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Unconditionality of general Franklin systems in $L^p[0,1]$, $1 < p < \infty$

by

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Abstract. By a general Franklin system corresponding to a dense sequence $\mathcal{T} = (t_n, n \geq 0)$ of points in $[0,1]$ we mean a sequence of orthonormal piecewise linear functions with knots $\mathcal{T}$, that is, the nth function of the system has knots $t_0, \ldots, t_n$. The main result of this paper is that each general Franklin system is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$.

1. INTRODUCTION

The classical Franklin system is a complete orthonormal system consisting of piecewise linear continuous functions with dyadic knots. It was introduced by Ph. Franklin [9] in 1928 as an example of a complete orthonormal system which is a basis in $C[0,1]$. Since then, this system has been studied by many authors from various points of view. In particular, an important tool in the study of the Franklin system is provided by the exponential estimates proved by Z. Ciesielski [4]. It is well known that the classical Franklin system is a basis in $C[0,1]$ and $L^p[0,1]$, $1 \leq p < \infty$, unconditional for $1 < p < \infty$ (S. V. Bochkarev [1]), a basis in $H^p[0,1]$, $1/2 \leq p \leq 1$, unconditional for $1/2 < p \leq 1$ (P. Wojtaszczyk [19] for $p = 1$, P. Sjölin and J. O. Strömberg [16] for general p), and the coefficients of functions with respect to the Franklin system give a characterization of Hölder classes in L^p-norms with exponent α, $0 < \alpha < 1 + 1/p$, BMO and VMO (Z. Ciesielski [4] and P. Wojtaszczyk [19]). Various generalizations of this system, like systems of orthonormal splines of higher order and regularity on $[0,1]$, and versions on $\mathbb{R}$ (see e.g. [18]), have also been studied.

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In this paper, we are interested in a generalization of the classical Franklin system obtained by passing to general sequences of knots. Thus, given a sequence $\mathcal{T} = (t_n, n \geq 0)$ of points in $[0,1]$ admitting at most double knots and dense in $[0,1]$, by a general Franklin system corresponding to $\mathcal{T}$ we mean the complete orthonormal system consisting of piecewise linear functions with knots $\mathcal{T}$ (see Section 2 for a more detailed description). Z. Ciesielski [3] has proved that the L^∞-norm of the orthogonal projection onto the space of piecewise linear functions with arbitrary knots does not exceed 3. This implies that each general Franklin system is a basis in $L^p[0,1]$, $1 \leq p < \infty$, and if all knots are simple (so that all functions from the system are continuous), it is a basis in $C[0,1]$. Various properties of these systems have been studied by Z. Ciesielski and A. Kamont [7], G. G. Gevorkyan and A. Kamont [10], G. G. Gevorkyan and A. A. Sahakian [11]; see also the survey article by Z. Ciesielski and A. Kamont [8].

In this paper, we are interested in the unconditionality of general Franklin systems in $L^p[0,1]$, $1 < p < \infty$. Recall that S. V. Bochkarev [1] has proved the unconditionality of the classical Franklin system in $L^p[0,1]$, $1 < p < \infty$. For general Franklin systems this question has been treated in [10] and [11], where some partial answers have been obtained, under additional conditions on the sequence of knots; those results, as well as methods of proof, are described in more detail in Section 2.1. Now, developing the method from [11], we prove that for any sequence of knots dense in $[0,1]$, the corresponding Franklin system is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$. Moreover, we show that each general Franklin system normalized in $L^p[0,1]$, $1 < p < \infty$, is a greedy basis in this space.

For comparison, recall that unconditionality in $L^p[0,1]$, $1 < p < \infty$, of each general Haar system (i.e. the orthonormal system consisting of piecewise constant functions with a given sequence of knots, dense in $[0,1]$) follows from D. L. Burkholder's results on boundedness of martingale transforms (see e.g. [2]). On the other hand, it is natural to ask whether one can obtain an analogous result for orthonormal spline systems of higher order and with arbitrary knots. It follows from the recent result of A. Yu. Shadrin [15] (uniform bound of L^∞-norms of orthogonal projections onto splines of higher order and with arbitrary knots, with the bound depending only on the order of the splines) that each such system is a basis in $L^p[0,1]$, $1 \leq p < \infty$, and $C[0,1]$. In fact, it is well known that in the case of dyadic knots orthonormal spline bases are unconditional in $L^p[0,1]$, $1 < p < \infty$ (see Z. Ciesielski [5]). Under some conditions on the sequence of partitions (in the terminology of [10], for quasi-dyadic strongly regular sequences of partitions), to get unconditionality, it is enough to use estimates from [6] and follow the scheme of proof from the dyadic case. However, in the general case the estimates from [6] are not sufficient.

The paper is organized as follows. In Section 2, we recall the definition of a general Franklin system and formulate the results: Theorem 2.1 and Corollary 2.2. In Section 2.1 we recall the results on unconditionality of general Franklin systems from [10] and [11] and comment on the method of proof. The basic properties of Franklin functions and Franklin systems needed for the proof are summarized in Section 3. The proofs of Theorem 2.1 and Corollary 2.2 are given in Section 4. Finally, Section 4.3 contains some comments and related results.

NOTATION. Throughout the paper, for a set $A \subset [0,1]$, we denote by χ_A the characteristic function of A, by $|A|$ the Lebesgue measure of A, and by A^{c} the complement of A; for $t \in [0,1]$, $\operatorname{dist}(t,A)$ is the distance from t to A. For a finite set B, $\#B$ denotes the number of elements of B. For a function $f : [0,1] \to \mathbb{R}$, $\mathcal{M}f$ is the Hardy–Littlewood maximal function of f. The notation $a \sim b$ means that there are positive constants c_1, c_2, independent of the variables appearing in a, b, such that $c_1 a \le b \le c_2 a$. Also, the usual abbreviations $x \vee y = \max(x,y)$, $x \wedge y = \min(x,y)$ are used.

2. DEFINITION OF A GENERAL FRANKLIN SYSTEM AND MAIN RESULTS

Let us begin by recalling the definitions of a general Franklin function and a general Franklin system.

Let $\sigma = (s_i, 0 \le i \le N)$ be a partition of $[0,1]$ admitting at most double knots, i.e., a sequence of points in $[0,1]$ such that

$$\begin{cases} 0 = s_0 < s_1 \le \cdots \le s_{N-1} < s_N = 1, \\ s_i < s_{i+2} \quad \text{for all } i,\ 0 \le i \le N-2. \end{cases} \tag{2.1}$$

Denote by $S(\sigma)$ the space of piecewise linear functions on $[0,1]$ with knots σ, that is, functions linear on each (s_i, s_{i+1}), left-continuous at each s_i (and right-continuous at $s_0 = 0$) and continuous at each s_i, $1 \le i \le N-1$, satisfying $s_{i-1} < s_i < s_{i+1}$. Each $f \in S(\sigma)$ has a unique representation

$$f = \sum_{i=0}^{N} a_i N_{\sigma,i}, \tag{2.2}$$

where

$$N_{\sigma,0}(t) = \frac{s_1 - t}{s_1 - s_0} \cdot \chi_{[s_0,s_1]}(t), \qquad N_{\sigma,N}(t) = \frac{t - s_{N-1}}{s_N - s_{N-1}} \cdot \chi_{[s_{N-1},s_N]}(t),$$

for i such that $s_{i-1} < s_i < s_{i+1}$ we have

$$N_{\sigma,i}(t) = \begin{cases} \dfrac{t - s_{i-1}}{s_i - s_{i-1}} & \text{for } t \in [s_{i-1}, s_i], \\ \dfrac{s_{i+1} - t}{s_{i+1} - s_i} & \text{for } t \in [s_i, s_{i+1}], \\ 0 & \text{otherwise}, \end{cases}$$

and for i such that $s_{i-1} = s_i$,

$$N_{\sigma,i-1}(t) = \frac{t - s_{i-2}}{s_{i-1} - s_{i-2}} \cdot \chi_{[s_{i-2},s_{i-1}]}(t), \qquad N_{\sigma,i}(t) = \frac{s_{i+1} - t}{s_{i+1} - s_i} \cdot \chi_{(s_i,s_{i+1}]}(t).$$

The coefficients a_i in (2.2) are given by: $a_0 = f(s_0)$; $a_N = f(s_N)$; $a_i = f(s_i)$ for i such that $s_{i-1} < s_i < s_{i+1}$; and $a_{i-1} = f(s_{i-1}) = f(s_i) = \lim_{t\to s_{i-1}-0} f(t)$ and $a_i = \lim_{t\to s_i+0} f(t)$ for i such that $s_{i-1} = s_i$.

Now, let $\sigma = (s_i, 0 \le i \le N)$ and $\sigma^* = (s_i^*, 0 \le i \le N+1)$ be a pair of partitions of $[0,1]$ satisfying (2.1) and such that σ^* is obtained from σ by adding one knot s^*. Note that s^* may be different from all points of σ (in this case, for some i, we have $s^* = s_i^*$ and $s_{i-1}^* < s_i^* < s_{i+1}^*$), or for some i, $s^* = s_i$ (then $s_{i-1}^* < s_i^* = s^* = s_{i+1}^* < s_{i+2}^*$). Now, there is a unique function $\varphi \in S(\sigma^*)$ such that φ is orthogonal to $S(\sigma)$ in $L^2[0,1]$, $\|\varphi\|_2 = 1$ and $\varphi(s^*) > 0$. This function is called the *general Franklin function corresponding to the pair of partitions* (σ, σ^*).

Now, we turn to sequences of partitions and general Franklin systems.

DEFINITION 2.1. Let $\mathcal{T} = (t_n, n \ge 0)$ be a sequence of points in $[0,1]$. The sequence $\mathcal{T}$ is called *admissible* if $t_0 = 0$, $t_1 = 1$, $t_n \in (0,1)$ for each $n \ge 2$, for each $t \in (0,1)$ there are at most two different indices $n_1 > n_2 \ge 2$ such that $t = t_{n_1} = t_{n_2}$, and $\mathcal{T}$ is dense in $[0,1]$.

For an admissible sequence of points $\mathcal{T} = (t_n, n \ge 0)$ and $n \ge 1$, let $\pi_n = (t_{n,i}, 0 \le i \le n)$ be the partition of $[0,1]$ obtained by the nondecreasing rearrangement of the sequence $(t_i, 0 \le i \le n)$, counting multiplicities. Note that each π_n satisfies (2.1), and π_n is obtained from π_{n-1} by adding one knot t_n.

DEFINITION 2.2. Let $\mathcal{T}$ be an admissible sequence of points. A *general Franklin system* with knots $\mathcal{T}$ is a sequence of functions $\{f_n, n \ge 0\}$ given by

$$f_0(t) = 1, \quad f_1(t) = \sqrt{3}\,(2t - 1),$$

and for $n \ge 2$, f_n is the general Franklin function corresponding to the pair of partitions (π_{n-1}, π_n).

It follows from the estimates of L^∞-norms of orthogonal projections onto piecewise linear functions (see [3]) that for each admissible sequence of knots, the corresponding Franklin system is a basis in $L^p[0,1]$, $1 \le p < \infty$. In addition, each continuous function on $[0,1]$ is a limit, in the uniform norm, of the sequence of its partial sums with respect to a general Franklin system, and if all knots in $\mathcal{T}$ are simple, then the corresponding general Franklin system is a basis in $C[0,1]$.

The main result of the present paper is the following:

THEOREM 2.1. *Let $\mathcal{T} = (t_n, n \geq 0)$ be an admissible sequence of knots in* $[0, 1]$. *Then the corresponding general Franklin system is an unconditional basis in each* $L^p[0, 1]$, $1 < p < \infty$.

REMARK 1. In fact, our proof gives more: for each p, $1 < p < \infty$, the unconditional basic constants for general Franklin systems are bounded by a constant C_p depending only on p. That is, for each p, there is a finite constant C_p such that for each admissible sequence $\mathcal{T}$, the corresponding Franklin system $\{f_n, n \geq 0\}$, each sequence of coefficients $\{a_n, n \geq 0\}$ and each choice of signs $\{\varepsilon_n, n \geq 0\}$, $\varepsilon_n \in \{-1, 1\}$,

$$\Big\| \sum_{n=0}^{\infty} \varepsilon_n a_n f_n \Big\|_p \leq C_p \Big\| \sum_{n=0}^{\infty} a_n f_n \Big\|_p.$$

The existence of C_p is just a consequence of the method of proof.

To formulate the next result, Corollary 2.2, we need to recall the concept of greedy basis (see S. V. Konyagin and V. N. Temlyakov [14]). Let $(X, \|\cdot\|)$ be a Banach space with a normalized basis $\mathcal{X} = (x_n, n \geq 0)$ (i.e. with $\|x_n\| = 1$). For $x \in X$ and $m \in \mathbb{N}$, let

$$\sigma_m(x) = \inf_{n_1,\ldots,n_m} \inf_{c_1,\ldots,c_m} \Big\| x - \sum_{i=1}^{m} c_i x_{n_i} \Big\|.$$

In addition, for $x = \sum_{n=0}^{\infty} a_n x_n$ and given $m \in \mathbb{N}$, let Λ_m be a subset of indices such that $\#\Lambda_m = m$ and

$$\min_{n \in \Lambda_m} |a_n| \geq \max_{n \notin \Lambda_m} |a_n|,$$

and put $G_m(x) = \sum_{n \in \Lambda_m} a_n x_n$. Clearly, $\sigma_m(x) \leq \|x - G_m(x)\|$. Following S. V. Konyagin and V. N. Temlyakov [14], a normalized basis $\mathcal{X} = (x_n, n \geq 0)$ of a Banach space $(X, \|\cdot\|)$ is called *greedy* if there is a constant $C > 0$ such that for all $m \in \mathbb{N}$ and $x \in X$,

$$\|x - G_m(x)\| \leq C\sigma_m(x). \tag{2.3}$$

Now, we have the following consequence of Theorem 2.1:

COROLLARY 2.2. *Let $\mathcal{T} = (t_n, n \geq 0)$ be an admissible sequence of knots in* $[0, 1]$ *with the corresponding general Franklin system* $\{f_n, n \geq 0\}$. *For given* p, $1 \leq p \leq \infty$, *let* $f_{n,p} = f_n/\|f_n\|_p$. *Then for each* p, $1 < p < \infty$, $\{f_{n,p}, n \geq 0\}$ *is a greedy basis in* $L^p[0, 1]$.

REMARK 2. For general Franklin systems normalized in $L^p[0, 1]$, $1 < p < \infty$, the constants in (2.3) can be chosen so that they depend on p, but not on the sequence of knots.

2.1. Earlier results and comments on the method of proof. As already mentioned, unconditionality in $L^p[0, 1]$, $1 < p < \infty$, of the classical

Franklin system (i.e. with dyadic knots) has been proved by S. V. Bochkarev [1]. In G. G. Gevorkyan and A. Kamont [10] and G. G. Gevorkyan and A. A. Sahakian [11] some partial answers to the question of unconditionality in $L^p[0,1]$, $1 < p < \infty$, of general Franklin systems have been obtained. In both [10] and [11], there are some assumptions on the structure and regularity of the sequence of knots under consideration. The first assumption is the quasi-dyadic structure of $\mathcal{T} = (t_n, n \geq 0)$. This means the following: consider a sequence of partitions $\mathcal{T}_j = \{\tau_{j,k}, 0 \leq k \leq 2^j\}$, $j \geq 0$, such that $0 = \tau_{j,0} < \tau_{j,1} < \cdots < \tau_{j,2^j} = 1$ and $\tau_{j+1,2k} = \tau_{j,k}$ for all j, k, $0 \leq k \leq 2^j$, i.e. between each pair of knots of $\mathcal{T}_j$, one new knot from $\mathcal{T}_{j+1}$ is inserted. Putting $t_0 = 0$, $t_1 = 1$ and $t_n = \tau_{j,2k-1}$ for $n = 2^j + k$ with $j \geq 0$ and $1 \leq k \leq 2^j$, we get an admissible sequence $\mathcal{T} = (t_n, n \geq 0)$ of simple knots with *quasi-dyadic structure.*

In addition, in [10] and [11], there are some regularity conditions imposed on the quasi-dyadic sequence under consideration. To describe these conditions, set $\Delta_{j,k} = [\tau_{j,k-1}, \tau_{j,k}]$ and observe that $\Delta_{j,k} = \Delta_{j+1,2k-1} \cup \Delta_{j+1,2k}$. In [10], the following *weak regularity condition* has been assumed: there is a constant γ, $0 < \gamma \leq 1/2$, such that for all j, k with $1 \leq k \leq 2^j$,

$$\gamma \leq \frac{|\Delta_{j+1,2k-1}|}{|\Delta_{j,k}|}, \frac{|\Delta_{j+1,2k}|}{|\Delta_{j,k}|} \leq 1 - \gamma.$$

This condition means that the newly inserted point $\tau_{j+1,2k-1}$ cannot be close to the endpoints of the interval $\Delta_{j,k}$ into which it is inserted.

In [11], the regularity condition has been weakened as follows: there is a constant M such that for any subsequence (j_l, k_{j_l}) with $j_l < j_{l+1}$ and $1 \leq k_{j_l} \leq 2^{j_l}$,

$$\sum_{l=1}^{\infty} |\Delta_{j_l,k_{j_l}}| \leq M \Big| \bigcup_{l=1}^{\infty} \Delta_{j_l,k_{j_l}} \Big|.$$

It can be seen that the above condition is equivalent to the following:

(∗) there is a constant $\zeta \geq 0$ such that for any $j_1 < \cdots < j_m$ and Δ_{j_i,k_i} such that $\Delta_{j_1,k_1} \supset \cdots \supset \Delta_{j_m,k_m}$, if $|\Delta_{j_m,k_m}| > |\Delta_{j_1,k_1}|/2$ then $m \leq \zeta$.

The method of proof in [11] is different than in [10], and turns out to be an important step towards proving unconditionality of general Franklin systems in $L^p[0,1]$, $1 < p < \infty$. This is done in the present paper by developing the method of [11], without any constraints on the structure or regularity of the sequence of knots.

The main new idea is a new choice of a "canonical" interval associated with a general Franklin function. In [11] (and also in [10]), the function f_n with $n = 2^j + k$ has been associated with the interval $\{n\} = \Delta_{j,k}$, i.e. the interval into which the point t_n is inserted, and all estimates, splittings and reorderings for a general Franklin system have been done with respect to positions of $\{n\}$ or t_n. The new choice of the canonical interval (called J_n) and its consequences (pointwise estimates etc.) are described in Section 3. The key property of these new intervals is Lemma 3.5, which can be regarded as condition $(*)$ for the intervals J_n. Note that condition $(*)$, which in [11] *has been assumed* for the intervals $\{n\}$, now *is a property* of the intervals J_n.

With this new choice of canonical intervals, we prove two technical estimates, Lemmas 4.2 and 4.3. Lemma 4.2 corresponds to inequalities (63) from [11], but with the splitting of a general Franklin system according to the position of the intervals J_n instead of the points t_n, and its proof is similar to the proof of (63) in [11]. Lemma 4.3 replaces Lemma 3 of [11]. In the notation of Lemma 4.3, Lemma 3 of [11] states that under condition $(*)$,

$$\sum_{n=n(V)} \left(\frac{1}{\theta}\right)^{pd_n(V)} \|f_n\|^p_{L^q(V)} \cdot \|f_n\|^p_{L^p(\widetilde{V}^c)} \le M_p,$$

where $1/p + 1/q = 1$. However, it can be seen that in the general case the above inequality does not hold, even for quasi-dyadic sequences of partitions (a counterexample can be constructed by considering the case of $J_n \subset V$). In comparison with the proof of Lemma 3 of [11], the proof of our Lemma 4.3 requires new techniques, like splitting the coefficients a_n with $J_n \subset V$ into three parts and treating each of them in a different way.

Once Lemmas 4.2 and 4.3 are proved, the remaining part of the proof is the same as in [11]. However, in [11], the parts of the argument which require condition $(*)$ and those which do not require $(*)$ are not clearly separated. Therefore, for the sake of completeness, we present that part of the proof as well.

3. BASIC PROPERTIES OF A GENERAL FRANKLIN SYSTEM

3.1. Properties of a single Franklin function. To simplify notation, assume for a while that

$$\pi = \{0 = \tau_{-k} < \tau_{-k+1} \le \cdots \le \tau_{-1} < \tau_1 \le \cdots \le \tau_{l-1} < \tau_l = 1\},$$

and $\pi^* = \pi \cup \{\tau\}$ with $\tau_{-1} < \tau = \tau_0 \le \tau_1$ (with $\tau_i < \tau_{i+2}$). The general Franklin function corresponding to (π, π^*) is defined in Section 2, but now we recall some more details of its construction. Moreover, we associate with a general Franklin function a "canonical" interval J.

For convenience, introduce the notation

$$\lambda_i = \tau_i - \tau_{i-1}. \tag{3.1}$$

First, consider the case when τ is a simple knot of π^*, i.e. $\tau_{-1} < \tau = \tau_0 < \tau_1$. In this case, the Franklin function φ is described as in Section 2.2 of [10]: let $G = G_{\pi^*} = [(N_{\pi^*,i}, N_{\pi^*,j}), -k \le i, j \le l]$ be the Gram matrix of the system $(N_{\pi^*,i}, -k \le i \le l)$, and let $G^{-1} = A = [a_{i,j}, -k \le i,j \le l]$. Then consider the function

$$(3.2)\quad \psi = \sum_{i=-k}^{l} \eta_i N_{\pi^*,i}, \quad \text{where} \quad \eta_i = -\frac{\lambda_1}{\lambda_0+\lambda_1}\, a_{i,-1} + a_{i,0} - \frac{\lambda_0}{\lambda_0+\lambda_1}\, a_{i,1}.$$

Representing the functions $N_{\pi,i}$, $i \neq 0$, as linear combinations of $N_{\pi^*,j}$ one can see that $N_{\pi,i} = N_{\pi^*,i}$ for $i \le -2$ and $i \ge 2$, and

$$N_{\pi,-1} = N_{\pi^*,-1} + \frac{\lambda_1}{\lambda_0+\lambda_1} N_{\pi^*,0}, \qquad N_{\pi,1} = N_{\pi^*,1} + \frac{\lambda_0}{\lambda_0+\lambda_1} N_{\pi^*,0}.$$

Using this, it is easy to see that $(\psi, N_{\pi,i}) = 0$ for all $i \neq 0$, and consequently $\varphi = \psi/\|\psi\|_2$.

In order to describe the choice of J, consider the following intervals:

$$(3.3)\qquad I = [\tau_{-1}, \tau_1], \quad I^- = [\tau_{-2}, \tau_0], \quad I^+ = [\tau_0, \tau_2],$$

$$(3.4)\qquad \nu = |I|, \quad \nu^- = |I^-|, \quad \nu^+ = |I^+|, \quad \mu = \min(\nu^-, \nu, \nu^+).$$

(For $k = 1$ or $l = 1$, we take $\tau_{-2} = 0$ or $\tau_2 = 1$, respectively.) Now, let $I^* = [\tau_{i^*}, \tau_{i^*+2}]$ be one of the intervals I^-, I, I^+ such that $\mu = |I^*|$, and consider its left and right parts $I^{*,l} = [\tau_{i^*}, \tau_{i^*+1}]$, $I^{*,r} = [\tau_{i^*+1}, \tau_{i^*+2}]$. Finally, let J be one of the intervals $I^{*,l}, I^{*,r}$ such that $|J| = \max(|I^{*,l}|, |I^{*,r}|)$.

Observe that with this choice of μ and J we have

$$(3.5)\qquad |J| \le \mu \le 2|J|.$$

For convenience, set

$$\tau^{-,-} = \tau_{-2}, \quad \tau^- = \tau_{-1}, \quad \tau^+ = \tau_1, \quad \tau^{+,+} = \tau_2.$$

Now, we turn to the case when τ is a double knot of π^*, i.e. $\tau_{-1} < \tau = \tau_0 = \tau_1 < \tau_2$. In this case we have $(N_{\pi^*,i}, N_{\pi^*,j}) = 0$ for all i, j such that $i \le 0$ and $j \ge 1$. Consequently, for the inverse matrix $A = G^{-1}$ (where G is the Gram matrix of the system $(N_{\pi^*,i}, -k \le i \le l)$) we also have $a_{i,j} = 0$ when $i \le 0$ and $j \ge 1$. Now, consider

$$(3.6)\qquad \psi = \sum_{i=-k}^{0} a_{i,0} N_{\pi^*,i} - \sum_{i=1}^{l} a_{i,1} N_{\pi^*,i}.$$

Since $N_{\pi,i} = N_{\pi^*,i}$ for $i \le -1$ and $i \ge 2$ and $N_{\pi,1} = N_{\pi^*,0} + N_{\pi^*,1}$, one can see that $(\psi, N_{\pi,i}) = 0$ for all $i \neq 0$, and consequently $\varphi = \psi/\|\psi\|_2$. To define the interval J, consider $I^- = [\tau_{-1}, \tau_0]$, $I^+ = [\tau_1, \tau_2]$ and put $\mu = \min(|I^-|, |I^+|)$. Now, we take as J one of I^-, I^+ such that $|J| = \mu$. Moreover, we put $\tau^- = \tau_{-1}$ and $\tau^+ = \tau_2$ ($\tau^{-,-}$, $\tau^{+,+}$ are not needed in this case).

In what follows, some pointwise estimates for a general Franklin function are needed. In the case of simple knots, the following estimates for a general Franklin function have been obtained in [10] and [11]. When double knots are allowed, the proof is analogous, but one has to consider the coefficients of φ from representation (2.2) instead of the values $\varphi(\tau_i)$, as done in [10] and [11].

PROPOSITION 3.1. *Let $\pi^* = \pi \cup \{\tau_0\}$ be as described above, and let φ be the general Franklin function corresponding to (π, π^*), $\varphi = \sum_{i=-k}^{l} \xi_i N_{\pi^*,i}$. If $\tau = \tau_0$ is a simple knot of π^*, then*

$$\tag{3.7} \begin{cases} \|\varphi\|_p \sim \mu^{1/p-1/2}, \quad 1 \le p \le \infty, \\ |\xi_{-1}| \sim \mu^{1/2}/\nu^-, \quad |\xi_0| \sim \mu^{1/2}/\nu, \quad |\xi_1| \sim \mu^{1/2}/\nu^+, \end{cases}$$

with the implied constants independent of (π, π^*) and p.

In addition, with $\varepsilon = (\sqrt{2}+1)/3$ and for some positive constant C in (a2), (b2), *independent of π and π^*,*

(a) *for $i \le -1$:*

$$\tag{a1} |\xi_{i-1}| \le \frac{2}{3} \frac{\tau_i - \tau_{i-1}}{\tau_i - \tau_{i-2}} |\xi_i|, \quad |\xi_{i-1}| \le \frac{|\xi_i|}{2},$$

$$\tag{a2} |\xi_i| \le C \left(\frac{2}{3}\right)^{|i|} \frac{\tau_{-1} - \tau_{-2}}{\tau_{-1} - \tau_{i-1}} \frac{\mu^{1/2}}{\nu^-},$$

$$\tag{a3} |\xi_{i-1}| \left(\frac{3}{2}\lambda_{i-1} + 2\lambda_i\right) \le |\xi_i|\lambda_i \le 2|\xi_{i-1}|(\lambda_{i-1} + \lambda_i),$$

$$\tag{a4} \begin{cases} \int_{\tau_{i-2}}^{\tau_{i-1}} |\varphi(t)|^p \, dt \le \varepsilon^p \int_{\tau_{i-1}}^{\tau_i} |\varphi(t)|^p \, dt, \quad 1 \le p < \infty, \\ \sup_{\tau_{i-2} \le t \le \tau_{i-1}} |\varphi(t)| \le \varepsilon \sup_{\tau_{i-1} \le t \le \tau_i} |\varphi(t)|, \end{cases}$$

and for $i \le i+s \le -1$:

$$\tag{a5} \begin{cases} \int_0^{\tau_i} |\varphi(t)|^p \, dt \le \dfrac{\varepsilon^{ps}}{1-\varepsilon^p} \int_{\tau_{i+s-1}}^{\tau_{i+s}} |\varphi(t)|^p \, dt, \\ \int_0^{\tau_i} |\varphi(t)|^p \, dt \le \dfrac{\varepsilon^{p|i|}}{1-\varepsilon^p} \|\varphi\|_p^p, \quad 1 \le p < \infty, \end{cases}$$

$$\tag{a6} \sup_{t \le \tau_i} |\varphi(t)| \le \varepsilon^s \sup_{\tau_{i+s-1} \le t \le \tau_{i+s}} |\varphi(t)|, \quad \sup_{t \le \tau_i} |\varphi(t)| \le \varepsilon^{|i|} \|\varphi\|_\infty,$$

(b) *for $i \ge 1$:*

$$\tag{b1} |\xi_{i+1}| \le \frac{2}{3} \frac{\tau_{i+1} - \tau_i}{\tau_{i+2} - \tau_i} |\xi_i|, \quad |\xi_{i+1}| \le \frac{|\xi_i|}{2},$$

$$\tag{b2} |\xi_i| \le C \left(\frac{2}{3}\right)^{|i|} \frac{\tau_2 - \tau_1}{\tau_{i+1} - \tau_1} \frac{\mu^{1/2}}{\nu^+},$$

$$\tag{b3} |\xi_{i+1}| \left(\frac{3}{2}\lambda_{i+2} + 2\lambda_{i+1}\right) \le |\xi_i|\lambda_{i+1} \le 2|\xi_{i+1}|(\lambda_{i+1} + \lambda_{i+2}),$$

$$
\text{(b4)} \qquad \begin{cases} \int\limits_{\tau_{i+1}}^{\tau_{i+2}} |\varphi(t)|^p \, dt \le \varepsilon^p \int\limits_{\tau_i}^{\tau_{i+1}} |\varphi(t)|^p \, dt, \quad 1 \le p < \infty, \\ \max\limits_{\tau_{i+1} \le t \le \tau_{i+2}} |\varphi(t)| \le \varepsilon \max\limits_{\tau_i \le t \le \tau_{i+1}} |\varphi(t)|, \end{cases}
$$

and for $1 \le i - s \le i$:

$$
\text{(b5)} \qquad \begin{cases} \int\limits_{\tau_i}^{1} |\varphi(t)|^p \, dt \le \dfrac{\varepsilon^{ps}}{1-\varepsilon^p} \int\limits_{\tau_{i-s}}^{\tau_{i-s+1}} |\varphi(t)|^p \, dt, \\ \int\limits_{\tau_i}^{1} |\varphi(t)|^p \, dt \le \dfrac{\varepsilon^{p|i|}}{1-\varepsilon^p} \, \|\varphi\|_p^p, \quad 1 \le p < \infty, \end{cases}
$$

$$
\text{(b6)} \qquad \sup_{\tau_i \le t} |\varphi(t)| \le \varepsilon^s \sup_{\tau_{i-s} \le t \le \tau_{i-s+1}} |\varphi(t)|, \quad \sup_{\tau_i \le t} |\varphi(t)| \le \varepsilon^{|i|} \|\varphi\|_\infty.
$$

If $\tau = \tau_0 = \tau_1$ *is a double knot of* π^*, *then*

$$
\text{(3.8)} \quad \|\varphi\|_p \sim \mu^{1/p-1/2}, \quad 1 \le p \le \infty, \quad |\xi_0| \sim \mu^{1/2}/\lambda_0, \quad |\xi_1| \sim \mu^{1/2}/\lambda_2,
$$

with the implied constants independent of (π, π^*) *and* p. *Moreover, inequalities* (a1) *and* (a3)–(a6) *hold for* $i \le 0$ *and* $i \le i+s \le 0$, *with* (a2) *replaced by*

$$
\text{(a2}'\text{)} \qquad |\xi_i| \le C \left(\frac{2}{3}\right)^{|i|} \frac{\mu^{1/2}}{\tau_0 - \tau_{i-1}},
$$

while inequalities (b1) *and* (b3)–(b6) *hold for* $i \ge 1$ *and* $i \ge i - s \ge 1$, *with* (b2) *replaced by*

$$
\text{(b2}'\text{)} \qquad |\xi_i| \le C \left(\frac{2}{3}\right)^{|i|} \frac{\mu^{1/2}}{\tau_{i+1} - \tau_1}.
$$

In both cases (*i.e. of* τ *being a simple or a double knot of* π^*) *we have* $|\xi_i| = (-1)^{|i|}\xi_i$ *and the following localization of the support of* φ: *if* $\tau_{i-1} = \tau_i \le \tau^-$ (*respectively*, $\tau^+ \le \tau_i = \tau_{i+1}$), *then* $\operatorname{supp}\varphi \subset [\tau_i, 1]$ (*respectively*, $\operatorname{supp}\varphi \subset [0, \tau_i]$).

Proof. First, consider the case when $\tau = \tau_0$ is a simple knot of π^*. If all knots of π^* are simple, the equivalences (3.7), the property $|\xi_i| = (-1)^{|i|}\xi_i$ and inequalities (a1), (b1), (a3), (b3) are contained in Proposition 2.3 of [10], and the main argument in the proof is representation (3.2) (cf. formulae (2.9), (2.10) of [10]) combined with the estimates for the entries of the matrix A (cf. Proposition 2.1 of [10], or Chapter 6.4 of [13])

Inequalities (a2), (b2) are obtained in Lemma 2 of [11]—more precisely, (a2), (b2) follow from (a1), (b1), (3.7) by repeated use of the following elementary inequality:

$$
\text{(3.9)} \qquad \frac{a}{(a+b)(a+c)} < \frac{1}{a+b+c} \quad \text{for } a, b, c > 0.
$$

Inequalities (a4), (b4) for $p = 1$ and $p = \infty$ are contained in Proposition 2.4 of [10]. The proof for $1 < p < \infty$ is similar to that for $p = 1$, but we sketch it for completeness. Let us give the proof of (a4); inequality (b4) is checked analogously. As the signs of ξ_{i-1} and ξ_i are opposite, we have

$$m_{i,p} = \int\limits_{\tau_{i-1}}^{\tau_i} |\varphi(t)|^p\,dt = \frac{\lambda_i}{p+1}\,\frac{|\xi_{i-1}|^{p+1} + |\xi_i|^{p+1}}{|\xi_{i-1}| + |\xi_i|}.$$

As $p \geq 1$, we get

$$|\xi_i|^p \geq \frac{|\xi_{i-1}|^{p+1} + |\xi_i|^{p+1}}{|\xi_{i-1}| + |\xi_i|} \geq \left(\frac{|\xi_{i-1}|^2 + |\xi_i|^2}{|\xi_{i-1}| + |\xi_i|}\right)^p \geq 2^p(\sqrt{2}-1)^p|\xi_i|^p,$$

and consequently

$$\frac{m_{i+1,p}}{m_{i,p}} \geq 2^p(\sqrt{2}-1)^p\,\frac{\lambda_{i+1}}{\lambda_i}\,\frac{|\xi_{i+1}|^p}{|\xi_i|^p}.$$

This inequality and (a3) give

$$\frac{m_{i+1,p}}{m_{i,p}} \geq 2^p(\sqrt{2}-1)^p\,\frac{\lambda_{i+1}}{\lambda_i}\left(2 + \frac{3}{2}\frac{\lambda_i}{\lambda_{i+1}}\right)^p \geq \frac{3}{4}\cdot 4^p(\sqrt{2}-1)^p \geq \varepsilon^{-p}.$$

Formulae (a5), (a6) and (b5), (b6) are just consequences of (a4), (b4), respectively.

When the double knots of π^* are allowed, but $\tau = \tau_0$ is a simple knot of π^*, the proofs are analogous to those for simple knots. More precisely, (3.7), the property $|\xi_i| = (-1)^{|i|}\xi_i$ and inequalities (a1), (a3), (b1), (b3) follow from represenation (3.2) and the properties of the matrix A (i.e. the Gram matrix, cf. Proposition 2.1 of [10]) in the same way as in the case of simple knots (cf. Proposition 2.3 of [10] and its proof), and the remaining inequalities are just their consequences; if the intervals appearing on the right-hand sides of (a4)–(a6) and (b4)–(b6) degenerate to a single point, the corresponding inequality follows from the localization of the support of φ.

The localization of $\operatorname{supp}\varphi$ when double knots are allowed follows from the orthogonality conditions. More precisely, by orthogonality of φ to $S(\pi)$ we have $(\varphi, N_{\pi,j}) = 0$ for all $j \neq 0$. Note that $N_{\pi,j} = N_{\pi^*,j}$ for $j \leq -2$ and $j \geq 2$, $N_{\pi,-1} = N_{\pi^*,-1} + \frac{\lambda_1}{\lambda_0+\lambda_1}N_{\pi^*,0}$, $N_{\pi,1} = N_{\pi^*,1} + \frac{\lambda_0}{\lambda_0+\lambda_1}N_{\pi^*,0}$. Now, calculating $(N_{\pi^*,i}, N_{\pi,j})$, we find that the orthogonality conditions take the following form:

$$\xi_{j-1}\lambda_j + 2\xi_j(\lambda_j + \lambda_{j+1}) + \xi_{j+1}\lambda_{j+1} = 0 \quad \text{for } j \leq -2 \text{ and } j \geq 2,$$

for $j = -1$:

$$\xi_{-2}\lambda_{-1} + \xi_{-1}\left(2(\lambda_{-1} + \lambda_0) + \frac{\lambda_0\lambda_1}{\lambda_0 + \lambda_1}\right) + \xi_0(\lambda_0 + 2\lambda_1) + \xi_1\frac{\lambda_1^2}{\lambda_0 + \lambda_1} = 0,$$

and for $j = 1$:

$$\xi_{-1}\frac{\lambda_0^2}{\lambda_0+\lambda_1} + \xi_0(2\lambda_0+\lambda_1) + \xi_1\left(2(\lambda_1+\lambda_2) + \frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\right) + \xi_2\lambda_2 = 0.$$

If $\lambda_i = 0$ for some $i \le -1$, then the equations with $-k \le j \le i-1$ contain only the variables $\xi_{-k},\dots,\xi_{i-1}$; as the matrix of this subsystem is the Gram matrix $[(N_{\pi,j_1}, N_{\pi,j_2}), -k \le j_1, j_2 \le i-1]$, we get $\xi_{-k} = \dots = \xi_{i-1} = 0$, $\varphi = \sum_{j=i}^{l} \xi_i N_{\pi^*,j}$ and consequently $\operatorname{supp}\varphi \subset [\tau_i, 1]$. The case $\lambda_i = 0$ for some $i \ge 2$ is analogous.

Finally, consider the case when $\tau = \tau_0 = \tau_1$ is a double knot of π^* (other double knots of π^* are also allowed). Then (3.8), the property $|\xi_i| = (-1)^{|i|}\xi_i$ and inequalities (a1), (a3), (b1) (b3) (for the appropriate range of indices) follow from representation (3.6) and the properties of the matrix A (cf. Proposition 2.1 of [10]) in the same way as they follow from representation (3.2) in the case when τ is a simple knot in π^*. Then the remaining properties (inequalities and localization of supports) are checked in the same way as in the case when τ is a simple knot. ∎

In what follows, we need some more estimates, in terms of the interval J. Before formulating Proposition 3.2, we introduce additional notation. For $x, y \in [0,1]$, we denote by $d_{\pi^*}(x,y)$ the number of points of π^* between x and y, counting multiplicities, i.e.

$$d_{\pi^*}(x,y) = \#\{i : x \wedge y \le \tau_i \le x \vee y\}.$$

By $d_{\pi^*}(x)$ we denote the number of points of π^* between x and J, counting multiplicities and endpoints of J, with the understanding that $d_{\pi^*}(x) = 0$ when $x \in J$. Similarly, for an interval $V \subset [0,1]$, by $d_{\pi^*}(V)$ we denote the number of points of π^* between V and J, counting multiplicities and endpoints of J or V, with the understanding that $d_{\pi^*}(V) = 0$ whenever $V \cap J \ne \emptyset$.

PROPOSITION 3.2. *Let $\pi^* = \pi \cup \{\tau_0\}$ be as described above, and let φ be the general Franklin function corresponding to (π, π^*), $\varphi = \sum_{i=-k}^{l} \xi_i N_{\pi^*,i}$. Then there is a constant $C > 0$, independent of π, π^*, such that*

$$(3.10)\qquad |\xi_i| \le C\left(\frac{2}{3}\right)^{d_{\pi^*}(\tau_i)} \frac{|J|^{1/2}}{|J| + \operatorname{dist}(\tau_i, J) + \tau_{i+1} - \tau_{i-1}} \quad \textit{for all } i.$$

Moreover, for each p, $1 \le p < \infty$, there is a constant C_p such that if x is to the left of J then

$$(3.11)\qquad \int_0^x |\varphi(t)|^p\,dt \le C_p\left(\frac{2}{3}\right)^{pd_{\pi^*}(x)} \frac{|J|^{p/2}}{(|J| + \operatorname{dist}(x,J))^{p-1}},$$

and if x is to the right of J then

$$\int_x^1 |\varphi(t)|^p\,dt \le C_p\left(\frac{2}{3}\right)^{p\,d_{\pi^*}(x)} \frac{|J|^{p/2}}{(|J|+\operatorname{dist}(x,J))^{p-1}}. \tag{3.12}$$

In addition,

$$\|\varphi\|_{L^p(J)} \sim \|\varphi\|_p \sim |J|^{1/p-1/2}, \qquad 1 \le p \le \infty, \tag{3.13}$$

with the implied constants independent of p, π, π^*.

REMARK. For comparison with (3.11), (3.12), it follows from (3.10) and the linearity of φ on each (τ_k, τ_{k+1}) that

$$|\varphi(x)| \le C\left(\frac{2}{3}\right)^{d_{\pi^*}(x)} \frac{|J|^{1/2}}{|J|+\operatorname{dist}(x,J)}. \tag{3.14}$$

Proof of Proposition 3.2. We present the proof in the case when $\tau = \tau_0$ is a simple knot of π^*. If $\tau = \tau_0 = \tau_1$ is a double knot of π^*, then the proof is analogous, but one should use the appropriate definition of J and the corresponding part of Proposition 3.1.

It should be clear that $|i| - 2 \le d_{\pi^*}(\tau_i) \le |i| + 2$.

We start by checking (3.10), and this is done by considering the possible choices of I^* and J. Recall that $\lambda_i = \tau_i - \tau_{i-1}$.

CASE 1: $J = [\tau_{-2}, \tau_{-1}]$. Then $I^* = [\tau_{-2}, \tau_0]$ is the shortest of I^-, I, I^+, and by the definition of I^* and J, $\lambda_{-1} \ge \lambda_0$ and $\lambda_{-1} \le \lambda_1$. Consequently, for $i = -1, 0, 1$ we have

$$\begin{aligned}
|J| + \operatorname{dist}(\tau_{-1}, J) + \tau_0 - \tau_{-2} &= 2\lambda_{-1} + \lambda_0 \sim \lambda_0 + \lambda_{-1} = \nu^-,\\
|J| + \operatorname{dist}(\tau_0, J) + \tau_1 - \tau_{-1} &= \lambda_{-1} + 2\lambda_0 + \lambda_1 \sim \lambda_1 + \lambda_0 = \nu,\\
|J| + \operatorname{dist}(\tau_1, J) + \tau_2 - \tau_0 &= \lambda_{-1} + \lambda_0 + 2\lambda_1 + \lambda_2 \sim \lambda_2 + \lambda_1 = \nu^+.
\end{aligned}$$

These equivalences combined with (3.7) give (3.10) for $i = -1, 0, 1$.

For $i < -1$ we have

$$\begin{aligned}
|J| + \operatorname{dist}(\tau_i, J) + \tau_{i+1} - \tau_{i-1} &= (\tau_{-1} - \tau_{-2}) + (\tau_{-2} - \tau_i) + (\tau_{i+1} - \tau_{i-1})\\
&\sim \tau_{-1} - \tau_{i-1},
\end{aligned}$$

which together with (a2) from Proposition 3.1 and (3.5) implies (3.10) for $i < -1$.

For $i > 1$ we have

$$\begin{aligned}
|J| + \operatorname{dist}(\tau_i, J) + \tau_{i+1} - \tau_{i-1} &= (\tau_{-1} - \tau_{-2}) + (\tau_i - \tau_{-1}) + (\tau_{i+1} - \tau_{i-1})\\
&\sim \tau_{i+1} - \tau_{-2} \sim \tau_{i+1} - \tau_0.
\end{aligned}$$

Using this, (b2) of Proposition 3.1, (3.5) and (3.9) with $a = \tau_2 - \tau_1$, $b = \tau_{i+1} - \tau_2$ and $c = \tau_1 - \tau_0$, we get (3.10) for $i > 1$.

CASE 2: $J = [\tau_{-1}, \tau_0]$. Then either $I^* = [\tau_{-2}, \tau_0]$ or $I^* = [\tau_{-1}, \tau_1]$. If $I^* = [\tau_{-2}, \tau_0]$, then I^- is the shortest of I^-, I, I^+, $\lambda_0 \ge \lambda_{-1}$, $\lambda_{-1} \le \lambda_1$ and

$\lambda_0 \le \lambda_1 + \lambda_2$. If $I^* = [\tau_{-1}, \tau_1]$, then I is the shortest of I^-, I, I^+, $\lambda_0 \ge \lambda_1$, $\lambda_{-1} \ge \lambda_1$ and $\lambda_0 \le \lambda_2$. Considering each of these cases separately, similarly to the previous case, we find that for $i = -1, 0, 1$,

$$|J| + \operatorname{dist}(\tau_{-1}, J) + \tau_0 - \tau_{-2} \sim \tau_0 - \tau_{-2} = \nu^-,$$
$$|J| + \operatorname{dist}(\tau_0, J) + \tau_1 - \tau_{-1} \sim \tau_1 - \tau_{-1} = \nu,$$
$$|J| + \operatorname{dist}(\tau_1, J) + \tau_2 - \tau_0 = (\tau_1 - \tau_{-1}) + (\tau_2 - \tau_0) \sim \tau_2 - \tau_0 = \nu^+,$$

which implies (3.10) for $i = -1, 0, 1$.

For $i < -1$ we have

$$|J| + \operatorname{dist}(\tau_i, J) + \tau_{i+1} - \tau_{i-1} = (\tau_0 - \tau_{-1}) + (\tau_{-1} - \tau_i) + (\tau_{i+1} - \tau_{i-1}) \sim \tau_0 - \tau_{i-1},$$

and (3.10) follows from (a2) of Proposition 3.1 and (3.9) with $a = \tau_{-1} - \tau_{-2}$, $b = \tau_0 - \tau_{-1}$ and $c = \tau_{-2} - \tau_{i-1}$.

For $i > 1$ we have

$$|J| + \operatorname{dist}(\tau_i, J) + \tau_{i+1} - \tau_{i-1} = (\tau_0 - \tau_{-1}) + (\tau_i - \tau_0) + (\tau_{i+1} - \tau_{i-1}) \sim \tau_{i+1} - \tau_{-1} \sim \tau_{i+1} - \tau_0,$$

and (3.10) follows from (b2) of Proposition 3.1 and (3.9) with $a = \tau_2 - \tau_1$, $b = \tau_{i+1} - \tau_2$ and $c = \tau_1 - \tau_0$.

The remaining cases $J = [\tau_0, \tau_1]$ and $J = [\tau_1, \tau_2]$ are treated analogously.

Now, we turn to the proof of (3.11). Take x to the left of J, and let i be such that $\tau_{i-1} < x \le \tau_i$. If $i \le -1$, then $|J| + \operatorname{dist}(x, J) \le |J| + \operatorname{dist}(\tau_i, J) + \lambda_i$. Using this inequality, (a5) and (a1) of Proposition 3.1, linearity of φ and (3.10) we get

$$\begin{aligned}\int_0^x |\varphi(t)|^p\,dt &\le C_p \int_{\tau_{i-1}}^{\tau_i} |\varphi(t)|^p\,dt \;\le\; C_p \lambda_i |\xi_i|^p \\ &\le C_p \left(\frac{2}{3}\right)^{p\cdot d_{\pi^*}(\tau_i)} \frac{\lambda_i |J|^{p/2}}{(|J| + \operatorname{dist}(\tau_i, J) + \lambda_{i+1} + \lambda_i)^p} \\ &\le C_p \left(\frac{2}{3}\right)^{p\cdot d_{\pi^*}(x)} \frac{|J|^{p/2}}{(|J| + \operatorname{dist}(x, J))^{p-1}}.\end{aligned}$$

Now, let $i > -1$; since x is to the left of J, we have $i \le 1$, and consequently $0 \le d_{\pi^*}(x) \le 2$. Consider the case $i = 0$. As x is to the left of J, we must have $J = [\tau_0, \tau_1]$ or $J = [\tau_1, \tau_2]$. In both cases it follows by (3.10) that

$$|\xi_{-1}|, |\xi_0| \le C \frac{|J|^{1/2}}{|J| + \operatorname{dist}(\tau_{-1}, J)}.$$

Since $\operatorname{dist}(\tau_{-1}, J) \geq \lambda_0$, using (3.11) for τ_{-1} we get

$$\int_0^x |\varphi(t)|^p\,dt \leq \int_0^{\tau_{-1}} |\varphi(t)|^p\,dt + \int_{\tau_{-1}}^{\tau_0} |\varphi(t)|^p\,dt$$
$$\leq C_p \frac{|J|^{p/2}}{(|J| + \operatorname{dist}(\tau_{-1}, J))^{p-1}} + \lambda_0 \max(|\xi_{-1}|^p, |\xi_0|^p)$$
$$\leq C_p \frac{|J|^{p/2}}{(|J| + \operatorname{dist}(\tau_{-1}, J))^{p-1}} \leq C_p \frac{|J|^{p/2}}{(|J| + \operatorname{dist}(x, J))^{p-1}}.$$

The remaining case $i = 1$ is treated similarly.

Inequality (3.12) follows by analogous arguments.

To check (3.13), note that $\sup_{t\in J} |\varphi(t)| \sim |J|^{-1/2}$. As φ is linear on J, this implies (3.13). ■

COROLLARY 3.3. *Let $\pi^* = \pi \cup \{\tau_0\}$ be as described above, and let φ be the general Franklin function corresponding to (π, π^*). Let $\chi_{J,2} = \chi_J/|J|^{1/2}$. Then there is a constant $C > 0$, independent of π, π^*, such that*

$$|\varphi(t)| \leq C\mathcal{M}\chi_{J,2}(t), \qquad |\chi_{J,2}(t)| \leq C\mathcal{M}\varphi(t).$$

Proof. The second inequality follows from (3.13) with $p = 1$: for $t \in J$,

$$\mathcal{M}\varphi(t) \geq \frac{1}{|J|} \int_J |\varphi(u)|\,du \geq C|J|^{-1/2} = C\chi_{J,2}(t).$$

Since $\mathcal{M}\chi_{J,2}(t) \sim |J|^{1/2}/(|J| + \operatorname{dist}(t, J))$, (3.14) implies the first inequality. ■

3.2. Properties of a general Franklin system. Let $\mathcal{T} = (t_n, n \geq 0)$ be an admissible sequence of points with the corresponding Franklin system $\{f_n, n \geq 0\}$. By I_n, I_n^*, J_n, μ_n, d_n etc. we denote the intervals and quantities defined above for a general Franklin function and corresponding to the function f_n and the partition π_n. In addition, the points $t_n^{-,-}$, t_n^-, t_n^+, $t_n^{+,+}$ correspond to t_n and π_{n-1} in the same way as $\tau^{-,-}$, τ^-, τ^+, $\tau^{+,+}$ correspond to τ and π in Section 3.1.

LEMMA 3.4. *Let $\mathcal{T} = (t_n, n \geq 0)$ be an admissible sequence of points with the corresponding Franklin system $\{f_n, n \geq 0\}$. Let $k, l \geq 0$ be such that $t_k \leq t_l$ and there is no $i \leq \max(k, l)$ with $t_i \in (t_k, t_l)$. For all such k, l we have*

$$\#\{n : J_n = [t_k, t_l]\} \leq 5.$$

Proof. First, note that if $t_k = t_l$, then $\#\{n : J_n = [t_k, t_l]\} = 0$. Consider the case $t_k < t_l$. If $J_n = [t_k, t_l]$, then one of the following must happen:

(i) $n = \max(k, l)$,
(ii) $n > \max(k, l)$ and $t_n \geq t_l$,

(iii) $n > \max(k,l)$ and $t_n \le t_k$.

Clearly, there is at most one n satisfying (i).

As only double knots are allowed, there is at most one n satisfying (ii) with $t_n = t_l$.

Now, we check that there is at most one n satisfying (ii) with $t_l < t_n$. Note that in such a case t_n must be a simple knot of π_n. Suppose that there are n_1, n_2 with $\max(k,l) < n_1 < n_2$ and $t_{n_1}, t_{n_2} > t_l$. Then we must have $t_l < t_{n_2} < t_{n_1}$, $t_{n_2}^+ \le t_{n_1}$, $t_{n_1}^- = t_{n_2}^- = t_l$ and $t_{n_1}^{-,-} = t_{n_2}^{-,-} = t_k$. Hence, by the definition of J_{n_1}, $|[t_k,t_l]| \ge |[t_l,t_{n_1}]|$. As $t_{n_2} > t_l$, this implies $|[t_k,t_{n_2}]| > |[t_l,t_{n_2}^+]|$, and consequently $J_{n_2} \ne [t_k,t_l]$. Thus, there is at most one $n > \max(k,l)$ with $J_n = [t_k,t_l]$ and $t_n > t_l$.

By analogous considerations, there are at most two n's satisfying (iii) with $J_n = [t_k,t_l]$. ■

Lemma 3.5. *Let $\mathcal{T} = (t_n, n \ge 0)$ be an admissible sequence of points with the corresponding Franklin system $\{f_n, n \ge 0\}$. Let $k,l \ge 0$ be such that $t_k \le t_l$ and there is no $i \le \max(k,l)$ with $t_i \in (t_k,t_l)$. For all such k,l we have*

$$\#\{n : J_n \subset [t_k,t_l] \text{ and } |J_n| > |[t_k,t_l]|/2\} \le 25.$$

Proof. Set $\Delta = [t_k,t_l]$ and

$$\kappa = \max\{n \in \mathbb{N} : \#\{i \le n : t_i \in \Delta\} \le 5\}.$$

If $n \le \kappa$ and $J_n \subset \Delta$, then $\#\{i \le n : t_i \in \Delta\} = j$, where $2 \le j \le 5$. These j points define $j-1$ intervals, but only one of them can have length $> |\Delta|/2$. Therefore, by Lemma 3.4,

$$\#\{n \le \kappa : J_n \subset \Delta \text{ and } |J_n| > |\Delta|/2\} \le 20.$$

It remains to consider $n > \kappa$. For such n, if $J_n \subset \Delta$, then one of the following must be satisfied:

(I) $t_n = t_k$ or $t_n = t_l$,
(II) $t_k < t_n^-$ and $t_n^+ < t_l$,
(III) $t_k = t_n^- < t_n$.
(IV) $t_n < t_n^+ = t_l$.
(V) $t_n < t_k$.
(VI) $t_n > t_l$.

Let us count the number of $n \in \mathbb{N}$ satisfying (I)–(VI).

Case I. There are at most two $n > \kappa$ satisfying (I).

This follows immediately from the fact that at most double knots are allowed.

Case II. There is no n such that $n > \kappa$, $J_n \subset \Delta$, $|J_n| > |\Delta|/2$, $t_k < t_n^-$ and $t_n^+ < t_l$.

First, consider the case when t_n is a simple knot of π_n. For such n we would have $t_k \le t_n^{-,-}$ and $t_n^{+,+} \le t_l$, so $I_n^-, I_n^+ \subset \Delta$. Then $|I_n^-|, |I_n^+| \ge |I_n^*| \ge |J_n| > |\Delta|/2$, so we would have $|I_n^- \cup I_n^+| = |I_n^-| + |I_n^+| > |\Delta|$, which is impossible.

If t_n is a double knot of π_n we would have $I_n^-, I_n^+ \subset \Delta$, $|I_n^-|, |I_n^+| \ge |J_n| > |\Delta|/2$, and consequently $|I_n^- \cup I_n^+| > |\Delta|$, which is impossible.

Case III. There is at most one n such that $n > \kappa$, $J_n \subset \Delta$, $|J_n| > |\Delta|/2$ and $t_k = t_n^- < t_n$.

First, note that t_n must be a simple knot of π_n: if t_n is a double knot, then $I_n^-, I_n^+ \subset \Delta$, which is impossible by the same argument as in case II.

Since t_n is a simple knot of π_n, we have $t_n^{-,-} \le t_k$, so J_n is one of $[t_n^-, t_n]$, $[t_n, t_n^+]$, $[t_n^+, t_n^{+,+}]$. However, we must have $J_n = [t_n, t_n^+]$: note that $I_n, I_n^+ \subset \Delta$. If $J_n = [t_n^-, t_n]$, then $|I_n^+| \ge |I_n^*| \ge |J_n| > |\Delta|/2$, so we would have $|J_n \cup I_n^+| > |\Delta|$, which is impossible, because $J_n \cup I_n^+ \subset \Delta$; the case $J_n = [t_n^+, t_n^{+,+}]$ is eliminated by an analogous argument.

Suppose that there is another n' with the same properties; clearly, we may assume that $n' > n$. Then we have $J_{n'} = [t_{n'}, t_{n'}^+]$, and as $t_{n'}^- = t_k = t_n^-$, $J_{n'} \subset [t_n^-, t_n]$. Moreover, we have $|J_n|, |J_{n'}| > |\Delta|/2$, which implies $|[t_n^-, t_n^+]| \ge |J_{n'}| + |J_n| > |\Delta|$, and this is impossible, because $[t_n^-, t_n^+] \subset \Delta$.

Case IV. There is at most one n such that $n > \kappa$, $J_n \subset \Delta$, $|J_n| > |\Delta|/2$ and $t_n < t_n^+ = t_l$.

Case IV is considered analogously to case III.

Case V. There are at most two n such that $n > \kappa$, $J_n \subset \Delta$, $|J_n| > |\Delta|/2$ and $t_n < t_k$.

Note that if $n > \kappa$, $t_n < t_k$ and t_n is a double knot of π_n, then $J_n \subset [0, t_k]$. Therefore, if $n > \kappa$ with $t_n < t_k$ and $J_n \subset [t_k, t_l]$, then t_n is a simple knot of π_n.

Observe that if n satisfies the conditions of case V then we must have $t_n^+ = t_k < t_n^{+,+}$, $I_n^* = I_n^+ = [t_n, t_n^{+,+}]$ and $J_n = [t_n^+, t_n^{+,+}]$. Moreover, by the definitions of I_n^* and J_n we get

$$|[t_n^-, t_n]| \ge |J_n| \ge |[t_n, t_n^+]|. \tag{3.15}$$

Now, suppose that there are at least three indices $n < n' < n''$ satisfying the conditions of case V. As $t_n^+ = t_{n'}^+ = t_{n''}^+ = t_k$, we must have

$$\Delta \supset J_n \supset J_{n'} \supset J_{n''}, \quad [t_n, t_n^+] \supset [t_{n'}^-, t_{n'}^+], \quad [t_{n'}, t_{n'}^+] \supset [t_{n''}^-, t_{n''}^+]. \tag{3.16}$$

Using (3.16) and (3.15) for n and n' we get

$$|[t_{n'}, t_{n'}^+]| \le |[t_{n'}^-, t_{n'}^+]|/2 \le |[t_n, t_n^+]|/2 \le |J_n|/2 \le |\Delta|/2. \tag{3.17}$$

Using (3.15) and (3.16) for n' and n'' we get

$$|\varDelta|/2 < |J_{n''}| \le |[t^-_{n''}, t_{n''}]| \le |[t_{n'}, t^+_{n'}]|,$$

which contradicts (3.17).

Case VI. There are at most two n such that $n > \kappa$, $J_n \subset \varDelta$, $|J_n| > |\varDelta|/2$ and $t_n > t_l$.

Case VI is treated analogously to Case V.

To complete the proof, note that if there is $n > \kappa$ satisfying (III) or (V), then there is no $n > \kappa$ satisfying (IV) or (VI), and conversely. ∎

Lemma 3.5 has the following consequence:

Corollary 3.6. *Let $\mathcal{T} = (t_n, n \ge 0)$ be an admissible sequence of points with the corresponding Franklin system $\{f_n, n \ge 0\}$. Let $\{n_s, s \ge 1\}$ be a subsequence of* $\mathbb{N}$ *such that $J_{n_s} \supset J_{n_{s+1}}$. Then for each $\gamma > 0$,*

$$\sum_{s\ge 1} |J_{n_s}|^\gamma \sim |J_{n_1}|^\gamma, \qquad \sum_{s=1}^{m} |J_{n_s}|^{-\gamma} \sim |J_{n_m}|^{-\gamma},$$

with the implied constants depending on γ, but independent of $\mathcal{T}$ and of the sequence $\{n_s, s \ge 1\}$.

4. PROOFS OF THE RESULTS

Let $\mathcal{T}$ be a fixed admissible sequence of knots with the corresponding general Franklin system $\{f_n, n \ge 0\}$. For $f \in L^p[0,1]$, $f = \sum_{n=0}^\infty a_n f_n$, define

$$Pf(t) = \Big(\sum_{n=0}^\infty a_n^2 f_n^2(t)\Big)^{1/2}, \qquad S^* f(t) = \sup_{n\ge 0} \Big|\sum_{i=0}^n a_i f_i(t)\Big|.$$

4.1. Technical estimates. As already mentioned, Lemma 4.2 below is a variant of inequalities (63) from [11], with the splitting of the set of indices done with respect to the position of the interval J_n, and the proof presented below is an adaptation of the proof of (63) from [11] to our splitting.

For the proof of Lemma 4.2, the following known property of polynomials is needed:

Proposition 4.1. *Let $k \in \mathbb{N}$ and $0 < \varrho < 1$ be fixed. There is a constant $C = C_{k,\varrho}$, depending only on k and ϱ, such that for every interval $[a,b]$, set $A \subset [a,b]$ with $|A| \ge \varrho |[a,b]|$ and polynomial Q of degree k,*

$$\max_{t\in[a,b]} |Q(t)| \le C_{k,\varrho} \sup_{t\in A} |Q(t)|, \qquad \int_a^b |Q(t)|\,dt \le C_{k,\varrho} \int_A |Q(t)|\,dt.$$

LEMMA 4.2. *Let* $f = \sum_{n=0}^{\infty} a_n f_n$ *and* $\lambda > 0$. *Let*

$$E_\lambda = \{t \in (0,1) : Pf(t) > \lambda\},$$

and let $V = (\alpha, \beta)$ *be an interval such that* $\mathcal{M}\chi_{E_\lambda}(\alpha) \le 1/4$ *and* $\mathcal{M}\chi_{E_\lambda}(\beta) \le 1/4$. *Moreover*, *let*

$$\Gamma = \{n \in \mathbb{N} : J_n \subset V\}, \qquad \Lambda = N \setminus \Gamma.$$

Then there is a constant $C > 0$ *such that for all* f, λ *and* V *as above*,

$$\int_{V^c} \sum_{n\in\Gamma} |a_n f_n(t)|\, dt \le C \int_V \Big(\sum_{n\in\Gamma} |a_n f_n(t)|^2\Big)^{1/2} dt, \tag{4.1}$$

$$\Big(\sum_{n\in\Lambda} |a_n f_n(t)|^2\Big)^{1/2} \le C\lambda \quad \text{for } t \in V. \tag{4.2}$$

REMARK. It follows from the proof that the constant C in Lemma 4.2 does not depend on $\mathcal{T}$.

Proof of Lemma 4.2. Let us begin with the proof of (4.1). We are going to estimate $\int_\beta^1(\ldots)$; the remaining integral $\int_0^\alpha(\ldots)$ can be treated analogously.

Note that by (b5) of Proposition 3.1 and (3.13) of Proposition 3.2, for each $n \in \Gamma$,

$$\int_\beta^1 |a_n f_n(t)|\, dt \le C\varepsilon^{d_n(\beta)} \int_{J_n} |a_n f_n(t)|\, dt.$$

Let J_n^l be the left half of J_n. As f_n is linear on J_n, and $J_n^l \subset J_n$ and $|J_n^l| = |J_n|/2$, we have $\int_{J_n} |f_n(t)| dt \le C \int_{J_n^l} |f_n(t)|\, dt$ (cf. Proposition 4.1). This implies

$$\begin{aligned} \int_\beta^1 |a_n f_n(t)|\, dt &\le C\varepsilon^{d_n(\beta)} \int_{J_n^l} |a_n f_n(t)|\, dt \\ &\le C\varepsilon^{d_n(\beta)} \int_{J_n^l} \Big(\sum_{n\in\Gamma} |a_n f_n(t)|^2\Big)^{1/2} dt. \end{aligned} \tag{4.3}$$

Let

$$\Gamma_s = \{n \in \Gamma : d_n(\beta) = s\}.$$

Note that $\Gamma_0 = \emptyset$: if $d_n(\beta) = 0$ then $\beta \in J_n$, but by the definition of the set Γ we have $J_n \subset V$. If $n \in \Gamma_s$ with $s \ge 1$, then there are exactly s points between β and J_n. This implies that, for fixed s, the intervals J_n with $n \in \Gamma_s$ can be grouped into packets, with intervals from one packet having a common right endpoint, and with maximal intervals from different packets disjoint. Therefore, by Lemma 3.5, each point $t \ne t_i$ belongs to at most 25 intervals J_n^l for $n \in \Gamma_s$. Hence

$$\sum_{n\in\Gamma_s} \int_\beta^1 |a_n f_n(t)|\,dt \le C\varepsilon^s \sum_{n\in\Gamma_s} \int_{J_n^l} \Big(\sum_{n\in\Gamma} |a_n f_n(t)|^2\Big)^{1/2} dt$$
$$\le C\varepsilon^s \int_V \Big(\sum_{n\in\Gamma} |a_n f_n(t)|^2\Big)^{1/2} dt,$$

so summing over $s \ge 1$ we get

$$\sum_{n\in\Gamma} \int_\beta^1 |a_n f_n(t)|\,dt \le C \int_V \Big(\sum_{n\in\Gamma} |a_n f_n(t)|^2\Big)^{1/2} dt.$$

The corresponding integral over $[0,\alpha]$ can be treated analogously. This completes the proof of (4.1).

Now, let us turn to the proof of (4.2). Let

$$\Lambda' = \{n \in \Lambda : \#(\pi_n \cap V) \le 1\}, \quad \Lambda'' = \Lambda \setminus \Lambda'.$$

First, note that there is a constant C such that

$$\Big(\sum_{n\in\Lambda'} |a_n f_n(t)|^2\Big)^{1/2} \le C\lambda \quad \text{for all } t \in V. \tag{4.4}$$

To see this, let γ be the first point of the sequence $\mathcal{T}$ falling into V. Then $\sum_{n\in\Lambda'} |a_n f_n(t)|^2$ is a polynomial of degree 2 on both (α,γ) and (γ,β). Since $\mathcal{M}\chi_{E_\lambda}(\alpha) \le 1/4$ and $\mathcal{M}\chi_{E_\lambda}(\beta) \le 1/4$, we have $|E_\lambda^{\rm c} \cap [\alpha,\gamma]| \ge \frac{3}{4}|[\alpha,\gamma]|$ and $|E_\lambda^{\rm c} \cap [\gamma,\beta]| \ge \frac{3}{4}|[\gamma,\beta]|$. Since $Pf(t) \le \lambda$ on $E_\lambda^{\rm c}$, inequality (4.4) on both (α,γ) and (γ,β) follows from Proposition 4.1.

Now, let $n \in \Lambda''$. Then, by the definition of Γ, $J_n \not\subset V$, and $V \not\subset J_n$, as V contains at least two knots of π_n. Thus, $\Lambda'' = \Lambda^- \cup \Lambda^+$, where

$$\Lambda^- = \{n \in \Lambda'' : J_n \subset [0,\alpha] \text{ or } \alpha \in J_n\},$$
$$\Lambda^+ = \{n \in \Lambda'' : J_n \subset [\beta,1] \text{ or } \beta \in J_n\}.$$

Consider the set Λ^+. We define inductively a sequence of points β_n and an associated splitting of Λ^+. Let $n_1 = \min \Lambda^+$, take

$$\beta_1 \in \pi_{n_1} \quad \text{such that} \quad \beta_1 < \beta \quad \text{and} \quad (\beta_1,\beta) \cap \pi_{n_1} = \emptyset$$

(note that $\alpha < \beta_1$), and set

$$\Lambda_1^+ = \{n \in \Lambda^+ : (\beta_1,\beta) \cap \pi_n = \emptyset\}.$$

Then we take $n_2 = \min \Lambda^+ \setminus \Lambda_1^+$; note that $\#((\beta_1,\beta) \cap \pi_{n_2}) \ge 1$, so we take

$$\beta_2 \in \pi_{n_2} \quad \text{with} \quad \beta_1 < \beta_2 < \beta \quad \text{and} \quad (\beta_2,\beta) \cap \pi_{n_2} = \emptyset,$$

and set

$$\Lambda_2^+ = \{n \in \Lambda^+ \setminus \Lambda_1^+ : (\beta_2,\beta) \cap \pi_n = \emptyset\}.$$

Observe that $\beta_2 \in \pi_n$ for all $n \ge n_2$.

Having defined points $\beta_1, \ldots, \beta_k$ and sets $\Lambda_1^+, \ldots, \Lambda_k^+$, we put $n_{k+1} = \min \Lambda^+ \setminus \bigcup_{i=1}^k \Lambda_i^+$ and note that $\#((\beta_k, \beta) \cap \pi_{n_{k+1}}) \geq 1$. Then we take

$$\beta_{k+1} \in \pi_{n_{k+1}} \quad \text{with} \quad \beta_k < \beta_{k+1} < \beta \quad \text{and} \quad (\beta_{k+1}, \beta) \cap \pi_{n_{k+1}} = \emptyset,$$

and set

$$\Lambda_{k+1}^+ = \Big\{ n \in \Lambda^+ \setminus \bigcup_{i=1}^k \Lambda_i^+ : (\beta_{k+1}, \beta) \cap \pi_n = \emptyset \Big\}.$$

Note that $\beta_k \in \pi_n$ for all $n \geq n_k$, and if $n \in \Lambda_l^+$ with $l \geq k$, then $n \geq n_k$.

Put $h_k = \sum_{n \in \Lambda_k^+} a_n f_n$. Observe that for each m and $n \in \bigcup_{i=1}^m \Lambda_i^+$, f_n is linear on (β_m, β). Since $\mathcal{M}\chi_{E_\lambda}(\beta) \leq 1/4$, it follows from Proposition 4.1 that there is an absolute constant C such that

$$P\Big(\sum_{i=1}^m h_i \Big)(t) \leq C\lambda \quad \text{for } t \in (\beta_m, \beta). \tag{4.5}$$

Now, consider Ph_m on (β_{k-1}, β_k) with $k \leq m$. Then, for $n \in \Lambda_m^+$, we have the following possibilities:

(i) $t_n > \beta_m$,
(ii) $t_n = \beta_m$,
(iii) $t_n < \beta_m$.

Case (i). Note that in this case $\beta_m \leq t_n^-$. Let $\tilde{h}_m$ be the function corresponding to the part of the sum defining h_m with n satisfying (i), and further let $h_{m,k}$ be the function corresponding to the part of the sum defining h_m with n satisfying (i) and all $\beta_k, \ldots, \beta_m$ being simple knots of f_n. As all f_n's appearing in $h_{m,k}$ are continuous at β_m, it follows from (4.5) that $P(h_{m,k})(\beta_m) \leq C\lambda$.

Note that all β_i with $i \leq m$ are knots of f_n. If all $\beta_k, \ldots, \beta_m$ are simple knots for f_n, then, since $\beta_m \leq t_n^-$, by Proposition 3.1 (cf. (a6) and (a1)),

$$|f_n(t)| \leq |f_n(\beta_k)| \leq C\varepsilon^{m-k} |f_n(\beta_m)| \quad \text{for } t \leq \beta_k.$$

If one of $\beta_k, \ldots, \beta_m$ is a double knot, then, since in case (i) we have $t_n^- \geq \beta_m$, $f_n(t) = 0$ for $t < \beta_k$ (cf. Proposition 3.1). Therefore

$$P(\widetilde{h}_m)(t) = P(h_{m,k})(t) \leq C\varepsilon^{m-k} P(h_{m,k})(\beta_m) \leq C\varepsilon^{m-k}\lambda \quad \text{for } t < \beta_k. \tag{4.6}$$

Case (ii). As at most double knots are allowed, this situation can happen at most twice. For $n \in \Lambda_m^+$ there are no points from π_n in (β_m, β). Since either $J_n \subset [\beta, 1]$ or $\beta \in J_n$, we now have two possibilities:

(ii-a) $\beta_m = t_n$ is the left endpoint of J_n (t_n may be either a simple or a double knot of π_n); then $\beta \in J_n$, and since $\mathcal{M}\chi_{E_\lambda}(\beta) \leq 1/4$, by Proposition 4.1, there is an absolute constant C such that

$$|a_n f_n(t)| \leq C\lambda \quad \text{for } t \in \operatorname{int} J_n.$$

(ii-b) $\beta_m = t_n$ is a simple knot of π_n and t_n^+ is the left endpoint of J_n. In this case $\beta \le t_n^+$. Moreover, by the definitions of the intervals I^* and J we have $I_n^* = [t_n, t_n^+] \cup J_n$ and $|J_n| \ge |[t_n, t_n^+]|$, i.e. $|J_n| \le |I_n^*| \le 2|J_n|$. Therefore, also in this case, using $\mathcal{M}\chi_{E_\lambda}(\beta) \le 1/4$ and Proposition 4.1, we get for some absolute constant C,

$$|a_n f_n(t)| \le C\lambda \quad \text{for } t \in \operatorname{int} J_n.$$

Now, combining cases (ii-a) and (ii-b) with (a6) of Proposition 3.1 and (3.13) we get

$$|a_n f_n(t)| \le C\varepsilon^{m-k}\lambda \quad \text{for } t < \beta_k. \tag{4.7}$$

Case (iii). Denote by h_m^* the function corresponding to the part of the sum defining h_m with n satisfying (iii). If $n > n_m$ and $t_n < \beta_m$ is a double knot of π_n, then $J_n \subset [0, \beta_m]$. Thus, if $t_n < \beta_m$ and we are in case $n \in \Lambda_m^+$, then t_n is a simple knot of π_n; moreover, we must have $\beta_m = t_n^+$, and t_n^+ is the left endpoint of J_n. These positions of t_n and J_n imply that β_m is a simple knot of f_n, f_n is continuous at β_m and moreover $|f_n(t_n^+)| \sim \|f_n\|_\infty \sim |J_n|^{-1/2}$ (cf. (3.7) and (3.13)). This and (4.5) imply that $P(h_m^*)(\beta_m) \le C\lambda$. Moreover, by the decay of Franklin functions from Proposition 3.1,

$$|f_n(t)| \le C\varepsilon^{m-k}|f_n(\beta_m)| \quad \text{for } t \le \beta_k.$$

Combining these facts we get

$$P(h_m^*)(t) \le C\varepsilon^{m-k}P(h_m^*)(\beta_m) \le C\varepsilon^{m-k}\lambda \quad \text{for } t \le \beta_k. \tag{4.8}$$

Putting together cases (i)–(iii), i.e. inequalities (4.6)–(4.8) (recall that there are at most 2 n's in case (ii)) we get

$$Ph_m(t) \le C\varepsilon^{m-k}\lambda \quad \text{for } t < \beta_k. \tag{4.9}$$

Now, let $t \in (\beta_s, \beta_{s+1})$. Using (4.5) and (4.9) we get

$$\begin{aligned}\sum_{n\in\Lambda^+} |a_n f_n(t)|^2 &\le \Big(P\Big(\sum_{k\le s} h_k\Big)(t)\Big)^2 + \sum_{k\ge s+1} (Ph_k(t))^2 \\ &\le C\lambda^2 + \sum_{k\ge s+1} C\varepsilon^{2(k-s-1)}\lambda^2 \le C\lambda^2.\end{aligned}$$

A similar argument, with the use of (4.9) only, gives an analogous inequality for $t \in (\alpha, \beta_1)$, while for $t \in (\beta_{\max}, \beta)$ (where $\beta_{\max} = \sup_{k\ge1}\beta_k$) it is enough to use (4.5). Finally, by left-continuity of f_n's, the required inequality holds for the points β_s as well.

The sum $\sum_{n\in\Lambda^-} |a_n f_n(t)|^2$ is treated analogously, which completes the proof of inequality (4.2). ∎

Lemma 4.3. *Let* $V = (\alpha, \beta) \subset (0,1)$, $f \in L^p[0,1]$ *with* $\operatorname{supp} f \subset V$, $f = \sum_{n=0}^\infty a_n f_n$ *and* $1 < p < 2$. *Let* $\theta = \sqrt{\varepsilon}$, *where* $\varepsilon = (\sqrt{2}+1)/3$. *Then* *there is a constant M_p, depending only on p, such that*

$$\sum_{n=n(V)}^{\infty} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_n|^p \|f_n\|^p_{L^p(\widetilde{V}^c)} \le M_p \|f\|_p^p,$$

where $n(V) = \min\{n : \pi_n \cap V \neq \emptyset\}$ and $\widetilde{V} = (\widetilde{\alpha}, \widetilde{\beta})$ with $\widetilde{\alpha} = \alpha - 2|V|$, $\widetilde{\beta} = \beta + 2|V|$.

Proof. Let q denote the conjugate exponent, $1/p + 1/q = 1$. Note that $0 < \theta < 1$.

We give the estimates for the part corresponding to $\int_0^{\widetilde{\alpha}} |f_n(t)|\, dt$; the other part is treated analogously.

Let $m \ge 0$ be fixed, and consider the set

$$T_{l,m} = \{n \in \mathbb{N} : n \ge n(V),\ \#([\widetilde{\alpha}, \alpha] \cap \pi_n) = m\}.$$

More precisely, $\#([\widetilde{\alpha}, \alpha] \cap \pi_n)$ counts knots with multiplicities, i.e.

$$\#([\widetilde{\alpha}, \alpha] \cap \pi_n) = \#\{i \le n : \widetilde{\alpha} \le t_i \le \alpha\}.$$

The "l" in $T_{l,m}$ and $T_{l,m}^{(i)}$ below means that we consider splitting of the set of indices suitable for the estimate of the "left part", i.e. the part corresponding to $\int_0^{\widetilde{\alpha}} |f_n(t)|\, dt$.

To simplify notation, let $x_1 \le \dots \le x_m$ be the points from the partitions π_n with $n \in T_{l,m}$ contained in $[\widetilde{\alpha}, \alpha]$.

We give the estimate of $\sum_{n \in T_{l,m}} (\dots)$. For this, we split $T_{l,m}$ into several subsets, according to the position of J_n. Observe that $T_{l,m}$ is finite—this follows just from the density of $\mathcal{T}$ in $[0,1]$.

$$\begin{aligned}
T_{l,m}^{(1)} &= \{n \in T_{l,m} : J_n \subset [\widetilde{\alpha}, \alpha]\},\\
T_{l,m}^{(2)} &= \{n \in T_{l,m} : \widetilde{\alpha} \in J_n,\ |J_n \cap [\widetilde{\alpha}, \alpha]| \ge |V|,\ J_n \not\subset [\widetilde{\alpha}, \alpha]\},\\
T_{l,m}^{(3)} &= \{n \in T_{l,m} : J_n \subset [0, \widetilde{\alpha}],\ \text{or } \widetilde{\alpha} \in J_n \text{ with}\\
&\qquad |J_n \cap [\widetilde{\alpha}, \alpha]| \le |V| \text{ and } J_n \not\subset [\widetilde{\alpha}, \alpha]\},\\
T_{l,m}^{(4)} &= \{n \in T_{l,m} : \alpha \in J_n,\ |J_n \cap [\widetilde{\alpha}, \alpha]| \ge |V|,\ J_n \not\subset [\widetilde{\alpha}, \alpha]\},\\
T_{l,m}^{(5)} &= \{n \in T_{l,m} : J_n \subset [\alpha, \widetilde{\beta}],\ \text{or } \alpha \in J_n \text{ with}\\
&\qquad |J_n \cap [\widetilde{\alpha}, \alpha]| \le |V| \text{ and } J_n \not\subset [\widetilde{\alpha}, \alpha]\},\\
T_{l,m}^{(6)} &= \{n \in T_{l,m} : J_n \subset [\widetilde{\beta}, 1],\ \text{or } \widetilde{\beta} \in J_n \text{ with } J_n \not\subset [\alpha, \widetilde{\beta}]\}.
\end{aligned}$$

CASE 1: $n \in T_{l,m}^{(1)}$. First, note that this case can appear only for $m \ge 2$. Observe that

$$\#T_{l,m}^{(1)} \le 11. \tag{4.10}$$

In fact, only the intervals $[x_1, x_2]$ and $[x_{m-1}, x_m]$ can be J_n for some $t_n \neq x_1, \dots x_m$, and only one of $[x_{i-1}, x_i]$, $3 \le i \le m-1$, can be J_n for t_n equal to the x_i, $1 \le i \le m$, which has been added as the last one. Inequality (4.10) follows now from Lemma 3.4. (Note that some $[x_{i-1}, x_i]$ can be J_n for some $n \notin T_{l,m}$, but we do not count it here.)

Moreover, by (a5) and (b5) of Proposition 3.1, for $n \in T_{l,m}^{(1)}$ we have

$$\int_0^{\widetilde{\alpha}} |f_n(t)|^p \, dt \le C_p \varepsilon^{p d_n(\widetilde{\alpha})} \|f_n\|_p^p, \qquad \int_V |f_n(t)|^q \, dt \le C_p \varepsilon^{q d_n(V)} \|f_n\|_q^q.$$

In addition, $d_n(\widetilde{\alpha}) + d_n(V) = m$ and $d_n(V) \le m$. The inequality

$$|a_n|^p \le \int_V |f(t)|^p dt \cdot \Big(\int_V |f_n(t)|^q dt \Big)^{p/q} \tag{4.11}$$

and (4.10) give

$$\sum_{n \in T_{m,l}^{(1)}} \left(\frac{1}{\theta}\right)^{p d_n(V)} |a_n|^p \int_0^{\widetilde{\alpha}} |f_n(t)|^p \, dt \le C \theta^{mp} \|f\|_p^p. \tag{4.12}$$

Case 2: $n \in T_{l,m}^{(2)}$. In this case $d_n(V) = m$. Moreover, if $n_0 < n_1 < \cdots < n_s$ are all elements of $T_{l,m}^{(2)}$, then $J_{n_0} \supset J_{n_1} \supset \cdots \supset J_{n_s}$. By the estimates of $\|f_n\|_p$ and pointwise estimates from Proposition 3.1, for $n \in T_{l,m}^{(2)}$ we have

$$\|f_n\|_p^p \sim |J_n|^{1-p/2}, \qquad \int_V |f_n(t)|^q \, dt \le C_q \varepsilon^{qm} |V| \, |J_n|^{-q/2}.$$

This and (4.11) give

$$|a_n|^p \|f_n\|_p^p \le C_p \varepsilon^{pm} \|f\|_p^p |V|^{p-1} |J_n|^{1-p}.$$

By the definition of $T_{l,m}^{(2)}$, $|J_{n_s}| \ge |V|$, so it follows from Corollary 3.6 that

$$\sum_{n \in T_{l,m}^{(2)}} |J_n|^{1-p} \sim |J_{n_s}|^{1-p} \le C_p |V|^{1-p}.$$

This gives

$$\sum_{n \in T_{l,m}^{(2)}} \left(\frac{1}{\theta}\right)^{p d_n(V)} |a_n|^p \|f_n\|_p^p \le C_p \theta^{mp} \|f\|_p^p. \tag{4.13}$$

Case 3: $n \in T_{l,m}^{(3)}$. Let

$$\alpha^* = \max\{\widetilde{\alpha}, \text{ right endpoints of } J_n\text{'s}, \, n \in T_{l,m}^{(3)}\}.$$

More precisely, if there is $n \in T_{l,m}^{(3)}$ with x_1 being the right endpoint of J_n (and with $\widetilde{\alpha} \in J_n$), then $\alpha^* = x_1$, otherwise $\alpha^* = \widetilde{\alpha}$.

Then $|V| \le |[\alpha^*, \alpha]| \le 2|V|$, and for all $n \in T_{l,m}^{(3)}$ we have

$$J_n \subset [0, \alpha^*], \qquad \#([\alpha^*, \alpha] \cap \pi_n) = m.$$

Consequently, for $n \in T_{l,m}^{(3)}$, $d_n(V) = m + d_n(\alpha^*) - \zeta$, where $\zeta \in \{0,1,2\}$ ($\zeta = 0$ when α^* is not a knot of π_n, $\zeta = 1$ when α^* is a simple knot of π_n, and $\zeta = 2$ when α^* is a double knot of π_n).

First, given $n \in T_{l,m}^{(3)}$, we need to estimate $\sup_{t \in V} |f_n(t)|$. For this, let Δ be an interval of linearity of f_n (with endpoints in π_n) such that $\Delta \cap V \neq \emptyset$. Note that either both V and Δ are to the right of J_n, or $J_n = \Delta$ (the latter can happen only for $m = 0$), and $d_n(\Delta) \ge d_n(V)$. It follows from the position of Δ that

$$\operatorname{dist}(\alpha, J_n) \le \operatorname{dist}(J_n, \Delta) + |\Delta|.$$

Therefore by Proposition 3.2 (note that $\varepsilon > 2/3$)

$$\begin{aligned}\sup_{t \in \Delta} |f_n(t)| &\le C\varepsilon^{d_n(\Delta)} \frac{|J_n|^{1/2}}{|J_n| + \operatorname{dist}(J_n, \Delta) + |\Delta|} \\ &\le C\varepsilon^{m + d_n(\alpha^*)} \frac{|J_n|^{1/2}}{|J_n| + \operatorname{dist}(\alpha, J_n)}.\end{aligned}$$

This implies that

$$\sup_{t \in V} |f_n(t)| = \max_{\Delta : V \cap \Delta \neq \emptyset} \sup_{t \in \Delta} |f_n(t)| \le C\varepsilon^{m + d_n(\alpha^*)} \frac{|J_n|^{1/2}}{|J_n| + \operatorname{dist}(\alpha, J_n)}.$$

Consequently,

$$\int_V |f_n(t)|^q dt \le C_p |V| \varepsilon^{q(m + d_n(\alpha^*))} \frac{|J_n|^{q/2}}{(|J_n| + \operatorname{dist}(\alpha, J_n))^q}.$$

Since $\|f_n\|_p^p \sim |J_n|^{1-p/2}$ (cf. (3.13)), the last inequality and (4.11) give

$$\left(\frac{1}{\theta}\right)^{pd_n(V)} |a_n|^p \|f_n\|_p^p \le C_p \|f\|_p^p \theta^{p(m + d_n(\alpha^*))} \frac{|J_n| \cdot |V|^{p-1}}{(|J_n| + \operatorname{dist}(\alpha, J_n))^p}. \tag{4.14}$$

Let J_n^l denote the left half of J_n. For fixed k, consider all $n \in T_{l,m}^{(3)}$ with $d_n(\alpha^*) = k$. For $k = 0$, the conditions of Case 3 imply that α^* is the right endpoint of J_n, and these intervals form a nested family, that is, they can be ordered so that $J_{n_1} \supset \cdots \supset J_{n_s}$. For $k > 0$, observe that if $n_1 < n_2$ with $d_{n_1}(\alpha^*) = k = d_{n_2}(\alpha^*)$, then all points of the partition π_{n_1} are also in π_{n_2}. Therefore, the right endpoint of J_{n_2} either coincides with the right endpoint of J_{n_1} (which implies that $J_{n_2} \subset J_{n_1}$), or it lies between the right endpoint of J_{n_1} and α^* (in this case, also the left endpoint of J_{n_2} must be between the right endpoint of J_{n_1} and α^*). Therefore, by Lemma 3.5, each $t \neq t_j$ can belong only to 25 intervals J_n^l with fixed m, k and $d_n(\alpha^*) = k$. In addition, for $t \in J_n^l$ we have

$$|J_n| + \operatorname{dist}(\alpha, J_n) = |J_n^l| + \operatorname{dist}(\alpha, J_n^l) \sim |J_n| + |t - \alpha| \geq |t - \alpha|.$$

Recall that $J_n \subset [0, \alpha^*]$. Therefore

$$\begin{aligned}\sum_{n \in T_{l,m}^{(3)} : d_n(\alpha^*) = k} \frac{|J_n| \cdot |V|^{p-1}}{(|J_n| + \operatorname{dist}(\alpha, J_n))^p} &\leq C_p \sum_{n \in T_{l,m}^{(3)} : d_n(\alpha^*) = k} \int_{J_n^l} \frac{|V|^{p-1}}{|t - \alpha|^p}\, dt \\ &\leq C_p |V|^{p-1} \int_{-\infty}^{\alpha^*} \frac{1}{|t - \alpha|^p}\, dt \\ &\leq C_p \frac{|V|^{p-1}}{(\alpha - \alpha^*)^{p-1}} \leq C_p.\end{aligned}$$

Using this, (4.14) and summing over k we get

$$\sum_{n \in T_{l,m}^{(3)}} \left(\frac{1}{\theta}\right)^{p d_n(V)} |a_n|^p \|f_n\|_p^p \leq C \|f\|_p^p \theta^{pm}. \tag{4.15}$$

CASE 4: $n \in T_{l,m}^{(4)}$. Note that we can ignore the cases $m = 0$ or $m = 1$ and $[\widetilde{\alpha}, \alpha] \cap \pi_n = \{\alpha\}$, since these situations are covered by respective $T_{l,m}^{(2)}$. Now, we have $d_n(V) = 0$. Since there is at least one point of π_n in V and at least one in $[\widetilde{\alpha}, \alpha)$, and $|J_n \cap [\widetilde{\alpha}, \alpha]| \geq |V|$, we have $|V| \leq |J_n| \leq 3|V|$. As $\alpha \in J_n$, the intervals J_n, $n \in T_{l,m}^{(4)}$, form a nested family. Therefore, by Lemma 3.5 we have $\# T_{l,m}^{(4)} \leq 50$. Moreover, by (a5) of Proposition 3.1, in this case

$$\int_0^{\widetilde{\alpha}} |f_n(t)|^p dt \leq C_p \varepsilon^{pm} \|f_n\|_p^p.$$

Combining these observations with (4.11) and with the formulae for $\|f_n\|_p$ and $\|f_n\|_q$ (cf. Proposition 3.1) we get

$$\sum_{n \in T_{l,m}^{(4)}} \left(\frac{1}{\theta}\right)^{p d_n(V)} |a_n|^p \int_0^{\widetilde{\alpha}} |f_n(t)|^p\, dt \leq C_p \varepsilon^{pm} \|f\|_p^p. \tag{4.16}$$

CASE 5: $n \in T_{l,m}^{(5)}$. Similarly to Case 3, let

$$\alpha' = \min\{\alpha,\ \text{left endpoints of } J_n\text{'s},\ n \in T_{l,m}^{(5)}\}.$$

Note that if there is $n \in T_{l,m}^{(5)}$ having x_m as the left endpoint of J_n, then $\alpha' = x_m$, otherwise $\alpha' = \alpha$. Then $|V| \leq |[\widetilde{\alpha}, \alpha']| \leq 2|V|$, and for all $n \in T_{l,m}^{(5)}$

we have

$$J_n \subset [\alpha', \widetilde{\beta}], \qquad \#([\widetilde{\alpha}, \alpha'] \cap \pi_n) = m.$$

Therefore, for $n \in T_{l,m}^{(5)}$, $d_n(\widetilde{\alpha}) = m + d_n(\alpha') - \zeta'$, where $\zeta' \in \{0,1,2\}$ and depends on the multiplicity of α' as a knot of π_n. In addition, $d_n(V) \le d_n(\alpha')$ and $|J_n| \le |[\alpha', \widetilde{\beta}]| \le 4|V|$.

Note that $\widetilde{\alpha}$ is now to the left of J_n. Moreover, $\operatorname{dist}(\widetilde{\alpha}, J_n) \ge |[\widetilde{\alpha}, \alpha']|$, which implies $|J_n| + \operatorname{dist}(\widetilde{\alpha}, J_n) \sim |V|$. Therefore, it follows from (3.11) that

$$\int_0^{\widetilde{\alpha}} |f_n(t)|^p dt \le C_p \left(\frac{2}{3}\right)^{p(m+d_n(\alpha'))} \frac{|J_n|^{p/2}}{|V|^{p-1}}. \tag{4.17}$$

For each $n \in T_{l,m}^{(5)}$, we decompose $[\alpha', \widetilde{\beta}]$ into a union of three disjoint intervals V_n^-, J_n, V_n^+ (V_n^-, V_n^+ are respectively the left and right parts of $[\alpha', \widetilde{\beta}] \setminus J_n$). Set

$$a_{n,1} = \int_{V_n^-} f(t) f_n(t)\, dt, \quad a_{n,2} = \int_{J_n} f(t) f_n(t)\, dt, \quad a_{n,3} = \int_{V_n^+} f(t) f_n(t)\, dt.$$

(Since $\operatorname{supp} f \subset [\alpha, \beta]$, it would be enough to consider the splitting of $V \setminus J_n$, but this would require more careful notation in what follows; thus, we choose the above splitting to simplify the notation.)

Let us start with the estimate of the part corresponding to $a_{n,2}$. Note that

$$|a_{n,2}|^p \le \|f_n\|_q^p \int_{J_n} |f(t)|^p\, dt.$$

For fixed k, consider $n \in T_{l,m}^{(5)}$ with $d_n(\alpha') = k$. Recall that if $n_1 < n_2$, then all points of the partition π_{n_1} are also in π_{n_2}. Therefore, for fixed k, the indices $n \in T_{l,m}^{(5)}$ with $d_n(\alpha') = k$ can be joined into packets, with the intervals J_n from one packet having a common left endpoint, and with maximal intervals from different packets disjoint. Note that the intervals from one packet form a nested family of intervals. Now, let J_{n_0} be one of these maximal intervals. Then, using (4.17) (recall that $\varepsilon > 2/3$) and Corollary 3.6 we get

$$\begin{aligned}
&\sum_{n \in T_{l,m}^{(5)},\, d_n(\alpha')=k,\, J_n \subset J_{n_0}} \left(\frac{1}{\theta}\right)^{p d_n(V)} |a_{n,2}|^p \int_0^{\widetilde{\alpha}} |f_n(t)|^p\, dt \\
&\quad \le \sum_{n \in T_{l,m}^{(5)},\, d_n(\alpha')=k,\, J_n \subset J_{n_0}} \left(\frac{1}{\theta}\right)^{pk} \int_{J_n} |f(t)|^p\, dt\, \|f_n\|_q^p \int_0^{\widetilde{\alpha}} |f_n(t)|^p\, dt \\
&\quad \le C_p \theta^{pk} \varepsilon^{pm} \int_{J_{n_0}} |f(t)|^p\, dt \sum_{n \in T_{l,m}^{(5)},\, d_n(\alpha')=k,\, J_n \subset J_{n_0}} \frac{|J_n|^{p/2} \cdot |J_n|^{p/2-1}}{|V|^{p-1}}
\end{aligned}$$

$$\leq C_p \theta^{pk} \varepsilon^{pm} \int_{J_{n_0}} |f(t)|^p \, dt \cdot \frac{|J_{n_0}|^{p-1}}{|V|^{p-1}}$$

$$\leq C_p \theta^{pk} \varepsilon^{pm} \int_{J_{n_0}} |f(t)|^p \, dt.$$

Summing over maximal intervals we get

$$\sum_{n \in T_{l,m}^{(5)},\, d_n(\alpha')=k} \Big(\frac{1}{\theta}\Big)^{pd_n(V)} |a_{n,2}|^p \int_0^{\widetilde{\alpha}} |f_n(t)|^p \, dt \leq C_p \theta^{pk} \varepsilon^{pm} \|f\|_p^p,$$

and summing over k yields

$$\sum_{n \in T_{l,m}^{(5)}} \Big(\frac{1}{\theta}\Big)^{pd_n(V)} |a_{n,2}|^p \int_0^{\widetilde{\alpha}} |f_n(t)|^p \, dt \leq C_p \varepsilon^{pm} \|f\|_p^p. \tag{4.18}$$

Now, we turn to the part corresponding to $a_{n,3}$. For fixed k, let $n_{k,j}$, $j \geq 1$, be the subsequence of all n's with $n \in T_{l,m}^{(5)}$ and $d_n(\alpha') = k$ (arranged in increasing order). Observe that if $n_1 < n_2$ are two such n's, then all knots of π_{n_1} are also knots of π_{n_2}, and either the left endpoint of J_{n_2} coincides with the left endpoint of J_{n_1} (in this case, $J_{n_2} \subset J_{n_1}$ and the right endpoint of J_{n_2} is in J_{n_1}—it may coincide with the right endpoint of J_{n_1}, but by Lemma 3.4, the number of such n's is at most 5), or the left endpoint of J_{n_2} is between α' and the left endpoint of J_{n_1} (in this case, also the right endpoint of J_{n_2} is between α' and the left endpoint of J_{n_1}).

Let $\gamma_{n_{k,j}}$ be the right endpoint of $J_{n_{k,j}}$, and in addition, let $\gamma_{n_{k,0}} = \widetilde{\beta}$. Note that $\gamma_{n_{k,i}}$ is a point of the partition $\pi_{n_{k,j}}$ for all $j \geq i \geq 1$, $\gamma_{n_{k,j+1}} \leq \gamma_{n_{k,j}}$ and $d_{n_{k,j}}(\gamma_{n_{k,i}}) \geq (j-i-5)/5$. Therefore for $j \geq i$, by Proposition 3.1(a5),

$$\int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} |f_{n_{k,j}}(t)|^q \, dt \leq C_q \varepsilon^{q d_{n_{k,j}}(\gamma_{n_{k,i}})} \|f_{n_{k,j}}\|_q^q \leq C_q \varepsilon^{q(j-i)/5} \|f_{n_{k,j}}\|_q^q.$$

Using this and the Hölder inequality we get, with $\kappa = \varepsilon^{1/10}$,

$$|a_{n_{k,j},3}|^p = \Big| \int_{\gamma_{n_{k,j}}}^{\beta} f(t) f_{n_{k,j}}(t) \, dt \Big|^p = \Big| \sum_{i=1}^{j} \kappa^{(j-i)} \kappa^{(i-j)} \int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} f(t) f_{n_{k,j}}(t) \, dt \Big|^p$$

$$\leq \Big(\sum_{i=1}^{j} \kappa^{q(j-i)} \Big)^{p/q} \cdot \sum_{i=1}^{j} \kappa^{p(i-j)} \Big| \int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} f(t) f_{n_{k,j}}(t) \, dt \Big|^p$$

$$\leq C_p \sum_{i=1}^{j} \kappa^{p(i-j)} \int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} |f(t)|^p \, dt \cdot \Big(\int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} |f_{n_{k,j}}(t)|^q \, dt \Big)^{p/q}$$

$$\le C_p \sum_{i=1}^{j} \kappa^{p(i-j)} \int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} |f(t)|^p\,dt \cdot \kappa^{2p(j-i)} \|f_{n_{k,j}}\|_q^p$$

$$= C_p \sum_{i=1}^{j} \kappa^{p(j-i)} \|f_{n_{k,j}}\|_q^p \int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} |f(t)|^p\,dt.$$

Recall that by Proposition 3.1(a5),

$$\int_0^{\widetilde{\alpha}} |f_{n_{k,j}}(t)|^p dt \le C_p \varepsilon^{p d_{n_{k,j}}(\widetilde{\alpha})} \|f_{n_{k,j}}\|_p^p \le C_p \varepsilon^{p(m+d_{n_{k,j}}(\alpha'))} \|f_{n_{k,j}}\|_p^p.$$

Combining these estimates, for fixed k we get

$$\sum_{n\in T_{l,m}^{(5)},\, d_n(\alpha')=k} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_{n,3}|^p \|f_n\|_{L^p(0,\widetilde{\alpha})}^p \le \sum_{j\ge 1} \left(\frac{1}{\theta}\right)^{pk} |a_{n_{k,j},3}|^p \|f_{n_{k,j}}\|_{L^p(0,\widetilde{\alpha})}^p$$

$$\le C_p \sum_{j\ge 1} \varepsilon^{pm} \theta^{pk} \|f_{n_{k,j}}\|_p^p \|f_{n_{k,j}}\|_q^p \sum_{i=1}^{j} \kappa^{p(j-i)} \int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} |f(t)|^p\,dt$$

$$\le C_p \varepsilon^{pm} \theta^{pk} \sum_{i\ge 1} \int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} |f(t)|^p\,dt \sum_{j\ge i} \kappa^{p(j-i)}$$

$$\le C_p \varepsilon^{pm} \theta^{pk} \sum_{i\ge 1} \int_{\gamma_{n_{k,i}}}^{\gamma_{n_{k,i-1}}} |f(t)|^p dt \le C_p \varepsilon^{pm} \theta^{pk} \|f\|_p^p.$$

Summing over $k \ge 0$ we get

$$\sum_{n\in T_{l,m}^{(5)}} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_{n,3}|^p \|f_n\|_{L^p(0,\widetilde{\alpha})}^p \le C_p \varepsilon^{pm} \|f\|_p^p. \tag{4.19}$$

It remains to estimate the part corresponding to $a_{n,1}$.

For fixed k and n with $d_n(\alpha') = k$, let $L_{0,n}, \ldots, L_{k,n}$ be the intervals of linearity of f_n contained between α' and J_n; in the case of double knots some of them may degenerate to a single point, or f_n may be 0 on some of these intervals. Note that $L_{0,n}$ has α' as its left endpoint, and if α' is not a point of π_n, then $L_{0,n}$ is not an interval of the partition π_n.

Define

$$b_{i,n} = \int_{L_{i,n}} f(t) f_n(t)\,dt.$$

Clearly, for n such that $d_n(\alpha) = k$ we have $a_{n,1} = \sum_{i=0}^{k} b_{i,k}$. By the Hölder inequality,

$$|b_{i,n}|^p = \Big| \int\limits_{L_{i,n}} f(t) f_n(t)\, dt \Big|^p \le \int\limits_{L_{i,n}} |f(t)|^p\, dt \Big(\int\limits_{L_{i,n}} |f_n(t)|^q dt \Big)^{p/q}.$$

By Proposition 3.2 (more precisely, directly by Proposition 3.2 in the case of $i \ge 1$, and by considering $\widetilde{L}_{0,n}$, the interval of linearity of f_n containing $L_{0,n}$, in case $i = 0$) we have

$$\sup_{t \in L_{i,n}} |f_n(t)| \le C \Big(\frac{2}{3}\Big)^{k-i} \frac{|J_n|^{1/2}}{|J_n| + |L_{i,n}| + \operatorname{dist}(J_n, L_{i,n})},$$

which implies

$$|b_{i,n}|^p \le C_p \Big(\frac{2}{3}\Big)^{p(k-i)} \int\limits_{L_{i,n}} |f(t)|^p\, dt \, \frac{|J_n|^{p/2} \cdot |L_{i,n}|^{p-1}}{(|J_n| + |L_{i,n}| + \operatorname{dist}(J_n, L_{i,n}))^p}. \tag{4.20}$$

Observe that

$$\begin{aligned} &\frac{|J_n|^{p/2} |L_{i,n}|^{p-1}}{(|L_{i,n}| + |J_n| + \operatorname{dist}(L_{i,n}, J_n))^p} \cdot \frac{|J_n|^{p/2}}{|V|^{p-1}} \\ &\qquad \le \frac{|J_n|}{|V|^{p-1}} \, \frac{(|L_{i,n}| + |J_n|)^{2(p-1)}}{(|L_{i,n}| + |J_n| + \operatorname{dist}(L_{i,n}, J_n))^p} \\ &\qquad \le \frac{|J_n|}{|V|^{p-1}} \, (|L_{i,n}| + |J_n| + \operatorname{dist}(L_{i,n}, J_n))^{p-2}. \end{aligned}$$

Combining this observation with (4.17) and (4.20) we get

$$\begin{aligned} &\Big(\frac{1}{\theta}\Big)^{p d_n(V)} |b_{i,n}|^p \int\limits_0^{\widetilde{\alpha}} |f_n(t)|^p\, dt \\ &\qquad \le C_p \Big(\frac{1}{\theta}\Big)^{pk} \Big(\frac{2}{3}\Big)^{p(k-i)} \Big(\frac{2}{3}\Big)^{p(m+k)} \int\limits_{L_{i,n}} |f(t)|^p\, dt \, \frac{|J_n|}{|V|^{p-1}} \\ &\qquad\quad \times (|L_{i,n}| + |J_n| + \operatorname{dist}(L_{i,n}, J_n))^{p-2} \\ &\qquad \le C_p \theta^{pk} \Big(\frac{2}{3}\Big)^{p(k-i)} \varepsilon^{pm} \int\limits_{L_{i,n}} |f(t)|^p\, dt \, \frac{|J_n|}{|V|^{p-1}} \\ &\qquad\quad \times (|L_{i,n}| + |J_n| + \operatorname{dist}(L_{i,n}, J_n))^{p-2}. \end{aligned}$$

For fixed k and i, consider the intervals $L_{i,n}$ for n satisfying $d_n(\alpha') = k$; observe that these intervals can be grouped into packets such that the intervals in one packet have a common left endpoint, and maximal intervals from different packets are disjoint. In addition, for $L_{i,n}$'s from one packet, the corresponding J_n's can again be grouped into subpackets with coinciding left endpoint (hence forming a family of nested intervals, i.e. they can be arranged so that $J_{n_1} \supset \cdots \supset J_{n_s}$), and with maximal intervals from different subpackets disjoint. Denoting by J_n^r the right half of J_n we note that by Lemma 3.5 each $t \neq t_s$ can belong to at most 25 intervals J_n^r corresponding to $L_{i,n}$'s from one packet. Moreover, denoting by u^* the common left endpoint of a packet of $L_{i,n}$'s, for $t \in J_n^r$ we have

$$|L_{i,n}| + |J_n| + \operatorname{dist}(L_{i,n}, J_n) \geq |t - u^*|.$$

As $p < 2$, this implies ($L_{i,n}^*$ denoting the maximal interval in the packet; also recall $J_n \subset [\alpha', \widetilde{\beta}]$) that

$$\begin{aligned}
&\sum_{n \text{ in one packet}} \left(\frac{1}{\theta}\right)^{pk} |b_{i,n}|^p \|f_n\|^p_{L^p(0,\widetilde{\alpha})} \\
&\quad \leq \frac{C_p \theta^{pk} \varepsilon^{pm}}{|V|^{p-1}} \left(\frac{2}{3}\right)^{p(k-i)} \sum_{n \text{ in one packet}} \int_{L_{i,n}} |f(t)|^p \, dt \\
&\quad \times |J_n| \cdot (|L_{i,n}| + |J_n| + \operatorname{dist}(L_{i,n}, J_n))^{p-2} \\
&\quad \leq \frac{C_p \theta^{pk} \varepsilon^{pm}}{|V|^{p-1}} \left(\frac{2}{3}\right)^{p(k-i)} \sum_{n \text{ in one packet}} \int_{L_{i,n}^*} |f(t)|^p \, dt \int_{J_n^r} |t - u^*|^{p-2} \, dt \\
&\quad \leq \frac{C_p \theta^{pk} \varepsilon^{pm}}{|V|^{p-1}} \left(\frac{2}{3}\right)^{p(k-i)} \int_{L_{i,n}^*} |f(t)|^p \, dt \int_{u^*}^{\widetilde{\beta}} |t - u^*|^{p-2} \, dt \\
&\quad \leq C_p \theta^{pk} \varepsilon^{pm} \left(\frac{2}{3}\right)^{p(k-i)} \int_{L_{i,n}^*} |f(t)|^p \, dt.
\end{aligned}$$

As for fixed k and i the maximal intervals $L_{i,n}^*$ are disjoint, we have

$$\sum_{\text{packets with } d_n(\alpha')=k} \int_{L_{i,n}^*} |f(t)|^p \, dt \leq \|f\|_p^p,$$

so that

$$\sum_{n \in T_{l,m}^{(5)},\, d_n(\alpha')=k} \left(\frac{1}{\theta}\right)^{pd_n(V)} |b_{i,n}|^p \|f_n\|^p_{L^p(0,\widetilde{\alpha})} \leq C_p \theta^{pk} \varepsilon^{pm} \left(\frac{2}{3}\right)^{p(k-i)} \|f\|_p^p. \tag{4.21}$$

To complete the estimate, note that by the Hölder inequality, for n with

$d_n(\alpha') = k$ and $\kappa = \sqrt{2/3}$,

$$|a_{n,1}|^p = \Big|\sum_{i=0}^{k} b_{i,n}\Big|^p = \Big|\sum_{i=0}^{k} \kappa^{k-i}\cdot\kappa^{i-k} b_{i,n}\Big|^p \le C_p \sum_{i=0}^{k} \kappa^{p(i-k)} |b_{i,n}|^p.$$

Combining this with (4.21) we get

$$\sum_{n\in T^{(5)}_{l,m},\, d_n(\alpha')=k} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_{n,1}|^p \|f_n\|^p_{L^p(0,\widetilde{\alpha})}$$

$$\le C_p \sum_{n\in T^{(5)}_{l,m},\, d_n(\alpha')=k} \left(\frac{1}{\theta}\right)^{pd_n(V)} \sum_{i=0}^{k} \kappa^{p(i-k)} |b_{i,n}|^p \|f_n\|^p_{L^p(0,\widetilde{\alpha})}$$

$$= C_p \sum_{i=0}^{k} \kappa^{p(i-k)} \sum_{n\in T^{(5)}_{l,m},\, d_n(\alpha')=k} \left(\frac{1}{\theta}\right)^{pd_n(V)} |b_{i,n}|^p \|f_n\|^p_{L^p(0,\widetilde{\alpha})}$$

$$\le C_p \sum_{i=0}^{k} \kappa^{p(i-k)} \theta^{pk}\, \varepsilon^{pm} \left(\frac{2}{3}\right)^{p(k-i)} \|f\|_p^p \le C_p \theta^{pk}\, \varepsilon^{pm} \|f\|_p^p.$$

Summing over k we get

$$\sum_{n\in T^{(5)}_{l,m}} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_{n,1}|^p \|f_n\|^p_{L^p(0,\widetilde{\alpha})} \le C_p \varepsilon^{pm} \|f\|_p^p.$$

Combining the last inequality with (4.18) and (4.19) yields

$$\sum_{n\in T^{(5)}_{l,m}} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_n|^p \|f_n\|^p_{L^p(0,\widetilde{\alpha})} \le C_p \varepsilon^{pm} \|f\|_p^p. \tag{4.22}$$

Case 6: $n \in \bigcup_{m=0}^{\infty} T^{(6)}_{l,m}$. To treat this case, observe that

$$\bigcup_{m=0}^{\infty} T^{(6)}_{l,m} \subset \bigcup_{s=0}^{\infty} T^{(2)}_{r,s} \cup T^{(3)}_{r,s},$$

where $T^{(i)}_{r,s}$ is the case symmetric to $T^{(i)}_{l,s}$, but corresponding to decomposition of the set

$$T_{r,s} = \{n \ge n(V) : \#(\pi_n \cap [\beta, \widetilde{\beta}]) = s\}.$$

(The "r" in $T_{r,s}$ and $T^{(i)}_{r,s}$ below indicates that the splitting of the set of indices under consideration is suitable for estimating the "right part", i.e. the part corresponding to $\int_{\widetilde{\beta}}^{1} |f_n(t)|\, dt$.) More precisely,

$$
\begin{aligned}
T_{r,s}^{(1)} &= \{n \in T_{r,s} : J_n \subset [\beta, \widetilde{\beta}]\},\\
T_{r,s}^{(2)} &= \{n \in T_{r,s} : \widetilde{\beta} \in J_n,\ |J_n \cap [\beta, \widetilde{\beta}]| \geq |V|,\ J_n \not\subset [\beta, \widetilde{\beta}]\},\\
T_{r,s}^{(3)} &= \{n \in T_{r,s} : J_n \subset [\widetilde{\beta}, 1],\ \text{or } \widetilde{\beta} \in J_n \text{ with}\\
&\qquad |J_n \cap [\beta, \widetilde{\beta}]| \leq |V| \text{ and } J_n \not\subset [\beta, \widetilde{\beta}]\},\\
T_{r,s}^{(4)} &= \{n \in T_{r,s} : \beta \in J_n,\ |J_n \cap [\beta, \widetilde{\beta}]| \geq |V|,\ J_n \not\subset [\beta, \widetilde{\beta}]\},\\
T_{r,s}^{(5)} &= \{n \in T_{r,s} : J_n \subset [\widetilde{\alpha}, \beta],\ \text{or } \beta \in J_n \text{ with}\\
&\qquad |J_n \cap [\beta, \widetilde{\beta}]| \leq |V| \text{ and } J_n \not\subset [\beta, \widetilde{\beta}]\},\\
T_{r,s}^{(6)} &= \{n \in T_{r,s} : J_n \subset [0, \widetilde{\alpha}],\ \text{or } \widetilde{\alpha} \in J_n \text{ with } J_n \not\subset [\widetilde{\alpha}, \beta]\}.
\end{aligned}
$$

The cases $T_{r,s}^{(i)}$, $i = 1, 2, 3, 4, 5$, are treated analogously to $T_{l,m}^{(i)}$. In particular, for $T_{r,s}^{(2)}$ and $T_{r,s}^{(3)}$ we obtain estimates analogous to (4.13) and (4.15). This gives

$$
\begin{aligned}
(4.23)\qquad \sum_{m=0}^{\infty} \sum_{n \in T_{l,m}^{(6)}} \left(\frac{1}{\theta}\right)^{pd_n(V)} & |a_n|^p \|f_n\|_{L^p(0,\widetilde{\alpha})}^p \\
&\leq \sum_{s=0}^{\infty} \sum_{n \in T_{r,s}^{(2)} \cup T_{r,s}^{(3)}} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_n|^p \|f_n\|_p^p \leq C_p \|f\|_p^p.
\end{aligned}
$$

To complete the proof, note that summing over $m \geq 0$ inequalities (4.12), (4.13), (4.15), (4.16), (4.22) and adding (4.23) we get

$$
\sum_{n \geq n(V)} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_n|^p \|f_n\|_{L^p(0,\widetilde{\alpha})}^p \leq C_p \|f\|_p.
$$

The second inequality, i.e.

$$
\sum_{n \geq n(V)} \left(\frac{1}{\theta}\right)^{pd_n(V)} |a_n|^p \|f_n\|_{L^p(\widetilde{\beta},1)}^p \leq C_p \|f\|_p,
$$

is obtained by analogous considerations. ■

4.2. Proofs of Theorem 2.1 and Corollary 2.2. We are ready to complete the proof of Theorem 2.1. Once we have proved Lemmas 4.2 and 4.3, the remaining part of the proof is the same as in [11], but we present it for the sake of completeness.

By the duality argument, it is enough to prove unconditionality of $\{f_n, n \geq 0\}$ in $L^p[0,1]$ with $1 < p < 2$. For this, we show that for each p, $1 < p < 2$, there are constants $C_p, c_p > 0$, depending only on p, such that for each $f \in L^p[0,1]$,

$$
(4.24)\qquad c_p \|Pf\|_p \leq \|f\|_p \leq C_p \|Pf\|_p.
$$

To prove the right-hand inequality in (4.24), let $f \in L^p[0,1]$, $f = \sum_{n=0}^{\infty} a_n f_n$, and

$$S^* f(t) = \sup_{m \ge 0} \Big| \sum_{n=0}^{m} a_n f_n(t) \Big|.$$

Without loss of generality, we may assume that the set $\{n \ge 0 : a_n \ne 0\}$ is finite.

Now, for fixed $\lambda > 0$, let

$$E_\lambda = \{t \in (0,1) : Pf(t) > \lambda\}, \qquad B_\lambda = \{t \in (0,1) : \mathcal{M}\chi_{E_\lambda}(t) > 1/4\}.$$

It follows from the properties of $\mathcal{M}$ that

$$|B_\lambda| \le C|E_\lambda|, \qquad B_\lambda = \bigcup_k V_k,$$

where $V_k = (\alpha_k, \beta_k)$ are nonoverlapping intervals, and moreover

$$\mathcal{M}\chi_{E_\lambda}(\alpha_k) \le 1/4, \qquad \mathcal{M}\chi_{E_\lambda}(\beta_k) \le 1/4.$$

Let Γ_k be the set Γ from Lemma 4.2 corresponding to V_k, and

$$\widetilde{\Gamma} = \bigcup_k \Gamma_k, \qquad \widetilde{\Lambda} = \mathbb{N} \setminus \widetilde{\Gamma}, \qquad \varphi_1 = \sum_{n \in \widetilde{\Gamma}} a_n f_n, \qquad \varphi_2 = \sum_{n \in \widetilde{\Lambda}} a_n f_n.$$

It follows from (4.1) that

$$\int_{B_\lambda^c} \sum_{n \in \widetilde{\Gamma}} |a_n f_n(t)|\, dt \le C \int_{B_\lambda} Pf(t)\, dt.$$

Using the above inequality and the fact that $Pf(t) \le \lambda$ for $t \notin E_\lambda$ we get

$$\begin{aligned} \psi_1(\lambda) = |\{t \in (0,1) : S^*\varphi_1(t) > \lambda/2\}| &\le |B_\lambda| + \frac{2}{\lambda} \int_{B_\lambda^c} S^*\varphi_1(t)\, dt \\ &\le |B_\lambda| + \frac{C}{\lambda} \int_{B_\lambda} Pf(t)\, dt \le |B_\lambda| + C|B_\lambda \setminus E_\lambda| + \frac{C}{\lambda} \int_{E_\lambda} Pf(t)\, dt, \end{aligned}$$

so

$$\psi_1(\lambda) \le C\Big(|E_\lambda| + \frac{1}{\lambda} \int_{E_\lambda} Pf(t)\, dt \Big). \tag{4.25}$$

Since $Pf(t) \le \lambda$ for $t \notin E_\lambda$, inequality (4.2) implies

$$P\varphi_2(t) \le C\lambda \quad \text{on } (0,1). \tag{4.26}$$

As $S^* g \le 64\mathcal{M} g$ (see [7, Theorem 4.1]), and $\mathcal{M}$ is of strong type $(2,2)$, we get

$$\begin{aligned} \psi_2(\lambda) = |\{t \in (0,1) : S^*\varphi_2(t) > \lambda/2\}| &\le \frac{128^2}{\lambda^2} \|\mathcal{M}\varphi_2\|_2^2 \\ &\le \frac{C}{\lambda^2} \|\varphi_2\|_2^2 = \frac{C}{\lambda^2} \|P\varphi_2\|_2^2 \\ &= \frac{C}{\lambda^2} \Big(\int_{E_\lambda} (P\varphi_2(t))^2\, dt + \int_{E_\lambda^c} (P\varphi_2(t))^2\, dt \Big). \end{aligned}$$

This inequality, combined with (4.26), gives

$$\psi_2(\lambda) \le C\Big(|E_\lambda| + \frac{1}{\lambda^2}\int\limits_{E_\lambda^c} (Pf(t))^2\,dt\Big). \tag{4.27}$$

Combining (4.25) and (4.27) we get

$$\begin{aligned}\psi(\lambda) &= |\{t\in(0,1) : S^*f(t) > \lambda\}| \le \psi_1(\lambda) + \psi_2(\lambda)\\ &\le C\Big(|E_\lambda| + \frac{1}{\lambda}\int\limits_{E_\lambda} Pf(t)\,dt + \frac{1}{\lambda^2}\int\limits_{E_\lambda^c}(Pf(t))^2\,dt\Big).\end{aligned}$$

This implies (recall that $1 < p < 2$)

$$\begin{aligned}\|S^*f\|_p^p &= p\int\limits_0^\infty \lambda^{p-1}\psi(\lambda)\,d\lambda\\ &\le Cp\Big(\int\limits_0^\infty \lambda^{p-1}|E_\lambda|\,d\lambda + \int\limits_0^\infty \lambda^{p-2}\int\limits_{E_\lambda} Pf(t)\,dt\,d\lambda\\ &\qquad + \int\limits_0^\infty \lambda^{p-3}\int\limits_{E_\lambda^c}(Pf(t))^2\,dt\,d\lambda\Big)\\ &\le C_p\Big(\|Pf\|_p^p + \int\limits_0^1 Pf(t)\int\limits_0^{Pf(t)} \lambda^{p-2}\,d\lambda\,dt\\ &\qquad + \int\limits_0^1 (Pf(t))^2\int\limits_{Pf(t)}^\infty \lambda^{p-3}\,d\lambda\,dt\Big)\\ &\le C_p\|Pf\|_p^p.\end{aligned}$$

This implies the right-hand inequality in (4.24).

Now, we turn to the proof of the left-hand inequality in (4.24). For this, it is enough to show that for each p, $1 < p < 2$, P is of weak type (p,p).

The following estimate will be needed (cf. Lemma 4 of [11]):

LEMMA 4.4. *For a given interval $V = (\alpha,\beta)$, let*

$$T_V f(t) = \begin{cases} u_{1,V}\varphi_{1,V}(t) + u_{2,V}\varphi_{2,V}(t) & \text{for } t\in V,\\ 0 & \text{otherwise},\end{cases}$$

where

$$\varphi_{1,V} = |V|^{-1/2}\cdot\chi_V, \qquad \varphi_{2,V}(t) = 2\sqrt{3}\cdot|V|^{-3/2}\Big(t - \frac{\alpha+\beta}{2}\Big)\chi_V(t),$$

and $u_{i,V} = \int_V f(t)\varphi_{i,V}(t)\,dt$, $i = 1,2$. For each p, λ and f, if $\int_V |f(t)|\,dt \le \lambda|V|$ then

$$\|T_V f\|_2^2 \le 4\lambda^2|V|, \qquad \|T_V f\|_p \le C_p\|f\|_{L^p(V)},$$

where C_p depends only on p.

It should be clear that on V, $T_V f$ is equal to the orthogonal projection of $f\chi_V$ onto the space of functions linear on V. An easy proof of Lemma 4.4 is omitted.

Let p, $1<p<2$, be fixed. We need to prove that there is a constant C_p, depending only on p, such that for each $f\in L^p[0,1]$ and $\lambda>0$,

$$|\{t\in(0,1): Pf(t)>\lambda\}|\le \frac{C_p}{\lambda^p}\,\|f\|_p^p. \tag{4.28}$$

Without loss of generality we can assume that $\|f\|_p=1$ and $\lambda>1$. Let

$$G_\lambda=\{t\in(0,1):\mathcal{M}f(t)>\lambda\}.$$

Then

$$|G_\lambda|\le C_p\frac{\|f\|_p^p}{\lambda^p},\qquad G_\lambda=\bigcup_k V_k, \tag{4.29}$$

where $V_k=(\alpha_k,\beta_k)$ are pairwise disjoint; in particular, $|G_\lambda|=\sum_k|V_k|$. Moreover,

$$|f(t)|\le\lambda\quad\text{a.e. on } G_\lambda^{\mathrm{c}},\qquad \int\limits_{V_k}|f(t)|\,dt\le\lambda|V_k|,\quad k=1,2,\ldots. \tag{4.30}$$

Let

$$h=f\cdot\chi_{G_\lambda^{\mathrm{c}}}+\sum_k T_{V_k}f,\qquad g=f-h.$$

Now, the parts corresponding to Ph and Pg are treated separately.

Using (4.29), (4.30) and Lemma 4.4 we get

$$\begin{aligned}\|h\|_2^2&=\int\limits_{G_\lambda^{\mathrm{c}}}f(t)^2\,dt+\sum_k\int\limits_{V_k}(T_{V_k}f)(t)^2\,dt\\&\le\lambda^{2-p}\int\limits_{G_\lambda^{\mathrm{c}}}|f(t)|^p\,dt+\lambda^2|G_\lambda|\le C_p\lambda^{2-p}\|f\|_p^p.\end{aligned}$$

The above inequality gives

$$|\{t\in(0,1):Ph(t)>\lambda/2\}|\le\frac{4}{\lambda^2}\,\|Ph\|_2^2=\frac{4}{\lambda^2}\,\|h\|_2^2\le C_p\,\frac{\|f\|_p^p}{\lambda^p}. \tag{4.31}$$

It remains to treat Pg. As $p<2$, we have

$$(Pg(t))^p=\Big(\sum_{n=0}^{\infty}a_n(g)^2f_n(t)^2\Big)^{p/2}\le\sum_{n=0}^{\infty}|a_n(g)|^p|f_n(t)|^p.$$

In addition, let $\widetilde{V}_k=(\widetilde{\alpha}_k,\widetilde{\beta}_k)$, where $\widetilde{\alpha}_k=\alpha_k-2|V_k|$, $\widetilde{\beta}_k=\beta_k+2|V_k|$ and $\widetilde{G}_\lambda=\bigcup_k\widetilde{V}_k$; observe that $|\widetilde{G}_\lambda|\le 5|G_\lambda|$. Now, we have

$$\begin{aligned}|\{t\in(0,1):Pg(t)>\lambda/2\}|&\le|\widetilde{G}_\lambda|+\frac{2^p}{\lambda^p}\int\limits_{\widetilde{G}_\lambda^{\mathrm{c}}}(Pg)^p(t)\,dt\\&\le C_p\,\frac{\|f\|_p^p}{\lambda^p}+\frac{2^p}{\lambda^p}\sum_n\int\limits_{\widetilde{G}_\lambda^{\mathrm{c}}}|a_n(g)|^p|f_n(t)|^p\,dt.\end{aligned} \tag{4.32}$$

Since by the definition of g and Lemma 4.4,

$$\|g\|_p^p = \sum_k \int_{V_k} |f(t) - T_{V_k} f(t)|^p \, dt \le C_p \sum_k \int_{V_k} |f(t)|^p \, dt \le C_p \|f\|_p^p, \tag{4.33}$$

it is enough to prove that

$$\sum_n \int_{\widetilde{G}_\lambda^{\mathrm{c}}} |a_n(g)|^p |f_n(t)|^p \, dt \le C_p \|g\|_p^p. \tag{4.34}$$

For this, put $g_k = g \cdot \chi_{V_k}$. Observe that the supports of g_k are disjoint, so $\|g\|_p^p = \sum_{k=1}^\infty \|g_k\|_p^p$ and $g = \sum_{k=1}^\infty g_k$, with the series convergent in $L^p[0,1]$. Therefore, for each $n \ge 0$, $a_n(g) = \sum_{k=1}^\infty a_n(g_k)$. Moreover, it follows from the definition of T_{V_k} that

$$\int_{V_k} g_k(t)(at+b) \, dt = 0 \quad \text{for all } a, b.$$

In particular, this implies that $a_n(g_k) = 0$ for $n < n(V_k)$. Thus, with θ as in Lemma 4.3 we have

$$\begin{aligned} |a_n(g)|^p &= \Big| \sum_{k\,:\, n \ge n(V_k)} a_n(g_k) \Big|^p \le \Big(\sum_{k\,:\, n \ge n(V_k)} \Big(\frac{1}{\theta}\Big)^{d_n(V_k)} |a_n(g_k)| \cdot \theta^{d_n(V_k)} \Big)^p \\ &\le \Big(\sum_{k\,:\, n \ge n(V_k)} \Big(\frac{1}{\theta}\Big)^{p d_n(V_k)} |a_n(g_k)|^p \Big) \cdot \Big(\sum_{k\,:\, n \ge n(V_k)} \theta^{q d_n(V_k)} \Big)^{p/q}. \end{aligned}$$

Note that if $n \ge n(V_k)$ then in V_k there is at least one point of π_n. This implies that for each $s \ge 0$ there are at most two k such that $n \ge n(V_k)$ and $d_n(V_k) = s$. Therefore

$$\Big(\sum_{k\,:\, n \ge n(V_k)} \theta^{q d_n(V_k)} \Big)^{p/q} \le C_p.$$

This and the previous inequality give

$$|a_n(g)|^p \le C_p \sum_{k\,:\, n \ge n(V_k)} \Big(\frac{1}{\theta}\Big)^{p d_n(V_k)} |a_n(g_k)|^p.$$

Recall that $\operatorname{supp} g_k \subset V_k$. Using the above inequality and Lemma 4.3 we get

$$\begin{aligned} &\sum_{n=0}^\infty \int_{\widetilde{G}_\lambda^{\mathrm{c}}} |a_n(g)|^p |f_n(t)|^p \, dt \\ &\qquad \le C_p \sum_{n=0}^\infty \sum_{k\,:\, n \ge n(V_k)} \Big(\frac{1}{\theta}\Big)^{p d_n(V_k)} \int_{\widetilde{G}_\lambda^{\mathrm{c}}} |a_n(g_k)|^p |f_n(t)|^p \, dt \end{aligned}$$

$$\le C_p \sum_k \sum_{n\ge n(V_k)} \Big(\frac{1}{\theta}\Big)^{pd_n(V_k)} |a_n(g_k)|^p \int_{V_k^c} |f_n(t)|^p\,dt$$

$$\le C_p \sum_k \|g_k\|_p^p = C_p\|g\|_p^p,$$

i.e. we have proved (4.34). Combining this with (4.32) and (4.33) we obtain

$$|\{t\in(0,1): Pg(t)>\lambda/2\}| \le C_p\,\frac{\|f\|_p^p}{\lambda^p}. \tag{4.35}$$

As $f=g+h$, it follows by (4.31) and (4.35) that

$$|\{t\in(0,1): Pf(t)>\lambda\}| \le C_p \frac{\|f\|_p^p}{\lambda^p},$$

i.e. P is of weak type (p,p) for each p, $1<p<2$. Since P is also of strong type $(2,2)$, by the Marcinkiewicz interpolation theorem it is of strong type (p,p) for each p, $1<p<2$, i.e. the left-hand inequality in (4.24) holds.

Since the constants from Lemmas 4.2 and 4.3 do not depend on $\mathcal{T}$, it follows from the method of proof that the uncoditional basic constant for general Franklin systems in $L^p[0,1]$ can be bounded by a constant C_p, depending on p, but not on the sequence $\mathcal{T}$ of knots.

This completes the proof of Theorem 2.1. ∎

Proof of Corollary 2.2. It has been proved by S. V. Konyagin and V. N. Temlyakov [14] that a normalized basis $\mathcal{X}=(x_n, n\ge 0)$ of a Banach space $(X,\|\cdot\|)$ is greedy if and only if it is unconditional and democratic, the latter meaning that for any two finite subsets of indices A,B with $\#A=\#B$,

$$\Big\|\sum_{n\in A} x_n\Big\| \sim \Big\|\sum_{n\in B} x_n\Big\|. \tag{4.36}$$

In addition, the constant C in (2.3) can be chosen so that it depends only on the unconditional basic constant of the basis $\mathcal{X}$ and the constants in the equivalence (4.36). Thus, it remains to check that $\{f_{n,p}, n\ge 0\}$ is democratic in $L^p[0,1]$, and that the constants in (4.36) can be chosen independent of $\mathcal{T}$. This is done by the usual argument using the unconditionality of $\{f_{n,p}, n\ge 0\}$ and the exponential decay of the lengths of nested J_n's. For this, let $\chi_{J_n,p}=|J_n|^{-1/p}\chi_{J_n}$. It follows from the unconditionality and normalization of $\{f_{n,p}, n\ge 0\}$ (cf. (3.7) and (3.8)), Corollary 3.3 and the maximal inequality of Fefferman and Stein (see e.g. Theorem 1, Chapter II of [17]) that for each sequence of coefficients $\{u_n, n\ge 0\}$,

$$\Big\|\sum_{n=0}^\infty u_n f_{n,p}\Big\|_p^p \sim \int_0^1 \Big(\sum_{n=0}^\infty |u_n f_{n,p}(t)|^2\Big)^{p/2} dt \sim \int_0^1 \Big(\sum_{n=0}^\infty |u_n \chi_{J_n,p}(t)|^2\Big)^{p/2} dt.$$

Moreover, it follows from Corollary 3.6 that for each t and each choice of indices $n_1 < \cdots < n_m$,

$$\Big(\sum_{i=1}^{m} |\chi_{J_{n_i},p}(t)|^2\Big)^{p/2} \sim (|J_{n_m}|^{-2/p})^{p/2} \sim \sum_{i=1}^{m} |\chi_{J_{n_i},p}(t)|^p.$$

Therefore

$$\Big\| \sum_{i=1}^{m} f_{n_i,p} \Big\|_p^p \sim \int_0^1 \sum_{i=1}^{m} |\chi_{J_{n_i},p}(t)|^p = m,$$

which proves that $\{f_{n,p}, n \geq 0\}$ is democratic in $L^p[0,1]$. It follows from the proof that the implied constants do not depend on $\mathcal{T}$. ∎

4.3. Final remarks

Remark 3 (The case of non-dense sequences $\mathcal{T}$). A general Franklin system discussed above is defined for an admissible sequence of knots $\mathcal{T}$ (cf. Definitions 2.1 and 2.2). In particular, Definition 2.1 requires the density of $\mathcal{T}$ in $[0,1]$. It should be clear that one can consider general Franklin systems corresponding to sequences $\mathcal{T}$ admitting at most double knots, but not necessarily dense in $[0,1]$. If $\mathcal{T}$ is not dense in $[0,1]$, then the corresponding Franklin system is not a basis in $L^p[0,1]$ (because it is not dense), but it is a basic sequence in this space. It follows from Theorem 2.1 and Remark 1 that for any finite collection of points $\mathcal{T} = \{t_n, 0 \leq n \leq m\}$ and the corresponding sequence of Franklin functions $\{f_n, 0 \leq n \leq m\}$, and for any choice of coefficients $\{a_n, 0 \leq n \leq m\}$ and signs $\varepsilon_n \in \{-1,1\}$, $0 \leq n \leq m$,

$$\Big\| \sum_{n=0}^{m} a_n f_n \Big\|_p \sim \Big\| \sum_{n=0}^{m} \varepsilon_n a_n f_n \Big\|_p,$$

with the implied constants depending only on p. Note that this implies that for each $\mathcal{T}$ admitting at most double knots, the corresponding Franklin system is an unconditional basic sequence in $L^p[0,1]$, $1 < p < \infty$, and the implied unconditional basic constants have a finite bound C_p, depending only on p.

Similarly, it follows from Corollary 2.2 (or more precisely, from the method of its proof) that for each $\mathcal{T}$ admitting at most double knots, the corresponding Franklin system, normalized in $L^p[0,1]$, is a greedy basis in its span in $L^p[0,1]$, $1 < p < \infty$, and the constants in inequality (2.3) can be chosen so that they depend only on p.

Remark 4 (Equivalence of general Franklin systems to subsequences of the dyadic Haar system). It has been shown in [12] that each general Haar system is equivalent in $L^p[0,1]$, $1 < p < \infty$, to a subsequence of the classical Haar system (i.e. the Haar system corresponding to dyadic knots). The same is true for the general Franklin systems.

The corresponding subsequence of the dyadic Haar system can be obtained as follows. Define $D_{j,k} = [(k-1)/2^j, k/2^j]$. Let $\mathcal{T} = (t_n, n \geq 0)$ be a sequence of points admitting at most double knots (not necessarily dense in $[0,1]$), with the corresponding general Franklin system $\{f_n, n \geq 0\}$. Consider the corresponding intervals J_n, $n \geq 0$. For $j \geq 0$, let

$$N_j = \{n \geq 0 : 1/2^{j+1} < |J_n| \leq 1/2^j\}.$$

Then for each $n \in N_j$ there is a dyadic interval $D_{j+2,k} \subset J_n$. Now, observe that for each k, $1 \leq k \leq 2^{j+2}$, the collection of intervals J_n such that $D_{j+2,k} \subset J_n$ is a nested family of intervals. It follows from Lemma 3.5 that for each j and k,

$$\#\{n \in N_j : D_{j+2,k} \subset J_n\} \leq 25.$$

Now, there are 32 different dyadic intervals $D_{j+7,l}$ included in $D_{j+2,k}$. This implies that it is possible to assign to each $n \in N_j$ a dyadic interval $D(n) \subset J_n$ of length $1/2^{j+7}$ in such a way that $D(n_1) \neq D(n_2)$ for $n_1 \neq n_2$. This means that $\{H_{D(n)}, n \geq 0\}$ is a permutation of a subsequence of the dyadic Haar system. Let $H_{D(n)}$ be the Haar function (normalized in $L^2[0,1]$) with support $D(n)$. By arguments similar to those used in the proof of Corollary 3.3, there is a constant $C > 0$, independent of $\mathcal{T}$, such that

$$|f_n(t)| \leq C\mathcal{M}H_{D(n)}(t), \qquad |H_{D(n)}| \leq C\mathcal{M}f_n(t).$$

Thus, using the Fefferman–Stein maximal inequality and unconditionality of both $\{H_n, n \geq 1\}$ and $\{f_n, n \geq 1\}$, for each $1 < p < \infty$ and any sequence of coefficients $\{a_n, n \geq 0\}$ we get

$$\Big\| \sum_{n=0}^{\infty} a_n f_n \Big\|_p \sim \| \sum_{n=0}^{\infty} a_n H_{D(n)} \|_p, \tag{4.37}$$

with the implied constants depending only on p. Moreover, one can replace in (4.37) the pair of systems $\{f_n, n \geq 0\}$ and $\{H_{D(n)}, n \geq 0\}$ by their L^p-normalized versions.

It follows from Corollary 3.3 and the Fefferman–Stein maximal inequality that for each p, $1 < p < \infty$, and each sequence $\mathcal{T}$ of at most double knots with the corresponding Franklin system $\{f_n, n \geq 0\}$,

$$\Big\| \Big(\sum_{n=0}^{\infty} |a_n f_n|^2 \Big)^{1/2} \Big\|_p \sim \Big\| \Big(\sum_{n=0}^{\infty} |a_n \chi_{J_n,2}|^2 \Big)^{1/2} \Big\|_p,$$

with the implied constants depending only on p. For completeness, we show that the above equivalence also holds for $p = 1$:

PROPOSITION 4.5. *There exist constants* $C_1, C_2 > 0$ *such that for each sequence* $\mathcal{T}$ *of points in* $[0,1]$, *admitting at most double knots*, *with the corresponding Franklin system* $\{f_n, n \geq 0\}$, *and any sequence of coefficients*

$\{a_n, n \ge 0\}$,

$$C_1\Big\|\Big(\sum_{n=0}^{\infty}|a_n\chi_{J_n,2}|^2\Big)^{1/2}\Big\|_1 \le \Big\|\Big(\sum_{n=0}^{\infty}|a_nf_n|^2\Big)^{1/2}\Big\|_1$$
$$\le C_2\Big\|\Big(\sum_{n=0}^{\infty}|a_n\chi_{J_n,2}|^2\Big)^{1/2}\Big\|_1.$$

For the proof of Proposition 4.5, we need the following

LEMMA 4.6. *There is a constant $C > 0$ such that for each sequence $\mathcal{T}$ of points in $[0,1]$, admitting at most double knots, with the corresponding Franklin system $\{f_n, n \ge 0\}$, and each interval $V = [\alpha, \beta] \subset [0,1]$,*

$$\sum_{n\,:\,J_n\subset V}|J_n|^{1/2}\int_{V^c}|f_n(t)|\,dt \le C|V|.$$

Proof of Lemma 4.6. Let us estimate the part of the sum corresponding to $\int_\beta^1 |f_n(t)|\,dt$. It follows from Proposition 3.1(b5) and the estimate of the L^1-norm of f_n (cf. (3.7) and (3.8), or (3.13)) that

$$\int_\beta^1 |f_n(t)|\,dt \le C\varepsilon^{d_n(\beta)}\|f_n\|_1 \le C\varepsilon^{d_n(\beta)}|J_n|^{1/2}.$$

Fix $k \ge 0$ and consider n such that $d_n(\beta) = k$. The corresponding intervals J_n can be arranged into packets, with the intervals from one packet having a common right endpoint and forming a nested collection of intervals, and with maximal intervals of different packets having disjoint interiors. As all these intervals are included in V, by Corollary 3.6 we get

$$\sum_{n\,:\,J_n\subset V,\,d_n(\beta)=k}|J_n|^{1/2}\int_\beta^1|f_n(t)|\,dt \le C\varepsilon^k\sum_{n\,:\,J_n\subset V,\,d_n(\beta)=k}|J_n| \le C\varepsilon^k|V|.$$

Summing over k yields

$$\sum_{n\,:\,J_n\subset V}|J_n|^{1/2}\int_\beta^1|f_n(t)|\,dt = \sum_{k\ge0}\ \sum_{n\,:\,J_n\subset V,\,d_n(\beta)=k}|J_n|^{1/2}\int_\beta^1|f_n(t)|\,dt$$
$$\le C\sum_{k\ge0}\varepsilon^k|V| \;=\; C|V|.$$

The part corresponding to $\int_0^\alpha |f_n(t)|\,dt$ is treated analogously, which completes the proof of Lemma 4.6. ∎

Proof of Proposition 4.5. For a given $\mathcal{T} = (t_n, n \ge 0)$, let $\mathcal{I}_\mathcal{T}$ be the family of intervals generated by $\mathcal{T}$, i.e.

$$\mathcal{I}_\mathcal{T} = \{[0,1]\} \cup \bigcup_{n\ge2}\{(t_n^-, t_n), (t_n, t_n^+)\},$$

where t_n^-, t_n^+ are as defined in Section 3.2. Consider the maximal function corresponding to $\mathcal{I}_\mathcal{T}$,

$$\mathcal{M}_\mathcal{T} f(t) = \sup_{I \in \mathcal{I}_\mathcal{T} \,:\, t \in I} \frac{1}{|I|} \int_I |f(u)|\, du.$$

Let us prove the right-hand inequality in Proposition 4.5. To this end, for a given sequence of coefficients $\{a_n, n \geq 0\}$, let

$$\begin{aligned}
F(t) &= \sum_{n=0}^{\infty} |a_n \chi_{J_n,2}(t)|^2, \\
E_r &= \{t : F(t) > 2^r\}, \quad r \in \mathbb{Z}, \\
B_r &= \{t : \mathcal{M}_\mathcal{T} \chi_{E_r}(t) > 1/2\}, \quad r \in \mathbb{Z}, \\
N_r &= \{n : \operatorname{int} J_n \subset B_r,\ \operatorname{int} J_n \not\subset B_{r+1}\}, \\
\psi_r(t) &= \Big(\sum_{n \in N_r} |a_n f_n(t)|^2 \Big)^{1/2}.
\end{aligned}$$

For $n \in N_r$ we have $|J_n \cap E_{r+1}^{\mathrm{c}}| \geq \frac{1}{2}|J_n|$, which implies

$$\int_{B_r \cap E_{r+1}^{\mathrm{c}}} \chi_{J_n,2}^2(t)\, dt \geq \int_{J_n \cap E_{r+1}^{\mathrm{c}}} \chi_{J_n,2}^2(t)\, dt \geq \frac{1}{2}.$$

Using this inequality and the fact that $F(t) \leq 2^{r+1}$ on E_{r+1}^{c} we get

$$\begin{aligned}
\|\psi_r\|_2^2 = \sum_{n \in N_r} a_n^2 &\leq 2 \int_{B_r \cap E_{r+1}^{\mathrm{c}}} \sum_{n \in N_r} a_n^2 \chi_{J_n,2}^2(t)\, dt \\
&\leq 2 \int_{B_r \cap E_{r+1}^{\mathrm{c}}} F(t)\, dt \leq 2^{r+2} |B_r|.
\end{aligned}$$

By the last inequality and Schwarz inequality,

$$\int_{B_r} \psi_r(t)\, dt \leq \|\chi_{B_r}\|_2 \cdot \|\psi_r\|_2 \leq 2^{1+r/2} |B_r|. \tag{4.38}$$

To estimate $\int_{B_r^{\mathrm{c}}} \psi_r(t)\, dt$, note that B_r is a union of some intervals from $\mathcal{I}_\mathcal{T}$. Any two intervals in $\mathcal{I}_\mathcal{T}$ are either disjoint, or one is included in the other. Let $\mathcal{I}_{r,\mathcal{T}}$ be the collection of maximal intervals of $\mathcal{I}_\mathcal{T}$ included in B_r. The intervals in $\mathcal{I}_{r,\mathcal{T}}$ are disjoint, so we have

$$B_r = \bigcup_{V \in \mathcal{I}_{r,\mathcal{T}}} V, \qquad |B_r| = \sum_{V \in \mathcal{I}_{r,\mathcal{T}}} |V|. \tag{4.39}$$

Observe that if $n \in N_r$, then $\operatorname{int} J_n \subset V$ for some $V \in \mathcal{I}_{r,\mathcal{T}}$. Moreover, for $n \in N_r$ we have $|a_n| \leq 2^{(r+1)/2} |J_n|^{1/2}$: if not, then $F(t) \geq |a_n \chi_{J_n,2}(t)|^2 > 2^{r+1}$ for $t \in J_n$, so $J_n \subset E_{r+1}$, contrary to the definition of N_r. Combining this fact with (4.39) and Lemma 4.6 we get

$$\begin{aligned}\int_{B_r^c} \psi_r(t)\,dt &\le \int_{B_r^c} \sum_{n\in N_r} |a_n f_n(t)|\,dt \\ &\le 2^{(r+1)/2} \sum_{V\in\mathcal{I}_{r,\mathcal{T}}} \sum_{n\in N_r\,:\,\mathrm{int}\,J_n\subset V} |J_n|^{1/2} \int_{V^c} |f_n(t)|\,dt \\ &\le C2^{r/2} \sum_{V\in\mathcal{I}_{r,\mathcal{T}}} |V| = C2^{r/2}|B_r|.\end{aligned}$$

Thus, putting together the last inequality and (4.38) we get

$$\int_0^1 \psi_r(t)\,dt \le C2^{r/2}|B_r|.$$

As $\mathcal{M}_{\mathcal{T}} f(t) \le \mathcal{M} f(t)$ and $\mathcal{M}$ is of weak type $(1,1)$, we have $|B_r| \le C|E_r|$. Therefore we obtain

$$\begin{aligned}\int_0^1 \Big(\sum_{n=0}^{\infty} |a_n f_n(t)|^2\Big)^{1/2} dt &\le \sum_{r\in\mathbb{Z}} \int_0^1 \psi_r(t)\,dt \\ &\le C\sum_{r\in\mathbb{Z}} 2^{r/2}|E_r| \le C\int_0^1 F^{1/2}(t)\,dt,\end{aligned}$$

which is the right-hand inequality in Proposition 4.5. It follows from the proof that the constant C can be chosen independent of $\mathcal{T}$.

The left-hand inequality in Proposition 4.5 follows by an analogous argument. ∎

Remark 5. Note that in Proposition 4.5, the sequence $\{\chi_{J_n,2}, n \ge 0\}$ can be replaced by the sequence of Haar functions $\{H_{D(n)}, n \ge 0\}$ from Remark 4.

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