STUDIA MATHEMATICA 207 (3) (2011)

Eigenvalues of Hille–Tamarkin operators and geometry of Banach function spaces

by

THOMAS KÜHN (Leipzig) and MIECZYSŁAW MASTYŁO (Poznań)

Abstract. We investigate how the asymptotic eigenvalue behaviour of Hille–Tamarkin operators in Banach function spaces depends on the geometry of the spaces involved. It turns out that the relevant properties are cotype p and p-concavity. We prove some eigenvalue estimates for Hille–Tamarkin operators in general Banach function spaces which extend the classical results in Lebesgue spaces. We specialize our results to Lorentz, Orlicz and Zygmund spaces and give applications to Fourier analysis. We are also able to show the optimality of our eigenvalue estimates in the Lorentz spaces $L_{2,q}$ with $1 \leq q < 2$ and in Zygmund spaces $L_p(\log L)_a$ with $2 \leq p < \infty$ and $a > 0$.

1. Introduction. The aim of this paper is to unify and extend the well-known classical results on eigenvalues of integral operators with Hille–Tamarkin kernels and weakly singular kernels. We introduce a class of more general Hille–Tamarkin kernels, containing as special cases not only the classical Hille–Tamarkin kernels, but also weakly singular kernels. Any integral operator with such a kernel acts in an appropriate Banach function space X. We study the question how the asymptotic eigenvalue behaviour of these operators depends on geometric properties of the underlying space X. It turns out that the relevant properties are cotype p and p-concavity $(2 \leq p < \infty)$. We obtain asymptotically optimal eigenvalue estimates for these general Hille–Tamarkin operators, expressed in terms of the cotype and concavity constants of X. In order to illustrate our general results, we give some examples in concrete spaces, namely in Lorentz spaces $L_{p,q}$ and in Zygmund spaces $L_p(\log L)_a$. We use similar techniques to those in our previous paper [14], where we investigated eigenvalues of integral operators with kernels of weakly singular type.

2010 *Mathematics Subject Classification*: Primary 47B06, 47G10; Secondary 47B10, 46E30.
Key words and phrases: eigenvalues, integral operators, Banach function spaces, Lorentz spaces, Orlicz spaces, Zygmund spaces, p-concavity, cotype p.

DOI: 10.4064/sm207-3-4

2. Preliminaries. First we fix some notation and collect a few definitions and results that will be needed in what follows. Throughout the paper we use standard notation from Banach space theory and operator theory as may be found e.g. in [15], [16], [13], [17] and [18]. These monographs may also serve as general references for more background on the subject of this paper.

2.1. Eigenvalues and operator ideals. Since we are concerned with eigenvalues, we consider only *complex* Banach spaces. The dual of a Banach space X will be denoted by X^*, and "operator" always means "bounded linear operator between Banach spaces".

The basic notion in the theory of eigenvalue distributions of Banach space operators is that of a *Riesz operator*; for the definition and a systematic treatment see [17, Chapter 26] and [13, Section 1.a]. In particular, compact and power-compact operators are Riesz. The spectrum of any Riesz operator $T\colon X \to X$ has no accumulation points except possibly zero, and all non-zero spectral values are eigenvalues of finite algebraic multiplicity. Therefore one can arrange the eigenvalues of T in a sequence $(\lambda_n(T))_{n=1}^{\infty}$ such that $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq |\lambda_n(T)| \geq \cdots \geq 0$ and each eigenvalue is repeated as many times as its multiplicity indicates. If T has less than n eigenvalues, we set $\lambda_k(T) = 0$ for $k \geq n$.

An important role in the spectral theory of Riesz operators in Banach spaces is played by operator ideals, in particular ideals generated by certain s-numbers (for instance Weyl numbers) and ideals of absolutely summing operators; for details we refer to [17]. Here we will use only p- and $(p, 2)$-summing operators; let us recall their definition.

Let $T\colon X \to Y$ be an operator between Banach spaces, $1 \leq p < \infty$ and $n \in \mathbb{N}$. Then we define $\pi_p^{(n)}(T)$ as the infimum of all constants $c > 0$ such that for all $x_1, \ldots, x_n \in X$ the inequality

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^p\Big)^{1/p} \leq c w_p(x_k) \tag{1}$$

holds, where $w_p(x_k) := \sup\{(\sum_{k=1}^n |\langle x_k, a\rangle|^p)^{1/p} : a \in X^*, \|a\| \leq 1\}$. We call T *absolutely p-summing* if $\pi_p(T) := \sup_{n\in\mathbb{N}} \pi_p^{(n)}(T) < \infty$. The class $\varPi_p(X, Y)$ of all such operators is a Banach space under the norm $\pi_p(T)$.

Let now $2 < p < \infty$. Replacing the quantity $w_p(x_k)$ in (1) by $w_2(x_k)$, we can define analogous norms $\pi_{p,2}^{(n)}(T)$ and $\pi_{p,2}(T)$, and obtain the class of *absolutely $(p, 2)$-summing* operators.

We now state the famous classical eigenvalue results for absolutely p- and $(p, 2)$-summing operators.

THEOREM 1 (see, e.g., [13, 2.b.1 and 2.a.9]).

(i) *Let* $1 \le p < \infty$, $q := \max(p, 2)$. *Then every absolutely* p*-summing operator* $T\colon X \to X$ *is power-compact, and its eigenvalues satisfy*

$$\Big(\sum_{k=1}^{\infty} |\lambda_k(T)|^q\Big)^{1/q} \le \pi_p(T). \tag{2}$$

(ii) *Let* $2 < p < \infty$. *Then every absolutely* $(p, 2)$*-summing operator* $T\colon X \to X$ *is power-compact, and its eigenvalues satisfy*

$$\sup_{k\in\mathbb{N}} k^{1/p}|\lambda_k(T)| \le 2e\pi_{p,2}(T). \tag{3}$$

2.2. Geometry of Banach function spaces. In this subsection we recall some basic geometric notions of Banach spaces and lattices, as may be found e.g. in [15] and [16]. We will not only work with Banach spaces, but also with quasi-Banach spaces and p-Banach spaces, $0 < p < 1$.

A *quasi-norm* on a vector space X over $\mathbb{K}$ is a map $\|\cdot\|_X$ satisfying:

(i) $\|x\|_X > 0$, $x \in X$, $x \neq 0$,
(ii) $\|\lambda x\|_X = |\lambda|\,\|x\|_X$, $\lambda \in \mathbb{K}$, $x \in X$,
(iii) $\|x + y\|_X \le C(\|x\|_X + \|y\|_X)$, $x, y \in X$,

for some constant $C \ge 1$ independent of x, y. A quasi-norm induces a locally bounded topology on X. A *quasi-Banach space* is a complete quasi-normed space. If we have in addition, for some $0 < p \le 1$,

$$\|x + y\|^p \le \|x\|^p + \|y\|^p, \qquad x, y \in X,$$

then X is called a *p-Banach space.* Notice that by a theorem due to Aoki and Rolewicz (see [19]) every quasi-norm is equivalent to a p-norm for some $0 < p \le 1$.

A Banach space X has *cotype* p, $2 \le p < \infty$, if there is a constant $c > 0$ such that for any finite family of elements $x_1, \ldots, x_n \in X$ the inequality

$$\Big(\sum_{k=1}^{n} \|x_k\|^p\Big)^{1/p} \le c\mathbb{E}\Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|$$

holds, where ε_k are i.i.d. Bernoulli random variables, i.e. $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$. The infimum over all possible constants $c > 0$ is denoted by $C_p(X)$ and called the *cotype p constant* of X.

Throughout the paper $(\Omega, \mu) := (\Omega, \mathcal{F}, \mu)$ denotes a complete σ-finite measure space. We let $L_0(\mu)$ denote the space of all equivalence classes of complex-valued $\mathcal{F}$-measurable functions on Ω equipped with the topology of convergence in measure on μ-finite sets. As usual, if $f, g \in L_0(\mu)$, then $|f| \le |g|$ means that $|f(\omega)| \le |g(\omega)|$ for μ-almost all $\omega \in \Omega$.

A quasi-Banach *function space* X on the measure space (Ω, μ) is a quasi-Banach space X which is a subspace of $L_0(\mu)$ such that there exists a strictly positive $h \in X$ and if $|f| \leq |g|$, where $g \in X$ and $f \in L_0(\mu)$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

The *Köthe dual* X' of a Banach function space X on (Ω, μ) is defined as the collection of all $f \in L_0(\mu)$ such that $\int_\Omega |fg| \, d\mu < \infty$ for every $g \in X$. Equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \leq 1} \int_\Omega |fg| \, d\mu,$$

X' is a Banach function space on (Ω, μ). The Köthe dual X' is a closed subspace of the dual Banach space X^* via the natural identification. It is well known (see, e.g., [12]) that $X^* \simeq X'$ if and only if X is order continuous (i.e. if $0 \leq f_n \in X$ and $f_n \downarrow 0$, then $\|f_n\|_X \to 0$).

We will use the notions of p-convexity and p-concavity, $0 < p < \infty$. These geometric properties make sense for *abstract quasi-Banach lattices*, but in our context of integral operators we will need them only for quasi-Banach *function spaces*. Given a quasi-Banach function space X and $n \in \mathbb{N}$, the *p-convexity constant with respect to n vectors* is defined as

$$M^{p,n}(X) := \sup \left\{ \left\| \Big(\sum_{k=1}^n |x_k|^p \Big)^{1/p} \right\| : \Big(\sum_{k=1}^n \|x_k\|^p \Big)^{1/p} \leq 1 \right\}$$

and the *p-concavity constant with respect to n vectors* as

$$M_{p,n}(X) := \sup \left\{ \Big(\sum_{k=1}^n \|x_k\|^p \Big)^{1/p} : \left\| \Big(\sum_{k=1}^n |x_k|^p \Big)^{1/p} \right\| \leq 1 \right\}.$$

The space X is called *p-convex* if

$$M^p(X) := \sup_{n \in \mathbb{N}} M^{p,n}(X) < \infty,$$

and *p-concave* if

$$M_p(X) := \sup_{n \in \mathbb{N}} M_{p,n}(X) < \infty.$$

Let $1 < p < \infty$ and $1/p + 1/q = 1$. It is well known (see [16, Prop. 1.d.4]) that a Banach lattice is p-convex (resp., p-concave) if and only if the dual X^* is q-concave (resp., q-convex). Moreover

$$M_q(X^*) = M^p(X) \quad (\text{resp., } M^q(X^*) = M_p(X)).$$

In the context of Banach function spaces, the well-known Köthe duality $(\ell_p^n(F))' = \ell_q^n(F')$ with equality of norms, where $1 \leq p < \infty$ and $1/p + 1/q$, implies the following relations. We omit the simple proofs.

Proposition 2. *Let $1 < p, q < \infty$ be such that $1/p + 1/q = 1$. If X is a Banach function space on a measure space (Ω, μ) and $n \in \mathbb{N}$, then*

(i) $M_{p,n}(X') = M^{q,n}(X)$.
(ii) $M^{p,n}(X') = M_{q,n}(X)$.

We note that every p-Banach space $(0 < p \leq 1)$ is p-convex, and the Lebesgue spaces $L_p(\Omega, \mathcal{F}, \mu)$ $(0 < p < \infty)$ are both p-convex and p-concave. Moreover, for Banach function spaces, cotype 2 coincides with 2-concavity, and for $2 < p < \infty$ we have the implications

$$p\text{-concave} \;\Rightarrow\; \text{cotype } p \;\Rightarrow\; (p+\varepsilon)\text{-concave for all } \varepsilon > 0.$$

2.3. Hille–Tamarkin operators. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $1 \leq p < \infty$ and $1/p + 1/p' = 1$. The classical Hille–Tamarkin operators are integral operators generated by $(\mu \otimes \mu)$-measurable kernels $k\colon \Omega \times \Omega \to \mathbb{C}$ with $k \in L_p[L_{p'}]$, i.e.

$$\|k\|_{L_p[L_{p'}]} := \Big(\int_\Omega \Big(\int_\Omega |k(x,t)|^{p'}\, d\mu(t) \Big)^{p/p'} d\mu(x) \Big)^{1/p} < \infty.$$

As shown in [9] (see also [13, 1.d.5]), the integral operator $T_k\colon L_p \to L_p$ defined by

$$T_k f(x) := \int_\Omega k(x,t) f(t)\, d\mu(t), \qquad x \in \Omega,$$

is absolutely p-summing, and hence, according to Theorem 1, it is a Riesz operator with eigenvalues in $\ell_{\max(p,2)}$. In this case k is even a strongly measurable $L_{p'}$-valued function (see [13, 3.a.2]). Kernels of the type $L_p[L_q]$ with additional regularity assumptions have been studied first by Carleman, and by Hille and Tamarkin [8]. Later on similar kernels, involving also Lorentz norms, have been considered by several authors (see, e.g., [3] and [4] and the references given therein).

We generalize this as follows. Let X be a Banach function space defined over a measure space $(\Omega, \mathcal{F}, \mu)$ with Köthe dual X'. The mixed space $X[X']$ is defined as the space of all $k \in L_0(\Omega \times \Omega, \mu \otimes \mu)$ such that $k(x,\cdot) \in X'$ for μ-almost all $x \in \Omega$ and $\Omega \ni x \mapsto \|k(x,\cdot)\|_{X'} \in X$. It is well known that $X[X']$ is a Banach function space on $(\Omega \times \Omega, \mu \otimes \mu)$ equipped with the norm

$$\|k\|_{X[X']} := \big\| \|k(x,\cdot)\|_{X'} \big\|_X.$$

(The inner norm in X' is taken with respect to the second variable t, and the outer norm in X with respect to the first variable x.) Then the corresponding integral operator T_k will be called (generalized) *Hille–Tamarkin operator*. Apart from the classical Hille–Tamarkin operators in Lebesgue spaces, this definition contains as special cases also the so-called *weakly singular operators* with kernels of the form

$$k(x,t) = \frac{\ell(x,t)}{|x-t|^\alpha}, \qquad x,t \in \Omega,\ x \neq t, \tag{4}$$

where Ω is a bounded subset of $\mathbb{R}^d$ equipped with (the restriction of) the Lebesgue measure, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, $\ell \in L_\infty(\Omega \times \Omega)$ and $0 < \alpha < d$. If $p := d/(d-\alpha) > 2$, then T_k is a Riesz operator in $L_{p,1}$ (or even in all L_r, $1 \le r \le \infty$) with eigenvalues in $\ell_{p,\infty}$ (see, e.g., [13, 3.a.11]). For the kernel (4) we have

$$k \in L_\infty[L_{p',\infty}] \subset L_{p,1}[L_{p',\infty}] = L_{p,1}[(L_{p,1})'].$$

3. Eigenvalues of Hille–Tamarkin operators. In this section we study the question which geometric properties of the underlying space X are "responsible" for the eigenvalue behaviour of Hille–Tamarkin operators with kernels in $X[X']$. First we give some general results, and then we consider concrete examples, namely when X is a Lorentz space $L_{p,q}$ or a Zygmund space $L_p(\log L)_a$.

3.1. General estimates. Let X be a Banach function space on (Ω, μ). The following simple observation will be useful. Every Hille–Tamarkin operator T_k with kernel $k \in X[X']$ has a factorization through $L_\infty(\mu)$ as follows:

$$T_k \colon X \xrightarrow{T_\ell} L_\infty \xrightarrow{M_g} X, \tag{5}$$

where M_g is the operator of pointwise multiplication with the function $g(x) := \|k(x,\cdot)\|_{X'}$ and T_ℓ is the integral operator with kernel

$$\ell(x,t) := \begin{cases} k(x,t)/g(x) & \text{if } g(x) > 0, \\ 0 & \text{if } g(x) = 0. \end{cases}$$

Notice that

$$\|g\|_X = \|k\|_{X[X']} \quad \text{and} \quad \|T_\ell : X \to L_\infty\| = 1. \tag{6}$$

First we investigate absolutely summing properties of multiplication operators.

Proposition 3. *Let X be a Banach function space and $g \in X$. Then the multiplication operator $M_g \colon L_\infty \to X$ is*

(i) *absolutely p-summing if X is p-concave, $1 \le p < \infty$,*
(ii) *absolutely $(p,2)$-summing if X has cotype p, $2 < p < \infty$.*

Proof. (i) Given $f_1, \ldots, f_n \in L_\infty$, the monotonicity of the norm in X implies

$$\Big(\sum_{k=1}^n \|M_g f_k\|_X^p\Big)^{1/p} \le M_p(X) \Big\| \Big(\sum_{k=1}^n |g f_k|^p\Big)^{1/p} \Big\|_X \le M_p(X) \|g\|_X \Big\| \Big(\sum_{k=1}^n |f_k|^p\Big)^{1/p} \Big\|_\infty.$$

Since

$$\Big\| \Big(\sum_{k=1}^n |f_k|^p\Big)^{1/p} \Big\|_\infty = w_p(f_k)$$

(see, e.g., [17, 17.3.8]), we obtain

$$\pi_p(M_g\colon L_\infty \to X) \le M_p(X)\|g\|_X.$$

(ii) Suppose now that X has cotype $p > 2$. Then X is r-concave for all $r > p$ (see [16, 1.f.9]), and we have the estimate

$$\begin{aligned}\Big(\sum_{k=1}^n \|M_g f_k\|_X^p\Big)^{1/p} &\le C_p(X)\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k g f_k\Big\|_X \le C_p(X)\Big(\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k g f_k\Big\|_X^r\Big)^{1/r}\\ &\le C_p(X)M_r(X)\Big\|\Big(\mathbb{E}\Big|\sum_{k=1}^n \varepsilon_k g f_k\Big|^r\Big)^{1/r}\Big\|_X.\end{aligned}$$

By Khintchine's inequality (see, e.g., [15, 2.b.3]) there is a constant $B_r > 0$ such that for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{C}$,

$$\Big(\mathbb{E}\Big|\sum_{k=1}^n \varepsilon_k a_k\Big|^r\Big)^{1/r} \le B_r\Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2}.$$

Using once again the monotonicity of the norm in X, we get

$$\begin{aligned}\Big\|\Big(\mathbb{E}\Big|\sum_{k=1}^n \varepsilon_k g f_k\Big|^r\Big)^{1/r}\Big\|_X &\le B_r\Big\|\Big(\sum_{k=1}^n |g f_k|^2\Big)^{1/2}\Big\|_X\\ &\le B_r\|g\|_X\Big\|\Big(\sum_{k=1}^n |f_k|^2\Big)^{1/2}\Big\|_\infty = B_r\|g\|_X w_2(f_k),\end{aligned}$$

which yields the desired result

$$\pi_{p,2}(M_g\colon L_\infty \to X) \le C_p(X)\, B_r\, M_r(X)\, \|g\|_X < \infty. \ \blacksquare$$

We remark that the proof of (i) even gives the estimate

$$\pi_p^{(n)}(M_g\colon L_\infty \to X) \le M_{p,n}(X)\|g\|_X, \quad n \in \mathbb{N}. \tag{7}$$

As an immediate consequence of Proposition 3, the factorization (5) and Theorem 1 we obtain the following general result.

Theorem 4. *Let X be a Banach function space having some finite cotype, and let $k \in X[X']$. Then the Hille–Tamarkin operator T_k is a Riesz operator in X, and its eigenvalues are in*

(i) $\ell_{\max(p,2)}$ *if X is p-concave, $1 \le p < \infty$,*
(ii) $\ell_{p,\infty}$ *if X has cotype p, $2 < p < \infty$.*

In Section 4 we will show that these results are best possible.

For more refined eigenvalue estimates we consider p-summing norms with finitely many vectors. The following result is due to Tomczak-Jaegermann [20] in the case $p = 2$, and to Zvavich [21] for $p > 2$. The estimate in (ii) was shown earlier by Johnson and Schechtman [10] for slightly larger m, namely $m \ge K_p\, n^{p/2} \log^3 n$, which would have been sufficient for our purposes.

Lemma 5. *Let X and Y be Banach spaces.*

(i) *For every $n \in \mathbb{N}$ and every rank n operator $T\colon X \to Y$ we have*

$$\pi_2(T) \le \sqrt{2}\,\pi_2^{(n)}(T).$$

(ii) *Let $2 < p < \infty$. Then there is a constant $K_p > 0$ such that for all $n \in \mathbb{N}$ and every operator $T\colon X \to Y$ with $\dim X = n$,*

$$\pi_p(T) \le 2\pi_p^{(m)}(T) \quad \text{for all } m \ge K_p\, n^{p/2} \log n.$$

Theorem 6. *Let X be a Banach space and $T\colon X \to X$ a Riesz operator. Then for all $n \in \mathbb{N}$,*

$$\Big(\sum_{k=1}^{n} |\lambda_k(T)|^2\Big)^{1/2} \le \sqrt{2}\,\pi_2^{(n)}(T), \tag{8}$$

and, if $2 < p < \infty$ and $m \ge K_p\, n^{p/2}\,(1 + \log n)$, then

$$\Big(\sum_{k=1}^{n} |\lambda_k(T)|^p\Big)^{1/p} \le 2\pi_p^{(m)}(T), \tag{9}$$

where the constant $K_p > 0$ is independent of n.

Proof. We show only (9), the proof of (8) is almost identical.

Given $n \in \mathbb{N}$, we choose an n-dimensional T-invariant subspace X_n of X such that the eigenvalues of the restriction $T_n = T|_{X_n}\colon X_n \to X_n$ are exactly $\lambda_1(T), \ldots, \lambda_n(T)$. This is always possible (see, e.g., [13, 1.a.6]). Let $J_n\colon X_n \hookrightarrow X$ be the canonical embedding. By Theorem 1, Lemma 5 and the injectivity of p-summing norms we obtain for $m \ge K_p\, n^{p/2}(1 + \log n)$ the desired inequality

$$\begin{aligned}
\Big(\sum_{k=1}^{n} |\lambda_k(T)|^p\Big)^{1/p} &= \Big(\sum_{k=1}^{n} |\lambda_k(T_n)|^p\Big)^{1/p} \le \pi_p(T_n\colon X_n \to X_n) \\
&= \pi_p(J_nT_n\colon X_n \to X) = \pi_p(TJ_n\colon X_n \to X) \\
&\le 2\pi_p^{(m)}(TJ_n\colon X_n \to X) \le 2\pi_p^{(m)}(T\colon X \to X). \ \blacksquare
\end{aligned}$$

Our next theorem gives refined eigenvalue estimates for general Hille–Tamarkin operators.

Theorem 7. *Let X be a Banach function space having some finite cotype, and let $k \in X[X']$ be a Hille–Tamarkin kernel. Then T_k is a Riesz operator in X, and its eigenvalues satisfy, for all $n \in \mathbb{N}$, the estimates*

$$\Big(\sum_{j=1}^{n} |\lambda_j(T_k)|^2\Big)^{1/2} \le \sqrt{2}\, M_{2,n}(X) \|k\|_{X[X']}$$

and, if $2 < p < \infty$ *and* $m \geq K_p\, n^{p/2}\,(1 + \log n)$,

$$\Big(\sum_{j=1}^{n} |\lambda_j(T_k)|^p\Big)^{1/p} \leq 2M_{p,m}(X)\|k\|_{X[X']},$$

where K_p *is the constant from Lemma* 5(ii).

Proof. From the factorization (5), taking also the relations (7) and (6) into account, we get

$$\begin{aligned}\pi_p^{(n)}(T_k\colon X \to X) &\leq \|T_\ell\colon X \to L_\infty\|\pi_p^{(n)}(M_g\colon L_\infty \to X)\\ &\leq M_{p,n}(X)\|g\|_X = M_{p,n}(X)\|k\|_{X[X']},\end{aligned}$$

and combining this with Theorem 6 ends the proof. ■

It is clear that this result is useful only for those Banach function spaces X which are not p-concave, but $(p+\varepsilon)$-concave for all $\varepsilon > 0$ and some $p > 2$. Below we will give examples where these estimates indeed can be applied and yield optimal results. We now introduce the corresponding spaces X.

3.2. Lorentz, Orlicz and Zygmund spaces. In this paper we will be mainly interested in some concrete classes of quasi-Banach function spaces which refine the scale of Lebesgue spaces L_p, namely Lorentz spaces $L_{p,q}$ and Zygmund spaces $L_p(\log L)_a$. Let us recall their definitions. For $f \in L_0(\mu)$ the *distribution function* d_f is defined as

$$d_f(s) := \mu\{\omega \in \Omega : |f(\omega)| > s\}, \qquad s \geq 0,$$

and the *non-increasing rearrangement* $f^*\colon [0, \mu(\Omega)) \to [0,\infty)$ as

$$f^*(t) := \inf\{s > 0 : d_f(s) \leq t\}.$$

The *Lorentz space* $L_{p,q}$, $0 < p < \infty, 0 < q \leq \infty$, consists of all $f \in L_0$ such that the quasi-norm

$$\|f\|_{L_{p,q}} := \begin{cases} \Big(\int_0^\infty (t^{1/p} f^*(t))^q \,\dfrac{dt}{t}\Big)^{1/q} & \text{if } 0 < q < \infty,\\ \sup\limits_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty,\end{cases}$$

is finite. For $1 < p < \infty$, $1 \leq q \leq \infty$, the quasi-norm $\|\cdot\|_{L_{p,q}}$ is equivalent to a norm. Moreover we have, with equivalence of (quasi-)norms,

$$(L_{p,q})^* = (L_{p,q})' = L_{p',q'}, \qquad 1 \leq q < \infty.$$

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a non-zero convex function with $\varphi(0) = 0$. Note that this implies in particular that φ is continuous and $\lim_{t\to\infty} \varphi(t) = \infty$. Let (Ω,μ) be a measure space. The *Orlicz space* L_φ consists of all $f \in L_0(\mu)$ such that for some $c > 0$,

$$\int_\Omega \varphi(|f|/c)\, d\mu < \infty.$$

It is a Banach space with respect to the Luxemburg norm

$$\|f\|_{L_\varphi} := \inf\Big\{c > 0 : \int_\Omega \varphi(|f|/c)\, d\mu \le 1\Big\}.$$

Clearly we can replace in this definition the function φ by any equivalent function. This leads to the same space with an equivalent (quasi-)norm.

Let $1 < p < \infty$ and $a \in \mathbb{R}$. The *Zygmund space* $L_p(\log L)_a$ is the Orlicz space generated by $\varphi(t) \asymp t^p \log(e+t)^{ap}$. In this case we have

$$(L_p(\log L)_a)^* = (L_p(\log L)_a)' = L_{p'}(\log L)_{-a}.$$

Note that the scales of Lorentz and Zygmund spaces are both refinements of the scale of Lebesgue spaces, since obviously $L_p = L_{p,p} = L_p(\log L)_0$. For fixed p the Lorentz spaces $L_{p,q}$ are increasing in q, while the Zygmund spaces $L_p(\log L)_a$ are decreasing in a. More information on duality and equivalent (quasi-)norms in Zygmund spaces can be found in [7, Section 2.6] and the references given therein.

3.3. Hille–Tamarkin operators in Lorentz spaces. The cotype and concavity behaviour of Lorentz spaces is well known. The following result is due to Creekmore [5].

LEMMA 8. *Let $1 < p < \infty$ and $1 \le q \le \infty$. The Lorentz spaces $L_{p,q}$, defined over arbitrary measure spaces, are*

(i) *q-concave if $p \le q$,*
(ii) *$(p+\varepsilon)$-concave for all $\varepsilon > 0$ but not p-concave if $q < p$,*
(iii) *of cotype $r = \max(p,q,2)$ if $p \ne 2$.*

Combining this with Theorem 4 we obtain the following result.

THEOREM 9. *Let $1 < p < \infty, 1 \le q < \infty$ and $X = L_{p,q}$. Then every Hille–Tamarkin kernel $k \in X[X']$ generates a Riesz operator T_k in X with eigenvalues in*

(i) *ℓ_q if $p \le q$ and $q > 2$,*
(ii) *$\ell_{p,\infty}$ if $q < p$ and $p > 2$,*
(iii) *ℓ_2 if $p \le q \le 2$ or $q < p < 2$.*

Now we pass to the interesting case $1 \le q < p = 2$, which is not covered by Theorem 9. Recall that the Lorentz spaces $L_{2,q}$ with $1 \le q < 2$ are neither 2-concave nor of cotype 2, but $(2+\varepsilon)$-concave for all $\varepsilon > 0$. In view of Theorem 7, our aim is to determine the 2-concavity constants with respect to n vectors of these spaces. One might expect that these constants grow moderately as $n \to \infty$, and in fact it will turn out that they are of logarithmic order in n. In the proofs we use the duality between p-concavity and p'-convexity for $1 < p < \infty$, and the r-convexification of Banach lattices, $r > 1$.

The following lemma might be known to specialists, but for completeness we include a proof.

LEMMA 10. *For every $0 < p < 1$ there exists a p-norm $\|\cdot\|_{(p)}$ on $L_{1,\infty}$ with*

$$\|f\|_{1,\infty} \le \|f\|_{(p)} \le \left(\frac{1}{1-p}\right)^{1/p} \|f\|_{1,\infty} \quad \textit{for all } f \in L_{1,\infty}. \tag{10}$$

Proof. Fix $0 < p < 1$. We claim that the functional $\|\cdot\|_{(p)}$ defined by

$$\|f\|_{(p)} := \sup\Big\{\Big(\mu(E)^{p-1} \int_E |f|^p \, d\mu\Big)^{1/p} : E \in \mathcal{F},\ 0 < \mu(E) < \infty\Big\}$$

satisfies (10). First observe that for every $f, g \in L_{1,\infty}$ and $E \in \mathcal{F}$ with $0 < \mu(E) < \infty$ we have

$$\int_E |f+g|^p \, d\mu \le \int_E |f|^p \, d\mu + \int_E |g|^p \, d\mu,$$

whence

$$\|f+g\|_{(p)}^p \le \|f\|_{(p)}^p + \|g\|_{(p)}^p,$$

whence $\|\cdot\|_{(p)}$ is a p-norm.

Next we show the first inequality in (10). For any $s > 0$ let

$$E_s := \{t \in \Omega : |f(t)| \ge s\}.$$

Then $\mu(E_s) < \infty$ and

$$\|f\|_{(p)} \ge \sup_{s>0} \mu(E_s)^{1-1/p} \Big(\int_{E_s} |f|^p \, d\mu\Big)^{1/p} \ge \sup_{s>0} s\mu(E_s) = \|f\|_{1,\infty}.$$

On the other hand, for any $E \in \mathcal{F}$ with $0 < \mu(E) < \infty$ we have

$$\int_E |f|^p \, d\mu \le \int_0^{\mu(E)} f^*(t)^p \, dt \le \|f\|_{1,\infty}^p \int_0^{\mu(E)} t^{-p} \, dt = \frac{\mu(E)^{1-p}}{1-p} \|f\|_{1,\infty}^p.$$

This implies

$$\|f\|_{(p)} \le \left(\frac{1}{1-p}\right)^{1/p} \|f\|_{1,\infty}$$

and completes the proof. ■

Using the above lemma we give an upper estimate of the 1-convexity constants of $L_{1,\infty}$ with respect to n vectors.

LEMMA 11. *The 1-convexity constants with respect to n vectors of the Lorentz space $L_{1,\infty}$ satisfy*

$$M^{1,n}(L_{1,\infty}) \le 4(1 + \log_2 n).$$

Proof. For $n = 1$ there is nothing to prove, so let $n \geq 2$ and $f_1, \ldots, f_n \in L_{1,\infty}$ be given. Combining Hölder's inequality with Lemma 10, we obtain, for all $0 < p < 1$,

$$\Big\| \sum_{j=1}^{n} |f_j| \Big\|_{1,\infty}^{p} \leq \Big\| \sum_{j=1}^{n} |f_j| \Big\|_{(p)}^{p} \leq \sum_{j=1}^{n} \|f_j\|_{(p)}^{p}$$
$$\leq \frac{1}{1-p} \sum_{j=1}^{n} \|f_j\|_{1,\infty}^{p} \leq \frac{n^{1-p}}{1-p} \Big(\sum_{j=1}^{n} \|f_j\|_{1,\infty} \Big)^{p}.$$

This implies

$$M^{1,n}(L_{1,\infty}) \leq n^{1/p-1} \left(\frac{1}{1-p} \right)^{1/p}.$$

Now we optimize over p. Let $p \in (0,1)$ be such that $1/p = 1 + 1/\log_2 n$. Then $1 + \log_2 n = 1/(1-p)$, and using Bernoulli's inequality we obtain

$$\begin{aligned} M^{1,n}(L_{1,\infty}) &\leq n^{1/\log_2 n} (1 + \log_2 n)^{1+1/\log_2 n} \\ &= 2(1 + \log_2 n) \cdot (1 + \log_2 n)^{1/\log_2 n} \leq 4(1 + \log_2 n). \ \blacksquare \end{aligned}$$

Remark. The following interpretation of the preceding lemma might be of independent interest. For a quasi-normed space $(X, \|\cdot\|)$ we define the *triangle constants with respect to n vectors* by

$$\tau_n(X) := \sup\Big\{ \Big\| \sum_{k=1}^{n} x_k \Big\| : x_k \in X, \sum_{k=1}^{n} \|x_k\| = 1 \Big\}.$$

If the quasi-norm is C-equivalent to a p-norm for some $C \geq 1$ and $0 < p \leq 1$, it is easy to check that $\tau_n(X) \leq C n^{1-p}$. That means the growth rate of $\tau_n(X)$ as $n \to \infty$ describes "how far" from a norm the given quasi-norm is. For quasi-Banach *function spaces* one clearly has $\tau_n(X) = M^{1,n}(X)$, whence the preceding lemma shows $\tau_n(L_{1,\infty}) \leq 4(1+\log_2 n)$. Moreover, if the underlying measure space is non-atomic, one even has $\tau_n(L_{1,\infty}) \asymp 1 + \log_2 n$.

Now we estimate the p-concavity constants with respect to n vectors of the Lorentz spaces $L_{p,q}$ with $1 \leq q < p < \infty$. We recall that the *r-convexification* (with $1 < r < \infty$) of a quasi-Banach function space E is the space $E^{(r)} := \{ f \in L_0 : |f|^r \in E \}$, equipped with the quasi-norm $\|f\|_{E^{(r)}} = \big\| |f|^r \big\|_E^{1/r}$. It is easy to check that for all $1 \leq p, r < \infty$ and $n \in \mathbb{N}$,

$$M^{rp,n}(E^{(r)}) = M^{p,n}(E)^{1/r} \quad \text{and} \quad M_{rp,n}(E^{(r)}) = M_{p,n}(E)^{1/r}. \tag{11}$$

In particular, for the r-convexification of Lorentz spaces we have

$$(L_{p,q})^{(r)} = L_{pr,qr} \quad \text{and} \quad \|f\|_{pr,qr} = \big\| |f|^r \big\|_{p,q}^{1/r}. \tag{12}$$

LEMMA 12. *Let $1 \le q < p < \infty$. There exists a constant $C_{p,q} > 0$ such that*

$$M_{p,n}(L_{p,q}) \le C_{p,q}\,(1+\log n)^{1/q-1/p}, \qquad n \in \mathbb{N}.$$

Proof. First observe that $(L_{1,\infty})^{(r)} = L_{r,\infty}$ with equality of quasi-norms. Therefore Lemma 10 implies that

$$M^{r,n}(L_{r,\infty}) \le (4(1+\log n))^{1/r}$$

for all $1 < r < \infty$ and $n \in \mathbb{N}$. Combining this estimate with Proposition 2 and the well-known fact that the Köthe dual $(L_{r,1})'$ is equal to $L_{r',\infty}$ with equivalence of quasi-norms, we conclude that for all $n \in \mathbb{N}$, all $1 < r < \infty$ and some constant $C_r > 0$,

$$M_{r,n}(L_{r,1}) \le C_r M^{r',n}(L_{r',\infty}) \le C_r(4(1+\log n))^{1-1/r}.$$

Take now $r = p/q > 1$ and apply the q-convexification procedure. In view of (12) we have $L_{p,q} = (L_{r,1})^{(q)}$, and by (11) we get

$$M_{p,n}(L_{p,q}) = M_{rq,n}((L_{r,1})^{(q)}) = M_{r,n}(L_{r,1})^{1/q} \le C_{p,q}(1+\log n)^{1/q-1/p}. \ \blacksquare$$

COROLLARY 13. *Let $1 \le q < p < \infty$, and let the Lorentz space $L_{p,q}$ be defined over an atomless measure space. For the p-concavity constants with respect to n vectors of $L_{p,q}$ one has*

$$M_{p,n}(L_{p,q}) \asymp (1+\log n)^{1/q-1/p}.$$

Proof. The upper estimate (for arbitrary measure spaces) was shown in the previous lemma. The lower estimate follows easily from the proof of Proposition 3.1 in [5]. ■

An immediate consequence of Lemma 12 and Theorem 7 is the following eigenvalue result for Hille–Tamarkin operators on $L_{2,q}$ in the missing case $1 \le q < 2$.

THEOREM 14. *Let $1 \le q < 2$ and $k \in L_{2,q}(L_{2,q'})$. Then the Hille–Tamarkin operator T_k is a Riesz operator in $L_{2,q}$, and its eigenvalues satisfy, for all $n \in \mathbb{N}$ with some constant $C_q > 0$, the estimate*

$$\Big(\sum_{j=1}^{n} |\lambda_j(T_k)|^2\Big)^{1/2} \le C_q\,(1+\log n)^{1/q-1/2}.$$

3.4. Hille–Tamarkin operators in Orlicz spaces. It is known (see, e.g., [11]) that an Orlicz space L_φ defined over an atomless measure space is p-concave if

$$c_p(\varphi) := \sup\Big\{\frac{\varphi(\lambda t)}{\lambda^p \varphi(t)} : \lambda > 1,\ t > 0\Big\} < \infty. \tag{13}$$

This is for instance the case if $t \mapsto \varphi(t^{1/p})$ is equivalent to a concave function. In particular, if

$$\varphi(t) \asymp t^p \log(e+t)^{ap}, \tag{14}$$

then condition (13) holds for $a \le 0$, and it fails for $a > 0$. Recall that the corresponding Orlicz spaces are the Zygmund spaces $L_p(\log L)_a$.

Our first result for Orlicz spaces is an immediate consequence of Theorem 4.

THEOREM 15. *Assume that the Orlicz function φ satisfies condition* (13) *for some $1 < p < \infty$. Then each Hille–Tamarkin kernel $k \in L_\varphi[L'_\varphi]$ generates a Riesz operator in L_φ with eigenvalues in $\ell_{\max(p,2)}$.*

In the context of Zygmund spaces we obtain the following corollary.

COROLLARY 16. *Let $X = L_p(\log L)_a$, where either $1 < p < 2$ and $a \in \mathbb{R}$, or $2 \le p < \infty$ and $a \le 0$. Then each Hille–Tamarkin kernel $k \in X[X']$ generates a Riesz operator in X with eigenvalues in $\ell_{\max(p,2)}$.*

Proof. We know that $L_p(\log L)_a = L_\varphi$ for any Orlicz function φ that satisfies condition (14). For $1 < p < 2$ and arbitrary $a \in \mathbb{R}$ this implies $c_2(\varphi) < \infty$, whence $L_p(\log L)_a$ is 2-concave. If $2 \le p < \infty$ and $a \le 0$, then $c_p(\varphi) < \infty$, thus $L_p(\log L)_a$ is p-concave. Applying Theorem 4 yields the result. ∎

Next we want to find the asymptotic eigenvalue behaviour of Hille–Tamarkin operators in Zygmund spaces in the case not covered by Corollary 16, i.e. when $2 \le p < \infty$ and $a > 0$. In this case $c_p(\varphi) = \infty$, but $c_{p+\varepsilon}(\varphi) < \infty$ for every $\varepsilon > 0$, whence $L_p(\log L)_a$ is $(p+\varepsilon)$-concave, and therefore the relevant eigenvalues are in $\bigcap_{\varepsilon>0} \ell_{p+\varepsilon}$. In order to improve this result, we estimate the p-concavity constants with respect to n vectors of $L_p(\log L)_a$.

This will be a consequence of the following lemma.

LEMMA 17. *Let L_φ be an Orlicz space over a measure space $(\Omega, \mathcal{F}, \mu)$, where $\varphi\colon [0,\infty) \to [0,\infty)$ is a convex function with $\varphi(0) = 0$ and $\varphi(1) = 1$. Let $1 \le p < \infty$. Then there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$M_{p,n}(L_\varphi) \le C \cdot c_{p,n}(\varphi)^{1/p},$$

where

$$c_{p,n}(\varphi) := \sup\left\{ \frac{\varphi(\lambda t)}{\lambda^p \varphi(t)} : t > 0,\ 1 \le \lambda \le n \right\}.$$

Proof. We give the proof only for $p = 1$, the general case is analogous. For simplicity we write $c_n(\varphi)$ instead of $c_{1,n}(\varphi)$. Let $f_1, \ldots, f_n \in L_\varphi$ with $\|f_1\|_{L_\varphi} \ge \cdots \ge \|f_n\|_{L_\varphi} > 0$, and define $f := \sum_{k=1}^n |f_k|$. Fix $\varepsilon \in (0,1)$. By homogeneity we can assume that $\|f\|_{L_\varphi} = 1 - \varepsilon < 1$, whence

$$\int_\Omega \varphi(f)\, d\mu \le 1. \tag{15}$$

Setting $a_k := (1-\varepsilon)\|f_k\|_{L_\varphi}$ for $1 \le k \le n$, we have

$$\int_\Omega \varphi(|f_k|/a_k)\, d\mu > 1. \tag{16}$$

Let $m < n$ be the integer with $a_m > 2/n \ge a_{m+1}$; if no such m exists set $m := n$. Then

$$\sum_{k=m+1}^{n} a_k \le 2\frac{n-m}{n} \le 2.$$

For $k = 1, \ldots, m$ we now consider the sets

$$A_k := \{x \in \Omega : |f_k(x)|/a_k \ge f(x)/2\} \quad \text{and} \quad B_k := \Omega \setminus A_k.$$

From inequality (16) we see that

$$\sum_{k=1}^{m} a_k \le \sum_{k=1}^{m} a_k \int_\Omega \varphi(|f_k|/a_k)\, d\mu = \sum_{k=1}^{m} a_k \Big(\int_{A_k} \cdots + \int_{B_k} \ldots\Big). \tag{17}$$

For $x \in B_k$ we have $|f_k(x)|/a_k \le f(x)/2$, and since φ is increasing and convex with $\varphi(0) = 0$ we get

$$\varphi(|f_k(x)|/a_k) \le \varphi(f(x)/2) \le \varphi(f(x))/2;$$

therefore, in view of (15), we arrive at

$$\sum_{k=1}^{m} a_k \int_{B_k} \varphi(|f_k|/a_k)\, d\mu \le \frac{1}{2}\sum_{k=1}^{m} a_k \int_\Omega \varphi(|f|)\, d\mu \le \frac{1}{2}\sum_{k=1}^{m} a_k.$$

Taking (17) into account, this implies

$$\sum_{k=1}^{m} a_k \le 2\sum_{k=1}^{m} a_k \int_{A_k} \varphi(|f_k|/a_k)\, d\mu. \tag{18}$$

Given any $x \in A_k$ we set

$$\lambda := \frac{2|f_k(x)|}{a_k f(x)} \quad \text{and} \quad t := \frac{f(x)}{2}.$$

By the definition of A_k, and since $a_k > 2/n$, we have $1 \le \lambda \le n$, whence

$$\varphi(|f_k(x)|/a_k) = \varphi(\lambda t) \le c_n(\varphi)\lambda\varphi(t).$$

Inserting this into (18), and using again (15) and $\varphi(f(x)/2) \le \varphi(f(x))/2$, we obtain

$$\sum_{k=1}^{m} a_k \le 2\sum_{k=1}^{m} a_k \int_{A_k} \varphi(|f_k(x)|/a_k)\,d\mu$$
$$\le 2c_n(\varphi)\sum_{k=1}^{m}\int_{A_k} a_k\cdot\frac{2|f_k(x)|}{a_k f(x)}\cdot\frac{\varphi(f(x))}{2}\,d\mu$$
$$\le 2c_n(\varphi)\int_{\Omega}\underbrace{\frac{\sum_{k=1}^{m}|f_k(x)|}{f(x)}}_{\le 1}\varphi(f(x))\,d\mu(x) \le 2c_n(\varphi).$$

Finally, putting everything together and letting $\varepsilon \to 0$, we obtain the desired estimate

$$\sum_{k=1}^{n}\|f_k\|_{L_\varphi} \le 2c_n(\varphi)+2 = 2(c_n(\varphi)+1)\Big\|\sum_{k=1}^{n}|f_k|\Big\|_{L_\varphi}. \ \blacksquare$$

As an immediate consequence we obtain the following estimate for the p-concavity constants of Zygmund spaces.

Lemma 18. *Let $1 \le p < \infty$ and $a > 0$. Then for all $n \in \mathbb{N}$,*

$$M_{p,n}(L_p(\log L)_a) \le C(1+\log n)^a.$$

Proof. Note that $L_p(\log L)_a$ coincides with the Orlicz space L_φ for any convex function $\varphi\colon [0,\infty) \to [0,\infty)$ with $\varphi(0)=0$, $\varphi(1)=1$ and

$$\varphi(t) \asymp t^p(\log(e+t))^{ap}.$$

An elementary calculation shows that

$$c_{p,n}(\varphi) \le C(1+\log n)^a,$$

which gives the desired estimate. ■

Together with Theorem 7 this implies the following eigenvalue result.

Theorem 19. *Let $2 \le p < \infty$, $a > 0$, $X = L_p(\log L)_a$ and $k \in X[X']$. Then the Hille–Tamarkin operator T_k is a Riesz operator in X, and its eigenvalues satisfy for all $n \in \mathbb{N}$ and some constant $C > 0$ the estimate*

$$\Big(\sum_{j=1}^{n}|\lambda_j(T_k)|^p\Big)^{1/p} \le C\,(1+\log n)^a.$$

Let us mention that eigenvalue estimates for *matrices* satisfying certain Orlicz norm conditions were given in [2] and [6]. However, the proof techniques in those papers are quite different.

3.5. Applications to Fourier analysis. Now we present some applications of our eigenvalue results to estimates of Fourier coefficients.

Let g be a 1-periodic function on $\mathbb{R}$ such that $g \in L_1([0,1])$, and consider the convolution kernel $k(x,t) := g(x-t)$, $x,t \in [0,1]$. Then it is well known and easy to verify that the eigenvalues of the integral operator $T_k : L_\infty \to L_\infty$ are exactly the Fourier coefficients of g,

$$\widehat{g}(n) := \int_0^1 g(x)e^{-2\pi i n x}\,dx, \qquad n \in \mathbb{Z}.$$

If $g \in X'$ for some Banach function space X over $[0,1]$, then obviously $k \in L_\infty[X'] \hookrightarrow X[X']$, whence $T_k : X \to X$ is indeed a Hille–Tamarkin operator. By the principle of related operators (see, e.g., [18, 3.3.4]) it has the same non-zero eigenvalues as $T_k : L_\infty \to L_\infty$.

Applying this observation to convolution operators generated by a function in the Lorentz space $L_{2,q}$ with $2 < q \le \infty$, we recover—as a special case of Theorem 9—a result due to Bochkarev [1].

Corollary 20. *Let $2 < q \le \infty$. Then there exists a constant $C_q > 0$ such that for every $f \in L_{2,q}([0,1])$ and each $n \in \mathbb{N}$,*

$$\Big(\sum_{k=1}^{n} |c_k|^2\Big)^{1/2} \le C_q(1+\log n)^{1/2-1/q}\, \|f\|_{L_{2,q}},$$

where $(c_k)_{k\in\mathbb{N}}$ is the non-increasing rearrangement of the sequence $(\widehat{f}(n))_{n\in\mathbb{Z}}$ of the Fourier coefficients of f.

Clearly the above estimate implies

$$|c_n| \le C_q n^{-1/2}(1+\log n)^{1/2-1/q}\, \|f\|_{L_{2,q}}.$$

We note that Bochkarev [1] even proved that this seemingly weaker estimate is still sharp. He showed that for every $2 < q \le \infty$ and $n \ge 2$ there exists a variant of the Rudin–Shapiro polynomials

$$P_{n,q}(t) = \sum_{k=0}^{n} c_k e^{2\pi i k t}, \qquad t \in [0,1],$$

which satisfies $\|P_{n,q}\|_{L_{2,q}} = 1$ and

$$\operatorname{card}(\{k \in \mathbb{N} : |c_k| \ge \alpha n^{-1/2}(\log n)^{1/2-1/q}\}) \ge \gamma n$$

for some positive constants α and γ.

This fact shows that our eigenvalue estimate for Hille–Tamarkin operators on Lorentz spaces $L_{2,q}$ with $1 \le q < 2$ presented in Theorem 14 is sharp. In the final section we will give a simpler example which also shows the optimality of Theorem 14.

Similarly we can apply the estimates from Theorem 19 for general Hille–Tamarkin operators on Zygmund spaces to the special case of convolution kernels, thus obtaining the following result for Fourier coefficients.

Corollary 21. *Let $1 < r \le 2$ and $a > 0$. Then there exists a constant $C_{r,a} > 0$ such that for every $f \in L_r(\log L)_{-a}([0,1])$ and $n \in \mathbb{N}$,*

$$\Big(\sum_{k=1}^{n} |c_k|^{r'}\Big)^{1/r'} \le C_{r,a}(1+\log n)^a \|f\|_{L_r(\log L)_a},$$

where $1/r + 1/r' = 1$, and (c_k) is the non-increasing rearrangement of the sequence $(\widehat{f}(n))$ of the Fourier coefficients of f.

4. Optimality of the eigenvalue results. We conclude the paper by showing the optimality of our eigenvalue results for Hille–Tamarkin operators in

- p-concave spaces and spaces of cotype p (Theorem 4),
- Lorentz spaces $L_{2,q}$ with $1 \le q < 2$ (Theorem 14),
- Zygmund spaces $L_p(\log L)_a$ with $2 \le p < \infty$ and $a > 0$ (Theorem 19).

For p-concave spaces we have the following simple example.

Example 1 (optimality in p-concave spaces, $2 \le p < \infty$). Let X be the sequence space ℓ_p, and let $(\sigma_n) \in \ell_p$ be any non-negative decreasing sequence. Then the diagonal operator $D\colon \ell_p \to \ell_p$, defined by $D(x_n) = (\sigma_n x_n)$ for all $(x_n) \in \ell_p$, can be viewed as a Hille–Tamarkin operator T_k with kernel $k \in \ell_p[\ell_{p'}]$. Clearly the eigenvalues of T_k are $\lambda_n(T_k) = \sigma_n$, whence relation (i) in Theorem 4 is optimal.

In cotype p spaces we can use convolution operators.

Example 2 (optimality in spaces of cotype p, $2 < p < \infty$). Let now X be the Lorentz space $L_{p,1}([0,1])$; then $X' = L_{p',\infty}([0,1])$. It is well-known that $L_{p,1}$ has cotype p (see [5]) and that the series

$$\sum_{n=1}^{\infty} n^{-1/p} e^{2\pi i n x}$$

converges a.e. to a function $g \in L_{p',\infty}([0,1])$ (see, e.g., [18, 6.5.8]). Hence the convolution kernel $k(x,t) = g(x-t)$ for all $x,t \in [0,1]$ belongs to $L_\infty[X'] \subset X[X']$, and for the eigenvalues of T_k we have $\lambda_n(T_k) = \widehat{g}(n) = n^{-1/p}$. This matches the upper bound given in part (ii) of Theorem 4.

For the construction of the remaining examples we need the following lemma.

Lemma 22. *Let $1 < r < \infty$ and $b \in \mathbb{R}$. Then there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$(19)\qquad \Big\|\sum_{j=1}^{n} j^{-1/2}e^{2\pi ijx}\Big\|_{L_{2,r}([0,1])} \le C(1+\log n)^{1/r},$$

$$(20)\qquad \Big\|\sum_{j=1}^{n} e^{2\pi ijx}\Big\|_{L_r(\log L)_b([0,1])} \le Cn^{1-1/r}(1+\log n)^{b}.$$

Proof. It is easy to check that for $0<|x|\le 1/2$,

$$B(x) := \sup_{m<n}\Big|\sum_{j=m}^{n} e^{2\pi ijx}\Big| \le \frac{1}{|\sin \pi x|} \le \frac{2}{|x|}.$$

Consequently, for the functions $f_n(x) := \sum_{j=1}^{n} e^{2\pi ijx}$ we have

$$|f_n(x)| \le g_n(x) := \min(n, 2|x|^{-1}), \quad 0<|x|\le 1/2.$$

The norm of g_n in the Zygmund space $L_r(\log L)_b$ (defined over the interval $[-1/2,1/2]$) can be easily calculated; it is of order $n^{1-1/r}(1+\log n)^b$. By the 1-periodicity of f_n and the monotonicity of the norm we obtain the desired estimate (20):

$$\|f_n\|_{L_r(\log L)_b([0,1])} \le Cn^{1-1/r}(1+\log n)^b.$$

The proof of (19) is similar. Let now $f_n(x) := \sum_{j=1}^{n} j^{-1/2}e^{2\pi ijx}$. For $|x|\le 1/n$ we use the trivial estimate

$$|f_n(x)| \le \sum_{j=1}^{n} j^{-1/2} \asymp n^{1/2}.$$

If $1/n<|x|\le 1/2$, let m be the unique integer with $1/m<|x|\le 1/(m-1)$. Then

$$|f_n(x)| \le \sum_{j=1}^{m-1} j^{-1/2} + \Big|\sum_{j=m}^{n} j^{-1/2}e^{2\pi ijx}\Big|.$$

The first summand is of order $m^{1/2}\asymp |x|^{-1/2}$, and the second one can be bounded via Abel's summation by

$$m^{-1/2}B(x) \le m^{-1/2}\cdot 2|x|^{-1} \le 2|x|^{-1/2}.$$

This implies, with some constant $C>0$ independent of n,

$$|f_n(x)| \le g_n(x) := C\min(n^{1/2}, |x|^{-1/2}), \qquad 0<|x|\le 1/2.$$

An elementary calculation shows that the norm of g_n in the Lorentz space $L_{2,r}$ (defined over the interval $[-1/2,1/2]$) is of order $(1+\log n)^{1/r}$. Hence, using again the 1-periodicity of f_n and the monotonicity of the norm, we get

$$\|f_n\|_{L_{2,r}([0,1])} \le C(1+\log n)^{1/r}. \ \blacksquare$$

Our final result shows the optimality of the estimates in Theorems 14 and 19 and Corollaries 20 and 21.

THEOREM 23. *Let (b_n) be a decreasing sequence with $\lim b_n = 0$.*

(i) *Let $1 \leq q < 2$ and $X = L_{2,q}([0,1])$. Then there is a kernel $k \in X[X']$ such that the eigenvalues of the integral operator $T_k : X \to X$ satisfy*

$$\limsup_{n\to\infty} \frac{(\sum_{j=1}^{n} |\lambda_j(T_k)|^2)^{1/2}}{b_n(1+\log n)^{1/q-1/2}} = \infty. \tag{21}$$

(ii) *Let $2 \leq p < \infty$, $a > 0$ and $X = L_p(\log L)_a([0,1])$. Then there exists a kernel $k \in X[X']$ such that the eigenvalues of $T_k : X \to X$ satisfy*

$$\limsup_{n\to\infty} \frac{(\sum_{j=1}^{n} |\lambda_j(T_k)|^p)^{1/p}}{b_n(1+\log n)^{a}} = \infty. \tag{22}$$

Proof. In both cases we use the same method: We construct appropriate 1-periodic functions g on $\mathbb{R}$ with $g \in X'$ and consider the convolution kernel $k(x,t) = g(x-t)$ on $[0,1] \times [0,1]$. Then $k \in L_\infty[X'] \subset X[X']$, since the underlying measure is finite. The functions g will be given by their Fourier series, so that we have direct control of the eigenvalues of T_k (= Fourier coefficients of g). First we choose an increasing sequence of natural numbers N_n such that $b_{N_n} \leq n^{-3}$.

(i) For the functions

$$g_n(x) := (1+\log N_n)^{-1/q'} \sum_{j=1}^{N_n} j^{-1/2} e^{2\pi i j x}$$

we have, with the constant C from (19), the uniform norm estimate

$$\|g_n\|_{L_{2,q'}([0,1])} \leq C.$$

Now the triangle inequality shows that the function

$$g := \sum_{n=1}^{\infty} n^{-2} g_n$$

belongs to $X' = L_{2,q'}([0,1])$. Since the Fourier coefficients of all g_n's are non-negative real, for $j = 1, \ldots, N_n$ we have

$$\lambda_j(T_k) = \widehat{g}(j) \geq n^{-2}\widehat{g_n}(j) = n^{-2}(1+\log N_n)^{-1/q'} j^{-1/2}.$$

By the choice of N_n and since $1/2 - 1/q' = 1/q - 1/2$ this gives the estimate

$$\frac{(\sum_{j=1}^{N_n} |\lambda_j(T_k)|^2)^{1/2}}{b_{N_n}(1+\log N_n)^{1/q-1/2}} \geq \frac{1}{n^2 b_{N_n}} \geq n,$$

which proves (21).

(ii) We proceed similarly to case (i), choosing now

$$g_n(x) := N_n^{-1/p}(1+\log N_n)^{a} \sum_{j=1}^{N_n} e^{2\pi i j x}.$$

With the constant C from (20), we again have a uniform norm estimate

$$\|g_n\|_{L_{p'}(\log L)_{-a}([0,1])} \le C.$$

The triangle inequality shows that the function $g := \sum_{n=1}^{\infty} n^{-2} g_n$ belongs to $L_{p'}(\log L)_{-a} = (L_p(\log L)_a)'$, and for the eigenvalues of T_k we obtain

$$\lambda_j(T_k) = \widehat{g}(j) \ge n^{-2}\widehat{g_n}(j) = n^{-2} N_n^{-1/p}(1+\log N_n)^a$$

for $j = 1, \dots, N_n$. This implies the estimate

$$\frac{(\sum_{j=1}^{N_n} |\lambda_j(T_k)|^p)^{1/p}}{b_{N_n}(1+\log N_n)^a} \ge \frac{1}{n^2 b_{N_n}} \ge n,$$

which proves the desired result (22). ∎

Acknowledgements. The first named author has been supported in part by Ministerio de Ciencia e Innovación, Spain, grant no. MTM2010-15814. The second named author was partially supported by the National Science Centre (NCN), Poland, grant no. 2011/01/B/ST1/06243.

References

[1] S. V. Bochkarev, *Inequalities for orthogonal series and a strengthening of the theorems of Carleman and Orlicz*, Dokl. Akad. Nauk 371 (2000), 17–20 (in Russian).

[2] B. Carl and A. Defant, *An elementary approach to an eigenvalue estimate for matrices*, Positivity 4 (2000), 131–141.

[3] B. Carl and T. Kühn, *Entropy and eigenvalues of certain integral operators*, Math. Ann. 268 (1984), 127–136.

[4] F. Cobos and T. Kühn, *On Hille–Tamarkin operators and Schatten classes*, Ark. Mat. 30 (1992), 217–220.

[5] J. Creekmore, *Type and cotype in Lorentz L_{pq} spaces*, Nederl. Akad. Wetensch. Indag. Math. 43 (1981), 145–152.

[6] A. Defant, M. Mastyło and C. Michels, *Orlicz norm estimates for eigenvalues of matrices*, Israel J. Math. 132 (2002), 45–59.

[7] D. E. Edmunds and H. Triebel, *Function Spaces, Entropy Numbers and Differential Operators*, Cambridge Univ. Press, Cambridge, 1996.

[8] E. Hille and J. Tamarkin, *On the characteristic values of linear integral equations*, Acta Math. 57 (1931), 1–76.

[9] W. B. Johnson, H. König, B. Maurey and J. R. Retherford, *Eigenvalues of p-summing and ℓ_p-type operators in Banach spaces*, J. Funct. Anal. 32 (1979), 353–380.

[10] W. B. Johnson and G. Schechtman, *Computing p-summing norms with few vectors*, Israel J. Math. 87 (1994), 19–31.

[11] A. Kamińska, *Indices, convexity and concavity in Musielak–Orlicz spaces*, Funct. Approx. Comment. Math. 26 (1998), 67–84.

[12] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, 2nd ed., Pergamon Press, Oxford-Elmsford, N.Y., 1982.

[13] H. König, *Eigenvalue Distribution of Compact Operators*, Birkhäuser, 1986.

[14] T. Kühn and M. Mastyło, *Weyl numbers and eigenvalues of abstract summing operators*, J. Math. Anal. Appl. 369 (2010), 408–422.

[15] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, Berlin, 1977.
[16] —, —, *Classical Banach Spaces II. Function Spaces*, Springer, Berlin, 1979.
[17] A. Pietsch, *Operator Ideals*, Deutscher Verlag Wiss., Berlin, 1978, and North-Holland, Amsterdam, 1980.
[18] —, *Eigenvalues and s-numbers*, Cambridge Stud. Adv. Math. 13, Cambridge Univ. Press, 1987.
[19] S. Rolewicz, *Metric Linear Spaces*, 2nd ed., Math. Appl. (East Eur. Ser.) 20, Reidel, Dordrecht, and PWN–Polish Sci. Publ., Warszawa, 1985.
[20] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite-Dimensional Operator Ideals*, Longman, 1989.
[21] A. Zvavitch, *A remark on p-summing norms of operators*, in: Trends in Banach Spaces and Operator Theory (Memphis, TN, 2001), Contemp. Math. 321, Amer. Math. Soc., Providence, RI, 2003, 371–378.
[22] A. Zygmund, *Trigonometric Series, Vol. II*, 2nd ed., Cambridge Univ. Press, 1968.

Thomas Kühn
Fakultät für Mathematik
und Informatik
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
D-04103 Leipzig, Germany
E-mail: kuehn@math.uni-leipzig.de

Mieczysław Mastyło
Faculty of Mathematics
and Computer Science
Adam Mickiewicz University
and
Institute of Mathematics
Polish Academy of Sciences (Poznań branch)
Umultowska 87
61-614 Poznań, Poland
E-mail: mastylo@amu.edu.pl

Received September 24, 2011 (7312)