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A class of solvable non-homogeneous differential operators on the Heisenberg group

by

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Abstract. In [8], we studied the problem of local solvability of complex coefficient second order left-invariant differential operators on the Heisenberg group $\mathbb{H}_n$, whose principal parts are "positive combinations of generalized and degenerate generalized sub-Laplacians", and which are homogeneous under the Heisenberg dilations. In this note, we shall consider the same class of operators, but in the presence of left invariant lower order terms, and shall discuss local solvability for these operators in a complete way. Previously known methods to study such non-homogeneous operators, as in [9] or [6], do not apply to these operators, and it is the main purpose of this article to introduce a new method, which should be applicable also in much wider settings.

1. Basic definitions and main results. Let ω denote the symplectic form on $\mathbb{R}^{2n}$ given by

$$\omega(z,z') := {}^{\mathrm{t}}z'Jz, \qquad J = J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The Heisenberg group $\mathbb{H}_n$ is $\mathbb{R}^{2n}\times\mathbb{R}$, endowed with the group law

$$(z,u)(z',u) = \big(z+z', u+u' - \tfrac{1}{2}\omega(z,z')\big). \tag{1.1}$$

The left-invariant vector fields

$$X_j := \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial u}, \qquad Y_j := \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial u},$$

$j=1,\ldots,n$, and $U := \partial/\partial u$ form a natural basis for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$. The only non-trivial commutation relations among these vector fields are $[X_j,Y_j]=U$, $j=1,\ldots,n$.

Denote by $\mathfrak{sp}(n,\mathbb{C})$ the symplectic Lie algebra, consisting of all complex $2n\times 2n$-matrices S satisfying

$${}^{\mathrm{t}}SJ + JS = 0.$$

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Let $S \in \mathfrak{sp}(n,\mathbb{C})$, denote by $A = (a_{jk})$ the symmetric matrix $A := SJ$, and put

$$\varDelta_S := \sum_{j,k=1}^{2n} a_{jk} V_j V_k, \tag{1.2}$$

where $V_j := X_j$, $V_{n+j} := Y_j$, $j = 1, \dots, n$.

In [5], the situation where the matrix S assumes a block diagonal form

$$S = \begin{pmatrix} \gamma_1 S_{(1)} & & \\ & \ddots & \\ & & \gamma_m S_{(m)} \end{pmatrix} \tag{1.3}$$

with respect to a suitable decomposition of $\mathbb{R}^{2n}$ into symplectic subspaces has been studied, under the assumptions that $\gamma_j \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $S_{(j)}^2 = -I$, $j = 1, \dots, m$. By means of Hörmander's criterion, it has been shown that for "most" of these matrices S, the operators $\varDelta_S +$ lower order terms are locally non-solvable.

There are only five exceptional classes of operators of the above type to which Hörmander's criterion does not apply and which are listed in [5]. In all these classes, each of the elementary blocks is of size 2×2. Moreover, according to the classification of normal forms in [7], after applying a suitable symplectic change of coordinates, one may assume that $S_{(j)}$ is either of the form

$$S_{(j)} = \begin{pmatrix} i\varepsilon_j\lambda_j & \lambda_j^2 - 1 \\ 1 & -i\varepsilon_j\lambda_j \end{pmatrix}, \qquad \text{“Type 1”}, \tag{1.4}$$

with $\lambda_j \in \{-1\} \cup [0, \infty[$ and $\varepsilon_j = 1$ if $|\lambda_j| \leq 1$, and $\varepsilon_j = \pm 1$ if $\lambda_j > 1$, or of the form

$$S_{(j)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \text{“Type 3”}. \tag{1.5}$$

Denote by σ_S the principal symbol of $-\varDelta_S$, and assume henceforth that $\operatorname{Re}\sigma_S \geq 0$. Then $\varDelta_S$ belongs to one of the major exceptional classes listed in [5]. It follows from [5] that $\varDelta_S$ is a positive combination of generalized sub-Laplacians and of degenerate generalized sub-Laplacians, i.e.

$$\begin{aligned} \varDelta_S = {} & \sum_{j=1}^{r} \gamma_j [(1 - \lambda_j^2) X_j^2 + Y_j^2 + i\lambda_j (X_j Y_j + Y_j X_j)] \\ & + i \sum_{j=r+1}^{n} \gamma_j (X_j^2 - Y_j^2), \end{aligned} \tag{1.6}$$

where $0 \leq r \leq n$, $|\lambda_j| \leq 1$, $\gamma_j \in \mathbb{C}^\times$ for $j = 1, \dots, r$, $\gamma_j > 0$ for $j = r+1, \dots, n$, and where for each $j = 1, \dots, r$ and every $\xi_j, \eta_j \in \mathbb{R}$,

$$\mathrm{Re}[\gamma_j[(1-\lambda_j^2)\xi_j^2+\eta_j^2+2i\lambda_j\xi_j\eta_j]]\geq 0, \tag{1.7}$$

provided we choose appropriate coordinates.

In this article, we deal with the operators of the form

$$L_\alpha := \varDelta_S + \sum_{j=1}^{n}(\beta_{j1}X_j+\beta_{j2}Y_j)+\alpha U + c, \tag{1.8}$$

where the coefficients β_{j1}, β_{j2}, α and c are complex. We assume in addition that

$$\mathrm{Re}\,\gamma_j > 0 \quad \text{for } j=1,\dots,r \tag{1.9}$$

(for a discussion of this condition, see Remark 1.2 in [8]).

Set

$$E^{\pm} := \Big\{\pm\sum_{j=1}^{r}\gamma_j(2l_j+1) : l_1,\dots,l_r\in\mathbb{N}\Big\}.$$

Moreover, denote by n_1, n_2^+ and n_2^- the number of "Type 1" blocks $S_{(j)}$ in S with $|\lambda_j|<1$, $\lambda_j=1$ and $\lambda_j=-1$, respectively, and by n_3 the number of "Type 3" blocks.

Since $S_{(j)}^2=-I$ and $\gamma_j\neq 0$, S is invertible, and so is the coefficient matrix A. We choose coordinates $z:=(z_1,\dots,z_n)$, with $z_j:=(x_j,y_j)$, corresponding to the block form (1.3) of S. With respect to these coordinates, the matrix A also assumes a block form

$$A=\begin{pmatrix} A_{(1)} & & \\ & \ddots & \\ & & A_{(n)}\end{pmatrix}.$$

Thus, arguing as in [5], we see that it is possible to eliminate the first order terms in the X_j's and Y_j's, by conjugating L_α by a multiplication operator. In fact

$$\begin{aligned} &L_\alpha(e^{-(1/2)\,{}^{\mathrm t}zA^{-1}\beta}f) \\ &\qquad = e^{-(1/2)\,{}^{\mathrm t}zA^{-1}\beta}\Big(\varDelta_S+i\alpha U+c-\frac14\sum_{j=1}^{n}{}^{\mathrm t}\beta_jA_{(j)}^{-1}\beta_j\Big)f, \end{aligned} \tag{1.10}$$

where ${}^{\mathrm t}\beta_j:=(\beta_{j1},\beta_{j2})$ and ${}^{\mathrm t}\beta:=({}^{\mathrm t}\beta_1,\dots,{}^{\mathrm t}\beta_n)$. Combining this observation with the results in [8], we obtain

THEOREM 1.1. *Suppose L_α, given by* (1.6) *and* (1.8), *satisfies* (1.7) *and* (1.9), *and assume that $c=\frac14\sum_{j=1}^{n}{}^{\mathrm t}\beta_jA_{(j)}^{-1}\beta_j$. Then the following hold*:

(I) *If $\alpha\notin E^+\cup E^-$, then L_α is locally solvable.*
(II) *If $\alpha\in E^-$, then L_α is locally solvable if and only if $n_2^++n_3>0$.*
(III) *If $\alpha\in E^+$, then L_α is locally solvable if and only if $n_2^-+n_3>0$.*

This theorem discusses local solvability of those operators which can be reduced to homogeneous ones. For the remaining cases, we shall prove the following positive result.

THEOREM 1.2. *Suppose* L_α, *given by* (1.6) *and* (1.8), *satisfies* (1.7) *and* (1.9), *and that* $c \neq \frac{1}{4}\sum_{j=1}^{n} {}^{\mathrm{t}}\beta_j A_{(j)}^{-1}\beta_j$. *Then* L_α *is locally solvable.*

REMARK 1.3. In Theorem 3.3 of [5], second order left-invariant differential operators on the Heisenberg group $\mathbb{H}_1$ were studied in a rather complete way. Following the notation of Theorem 3.3 in [5], the only case which was left open was the case where Δ_S is of type (a.2) and $c \neq (\beta_1^2 - 2i\varepsilon\beta_1\beta_2)/4$. Our main result shows that in this case the operator L is locally solvable.

2. Proof of Theorem 1.2. We apply the usual convention that C is a constant whose value may change from line to line.

In view of (1.10), we may assume that $\beta = 0$, i.e. that $L_\alpha = \Delta_S + i\alpha U + c$, where $c \neq 0$. Denote by f^μ the partial Fourier transform of f along the center of $\mathbb{H}_n$, i.e.

$$f^\mu(z) := \int\limits_{\mathbb{R}} f(z,u)e^{-i\mu u}\,du, \qquad \mu \in \mathbb{R}^\times.$$

Moreover, for suitable functions or distributions φ, ψ on $\mathbb{R}^{2n}$, define the *μ-twisted convolution* of φ and ψ by

$$\varphi \times_\mu \psi(z) := \int\limits_{\mathbb{R}^{2n}} \varphi(z-z')\psi(z')e^{i(\mu/2)\omega(z-z',z')}\,dz'.$$

Then one easily verifies that, for suitable distributions f, g on $\mathbb{H}_n$,

$$(f * g)^\mu = f^\mu \times_\mu g^\mu.$$

Denote by $\mathfrak{u}(\mathfrak{h}_n)$ the universal enveloping algebra of $\mathfrak{h}_n$. Then $\mathfrak{u}(\mathfrak{h}_n)$ can be identified with the associative algebra of all left-invariant differential operators on $\mathbb{H}_n$. If $D \in \mathfrak{u}(\mathfrak{h}_n)$, we define the partial Fourier transform of D as the partial differential operator D^μ on $\mathbb{R}^{2n}$ which is given by the formula

$$D^\mu f^\mu = (Df)^\mu, \qquad f \in S(\mathbb{H}_n),$$

where $S(\mathbb{H}_n)$ denotes the Schwartz space on $\mathbb{H}_n$. In particular, we then have $L_\alpha^\mu = \Delta_S^\mu - \alpha\mu + c$.

Let $f \in S(\mathbb{H}_n)$. We intend to solve the equation

$$L_\alpha v = f.$$

Proceeding formally as in [5], [8], we put

$$\Gamma_{t,S}^\mu(z) := \frac{|\mu|^n}{(4\pi)^n \prod_{j=1}^n \sinh(\gamma_j t)} e^{-(|\mu|/4)\sum_{j=1}^n \coth(\gamma_j t)\, {}^{\mathrm{t}}z_j J_1 S_{(j)} z_j}, \qquad \mu \in \mathbb{R}^\times.$$

From [5], we have

$$|\mu|\frac{d}{dt}(\varphi \times_\mu \Gamma^\mu_{t,S}) = \Delta^\mu_S(\varphi \times_\mu \Gamma^\mu_{t,S})$$

for every $\varphi \in S(\mathbb{R}^{2n})$. Therefore

$$\begin{aligned}(2.1)\quad |\mu|\frac{d}{dt}[e^{-\alpha(\operatorname{sgn}\mu)t+(c/|\mu|)t}(\varphi \times_\mu \Gamma^\mu_{t,S})] \\ = L^\mu_\alpha[e^{-\alpha(\operatorname{sgn}\mu)t+(c/|\mu|)t}(\varphi \times_\mu \Gamma^\mu_{t,S})].\end{aligned}$$

So a fundamental solution for $L_\alpha = \Delta_S + i\alpha U + c$ can be formally defined by

$$(2.2)\qquad F_\alpha(z,u) := -\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}^\times} e^{-\alpha(\operatorname{sgn}\mu)t+(c/|\mu|)t}\Gamma^\mu_{t,S}(z)e^{i\mu u}\,\frac{d\mu}{|\mu|}\,dt.$$

On a formal level, this suggests that a solution to the equation $L_\alpha v = f$ is given by

$$\begin{aligned}(2.3)\quad v(z,u) &= f * F_\alpha(z,u) \\ &= -\frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}^\times} e^{-a(\operatorname{sgn}\mu)t+(c/|\mu|)t} f^\mu \times_\mu \Gamma^\mu_{t,S}(z)e^{i\mu u}\,\frac{d\mu}{|\mu|}\,dt.\end{aligned}$$

However, the integral (2.3) will in general not be convergent. A main and principal obstacle is the exponential growth of the factor $e^{(c/|\mu|)t}$ as $\mu \to 0$, in the case where $\operatorname{Re} c > 0$.

On the other hand, in order to prove local solvability, we do not really have to construct a fundamental solution, and are even allowed to ignore "low frequencies" (compare also [1]), as the following auxiliary result shows.

Fix a cut-off function $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \chi \le 1$, $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| > 2$, and write $\chi_B(x) := \chi(x/B)$ if $B > 0$. We introduce Fourier multiplier operators Q_B and P_B on $L^2(\mathbb{R}^d)$ by

$$(2.4)\qquad \widehat{Q_B f} := \chi_B\widehat{f},\quad P_B f = (I - Q_B)f,\qquad f \in L^2(\mathbb{R}^d).$$

Proposition 2.1. *There exists a constant $K > 1$, depending only on the dimension d, such that the following holds: If r and B are positive numbers such that $r < 1/(KB)$, then for every function $f \in L^2(\mathbb{R}^d)$ which is supported in the ball $B_r(0) := \{x \in \mathbb{R}^d : |x| < r\}$, we have*

$$\|f\|_2 \le 2\|\chi_{2r}(P_B f)\|_2.$$

Proof. Denote by η_B the inverse Fourier transform of χ_B. Then for every $N > 0$ we find a constant $C = C_N > 0$ such that

$$|\eta_B(x)| \le CB^d(1 + |Bx|)^{-N}.$$

Assume that $f \in L^2(\mathbb{R}^d)$ and $\operatorname{supp} f \subset B_r(0)$. If $|x| \ge 2r$, then

$$|(P_B f)(x)| = |f(x) - \eta_B * f(x)| = |\eta_B * f(x)| = \Big| \int\limits_{|y|\le r} f(y)\eta_B(x-y)\,dy\Big|$$
$$\le CB^d \|f\|_2 \Big(\int\limits_{|y|\le r} (1+B|x-y|)^{-2N}\,dy\Big)^{1/2}$$
$$\le CB^d r^{d/2}\|f\|_2 (1+B|x|)^{-N}.$$

Thus, if $r < 1/(KB)$, and if we choose N such that $d > N > d/2$, we obtain

$$\Big(\int\limits_{|x|\ge 2r} |(P_B f)(x)|^2\,dx\Big)^{1/2} \le CB^d r^{d/2}\|f\|_2 \Big(\int\limits_{|x|\ge 2r} |Bx|^{-2N}\,dx\Big)^{1/2}$$
$$= CB^{d-N} r^{d-N}\|f\|_2 \le CK^{-(d-N)}\|f\|_2.$$

On the other hand, since $f = \chi_r f$, we have

$$|\widehat{f}(\xi)| = |\widehat{\chi}_r * \widehat{f}(\xi)| \le \|\widehat{f}\|_2 \cdot \|\widehat{\chi}_r\|_2 \le Cr^{d/2}\|f\|_2$$

for every $\xi \in \mathbb{R}^d$. Therefore, by Plancherel's formula,

$$\|Q_B f\|_2 \le \Big(\int\limits_{|\xi|\le 2B} |\widehat{f}(\xi)|^2\,d\xi\Big)^{1/2} \le CB^{d/2} r^{d/2}\|f\|_2 \le CK^{-d/2}\|f\|_2.$$

Choosing K sufficiently large, we thus obtain

$$\|f\|_2 \le \|Q_B f\|_2 + \|P_B f\|_2 \le \frac{1}{2}\|f\|_2 + \|\chi_{2r}(P_B f)\|_2.$$

This implies Proposition 2.1. ∎

We shall apply the proposition above to the central variable u on the Heisenberg group. To this end, we define for $B > 0$ the operator $\mathcal{P}_B$ on $L^2(\mathbb{H}_n)$ by means of the formula

$$(\mathcal{P}_B f)^\mu := [1 - \chi_B(\mu)] f^\mu, \qquad \mu \in \mathbb{R}^\times,\ f \in S(\mathbb{H}_n).$$

Because of the blow-up of the integrated function that appears in (2.3), we shall try to solve the equation $L_\alpha v = \mathcal{P}_B f$ instead of $L_\alpha v = f$. Consider the distribution H_α, which is formally defined by

$$\langle H_\alpha, \varphi\rangle := \frac{-1}{2\pi} \int_0^\infty \int (1-\chi_B(\mu)) e^{-\alpha(\operatorname{sgn}\mu)t + (c/|\mu|)t} \langle \Gamma^\mu_{t,S}, \varphi^{-\mu}\rangle \frac{d\mu}{|\mu|}\,dt, \qquad \varphi \in \mathcal{S}(\mathbb{H}_n). \tag{2.5}$$

PROPOSITION 2.2. *Let* $A := \sum_j \operatorname{Re}\gamma_j$. *If* $|\operatorname{Re}\alpha| + |\operatorname{Re} c|/B < A$, *then* H_α *is a well-defined tempered distribution on* $\mathbb{H}_n$, *satisfying*

$$L_\alpha(f * H_\alpha) = \mathcal{P}_B f \quad \textit{for every } f \in \mathcal{S}(\mathbb{H}_n). \tag{2.6}$$

Moreover, if $A > |\operatorname{Re} c|/B$, *then* $\{H_\alpha\}_{|\operatorname{Re}\alpha| < A - |\operatorname{Re} c|/B}$ *is an analytic family of tempered distributions.*

Proof. We closely follow [5], §§7.3, 7.4. If we put $\check{g}(z) := g(-z)$, then

$$\langle \Gamma^\mu_{t,S}, \varphi^{-\mu}\rangle = \frac{1}{(2\pi)^n}\langle \widehat{\Gamma^\mu_{t,S}}, \widehat{\check{\varphi}^{-\mu}}\rangle,$$

where $\widehat{\Gamma^\mu_{t,S}}$ is given by

$$\widehat{\Gamma^\mu_{t,S}}(\zeta) = \frac{1}{\sigma(t)} e^{-\frac{1}{|\mu|}q_t(\zeta)},$$

with

$$\begin{aligned}
\sigma(t) &:= \prod_{j=1}^n \cosh(\gamma_j t),\\
q_t(\zeta) &:= \sum_{j=1}^r \tanh(\gamma_j t)[(1-\lambda_j^2)\xi_j^2 + \eta_j^2 + 2i\lambda_j\xi_j\eta_j] + i\sum_{j=r+1}^n \tanh(\gamma_j t)(\xi_j^2 - \eta_j^2)\\
&=: \sum_{j=1}^n \tanh(\gamma_j t) q_j(\zeta_j)
\end{aligned}$$

if $\zeta_j = (\xi_j, \eta_j)$. Since $\operatorname{Re} q_t \ge 0$ (see [5]), we thus have

$$|e^{-\alpha(\operatorname{sgn}\mu)t + (c/|\mu|)t}\,\widehat{\Gamma^\mu_{t,S}}(\zeta)| \le Ce^{-(-|\operatorname{Re}\alpha| - |\operatorname{Re} c|/|\mu| + A)t},$$

and consequently, for $|\mu| \ge B$,

$$\int_0^\infty |e^{-\alpha(\operatorname{sgn}\mu)t + (c/|\mu|)t}\,\widehat{\Gamma^\mu_{t,S}}(\zeta)|\,dt \le C\frac{|\mu|}{(A - |\operatorname{Re}\alpha|)|\mu| - |\operatorname{Re} c|} \qquad \text{for every } \zeta \in \mathbb{R}^{2n}.$$

This shows that the integral defining $\langle H_\alpha, \varphi\rangle$ is absolutely convergent, and that

$$|\langle H_\alpha, \varphi\rangle| \le C \int_{|\mu|\ge B}\int_{\mathbb{R}^{2n}} \left|\frac{\widehat{\check{\varphi}^{-\mu}}(\zeta)}{(A - |\operatorname{Re}\alpha|)|\mu| - |\operatorname{Re} c|}\right| d\zeta\,d\mu \le \|\varphi\|_{\mathcal{S}}$$

for some Schwartz norm $\|\cdot\|_{\mathcal{S}}$. Moreover, the mapping $\alpha \mapsto \langle H_\alpha, \varphi\rangle$ is analytic for $|\operatorname{Re}\alpha| < A - |\operatorname{Re} c|/B$. Finally, from (2.1) we obtain

$$L_\alpha(f * H_\alpha)(z,u) = \frac{1}{2\pi}\int (1 - \chi_B(\mu)) f^\mu(z) e^{i\mu u}\,d\mu = \mathcal{P}_B f(z,u). \ \blacksquare$$

Fix now B such that $b_0 := A - |\operatorname{Re} c|/B > 0$, and put $\gamma := \sum_j \gamma_j$, $a := \min_j \operatorname{Re}\gamma_j$ and $b_m := b_0 + 2ma$, $m \in \mathbb{N}$, so that $b_m \to \infty$ as $m \to \infty$. In particular, the domains $\Sigma_m := \{\alpha \in \mathbb{C} : |\operatorname{Re}\alpha| < b_m\}$, $m \in \mathbb{N}$, cover the complex plane.

The analytic family of distributions H_α will in general not extend from Σ_0 to a larger strip Σ_m. However, arguing as in [5], by means of integrations by parts in the t-variable, we can prove the following

PROPOSITION 2.3. *Let $m \in \mathbb{N}$. Then there exist an analytic family of tempered distributions $\{H_\alpha^m\}_{\alpha\in\Sigma_m}$ on $\mathbb{H}_n$ and a family of non-trivial differential operators $Q_\alpha = \sum_k a_k(\alpha)U^k$ whose coefficients $a_k(\alpha)$ depend polynomially on α, such that*

$$L_\alpha(f * H_\alpha^m) = Q_\alpha \mathcal{P}_B f \quad \text{for } f \in \mathcal{S}(\mathbb{H}_n),\ \alpha \in \Sigma_m. \tag{2.7}$$

Proof. First re-write H_α for $\alpha \in \Sigma_0$. Let $\varepsilon(\mu) := \operatorname{sgn}\mu$. If $J = (j_1, \ldots \ldots, j_m)$ is a multi-index of length $\ell(J) = m$ in $\{1, \ldots, n\}^m$, we set

$$\varrho_J(\mu) := \prod_{k=0}^{m} \Big(\varepsilon(\mu)\alpha + \gamma + \sum_{l\le k} 2\gamma_{j_l} - \frac{c}{|\mu|}\Big)$$

(the factor with $k = 0$ is taken to be $(\varepsilon(\mu)\alpha + \gamma - c/|\mu|)$). Put $M_j := \gamma_j q_j$, and

$$M_J := M_{j_1} \ldots M_{j_m}$$

(where $M_\emptyset := 1$, with $\ell(\emptyset) = 0$). Then, after m integrations by parts following the lines in [5], we find that there are functions β_J on $\{-1, +1\}$ for $\ell(J) < m$, and functions $K_{\mu,J}(t)$ for $\ell(J) = m$, such that

$$\begin{aligned}\langle H_\alpha, f\rangle = &\sum_{\ell(J)<m} \int \frac{\beta_J(\varepsilon(\mu))}{\varrho_{\widetilde J}(\mu)} M_J(0) f^{-\mu}(0) \frac{1-\chi_B(\mu)}{|\mu|^{\ell(J)+1}}\, d\mu \\ &+ \sum_{\ell(J)=m} \int_0^\infty \int K_{\mu,J}(t) \langle e^{-q_t/|\mu|}, M_J \widehat{\check f^{-\mu}}\rangle \frac{1-\chi_B(\mu)}{|\mu|^{m+1}}\, d\mu\, dt,\end{aligned} \tag{2.8}$$

and

$$K_{\mu,J}(t) \sim \frac{c_J}{\varrho_{\widetilde J}(\mu)} e^{-(\varepsilon(\mu)\alpha+\gamma+\sum_{l=1}^m 2\gamma_{j_l} - c/|\mu|)t}, \tag{2.9}$$

where $\widetilde J$ is obtained from J by deleting the last index j_m.

In order to extend the definition of H_α by means of (2.8) and (2.9) to $\alpha \in \Sigma_m$, we have to "remove" the zeros of the functions $\varrho_{\widetilde J}$. To this end, put

$$h_{\alpha,J}(\mu) := \prod_{k=0}^{m} \Big[\Big(\Big(\alpha + \gamma + \sum_{l\le k} 2\gamma_{j_l}\Big)\mu - c\Big)\Big(\Big(\alpha - \gamma - \sum_{l\le k} 2\gamma_{j_l}\Big)\mu - c\Big)\Big]$$

and

$$Q_\alpha(i\mu) := \prod_{\ell(J)\le m} h_{\alpha,\widetilde J}(\mu).$$

Then Q_α is a non-trivial polynomial, since $c \ne 0$. Moreover, $Q_\alpha(i\mu)/\varrho_{\widetilde J}(\mu)$ is a smooth function away from $\mu = 0$ which grows at most polynomially. We therefore define a tempered distribution H_α^m by the formula

$$\text{(2.10)}\qquad \langle H_\alpha^m,\varphi\rangle := \frac{-1}{2\pi}\int_0^\infty\!\!\int (1-\chi_B(\mu))e^{-\alpha(\operatorname{sgn}\mu)t+(c/|\mu|)t} \times \langle \Gamma_{t,S}^\mu,\varphi^{-\mu}\rangle Q_\alpha(i\mu)\frac{d\mu}{|\mu|}\,dt,\qquad \varphi\in\mathcal{S}(\mathbb{H}_n).$$

Replacing H_α by H_α^m in the preceding argument, we obtain

$$\text{(2.11)}\qquad \langle H_\alpha^m,f\rangle = \sum_{\ell(J)<m}\int \frac{Q_\alpha(i\mu)\beta_J(\varepsilon(\mu))}{\varrho_{\tilde J}(\mu)}M_J(0)f^{-\mu}(0)\frac{1-\chi_B(\mu)}{|\mu|^{\ell(J)+1}}\,d\mu + \sum_{\ell(J)=m}\int_0^\infty\!\!\int Q_\alpha(i\mu)K_{\mu,J}(t)\langle e^{-q_t/|\mu|},M_J\widehat{\check{f}^{-\mu}}\rangle\frac{1-\chi_B(\mu)}{|\mu|^{m+1}}\,d\mu\,dt.$$

From (2.9), one finds that the right-hand side of (2.11) converges absolutely even for $\alpha\in\Sigma_m$, and that it defines an analytic family of tempered distributions H_α^m for $\alpha\in\Sigma_m$. Moreover, if $\alpha\in\Sigma_0$, we may argue as in the proof of Proposition 2.2 in order to verify (2.7). Finally, since both sides of (2.7) are analytic in $\alpha\in\Sigma_m$, we see that, by analytic continuation, (2.7) remains valid for every $\alpha\in\Sigma_m$. ∎

Now, in order to prove Theorem 1.2, if $\alpha\in\mathbb{C}$ we choose $m\in\mathbb{N}$ such that $\alpha\in\Sigma_m$, and then $H_\alpha^m\in\mathcal{S}'(\mathbb{H}_n)$ and the bi-invariant differential Q_α according to Proposition 2.3. Let $\mathcal{H}_\alpha f := f * H_\alpha^m$ for $f\in\mathcal{S}(\mathbb{H}_n)$.

If we denote by ${}^{\mathrm t}L_\alpha$, ${}^{\mathrm t}\mathcal{H}_\alpha$ and ${}^{\mathrm t}\mathcal{P}_B$ the formal transposes of L_α, $\mathcal{H}_\alpha$ and $\mathcal{P}_B$ on $\mathcal{S}(\mathbb{H}_n)$, respectively, then by (2.7),

$$\text{(2.12)}\qquad {}^{\mathrm t}\mathcal{H}_\alpha\,{}^{\mathrm t}L_\alpha f = \mathcal{P}_B\,{}^{\mathrm t}Q_\alpha f,\qquad f\in\mathcal{S}(\mathbb{H}_n),$$

since ${}^{\mathrm t}\mathcal{P}_B = \mathcal{P}_B$.

Denote by B_r the set of all $(z,u)\in\mathbb{H}_n$ such that $|z|\le r$, $|u|\le r$. Applying Proposition 2.1 to the central variable u, keeping z fixed, we see that there is an $r>0$ such that

$$\|\varphi\|_2 \le C\|\chi_{B_{2r}}(\mathcal{P}_B\varphi)\|_2$$

for every $\varphi\in C_0^\infty(\mathbb{H}_n)$ supported in B_r. Replacing φ by ${}^{\mathrm t}Q_\alpha\varphi$, and applying (2.12), we get

$$\|{}^{\mathrm t}Q_\alpha\varphi\|_2 \le C\|\chi_{B_{2r}}(\mathcal{P}_B\,{}^{\mathrm t}Q_\alpha\varphi)\|_2 = \|\chi_{B_{2r}}({}^{\mathrm t}\mathcal{H}_\alpha\,{}^{\mathrm t}L_\alpha\varphi)\|_2.$$

Consequently,

$$\|{}^{\mathrm t}Q_\alpha\varphi\|_2 \le C\sup_{x\in B_{2r}}|({}^{\mathrm t}\mathcal{H}_\alpha({}^{\mathrm t}L_\alpha\varphi))(x)| = C\sup_{x\in B_{2r}}\Big|\int_{\mathbb{H}_n}H_\alpha^m(y)({}^{\mathrm t}L_\alpha\varphi)(xy)\,dy\Big| \le C\sup_{x\in B_{2r}}\|{}^{\mathrm t}L_\alpha\varphi(x\,\cdot)\|_{(k)}$$

for some Sobolev norm $\|\cdot\|_{(k)}$. But, since for $x\in B_r$, all functions ${}^{\mathrm t}L_\alpha\varphi(x\,\cdot)$ are supported in a fixed compact subset, we may find an elliptic, left-invariant differential operator D such that

$$\|{}^{\mathrm{t}}L_\alpha\varphi(x\,\cdot)\|_{(k)} \le \|D[{}^{\mathrm{t}}L_\alpha\varphi(x\,\cdot)]\|_2 = \|(D\,{}^{\mathrm{t}}L_\alpha)\varphi(x\,\cdot)\|_2 = \|D\,{}^{\mathrm{t}}L_\alpha\varphi\|_2,$$

hence

$$\|{}^{\mathrm{t}}Q_\alpha\varphi\|_2 \le C\|D\,{}^{\mathrm{t}}L_\alpha\varphi\|_2,$$

whenever $\operatorname{supp}\varphi \subset B_r$.

The operator ${}^{\mathrm{t}}Q_\alpha$ is a constant coefficient operator acting on the variable u only, and thus has a fundamental solution. So, arguing as before we see that there exists a bi-invariant differential operator R such that

$$\|\varphi\|_2 \le C\|R\,{}^{\mathrm{t}}Q_\alpha\varphi\|_2$$

if $\operatorname{supp}\varphi \subset B_r$. But, R commutes with D and ${}^{\mathrm{t}}Q_\alpha$, and so we obtain

$$\|\varphi\|_2 \le C\|RD\,{}^{\mathrm{t}}L_\alpha\varphi\|_2 \le \|{}^{\mathrm{t}}L_\alpha\varphi\|_{(l)}$$

for a suitable Sobolev norm $\|\cdot\|_{(l)}$. This implies that L_α is locally solvable (see e.g. [3]).

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