STUDIA MATHEMATICA 207 (3) (2011)

# Infinitely divisible cylindrical measures on Banach spaces

by

Markus Riedle (London)

**Abstract.** In this work infinitely divisible cylindrical probability measures on arbitrary Banach spaces are introduced. The class of infinitely divisible cylindrical probability measures is described in terms of their characteristics, a characterisation which is not known in general for infinitely divisible Radon measures on Banach spaces. Further properties of infinitely divisible cylindrical measures such as continuity are derived. Moreover, the classification result enables us to deduce new results on genuine Lévy measures on Banach spaces.

**1. Introduction.** Probability theory in Banach spaces has been extensively studied since 1960 and several monographs are dedicated to this field, e.g. de Araujo and Giné [7], Ledoux and Talagrand [11] and Vakhaniya et al. [21]. This area is closely related to the theory of Banach space geometry and it has applications not only in probability theory but also in operator theory, harmonic analysis and C*-algebras.

Cylindrical stochastic processes in Banach spaces appear naturally as the driving noise in stochastic differential equations in infinite dimensions, such as interest rate models. Up to now, cylindrical Wiener processes are the standard examples of the driving noise, which restricts the noise to a Gaussian perturbation with continuous paths. A natural non-Gaussian and discontinuous generalisation is introduced by *cylindrical Lévy processes*. The notion of cylindrical Lévy process appears for the first time in Peszat and Zabczyk [16] and it is studied by Brzeźniak et al. [3], Brzeźniak and Zabczyk [4] and Priola and Zabczyk [17]. The first systematic introduction to cylindrical Lévy processes appears in Applebaum and Riedle [1].

The introduction of cylindrical Lévy processes in [1] is based on the theory of cylindrical or generalised processes and cylindrical measures (see for example Schwartz [20] or Vakhaniya et al. [21]). The approach in [1] is

2010 *Mathematics Subject Classification*: Primary 46G12; Secondary 46B09, 60B11, 60G20.
*Key words and phrases*: infinitely divisible measure, Lévy measure, cylindrical measure, cylindrical random variable.

DOI: 10.4064/sm207-3-2

inspired by the analogous definition for cylindrical Wiener processes (see Kallianpur and Xiong [10], Métivier and Pellaumail [14] or Riedle [18]). In the same way as cylindrical Wiener processes are related to the class of Gaussian cylindrical measures, the introduction of cylindrical Lévy processes in [1] leads to the new class of infinitely divisible cylindrical measures which have not been considered so far. Since the article [1] focused on cylindrical Lévy processes and their stochastic integral, no further properties of infinitely divisible cylindrical measures were derived. In this work we give a rigorous introduction of infinitely divisible cylindrical measures in Banach spaces and derive some of their fundamental properties. Some of our results also give a new insight on genuine infinitely divisible Radon measures on Banach spaces.

The main result is a characterisation of the class of infinitely divisible cylindrical measures in a Banach space in terms of a triplet $(p, q, \nu)$ where $p, q$ are some functions and $\nu$ is a cylindrical measure. This result is surprising since for infinitely divisible Radon measures in Banach spaces such a classification is not known in general (see de Araujo and Giné [7]). Furthermore, since in analogy to the characteristics of Lévy processes the triplet describes the deterministic drift, the covariance structure of the Gaussian part and the jump distribution, it provides the construction of an infinitely divisible cylindrical random variable for given data specifying these properties. Moreover, this main result enables us to derive the following two important conclusions.

The first one concerns the following problem: even in the finite-dimensional case, a probability measure on $\mathbb{R}^2$ such that all image measures under linear projections to $\mathbb{R}$ are infinitely divisible might not be infinitely divisible (see Giné and Hahn [8] and Marcus [13]). However, a question left open is if a probability measure on an infinite-dimensional space is infinitely divisible whenever all its linear projections to $\mathbb{R}^n$ for all finite dimensions $n \in \mathbb{N}$ are infinitely divisible. By the characterisation of the set of infinitely divisible cylindrical measures mentioned above we are able to answer this question in the affirmative.

The second conclusion of our main result concerns the characterisation of Lévy measures in Banach spaces. In a Hilbert space $H$ it is well known that a $\sigma$-finite measure $\nu$ is the Lévy measure of an infinitely divisible Radon measure if and only if

$$\int_H (\|u\|^2 \wedge 1)\, \nu(du) < \infty \tag{1.1}$$

(see for example Parthasarathy [15]). Although this integrability condition can be used to classify the type and cotype of Banach spaces (see de Araujo and Giné [6]), in general Banach spaces such an explicit description of infinitely divisible measures in terms of the Lévy measure $\nu$ is not known. Even worse, the condition (1.1) might be neither sufficient nor necessary for a $\sigma$-finite measure $\nu$ on an arbitrary Banach space $U$ to guarantee that there exists an infinitely divisible measure with characteristics $(0,0,\nu)$ (see for example $U = C[0,1]$ in Araujo [5]). However, we show in the last part of this work that a $\sigma$-finite measure $\nu$ satisfying the weaker condition

$$\int_U (|\langle u,a\rangle|^2 \wedge 1)\,\nu(du) < \infty \quad \text{for all } a \in U^* \tag{1.2}$$

always generates an infinitely divisible cylindrical measure $\mu$. This result reduces the question whether a $\sigma$-finite measure $\nu$ generates an infinitely divisible Radon measure to the question whether the infinitely divisible cylindrical measure $\mu$ extends to a Radon measure.

**2. Preliminaries.** For a measure space $(S,\mathcal{S},\mu)$ we denote by $L^p_\mu(S,\mathcal{S})$, $p \ge 0$, the space of equivalence classes of measurable functions $f : S \to \mathbb{R}$ which satisfy $\int |f(s)|^p\,\mu(ds) < \infty$.

Let $U$ be a Banach space with dual $U^*$. The dual pairing is denoted by $\langle u,a\rangle$ for $u \in U$ and $a \in U^*$. The Borel $\sigma$-algebra in $U$ is denoted by $\mathcal{B}(U)$ and the closed unit ball at the origin by $B_U := \{u \in U : \|u\| \le 1\}$.

For every $a_1,\dots,a_n \in U^*$ and $n \in \mathbb{N}$ we define a linear map

$$\pi_{a_1,\dots,a_n} : U \to \mathbb{R}^n, \qquad \pi_{a_1,\dots,a_n}(u) = (\langle u,a_1\rangle,\dots,\langle u,a_n\rangle).$$

Let $\Gamma$ be a subset of $U^*$. Sets of the form

$$Z(a_1,\dots,a_n;B) := \{u \in U : (\langle u,a_1\rangle,\dots,\langle u,a_n\rangle) \in B\} = \pi^{-1}_{a_1,\dots,a_n}(B),$$

where $a_1,\dots,a_n \in \Gamma$ and $B \in \mathcal{B}(\mathbb{R}^n)$, are called *cylindrical sets*. The set of all cylindrical sets is denoted by $\mathcal{Z}(U,\Gamma)$ and it is an algebra. The generated $\sigma$-algebra is denoted by $\mathcal{C}(U,\Gamma)$ and called the *cylindrical $\sigma$-algebra with respect to* $(U,\Gamma)$. If $\Gamma = U^*$ we write $\mathcal{Z}(U) := \mathcal{Z}(U,\Gamma)$ and $\mathcal{C}(U) := \mathcal{C}(U,\Gamma)$.

A function $\mu : \mathcal{Z}(U) \to [0,\infty]$ is called a *cylindrical measure on* $\mathcal{Z}(U)$ if for each finite subset $\Gamma \subseteq U^*$ the restriction of $\mu$ to the $\sigma$-algebra $\mathcal{C}(U,\Gamma)$ is a measure. A cylindrical measure is called *finite* if $\mu(U) < \infty$, and a *cylindrical probability measure if* $\mu(U) = 1$.

For every function $f : U \to \mathbb{C}$ which is measurable with respect to $\mathcal{C}(U,\Gamma)$ for a finite subset $\Gamma \subseteq U^*$ the integral $\int f(u)\,\mu(du)$ is well defined as a complex valued Lebesgue integral if it exists. In particular, the *characteristic function* $\varphi_\mu : U^* \to \mathbb{C}$ of a finite cylindrical measure $\mu$ is defined by

$$\varphi_\mu(a) := \int_U e^{i\langle u,a\rangle}\,\mu(du) \quad \text{for all } a \in U^*.$$

In contrast to probability measures on $\mathcal{B}(U)$ there exists an analogue of Bochner's theorem for cylindrical probability measures (cf. [21, Prop. VI.3.2]):

a function $\varphi : U^* \to \mathbb{C}$ with $\varphi(0) = 1$ is the characteristic function of a cylindrical probability measure if and only if it is positive-definite and continuous on every finite-dimensional subspace.

For every $a_1, \ldots, a_n \in U^*$ we obtain an image measure $\mu \circ \pi^{-1}_{a_1,\ldots,a_n}$ on $\mathcal{B}(\mathbb{R}^n)$. Its characteristic function $\varphi_{\mu\circ\pi^{-1}_{a_1,\ldots,a_n}}$ is determined by that of $\mu$:

$$\varphi_{\mu\circ\pi^{-1}_{a_1,\ldots,a_n}}(t) = \varphi_\mu(t_1 a_1 + \cdots + t_n a_n) \tag{2.1}$$

for all $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$.

If $\mu_1$ and $\mu_2$ are cylindrical probability measures on $\mathcal{Z}(U)$ their convolution is the cylindrical probability measure defined by

$$(\mu_1 * \mu_2)(Z) = \int_U \mu_1(Z - u)\, \mu_2(du)$$

for each $Z \in \mathcal{Z}(U)$. Indeed if $Z = \pi^{-1}_{a_1,\ldots,a_n}(B)$ for some $a_1, \ldots, a_n \in U^*, B \in \mathcal{B}(\mathbb{R}^n)$, then it is easily verified that

$$(\mu_1 * \mu_2)(Z) = (\mu_1 \circ \pi^{-1}_{a_{a_1,\ldots,a_n}}) * (\mu_2 \circ \pi^{-1}_{a_{a_1,\ldots,a_n}})(B).$$

A standard calculation yields $\varphi_{\mu_1 * \mu_2} = \varphi_{\mu_1}\varphi_{\mu_2}$. For more information about convolution of cylindrical probability measures, see [19]. The $k$-fold convolution of a cylindrical probability measure $\mu$ with itself is denoted by $\mu^{*k}$.

**3. Infinitely divisible cylindrical measures.** For later reference, we begin with the well understood class of infinitely divisible measures on $\mathbb{R}$. A probability measure $\zeta$ on $\mathcal{B}(\mathbb{R})$ is called *infinitely divisible* if for every $k \in \mathbb{N}$ there exists a probability measure $\zeta_k$ such that $\zeta = (\zeta_k)^{*k}$. It is well known that infinitely divisible probability measures on $\mathcal{B}(\mathbb{R})$ are characterised by their characteristic function. The characteristic function is unique but its specific representation depends on the chosen truncation function.

DEFINITION 3.1. A *truncation function* is any measurable function $h : \mathbb{R} \to \mathbb{R}$ which is bounded and satisfies $h = \mathrm{Id}$ in a neighborhood $D(h)$ of 0.

Given a truncation function $h$ a probability measure $\zeta$ on $\mathcal{B}(\mathbb{R})$ is infinitely divisible if and only if its characteristic function is of the form

$$\varphi_\zeta : \mathbb{R} \to \mathbb{C}, \quad \varphi_\zeta(t) = \exp\Big(imt - \tfrac{1}{2}r^2t^2 + \int_{\mathbb{R}} \widetilde{\psi}_h(s,t)\, \eta(ds)\Big) \tag{3.1}$$

for some constants $m \in \mathbb{R}$, $r \geq 0$ and a *Lévy measure* $\eta$, which is a $\sigma$-finite measure $\eta$ on $\mathcal{B}(\mathbb{R})$ with $\eta(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (|s|^2 \wedge 1)\, \eta(ds) < \infty.$$

The function $\widetilde{\psi}_h$ is defined by

$$\widetilde{\psi}_h : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad \widetilde{\psi}_h(s,t) := e^{ist} - 1 - ith(s).$$

In this situation we call the triplet $(m,r,\eta)_h$ the *characteristics of* $\zeta$. If $h'$ is another truncation function then $(m',r,\eta)_{h'}$ is the characteristics of $\zeta$ with respect to $h'$, where

$$m' := m + \int_{\mathbb{R}} (h'(s) - h(s))\,\eta(ds).$$

The integral exists because $h'(s) - h(s) = 0$ for $s \in D(h') \cap D(h)$ and $h$ and $h'$ are both bounded. From Bochner's theorem and Schoenberg's correspondence (see [21, Ch. IV.1.4]) it follows that the function

$$t \mapsto -\int_{\mathbb{R}} \tilde{\psi}_h(s,t)\,\eta(ds)$$

is negative-definite for all Lévy measures $\eta$. By choosing $\eta = \delta_{s_0}$, where $\delta_{s_0}$ denotes the Dirac measure at $s_0$ for a constant $s_0 \in \mathbb{R}$, we conclude that

(3.2) $\qquad t \mapsto -\tilde{\psi}_h(s_0,t)$ is negative-definite for all $s_0 \in \mathbb{R}$.

Now, we turn to the general situation of an arbitrary Banach space $U$. A Radon probability measure $\mu$ on $\mathcal{B}(U)$ is called *infinitely divisible* if for each $k \in \mathbb{N}$ there exists a Radon probability measure $\mu_k$ such that $\mu = (\mu_k)^{*k}$. We generalise this definition to cylindrical measures:

Definition 3.2. A cylindrical probability measure $\mu$ on $\mathcal{Z}(U)$ is called *infinitely divisible* if for each $k \in \mathbb{N}$ there exists a cylindrical probability measure $\mu_k$ such that $\mu = (\mu_k)^{*k}$.

Bochner's theorem for cylindrical probability measures [21, Prop. VI.3.2] implies that a cylindrical probability measure $\mu$ on $\mathcal{Z}(U)$ is infinitely divisible if and only if for every $k \in \mathbb{N}$ there exists a characteristic function $\varphi_{\mu_k}$ of a cylindrical probability measure $\mu_k$ such that

$$\varphi_\mu(a) = (\varphi_{\mu_k}(a))^k \quad \text{for all } a \in U^*.$$

One might conjecture that a cylindrical probability measure $\mu$ is infinitely divisible if every image measure $\mu \circ a^{-1}$ is infinitely divisible for all $a \in U^*$. But this is wrong already in the case $U = \mathbb{R}^2$ as shown by Giné and Hahn [8] and Marcus [13]. They constructed a probability measure $\mu$ on $\mathcal{B}(\mathbb{R}^2)$ such that all projections $\mu \circ a^{-1}$ are infinitely divisible for all linear functions $a : \mathbb{R}^2 \to \mathbb{R}$ but $\mu$ is not infinitely divisible. However, in infinite dimensions one can require that all finite-dimensional projections are infinitely divisible.

Definition 3.3. A cylindrical probability measure $\mu$ on $\mathcal{Z}(U)$ is called *weakly infinitely divisible* if

$$\mu \circ \pi^{-1}_{a_1,\dots,a_n} \text{ is infinitely divisible for all } a_1,\dots,a_n \in U^* \text{ and } n \in \mathbb{N}.$$

A cylindrical probability measure $\mu$ is weakly infinitely divisible if and only if for each $k \in \mathbb{N}$ and all $a_1,\dots,a_n \in U^*$, $n \in \mathbb{N}$ there exists a characteristic function $\varphi_{\xi_{k,a_1,\dots,a_n}}$ of a probability measure $\xi_{k,a_1,\dots,a_n}$ on $\mathcal{B}(\mathbb{R}^n)$

such that

$$\varphi_{\mu\circ\pi^{-1}_{a_1,\dots,a_n}}(t) = (\varphi_{\xi_{k,a_1,\dots,a_n}}(t))^k \quad \text{for all } t \in \mathbb{R}^n. \tag{3.3}$$

It follows that every infinitely divisible cylindrical probability measure $\mu$ is also weakly infinitely divisible since for each $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ we have

$$\begin{aligned}\varphi_{\mu\circ\pi^{-1}_{a_1,\dots,a_n}}(t) &= \varphi_\mu(t_1a_1 + \dots + t_na_n) = (\varphi_{\mu_k}(t_1a_1 + \dots + t_na_n))^k \\ &= (\varphi_{\mu_k\circ\pi^{-1}_{a_1,\dots,a_n}}(t))^k\end{aligned}$$

for all $a_1, \dots, a_n \in U^*$ and all $k \in \mathbb{N}$. We will later see that the converse is also true, i.e. that the concepts of Definitions 3.2 and 3.3 coincide.

If $\mu$ is a weakly infinitely divisible cylindrical measure then $\mu \circ a^{-1}$ is an infinitely divisible measure in $\mathcal{B}(\mathbb{R})$, and thus

$$\begin{aligned}\varphi_\mu(a) &= \varphi_{\mu\circ a^{-1}}(1) \\ &= \exp\Big(im_a - \tfrac{1}{2}r_a^2 + \int_{\mathbb{R}} (e^{is} - 1 - is\,\mathbb{1}_{B_{\mathbb{R}}}(s))\,\eta_a(ds)\Big)\end{aligned} \tag{3.4}$$

for some constants $m_a \in \mathbb{R}$, $r_a \geq 0$ and a Lévy measure $\eta_a$ on $\mathcal{B}(\mathbb{R})$. For infinitely divisible cylindrical measures, this representation can be significantly improved, as shown in Applebaum and Riedle [1]. The same proof establishes the result for weakly infinitely divisible cylindrical measures in the following theorem.

Theorem 3.4. *Let $\mu$ be a weakly infinitely divisible cylindrical probability measure on $\mathcal{Z}(U)$. Then its characteristic function $\varphi_\mu : U^* \to \mathbb{C}$ is given by*

$$\begin{aligned}&\varphi_\mu(a) \\ &\quad = \exp\Big(iw(a) - \tfrac{1}{2}q(a) + \int_U (e^{i\langle u,a\rangle} - 1 - i\langle u,a\rangle\,\mathbb{1}_{B_{\mathbb{R}}}(\langle u,a\rangle))\,\nu(du)\Big),\end{aligned} \tag{3.5}$$

*where $w : U^* \to \mathbb{R}$ is a mapping, $q : U^* \to \mathbb{R}$ is a quadratic form and $\nu$ is a cylindrical measure on $\mathcal{Z}(U)$ such that $\nu \circ \pi^{-1}_{a_1,\dots,a_n}$ is the Lévy measure on $\mathcal{B}(\mathbb{R}^n)$ of $\mu \circ \pi^{-1}_{a_1,\dots,a_n}$ for all $a_1, \dots, a_n \in U^*$, $n \in \mathbb{N}$.*

It is natural to call the measure $\nu$ appearing in (3.5) a *cylindrical Lévy measure* as we do in the following definition. However, it turns out that it is sufficient to require only that the image measures under all one-dimensional linear projections to $\mathbb{R}$ are Lévy measures and it is not necessary to consider the image measures under all linear projections to $\mathbb{R}^n$ for all finite-dimensions $n$.

Definition 3.5. A cylindrical measure $\nu : \mathcal{Z}(U) \to [0, \infty]$ is called a *cylindrical Lévy measure* if $\nu \circ a^{-1}$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$ for all $a \in U^*$.

From (3.5) we can easily derive a representation of the characteristic function $\varphi_\mu$ of a weakly infinitely divisible cylindrical probability measure $\mu$ for an arbitrary truncation function $h$. Since $h = \mathrm{Id}$ on $D(h)$ one can define

$$p : U^* \to \mathbb{R}, \qquad p(a) := w(a) + \int_U \big(h(\langle u, a\rangle) - \langle u, a\rangle \, \mathbb{1}_{B_\mathbb{R}}(\langle u, a\rangle)\big)\, \nu(du).$$

It follows from (3.5) that

$$\varphi_\mu(a) = \exp\Big(ip(a) - \tfrac{1}{2}q(a) + \int_U \psi_h(\langle u, a\rangle)\, \nu(du)\Big), \tag{3.6}$$

where the kernel function $\psi_h$ is defined by

$$\psi_h : \mathbb{R} \to \mathbb{C}, \qquad \psi_h(t) := e^{it} - 1 - ih(t),$$

for an arbitrary truncation function $h$.

Definition 3.6. Let $h$ be a truncation function and let $\mu$ be a weakly infinitely divisible cylindrical probability measure on $\mathcal{Z}(U)$ with characteristic function (3.6). Then we call the triplet $(p, q, \nu)_h$ the *cylindrical characteristics* of $\mu$.

Analogously to the one-dimensional situation described after Definition 3.1 one can convert the cylindrical characteristics $(p, q, \nu)_h$ into $(p', q, \nu)_{h'}$ if $h'$ is another truncation function.

It follows from (3.6) that the characteristic function $\varphi_{\mu\circ a^{-1}}$ of the probability measure $\mu \circ a^{-1}$ on $\mathcal{B}(\mathbb{R})$ is for all $t \in \mathbb{R}$ given by

$$\begin{aligned} \varphi_{\mu\circ a^{-1}}(t) &= \varphi_\mu(at) \\ &= \exp\Big(ip(at) - \tfrac{1}{2}q(a)t^2 + \int_\mathbb{R} \psi_h(st)\, (\nu \circ a^{-1})(ds)\Big). \end{aligned} \tag{3.7}$$

This representation of $\varphi_{\mu\circ a^{-1}}$ does not coincide with the representation (3.1) because the functions $\widetilde{\psi}_h$ and $\psi_h$ do not coincide. Thus, we cannot directly read off the characteristics of $\mu \circ a^{-1}$ from (3.7).

Lemma 3.7. *Let $\mu$ be a weakly infinitely divisible cylindrical probability measure on $\mathcal{Z}(U)$ with cylindrical characteristics $(p, q, \nu)_h$ for a truncation function $h$. Then $\mu \circ a^{-1}$ has the characteristics $(p(a), q(a), \nu \circ a^{-1})_h$ for all $a \in U^*$.*

*Proof.* As above we can rewrite the characteristic function $\varphi_{\mu\circ a^{-1}}$ in the form (3.7) for the given truncation function $h$. In order to write $\varphi_{\mu\circ a^{-1}}$ in the standard form (3.1), we introduce the function $\widetilde{p} : U^* \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{p}(a, t) := \begin{cases} p(at) + \int_\mathbb{R} (th(s) - h(st))(\nu \circ a^{-1})(ds) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Note that the integral is well defined because for each $t \neq 0$ we have

$$th(s) - h(st) = 0 \quad \text{for all } s \in D(h) \cap \frac{1}{t} D(h)$$

and because $h$ is bounded. By defining the function

$$\widetilde{\psi}_h : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \qquad \widetilde{\psi}_h(s,t) = e^{ist} - 1 - ith(s),$$

we can rewrite the characteristic function (3.7) of $\mu \circ a^{-1}$ for all $t \in \mathbb{R}$:

$$\varphi_{\mu \circ a^{-1}}(t) = \exp\Big( i\widetilde{p}(a,t) - \tfrac{1}{2} q(a) t^2 + \int_{\mathbb{R}} \widetilde{\psi}_h(s,t) \, (\nu \circ a^{-1})(ds) \Big).$$

By Theorem 3.4 the Lévy measure of the infinitely divisible probability measure $\mu \circ a^{-1}$ is given by $\nu \circ a^{-1}$ for each $a \in U^*$. Thus, there exist some constants $m_a \in \mathbb{R}$ and $r_a \geq 0$ such that $(m_a, r_a, \nu \circ a^{-1})_h$ is the characteristics of $\mu \circ a^{-1}$. For all $t \in \mathbb{R}$ it follows that

$$\begin{aligned} \varphi_{\mu \circ a^{-1}}(t) &= \exp\Big( i\widetilde{p}(a,t) - \tfrac{1}{2} q(a) t^2 + \int_{\mathbb{R}} \widetilde{\psi}_h(s,t) \, (\nu \circ a^{-1})(ds) \Big) \\ &= \exp\Big( i m_a t - \tfrac{1}{2} r_a^2 t^2 + \int_{\mathbb{R}} \widetilde{\psi}_h(s,t) \, (\nu \circ a^{-1})(ds) \Big), \end{aligned}$$

which results in $\widetilde{p}(a,t) = m_a t = \widetilde{p}(a,1)t = p(a)t$. Consequently, we have

$$\varphi_{\mu \circ a^{-1}}(t) = \exp\Big( i p(a) t - \tfrac{1}{2} q(a) t^2 + \int_{\mathbb{R}} \widetilde{\psi}_h(s,t) \, (\nu \circ a^{-1})(ds) \Big),$$

which completes the proof. ■

Recalling the Lévy–Khintchine decomposition for infinitely divisible measures we could expect from (3.6) that

$$a \mapsto \exp(ip(a)), \qquad a \mapsto \exp\Big( \int_U \psi_h(\langle u, a \rangle) \, \nu(du) \Big),$$

are the characteristic functions of some cylindrical measures on $\mathcal{Z}(U)$. But the following example shows that we cannot separate the drift part $p$ and the integral term with respect to the cylindrical Lévy measure $\nu$ in order to obtain cylindrical measures.

Example 3.8. Let $\ell : U^* \to \mathbb{R}$ be a linear but not necessarily continuous functional and $\lambda > 0$ a constant. We will see later in Example 3.11 that

$$\varphi : U^* \to \mathbb{C}, \qquad \varphi(a) := \exp(\lambda(e^{i\ell(a)} - 1)),$$

is the characteristic function of an infinitely divisible cylindrical probability measure. In order to write $\varphi$ in the form (3.6) let $\nu$ be the cylindrical measure on $\mathcal{Z}(U)$ defined by

$$\nu(Z(a_1, \dots, a_n; B)) := \begin{cases} \lambda & \text{if } (\ell(a_1), \dots, \ell(a_n)) \in B, \\ 0 & \text{else,} \end{cases}$$

for every $a_1, \dots, a_n \in U^*$, $B \in \mathcal{B}(\mathbb{R}^n)$ and $n \in \mathbb{N}$. Then we can represent $\varphi$ by

$$\varphi(a) = \exp\Big(ip(a) + \int_U \psi_h(\langle u, a\rangle)\,\nu(du)\Big),$$

where $p(a) := \lambda h(\ell(a))$. Since $a \mapsto \exp(ip(a))$ is not positive-definite in general there does not exist a cylindrical measure with this function as its characteristic function.

Example 3.8 leads us to the insight that some necessary conditions guaranteeing the existence of an infinitely divisible cylindrical probability measure with cylindrical characteristics $(p, 0, \nu)$ rely on the interplay of the entries $p$ and $\nu$. The following result gives some properties of the entries $p$, $q$ and $\nu$ of the cylindrical characteristics, but also the interplay of $p$ and $\nu$.

LEMMA 3.9. *Let $\mu$ be a weakly infinitely divisible cylindrical probability measure on $\mathcal{Z}(U)$ with cylindrical characteristics $(p, q, \nu)_h$ for a continuous truncation function $h$. Then:*

(a) *The function*

$$a \mapsto \kappa(a) := -\Big(ip(a) + \int_U \psi_h(\langle u, a\rangle)\,\nu(du)\Big)$$

*is negative-definite.*

(b) *For every sequence $a_n \to a$ in a finite-dimensional subspace $V \subseteq U^*$ equipped with $\|\cdot\|_{U^*}$ we have:*

(i) $p(a_n) \to p(a)$;

(ii) $q(a_n) \to q(a)$;

(iii) $(|s|^2 \wedge 1)\,(\nu \circ a_n^{-1})(ds) \to (|s|^2 \wedge 1)\,(\nu \circ a^{-1})(ds)$ *weakly.*

*Proof.* (a) Let $Z$ be a cylindrical random variable on a probability space $(\Omega, \mathcal{A}, P)$ with cylindrical distribution $\mu$. As in Theorem 3.9 of [1], it follows that there exist two cylindrical random variables $W$ and $X$ such that $Z = W + X$ $P$-a.s. where the cylindrical distributions $\mu_1$ of $W$ and $\mu_2$ of $X$ have the characteristic functions $\varphi_1$ and $\varphi_2$ given by

$$\varphi_1(a) := \exp(-\tfrac{1}{2}q(a)), \qquad \varphi_2(a) := \exp(-\kappa(a)).$$

For fixed $a_1, \dots, a_n \in U^*$ the $\mathbb{R}^n$-valued random variable $(Za_1, \dots, Za_n)$ is infinitely divisible since $\mu$ is assumed to be weakly infinitely divisible and the $\mathbb{R}^n$-valued random variable $(Wa_1, \dots, Wa_n)$ is also infinitely divisible as it is Gaussian. Thus, the $\mathbb{R}^n$-valued random variable $(Xa_1, \dots, Xa_n)$ is infinitely divisible, that is, the cylindrical measure $\mu_2$ is weakly infinitely divisible.

We show (a) by applying Schoenberg's correspondence (see [21, Property(h), p. 192]), for which we have to show that $a \mapsto \exp(-k^{-1}\kappa(a))$ is positive-definite for all $k \in \mathbb{N}$ and that $\kappa$ is Hermitian, i.e. $\overline{\kappa(-a)} = \kappa(a)$ for all $a \in U^*$. To prove positive-definiteness, fix $k \in \mathbb{N}$, $a_1, \dots, a_n \in U^*$ and $z_1, \dots, z_n \in \mathbb{C}$ and let $e_i$ denote the $i$th unit vector in $\mathbb{R}^n$. Since $\mu_2$ is weakly infinitely divisible there exists a characteristic function $\varphi_{\xi_{k,a_1,\dots,a_n}}$ of a probability measure $\xi_{k,a_1,\dots,a_n}$ on $\mathcal{B}(\mathbb{R}^n)$ such that

$$\varphi_{\mu_2 \circ \pi^{-1}_{a_1,\dots,a_n}}(t) = (\varphi_{\xi_{k,a_1,\dots,a_n}}(t))^k \quad \text{for all } t \in \mathbb{R}^n.$$

Consequently, we have

$$\sum_{i,j=1}^n z_i \bar{z}_j \exp\Big(-\frac{1}{k}\kappa(a_i - a_j)\Big) = \sum_{i,j=1}^n z_i \bar{z}_j (\varphi_{\mu_2}(a_i - a_j))^{1/k}$$
$$= \sum_{i,j=1}^n z_i \bar{z}_j (\varphi_{\mu_2 \circ \pi^{-1}_{a_1,\dots,a_n}}(e_i - e_j))^{1/k} = \sum_{i,j=1}^n z_i \bar{z}_j \varphi_{\xi_{k,a_1,\dots,a_n}}(e_i - e_j) \geq 0,$$

where the inequality follows from the fact that $\varphi_{\xi_{k,a_1,\dots,a_n}}$ is a characteristic function on $\mathbb{R}^n$.

Next, we want show that $\kappa$ is Hermitian. Since rewriting the characteristic function of $\mu_2$ for different truncation functions does not affect $\kappa$, we can fix $h(s) = s\,\mathbb{1}_{B_{\mathbb{R}}}(s)$ for $s \in \mathbb{R}$, which yields $\widetilde{\psi}_h(-s,t) = \widetilde{\psi}_h(s,-t)$ for all $s, t \in \mathbb{R}$. By Lemma 3.7, for all $t \in \mathbb{R}$ we obtain

$$\begin{aligned}\varphi_{\mu_2 \circ a^{-1}}(t) &= \varphi_{\mu_2 \circ (-a)^{-1}}(-t)\\ &= \exp\Big(ip(-a)(-t) + \int_{\mathbb{R}} \widetilde{\psi}_h(s,-t)\,(\nu \circ (-a)^{-1})(ds)\Big)\\ &= \exp\Big(ip(-a)(-t) + \int_{\mathbb{R}} \widetilde{\psi}_h(-s,-t)\,(\nu \circ a^{-1})(ds)\Big)\\ &= \exp\Big(ip(-a)(-t) + \int_{\mathbb{R}} \widetilde{\psi}_h(s,t)\,(\nu \circ a^{-1})(ds)\Big),\end{aligned}$$

which implies $p(-a) = -p(a)$. It follows that

$$\begin{aligned}\overline{\kappa(-a)} &= \overline{-ip(-a)} - \int_U \overline{\psi_h(\langle u, -a\rangle)}\,\nu(du)\\ &= -ip(a) - \int_U \psi_h(\langle u, a\rangle)\,\nu(du) = \kappa(a)\end{aligned}$$

for all $a \in U^*$, which completes the proof of (a).

To see (b), let $a_n \to a$ in a finite-dimensional subspace $V \subseteq U^*$ and let the truncation function $h$ be continuous. Then Bochner's theorem implies that

$$\lim_{n\to\infty} \varphi_{\mu \circ a_n^{-1}}(t) = \lim_{n\to\infty} \varphi_\mu(ta_n) = \varphi_\mu(ta) = \varphi_{\mu \circ a^{-1}}(t) \tag{3.8}$$

for all $t \in \mathbb{R}$. By Lemma 3.7 the measures $\mu \circ a_n^{-1}$ are infinitely divisible with characteristics $(p(a_n), q(a_n), \nu \circ a_n^{-1})$. It follows from (3.8) that the infinitely divisible measures with characteristics $(p(a_n), q(a_n), \nu \circ a_n^{-1})$ converge weakly to $\mu \circ a^{-1}$ which has the characteristics $(p(a), q(a), \nu \circ a^{-1})$. Applying Theorem VII.2.9 and Remark VII.2.10 (p. 396) in Jacod and Shiryaev [9], which characterises the weak convergence of infinitely divisible measures in terms of their characteristics, implies $p(a_n) \to p(a)$ and

$$q(a_n)\, \delta_0(ds) + (|s|^2 \wedge 1)\, (\nu \circ a_n^{-1})(ds) \\ \to q(a)\, \delta_0(ds) + (|s|^2 \wedge 1)\, (\nu \circ a^{-1})(ds) \quad \text{weakly.}$$

But since $q$ is a quadratic form and therefore it is continuous on a finite-dimensional space, we have $q(a_n) \to q(a)$, which is property (ii) and which implies (iii). ∎

THEOREM 3.10. *Let $\nu : \mathcal{Z}(U) \to [0, \infty]$ be a given set function, let $p, q : U^* \to \mathbb{R}$ be given functions and let $h$ be a continuous truncation function. Then the following are equivalent:*

(a) *There exists an infinitely divisible cylindrical probability measure $\mu$ with cylindrical characteristics $(p, q, \nu)_h$.*
(b) *The following are satisfied:*
   (1) *$p(0) = 0$ and $p(a_n) \to p(a)$ for every sequence $a_n \to a$ in a finite-dimensional subspace $V \subseteq U^*$ equipped with $\|\cdot\|_{U^*}$;*
   (2) *$q : U^* \to \mathbb{R}$ is a quadratic form;*
   (3) *$\nu$ is a cylindrical Lévy measure;*
   (4) *the mapping $a \mapsto \kappa(a) := -(ip(a) + \int_U \psi_h(\langle u, a\rangle)\, \nu(du))$ is negative-definite.*

*In this situation, the characteristic function of $\mu$ is given by*

$$\varphi_\mu : U^* \to \mathbb{C}, \qquad \varphi_\mu(a) = \exp\Big(ip(a) - \tfrac{1}{2}q(a) + \int_U \psi_h(\langle u, a\rangle)\, \nu(du)\Big)$$

*and $\mu = \mu_1 * \mu_2$ where $\mu_1$ and $\mu_2$ are cylindrical probability measures with characteristic functions $\varphi_{\mu_1}(a) = \exp(-\frac{1}{2}q(a))$ and $\varphi_{\mu_2}(a) = \exp(-\kappa(a))$.*

*Proof.* (a)⇒(b): Properties (2) and (3) are stated in Theorem 3.4, and (1) and (4) are derived in Lemma 3.9. The property $p(0) = 0$ is an immediate consequence of Bochner's theorem, as is the fact that $q(0) = 0$.

(b)⇒(a): Property (2) implies that

$$\varphi_1 : U^* \to \mathbb{C}, \qquad \varphi_1(a) := e^{-\frac{1}{2}q(a)},$$

is the characteristic function of a Gaussian cylindrical probability measure $\mu_1$ (see [18] or [21, p. 393]). Since also $k^{-1}q$ is a quadratic form for every $k \in \mathbb{N}$ it follows that $(\varphi_1)^{1/k}$ is the characteristic function of a cylindrical measure, which verifies that $\mu_1$ is infinitely divisible. Thus, it remains to establish that

$$\varphi_2 : U^* \to \mathbb{C}, \qquad \varphi_2(a) := e^{-\kappa(a)},$$

is the characteristic function of an infinitely divisible cylindrical measure.

For that purpose we show that the function

$$\varphi_k : U^* \to \mathbb{C}, \qquad \varphi_k(a) := \exp\bigg(\frac{1}{k}\Big(ip(a) + \int_U \psi_h(\langle u, a\rangle)\,\nu(du)\Big)\bigg),$$

is the characteristic function of a cylindrical probability measure for each $k \in \mathbb{N}$. The case $k = 1$ shows that there exists a cylindrical measure $\mu$ with characteristic function $\varphi_1$, and the cases $k \geq 1$ show that $\mu$ is infinitely divisible. Note first that the integral in the definition of $\varphi_k$ exists and is finite because of condition (3).

Obviously, $\varphi_k(0) = 1$ by (1) and (3). Property (4) implies by Schoenberg's correspondence for functions on Banach spaces (property (h), p. 192 in [21]) that $\varphi_k$ is positive-definite. In order to verify the last condition of Bochner's theorem let $V \subseteq U^*$ be a finite-dimensional subspace, say $V = \operatorname{span}\{b_1, \ldots, b_d\}$ for $b_1, \ldots, b_d \in U^*$, and suppose $a_n \to a_0$ in $V$ as $n \to \infty$. Then $(U, \mathcal{Z}(U, \{b_1, \ldots, b_d\}), \nu)$ is a measure space. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and define

$$g_n : U \to \mathbb{R}, \qquad g_n(u) := f(\langle u, a_n\rangle)(|\langle u, a_n\rangle|^2 \wedge 1),$$

for $n \in \mathbb{N} \cup \{0\}$. By (3), each $g_n$ is in $L^1_\nu(U, \mathcal{Z}(U, \{b_1, \ldots, b_d\}))$ and

$$|g_n(u)| \leq \|f\|_\infty (1 + c)(|\langle u, a_0\rangle|^2 \wedge 1)$$

for a constant $c > 0$. Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_U g_n(u)\,\nu(du) = \int_U g_0(u)\,\nu(du),$$

which shows that

$$(3.9) \qquad (|s|^2 \wedge 1)(\nu \circ a_n^{-1})(ds) \to (|s|^2 \wedge 1)(\nu \circ a^{-1})(ds) \quad \text{weakly.}$$

Condition (3) guarantees that for each $a \in U^*$,

$$\varphi_{\mu_a} : \mathbb{R} \to \mathbb{C}, \qquad \varphi_{\mu_a}(t) = \exp\Big(ip(a)t + \int_{\mathbb{R}} \widetilde{\psi}_h(s, t)\,(\nu \circ a^{-1})(ds)\Big),$$

is the characteristic function of an infinitely divisible probability measure, say $\mu_a$, on $\mathcal{B}(\mathbb{R})$ with characteristics $(p(a), 0, \nu \circ a^{-1})_h$. Then condition (1) together with the weak convergence in (3.9) imply by Theorem VII.2.9 and Remark VII.2.10 (p. 396) in [9] that $\varphi_{\mu_{a_n}}(t) \to \varphi_{\mu_{a_0}}(t)$ for all $t \in \mathbb{R}$. Because $\varphi_k(a) = (\varphi_{\mu_a}(1))^{1/k}$ for all $a \in U^*$ and $k \in \mathbb{N}$, the functions $\varphi_k$ are continuous on every finite-dimensional subspace, which is the last condition in Bochner's theorem.

The remaining part follows directly from the proof of (b). ■

EXAMPLE 3.11. Now we can show that the function $\varphi$ in Example 3.8 is in fact the characteristic function of an infinitely divisible cylindrical measure. The linearity of $\ell$ implies that the mapping $a \mapsto p(a) = \lambda h(\ell(a))$ is continuous on each finite-dimensional subspace of $U^*$ if the truncation function $h$ is continuous. The measure $\nu$ satisfies $\nu \circ a^{-1} = \lambda \delta_{\ell(a)}$ for each $a \in U^*$ and is therefore a cylindrical Lévy measure. Since (3.2) implies that

$$f : \mathbb{R} \to \mathbb{C}, \qquad f(t) := -\lambda(e^{it} - 1),$$

is negative-definite, it follows for $z_1, \dots, z_n \in \mathbb{C}$, $a_1, \dots, a_n \in U^*$ that

$$\sum_{i,j=1}^{n} z_i \bar{z}_j \kappa(a_i - a_j) = \sum_{i,j=1}^{n} z_i \bar{z}_j f(\ell(a_i) - \ell(a_j)) \le 0.$$

Thus, $\kappa$ is negative-definite, which proves the claim due to Theorem 3.10.

For a given cylindrical Lévy measure $\nu$ there does not exist in general an infinitely divisible cylindrical probability measure with cylindrical characteristics $(0, 0, \nu)$ (see Example 3.8). But one might be able to construct a function $p : U^* \to \mathbb{R}$ such that there exists a cylindrical probability measure with cylindrical characteristics $(p, 0, \nu)$.

The following example shows the construction of the function $p$ for a given cylindrical Lévy measure $\nu$ with weak second moments. In Section 5 we consider the case where the cylindrical Lévy measure extends to a $\sigma$-finite measure on $\mathcal{B}(U)$.

EXAMPLE 3.12. Let $\nu$ be a cylindrical Lévy measure which satisfies

$$\int_U |\langle u, a\rangle|^2 \, \nu(du) < \infty \qquad \text{for all } a \in U^*.$$

The existence of the weak second moments enables us to define

$$p : U^* \to \mathbb{R}, \qquad p(a) := \int_U (h(\langle u, a\rangle) - \langle u, a\rangle) \, \nu(du),$$

for a continuous truncation function $h$. With a careful analysis similar to the one in the proof of Theorem 5.1 it can be shown that $p$ is continuous on every finite-dimensional subspace of $U^*$. From (3.2) it follows that

$$f : \mathbb{R} \to \mathbb{C}, \qquad f(t) := -(e^{it} - 1 - it),$$

is negative-definite. For $z_1, \dots, z_n \in \mathbb{C}$, $a_1, \dots, a_n \in U^*$ we have

$$\sum_{i,j=1}^{n} -z_i \bar{z}_j \Big( ip(a_i - a_j) + \int_U \psi_h(\langle u, a_i - a_j\rangle) \, \nu(du) \Big) = \int_U \sum_{i,j=1}^{n} z_i \bar{z}_j f(\langle u, a_i\rangle - \langle u, a_j\rangle) \, \nu(du) \le 0.$$

Theorem 3.10 shows that there exists an infinitely divisible cylindrical measure with cylindrical characteristics $(p, 0, \nu)_h$.

We finish this section by establishing that our two Definitions 3.2 and 3.3 of infinite divisibility for cylindrical measures coincide. In particular, this result enables us to show that a Radon measure is infinitely divisible if all its finite-dimensional projections are infinitely divisible.

THEOREM 3.13.

(a) *A cylindrical probability measure $\mu$ on $\mathcal{Z}(U)$ is infinitely divisible if and only if it is weakly infinitely divisible.*
(b) *A Radon probability measure $\mu$ on $\mathcal{B}(U)$ is infinitely divisible if and only if $\mu \circ \pi_{a_1,\dots,a_n}^{-1}$ is an infinitely divisible probability measure for all $a_1, \dots, a_n \in U^*$, $n \in \mathbb{N}$.*

*Proof.* (a) If $\mu$ is weakly infinitely divisible then Theorem 3.4 and Lemma 3.9 guarantee that the cylindrical characteristics of $\mu$ satisfy the conditions in Theorem 3.10.

(b) Suppose that all finite-dimensional projections of the Radon measure $\mu$ are infinitely divisible. Then the restriction of $\mu$ to $\mathcal{Z}(U)$ is a weakly infinitely divisible cylindrical measure and it follows from (a) that for each $k \in \mathbb{N}$ there exists a cylindrical probability measure $\mu_k$ such that $\mu = \mu_k^{*k}$. Theorem 1 in [19] implies that there exists $\ell$ in the algebraic dual $U^{*\prime}$ of $U^*$ such that $\mu_k * \delta_\ell$ is a Radon probability measure where

$$\delta_\ell(Z) := \begin{cases} 1 & \text{if } (\ell(a_1), \dots, \ell(a_n)) \in B, \\ 0 & \text{otherwise,} \end{cases}$$

for every $Z := Z(a_1, \dots, a_n; B) \in \mathcal{Z}(U)$. Since

$$\mu * \delta_\ell^{*k} = \mu_k^{*k} * \delta_\ell^{*k} = (\mu_k * \delta_\ell)^{*k}$$

and the right hand side is Radon it follows from [2, Prop. 7.14.50] that $\delta_\ell^{*k}$ is a Radon probability measure, which implies $\ell \in U$ by considering the characteristic functions. Since then $\mu_k * \delta_\ell$ and $\delta_\ell$ are Radon probability measures, a further application of [2, Prop.7.14.50] implies that $\mu_k$ is a Radon probability measure, which shows that $\mu$ is infinitely divisible. ∎

**4. Continuous infinitely divisible cylindrical measures.** Continuity of cylindrical measures is defined with respect to an arbitrary vector topology $\mathcal{O}$ in $U^*$. We assume here that the topological space $(U^*, \mathcal{O})$ is first countable, that is, the neighborhood system of every point in $U^*$ has a countable base. In such spaces, convergence is equivalent to sequential convergence. In particular, $U^*$ equipped with the norm topology is first countable.

DEFINITION 4.1. A cylindrical probability measure $\mu$ on $\mathcal{Z}(U)$ is called *$\mathcal{O}$-continuous* if for each $\varepsilon > 0$ there exists a neighborhood $N$ of 0 such that

$$\mu(\{u \in U : |\langle u, a\rangle| \geq 1\}) \leq \varepsilon$$

for all $a \in N$. If $\mathcal{O}$ is the norm topology we say $\mu$ *is continuous.*

A cylindrical probability measure $\mu$ is $\mathcal{O}$-continuous if and only if its characteristic function $\varphi_\mu : U^* \to \mathbb{C}$ is continuous in the topology $\mathcal{O}$ (see [20, Th. II.3.1]). This enables us to derive the following criterion:

LEMMA 4.2. *Let $\mu$ be an infinitely divisible cylindrical probability measure on $\mathcal{Z}(U)$ with cylindrical characteristics $(p, q, \nu)_h$ for a continuous truncation function $h$. Then the following are equivalent:*

(a) *$\mu$ is $\mathcal{O}$-continuous.*
(b) *For every sequence $a_n \to a$ in $(U^*, \mathcal{O})$ we have:*
   (i) *$p(a_n) \to p(a)$;*
   (ii) *$q(a_n)\,\delta_0(ds) + (|s|^2 \wedge 1)\,(\nu \circ a_n^{-1})(ds)$*
   *$\to q(a)\delta_0(ds) + (|s|^2 \wedge 1)\,(\nu \circ a^{-1})(ds)$ weakly.*

*Proof.* The cylindrical measure $\mu$ is $\mathcal{O}$-continuous if and only if its characteristic function $\varphi_\mu : U^* \to \mathbb{C}$ is continuous in $(U^*, \mathcal{O})$, or equivalently, $\varphi_\mu : U^* \to \mathbb{C}$ is sequentially continuous. It follows as in the proof of Theorem 3.10 that

$$\begin{aligned}&\varphi_\mu(a_n) \to \varphi_\mu(a) \text{ for all sequences } a_n \to a \text{ in } (U^*, \mathcal{O})\\ &\qquad \Leftrightarrow \varphi_{\mu\circ a_n^{-1}}(t) \to \varphi_{\mu\circ a^{-1}}(t) \text{ for all sequences } a_n \to a \text{ in } (U^*, \mathcal{O}),\ t \in \mathbb{R}.\end{aligned}$$

By applying Theorem VII.2.9 and Remark VII.2.10 in [9] and Lemma 3.7 the right hand side is equivalent to conditions (i) and (ii) in (b), which completes the proof. ■

In Lemma 4.2 it does not follow from (a) that we can consider separately the quadratic form $q$ and the term depending on the cylindrical Lévy measure $\nu$ in condition (b). This is due to the well known fact that a sequence of infinitely divisible measures on $\mathcal{B}(\mathbb{R})$ can converge weakly in such a way that the small jumps contribute to the Gaussian part in the limit. But since an infinitely divisible cylindrical measure $\mu$ is the convolution of two other infinitely divisible cylindrical measures it is of interest whether the continuity of $\mu$ is inherited by the convolution cylindrical measures.

DEFINITION 4.3. An infinitely divisible cylindrical probability measure with cylindrical characteristics $(p, q, \nu)_h$ is called *regularly $\mathcal{O}$-continuous* if the infinitely divisible cylindrical probability measures with cylindrical characteristics $(0, q, 0)_h$ and $(p, 0, \nu)_h$ are $\mathcal{O}$-continuous.

Lemma 4.4. *Let $h$ be a continuous truncation function. For an $\mathcal{O}$-continuous infinitely divisible cylindrical probability measure $\mu$ with cylindrical characteristics $(p,q,\nu)_h$ the following are equivalent:*

(a) *$\mu$ is regularly $\mathcal{O}$-continuous.*
(b) *$q : U^* \to \mathbb{R}$ is continuous in $(U^*,\mathcal{O})$.*
(c) *For every sequence $a_n \to a$ in $(U^*,\mathcal{O})$ we have:*
   (i) *$p(a_n) \to p(a)$;*
   (ii) *$(|s|^2 \wedge 1)\,(\nu \circ a_n^{-1})(ds) \to (|s|^2 \wedge 1)\,(\nu \circ a^{-1})(ds)$ weakly.*

*Proof.* Let $\varphi_\mu$ be the characteristic function of $\mu$. Then $\varphi_\mu = \varphi_1 \cdot \varphi_2$ where $\varphi_1$ is the characteristic function of the cylindrical measure $\mu_1$ with cylindrical characteristics $(0,q,0)_h$ and $\varphi_2$ is the characteristic function of the cylindrical measure $\mu_2$ with cylindrical characteristics $(p,0,\nu)_h$. Since the characteristic function of an infinitely divisible measure does not vanish at any point, the continuity of $\varphi_1$ and $\varphi_\mu$ results in the continuity of $\varphi_2$, and analogously if $\varphi_2$ and $\varphi_\mu$ are continuous then so is $\varphi_1$. Thus, $\mu$ is regularly $\mathcal{O}$-continuous if and only if either $\mu_1$ or $\mu_2$ is $\mathcal{O}$-continuous.

Applying Lemma 4.2 to $\mu_1$ shows the equivalence (a)⇔(b), and applying Lemma 4.2 to $\mu_2$ shows the equivalence (a)⇔(c). ∎

Remark 4.5. If $U^*$ is equipped with the norm topology then (b) in Lemma 4.4 can be replaced by

(b′) there exists a positive, symmetric operator $Q : U^* \to U^{**}$ such that $q(a) = \langle a, Qa\rangle$ for all $a \in U^*$.

*Proof.* According to Proposition IV.4.2 in [21] there exist a probability space $(\Omega,\mathcal{A},P)$ and a cylindrical random variable $X : U^* \to L_P^0(\Omega,\mathcal{A})$ with cylindrical distribution $(0,q,0)$ and with characteristic function $a \mapsto \varphi(a) = \exp(-\frac{1}{2}q(a))$. If $q$ is continuous Proposition VI.5.1 in [21] implies that the mapping $X : U^* \to L_P^0(\Omega,\mathcal{A})$ is continuous. Consequently, it follows from Theorem 4.7 in [1] that $(Qa)b := E[(Xa)(Xb)]$ for $a, b \in U^*$ defines a positive, symmetric operator $Q : U^* \to U^{**}$. Obviously, $q(a) = (Qa)a$ for each $a \in U^*$. ∎

Example 4.6. Let $(\Omega,\mathcal{A},P)$ be a probability space and let $L := (L(t) : t \geq 0)$ be a cylindrical process, that is, the mappings $L(t) : U^* \to L_P^0(\Omega,\mathcal{A})$ are linear. In Applebaum and Riedle [1] we call $L$ a *cylindrical Lévy process* if

$$((L(t)a_1,\dots,L(t)a_n) : t \geq 0)$$

is a Lévy process in $\mathbb{R}^n$ for all $a_1,\dots,a_n \in U^*$, $n \in \mathbb{N}$. If $L$ is a cylindrical Lévy process we derive in [1] that it can be decomposed as

$$L(t) = W(t) + Y(t) \quad \text{for all } t \geq 0 \ P\text{-a.s.},$$

where $(W(t) : t \geq 0)$ and $(Y(t) : t \geq 0)$ are cylindrical processes. Their characteristic functions are for all $a \in U^*$ given by

$$\varphi_{W(t)}(a) := E[\exp(iW(t)a)] = \exp(-\tfrac{1}{2}q(a)t)$$

for a quadratic form $q : U^* \to \mathbb{R}$ and

$$\varphi_{Y(t)}(a) := E[\exp(iY(t)a)] = \exp\Big(t\Big(ip(a) + \int_U \psi_h(\langle u, a\rangle)\,\nu(du)\Big)\Big)$$

for a mapping $p : U^* \to \mathbb{R}$ and a cylindrical Lévy measure $\nu$. Obviously, $(p, q, \nu)_h$ is the triplet of cylindrical characteristics of an infinitely divisible cylindrical measure $\mu$. If $\mu$ is regularly continuous, i.e. the cylindrical measures with the characteristic functions $\varphi_{W(1)}$ and $\varphi_{L(1)}$ are continuous, it follows that also the mappings

$$W(t) : U^* \to L^0_P(\Omega, \mathcal{A}), \qquad Y(t) : U^* \to L^0_P(\Omega, \mathcal{A})$$

are continuous (see [21, Prop.VI.5.1]). Moreover, according to Remark 4.5 the quadratic form $q$ is of the form $q(a) = \langle a, Qa\rangle$ for all $a \in U^*$ and some symmetric, positive operator $Q : U^* \to U^{**}$. If $Q(U^*) \subseteq U$ then $W$ is a cylindrical Wiener process in the strong sense as is usually considered in the literature (see Riedle [18]).

**5. Lévy measures on Banach spaces.** In this section we consider the situation where the cylindrical Lévy measure $\nu$ extends to a $\sigma$-finite measure on $\mathcal{B}(U)$, also denoted by $\nu$. The unit ball is denoted by $B_U := \{u \in U : \|u\| \leq 1\}$.

THEOREM 5.1. *Let $\nu$ be a cylindrical Lévy measure which extends to a $\sigma$-finite measure on $\mathcal{B}(U)$ with $\nu(B_U^c) < \infty$. Then there exists a regularly continuous infinitely divisible cylindrical probability measure $\mu$ with cylindrical characteristics $(d_\nu, 0, \nu)_h$, where*

$$d_\nu : U^* \to \mathbb{R}, \qquad d_\nu(a) := \int_U (h(\langle u, a\rangle) - \langle u, a\rangle\, \mathbb{1}_{B_U}(u))\,\nu(du).$$

*Proof.* First we show that the integral in the definition of $d_\nu$ is well defined for every truncation function $h$. Choose a constant $c > 0$ such that

$$\{t \in \mathbb{R} : |t| \leq c\} \subseteq D(h)$$

and define for every $a \in U^*$ the set

$$D(a) := \{v \in U : |\langle v, a\rangle| \leq c\}.$$

For the integrand $f_a(u) := h(\langle u, a\rangle) - \langle u, a\rangle\, \mathbb{1}_{B_U}(u)$ it follows for every $u \in U$ that

$$f_a(u) \neq 0 \;\Rightarrow\; u \in (D(a) \cap B_U^c) \cup (D^c(a) \cap B_U) \cup (D^c(a) \cap B_U^c).$$

But on these three domains we obtain

$$\int_{D(a)\cap B_U^c} |f_a(u)|\,\nu(du) \le \int_{B_U^c} c\,\nu(du) = c\,\nu(B_U^c) < \infty,$$

and

$$\begin{aligned}\int_{D^c(a)\cap B_U} |f_a(u)|\,\nu(du) &\le \int_{c<|\langle u,a\rangle|\le \|a\|} |h(\langle u,a\rangle) - \langle u,a\rangle|\,\nu(du)\\ &= \int_{c<|s|\le\|a\|} |h(s)-s|\,(\nu\circ a^{-1})(ds)\\ &\le (\|h\|_\infty + \|a\|)\,(\nu\circ a^{-1})(\{s\in\mathbb{R} : |s|>c\}) < \infty,\end{aligned}$$

because $\nu\circ a^{-1}$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$, and

$$\int_{D^c(a)\cap B_U^c} |f_a(u)|\,\nu(du) \le \|h\|_\infty \int_{B_U^c} \nu(du) = \|h\|_\infty \nu(B_U^c) < \infty.$$

Now we choose the truncation function $h$ to be continuous and show by a similar decomposition that $d_\nu$ is continuous. Let $a_n \to a$ in $U^*$ and choose a constant $c > 0$ such that

$$\{t\in\mathbb{R} : |t| \le c+\varepsilon\} \subseteq D(h)$$

for a constant $\varepsilon > 0$. Let $D(a) = \{v\in U : |\langle v,a\rangle| \le c\}$. Since for every $u \in B_U$ we have

$$|\langle u,a_n\rangle - \langle u,a\rangle| \le \|a_n - a\|,$$

we can conclude that there exists $n_0\in\mathbb{N}$ such that $u\in D(a)\cap B_U$ implies that $\langle u,a\rangle, \langle u,a_n\rangle \in D(h)$ for every $n \ge n_0$. Consequently, we have for $f_{a,n}(u) := h(\langle u,a_n\rangle) - h(\langle u,a\rangle) - (\langle u,a_n\rangle - \langle u,a\rangle)\,\mathbb{1}_B(u)$ and $n\ge n_0$ the implication

$$f_{a,n}(u) \ne 0 \;\Rightarrow\; u\in (D(a)\cap B_U^c)\cup(D^c(a)\cap B_U)\cup(D^c(a)\cap B_U^c).$$

As above it can be shown that $f_{a,n}$ is dominated by an integrable function on all three sets, and therefore Lebesgue's dominated convergence theorem shows that $d_\nu$ is continuous.

It follows for $h'(s) := s\,\mathbb{1}_{B_\mathbb{R}}(s)$ from (3.2) that

$$f:\mathbb{R}\to\mathbb{C},\qquad f(s_0,t) := -\widetilde{\psi}_{h'}(s_0,t),$$

is negative-definite for each $s_0\in\mathbb{R}$. For $z_1,\dots,z_n\in\mathbb{C}$, $a_1,\dots,a_n\in U^*$ we have

$$\begin{aligned}&\sum_{i,j=1}^n -z_i\bar z_j\Big(id_\nu(a_i-a_j) + \int_U \psi_h(\langle u,a_i-a_j\rangle)\,\nu(du)\Big)\\ &\qquad= \sum_{i,j=1}^n -z_i\bar z_j \int_U (e^{i\langle u,a_i-a_j\rangle} - 1 - i\langle u,a_i-a_j\rangle\,\mathbb{1}_{B_U}(u))\,\nu(du)\end{aligned}$$

$$= \sum_{i,j=1}^{n} -z_i\bar{z}_j \int_U \left( e^{i\frac{\langle u,a_i-a_j\rangle}{\|u\|}\|u\|} - 1 - i\frac{\langle u, a_i - a_j\rangle}{\|u\|}\|u\| \, \mathbb{1}_{B_{\mathbb{R}}}(\|u\|)\right) \nu(du)$$
$$= \int_U \sum_{i,j=1}^{n} z_i\bar{z}_j f\left(\|u\|, \frac{1}{\|u\|}(\langle u, a_i\rangle - \langle u, a_j\rangle)\right) \nu(du) \le 0.$$

Theorem 3.10 implies that there exists an infinitely divisible cylindrical probability measure $\mu$ with cylindrical characteristics $(d_\nu, 0, \nu)_h$. In order to show that $\mu$ is continuous let $a_n \to a_0$ in $U^*$. For a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ define

$$g_n : U \to \mathbb{R}, \qquad g_n(u) := f(\langle u, a_n\rangle)(|\langle u, a_n\rangle|^2 \wedge 1)$$

for $n \in \mathbb{N} \cup \{0\}$. It follows that each $g_n$ is in $L^1_\nu(U, \mathcal{B}(U))$ and

$$|g_n(u)| \le \|f\|_\infty(1 + c)(|\langle u, a_0\rangle|^2 \wedge 1)$$

for a constant $c > 0$. Lebesgue's dominated convergence theorem implies

$$\lim_{n\to\infty} \int_U g_n(u)\, \nu(du) = \int_U g_0(u)\, \nu(du),$$

which shows that

$$(5.1) \qquad (|s|^2 \wedge 1)\, (\nu \circ a_n^{-1})(ds) \to (|s|^2 \wedge 1)\, (\nu \circ a_0^{-1})(ds) \quad \text{weakly.}$$

Lemma 4.2 implies that $\mu$ is continuous and thus $\mu$ is regular continuous by Lemma 4.4. ∎

A cylindrical Lévy measure which extends to a $\sigma$-finite measure on $\mathcal{B}(U)$ is an obvious candidate to be a Lévy measure in the usual sense. We recall the definition from Linde [12]: a $\sigma$-finite measure $\nu$ on $\mathcal{B}(U)$ is called a *Lévy measure* if

(a) $\int_U (\langle u, a\rangle^2 \wedge 1)\, \nu(du) < \infty$ for all $a \in U^*$.
(b) There exists a Radon probability measure $\mu$ on $\mathcal{B}(U)$ with characteristic function

$$(5.2) \quad \varphi_\mu : U^* \to \mathbb{C}, \quad \varphi_\mu(a) = \exp\left(\int_U (e^{i\langle u,a\rangle} - 1 - i\langle u, a\rangle \, \mathbb{1}_{B_U}(u))\, \nu(du)\right).$$

In fact, this is rather a result (Theorem 5.4.8) in Linde [12] than his definition. Note furthermore that this definition includes already the requirement that a Radon probability measure on $\mathcal{B}(U)$ exists with the corresponding characteristic function. In general, no conditions on a measure $\nu$ are known which guarantee that $\nu$ is a Lévy measure. In particular, the condition

$$\int_U (\|u\|^2 \wedge 1)\, \nu(du) < \infty$$

is sufficient and necessary in Hilbert spaces, but in general spaces it is neither sufficient nor necessary, e.g. in the space of continuous functions on $[0, 1]$ (see Araujo [5]).

Corollary 5.2. *Let $\nu$ be a $\sigma$-finite measure on $\mathcal{B}(U)$ and $h$ be a truncation function. Then the following are equivalent:*

(a) *$\nu$ is a Lévy measure.*
(b) *There exists an infinitely divisible cylindrical probability measure $\mu$ with cylindrical characteristics $(d_\nu, 0, \nu)_h$ which extends to a Radon measure on $\mathcal{B}(U)$.*

*In this situation, the Radon probability measure with characteristic function* (5.2) *corresponding to the Lévy measure $\nu$ coincides with the Radon extension of $\mu$.*

*Proof.* It is easily seen that the characteristic function of the cylindrical measure with cylindrical characteristics $(d_\nu, 0, \nu)_h$ is of the form (5.2). Consequently, (b) implies (a). If $\nu$ is a Lévy measure, Proposition 5.4.5 in [12] guarantees that $\nu(B_U^c) < \infty$. Theorem 5.1 implies that there exists a cylindrical probability measure with cylindrical characteristics $(d_\nu, 0, \nu)_h$ which extends to a Radon probability measure, because its characteristic function is of the form (5.2). ∎

Remark 5.3. If $\nu$ is a Lévy measure and $\mu$ the infinitely divisible Radon probability measure with characteristic function (5.2), one calls the triplet $(0, 0, \nu)$ the characteristics of $\mu$. However, according to Corollary 5.2 the measure $\mu$ considered as an infinitely divisible cylindrical probability measure has the cylindrical characteristics $(d_\nu, 0, \nu)_h$. Even if we choose the truncation function to be $s \mapsto h(s) := s\,\mathbb{1}_{B_\mathbb{R}}(s)$, the entry $d_\nu$ does not vanish. This asymmetry illustrates the interaction of the components $p$ and $\nu$ of the cylindrical characteristics $(p, 0, \nu)$ of an infinitely divisible cylindrical measure. Even if $\nu$ is a Lévy measure and $p = d_\nu$ then the function

$$a \mapsto \kappa(a) := -\Big(\int_U (e^{-i\langle u,a\rangle} - 1 - i\langle u, a\rangle\,\mathbb{1}_{B_U}(u))\,\nu(du)\Big)$$

is negative-definite by Bochner's theorem and Schoenberg's correspondence. But although

$$\kappa(a) = -\Big(i d_\nu(a) + \int_U \psi_h(\langle u, a\rangle)\,\nu(du)\Big),$$

none of the summands in this representation is negative-definite in general.

In general, condition (b) in Corollary 5.2 may be verified by applying Prokhorov's theorem [21, Th. VI.3.2], and proving that the cylindrical measure $\mu$ is tight. In Sazonov spaces this is simplified:

Remark 5.4. If $U$ is a Sazonov space then condition (b) in Corollary 5.2 can be replaced by:

(b′) (i) there exists an infinitely divisible cylindrical probability measure $\mu$ with cylindrical characteristics $(d_\nu, 0, \nu)$;

(ii) $a \mapsto \kappa(a) = -(id_\nu(a) + \int_U \psi_h(\langle u, a\rangle)\, \nu(du))$ is continuous in an admissible topology.

EXAMPLE 5.5. If $U$ is a Hilbert space then the Sazonov topology is admissible. If $(d_\nu, 0, \nu)_h$ are cylindrical characteristics the function $\kappa$ is necessarily negative-definite by Theorem 3.10 and if it is also continuous in the Sazonov topology one obtains the well known Lévy–Khintchine formula in Hilbert spaces (see [15, Th. 6.4.10]).

**Acknowledgements.** The author thanks Dave Applebaum for some very helpful comments and discussions.

## References

[1] D. Applebaum and M. Riedle, *Cylindrical Lévy processes in Banach spaces*, Proc. London Math. Soc. 101 (2010), 697–726.
[2] V. I. Bogachev, *Measure Theory. Vols. I, II*, Springer, Berlin, 2007.
[3] Z. Brzeźniak, B. Goldys, P. Imkeller, S. Peszat, E. Priola, and J. Zabczyk, *Time irregularity of generalized Ornstein–Uhlenbeck processes*, C. R. Math. Acad. Sci. Paris 348 (2010), 273–276.
[4] Z. Brzeźniak and J. Zabczyk, *Regularity of Ornstein–Uhlenbeck processes driven by a Lévy white noise*, Potential Anal. 32 (2010), 153–188.
[5] A. P. de Araujo, *On infinitely divisible laws in* $C[0,1]$, Proc. Amer. Math. Soc. 51 (1975), 179–185.
[6] A. P. de Araujo and E. Giné, *Type, cotype and Lévy measures in Banach spaces*, Ann. Probab. 6 (1978), 637–643.
[7] —, —, *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, Chichester, 1980.
[8] E. Giné and M. G. Hahn, *On stability of probability laws with univariate stable marginals*, Z. Wahrsch. Verw. Gebiete 64 (1983), 157–165.
[9] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin, 2003.
[10] G. Kallianpur and J. Xiong, *Stochastic Differential Equations in Infinite Dimensional Spaces*, Inst. Math. Statist., Hayward, CA, 1996.
[11] M. Ledoux and M. Talagrand, *Probability in Banach Spaces. Isoperimetry and Processes*, Springer, Berlin, 1991.
[12] W. Linde, *Infinitely Divisible and Stable Measures on Banach Spaces*, Teubner, Leipzig, 1983.
[13] D. J. Marcus, *Non-stable laws with all projections stable*, Z. Wahrsch. Verw. Gebiete 64 (1983), 139–156.
[14] M. Métivier and J. Pellaumail, *Stochastic Integration*, Academic Press, New York, 1980.
[15] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.
[16] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise. An Evolution Equation Approach*, Cambridge Univ. Press, Cambridge, 2007.
[17] E. Priola and J. Zabczyk, *Structural properties of semilinear SPDEs driven by cylindrical stable processes*, Probab. Theory Related Fields 149 (2011), 97–137.

[18] M. Riedle, *Cylindrical Wiener processes*, in: Séminaire de Probabilités XLIII, Lecture Notes in Math. 2006, Springer, Berlin, 2011, 191–214.
[19] J. Rosiński, *On the convolution of cylindrical measures*, Bull. Polish Acad. Sci. Math. 30 (1982), 379–383.
[20] L. Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, Oxford Univ. Press, London, 1973.
[21] N. N. Vakhaniya, V. I. Tarieladze, and S. A. Chobanyan, *Probability Distributions on Banach Spaces*, Reidel, Dordrecht, 1987.

Markus Riedle
Department of Mathematics
King's College London
London WC2R 2LS, UK
E-mail: markus.riedle@kcl.ac.uk

*Received August 2, 2011* (7267)