^{(r)}$. We get (5) by following the proof of the known inequalities (cf. [1, Ch. 2, Theorem 5.6])

$$u^j\sup_x|f^{(j)}(x)| \le C(\sup_x|f(x)| + u^r\sup_x|f^{(r)}(x)|), \quad j = 1,\ldots,r-1, \tag{6}$$

where $u > 0$ is arbitrary and C depends only on r. For the reader's convenience, we give a proof of (5). Let $1 = \lambda_1 < \ldots < \lambda_{r-1} = 2$. It follows from the Taylor formula that

$$f(x+\lambda_l x) = \sum_{j=0}^{r-1}\frac{(\lambda_l x)^j}{j!}f^{(j)}(x) + \int_0^{\lambda_l x}\frac{(\lambda_l x - t)^{r-1}}{(r-1)!}f^{(r)}(x+t)\,dt$$

for $l = 1,\ldots,r-1$. We denote the remainder term by $R_r(x,l)$. We have a linear system

$$\sum_{j=1}^{r-1} \lambda_l^j \frac{x^j}{j!} f^{(j)}(x) = f(x+\lambda_l x) - f(x) - R_r(x,l), \qquad l = 1,\dots,r-1.$$

Since the Vandermonde determinant is nonzero, this system can be solved for $x^j f^{(j)}(x)/j!$. Further, the remainder term $R_r(x,l)$ is estimated as follows:

$$|R_r(x,l)| \le C(\sup_x |x|^r |f^{(r)}(x)|) \int_0^{\lambda_l} \frac{(\lambda_l - u)^{r-1}}{(1+u)^r}\, du$$

for $l = 1,\dots,r-1$, where C depends only on r, which leads to (5).

5. Proof of the Theorem. We first discuss defining the value $\mathcal{H}_\phi f$ for $f \in H^p(\mathbb{R})$, $0 < p < 1$. We use the fact that a function of the Lipschitz space $\Lambda_{1/p-1}(\mathbb{R})$ defines a continuous linear functional on $H^p(\mathbb{R})$ (cf. [2, III.5]). Let $0 < p < 1$ and r be the smallest integer such that $r > 1/p - 1/2$. Suppose that ϕ satisfies the same conditions as in the Lemma.

We put $(\widehat{\phi})_t(x) = \widehat{\phi}(xt)$. Then we have $|(d/dx)^j (\widehat{\phi})_t(x)| \le A_j |t|^j$, $j = 0,1,\dots,r$, where $A_j = \sup_x |\widehat{\phi}^{(j)}(x)|$. The constants A_j are finite, which follows from (6). Hence

$$\|(\widehat{\phi})_t\|_{\Lambda_{1/p-1}(\mathbb{R})} \le C(1+|t|^r). \tag{7}$$

This implies $(\widehat{\phi})_t \in \Lambda_{1/p-1}(\mathbb{R})$, that is, for every $t \in \mathbb{R}$ the function $(\widehat{\phi})_t$ defines a continuous linear functional of $H^p(\mathbb{R})$ and

$$|\langle f, (\widehat{\phi})_t \rangle| \le C(1+|t|^r)\|f\|_{H^p(\mathbb{R})} \quad \text{for } f \in H^p(\mathbb{R}),$$

where $\langle \cdot,\cdot \rangle$ is the duality paring between $H^p(\mathbb{R})$ and $\Lambda_{1/p-1}(\mathbb{R})$, and C is independent of f and t. We define $\mathcal{H}_\phi f$ for $f \in H^p(\mathbb{R})$ as the inverse Fourier transform of the tempered function $\langle f, (\widehat{\phi})_t \rangle$ with respect to the variable t, that is, $\widehat{\mathcal{H}_\phi f}(t) = \langle f, (\widehat{\phi})_t \rangle$. This coincides with the original definition when $f, \phi \in L^1(\mathbb{R})$ since

$$\widehat{\mathcal{H}_\phi f}(t) = \int_{\mathbb{R}} \widehat{f}(t\xi)\phi(\xi)\, d\xi = \int_{\mathbb{R}} f(x)\widehat{\phi}(tx)\, dx.$$

We turn to the proof of the Theorem. Let $0 < p \le 1$ and r be the smallest integer such that $r > 1/p - 1/2$. Let $f \in H^p(\mathbb{R})$. We have an atomic decomposition $f = \sum_{j=0}^\infty \lambda_j a_j$, where $\sum_{j=0}^\infty |\lambda_j|^p \le C\|f\|_{H^p(\mathbb{R})}^p$ and a_j is a $(p,2,r-1)$-atom. By the Lemma, we have

$$\sum_{j=0}^\infty N(\lambda_j \mathcal{H}_\phi a_j)^p = \sum_{j=0}^\infty |\lambda_j|^p N(\mathcal{H}_\phi a_j)^p \le C \sum_{j=0}^\infty |\lambda_j|^p \le C\|f\|_{H^p(\mathbb{R})}^p.$$

Thus, the series $\sum_{j=0}^{\infty} \lambda_j \mathcal{H}_\phi a_j$ converges to a tempered distribution g in $\mathcal{S}'$ and $\|g\|_{H^p(\mathbb{R})} \le C\|f\|_{H^p(\mathbb{R})}$, where $\mathcal{S}$ is the Schwartz space. It is enough to show that $g = \mathcal{H}_\phi f$ in $\mathcal{S}'$. Let $\psi \in \mathcal{S}$. It follows that

$$\begin{aligned}(g, \widehat{\psi}) &= \sum_{j=0}^{\infty} \lambda_j (\mathcal{H}_\phi a_j, \widehat{\psi}) = \sum_{j=0}^{\infty} \lambda_j (\widehat{\mathcal{H}_\phi a_j}, \psi) \\ &= \sum_{j=0}^{\infty} \lambda_j \int_{\mathbb{R}} \widehat{\mathcal{H}_\phi a_j}(t)\psi(t)\,dt = \sum_{j=0}^{\infty} \lambda_j \int_{\mathbb{R}} \langle a_j, (\widehat{\phi})_t\rangle \psi(t)\,dt,\end{aligned}$$

where $(\cdot,\cdot)$ is the duality pairing between $\mathcal{S}'$ and $\mathcal{S}$. By the fact that $|\langle a_j, (\widehat{\phi})_t\rangle| \le C\|(\widehat{\phi})_t\|_{\Lambda_{1/p-1}(\mathbb{R})}$ and by (7), we can change the order of the sum and integral in the last term. It follows that

$$\begin{aligned}(g, \widehat{\psi}) &= \int_{\mathbb{R}} \sum_{j=0}^{\infty} \lambda_j \langle a_j, (\widehat{\phi})_t\rangle \psi(t)\,dt = \int_{\mathbb{R}} \langle f, (\widehat{\phi})_t\rangle \psi(t)\,dt \\ &= (\widehat{\mathcal{H}_\phi f}, \psi) = (\mathcal{H}_\phi f, \widehat{\psi}).\end{aligned}$$

Therefore, we have $g = \mathcal{H}_\phi f$ in $\mathcal{S}'$, which completes the proof of the Theorem.

Finally, we prove the Corollary. We put $\phi_\alpha(\xi) = \alpha(1-\xi)^{\alpha-1}\chi_{(0,1)}(\xi)$. Then $\mathcal{C}_\alpha = \mathcal{H}_{\phi_\alpha}$. Trivially, $\phi_\alpha \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} |\xi|^{-1/2}|\phi_\alpha(\xi)|\,d\xi < \infty$. We check condition (ii) of the Lemma for $\widehat{\phi}_\alpha$. We have

$$\widehat{\phi}_\alpha(x) = \alpha!(-ix)^{-\alpha}\left(e^{-ix} - \sum_{j=0}^{\alpha-1} \frac{(-ix)^j}{j!}\right).$$

We easily see that $\widehat{\phi}_\alpha \in C^\infty(\mathbb{R})$ and

$$\sup_x |x|^\alpha |\widehat{\phi}_\alpha^{(\alpha)}(x)| < \infty, \qquad \sup_x |x|^\alpha |\widehat{\phi}_\alpha^{(2\alpha)}(x)| < \infty$$

for $\alpha = 1, 2, \ldots$ Therefore, the Theorem yields the Corollary.

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Department of Mathematics
General Education Hall
Kanazawa University
Kanazawa 920-1192, Japan
E-mail: kanjin@kenroku.kanazawa-u.ac.jp

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