

Time-dependent Schrödinger perturbations of transition densities

by

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This work is dedicated to Professor Jan Kiszyński

Abstract. We construct the fundamental solution of $\partial_t - \Delta_y - q(t, y)$ for functions q with a certain integral space-time relative smallness, in particular for those satisfying a relative Kato condition. The resulting transition density is comparable to the Gaussian kernel in finite time, and it is even asymptotically equal to the Gaussian kernel (in small time) under the relative Kato condition.

The result is generalized to arbitrary strictly positive and finite time-nonhomogeneous transition densities on measure spaces.

We also discuss specific applications to Schrödinger perturbations of the fractional Laplacian in view of the fact that the 3P Theorem holds for the fundamental solution corresponding to the operator.

1. Main results and overview. Let d be a natural number. The Gaussian kernel on \mathbb{R}^d is defined as

$$g(s, x, t, y) = \frac{1}{(4\pi(t-s))^{d/2}} \exp\left(-\frac{|y-x|^2}{4(t-s)}\right) \quad \text{if } -\infty < s < t < \infty,$$

and we let $g(s, x, t, y) = 0$ if $s \geq t$. Here $x, y \in \mathbb{R}^d$ are arbitrary. It is well-known that g is a time-homogeneous transition density with respect to the Lebesgue measure, dz , on \mathbb{R}^d . In particular, for $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} g(s, x, u, z)g(u, z, t, y) dz = g(s, x, t, y) \quad \text{if } s < u < t.$$

2000 *Mathematics Subject Classification*: 47A55, 60J35, 60J57.

Key words and phrases: relative Kato condition, conditional process.

The first and third authors were partially supported by KBN.

We will consider a Borel measurable function $q : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, and numbers $h > 0$ and $0 \leq \eta < 1$, such that for all $x, y \in \mathbb{R}^d$ and $s < t \leq s + h$,

$$(1) \quad \int_s^t \int_{\mathbb{R}^d} \frac{g(s, x, u, z)g(u, z, t, y)}{g(s, x, t, y)} |q(u, z)| dz du \leq \eta.$$

THEOREM 1. *There is a unique continuous transition density \tilde{g} such that*

$$(2) \quad \int_s^\infty \int_{\mathbb{R}^d} \tilde{g}(s, x, t, y) [\partial_t \phi(t, y) + \Delta_y \phi(t, y) + q(t, y) \phi(t, y)] dy dt = -\phi(s, x),$$

and

$$\frac{\tilde{g}(s, x, t, y)}{g(s, x, t, y)} \leq \frac{1}{1 - \eta} \exp \left\{ \frac{\eta}{(1 - \eta)h} (t - s) \right\}.$$

Here $s < t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ are arbitrary.

We consider $\Delta + q$ as an additive perturbation of the Laplacian Δ by the operator of multiplication by q (i.e. a Schrödinger perturbation). According to (2), \tilde{g} is the integral kernel of a left inverse of $-(\partial_t + \Delta_y + q)$. Put differently, $f : (t, y) \mapsto \tilde{g}(s, x, t, y)$ solves $(\partial_t - \Delta_y - q)f = \delta_{(s,x)}$ in the sense of distributions. Thus, \tilde{g} is a fundamental solution of $\partial_t - \Delta_y - q$ ([31]).

As we will see, \tilde{g} is constructed by means of g and q only, without referring to Δ . A similar procedure applies to the fundamental solution of the fractional Laplacian $\Delta^{\alpha/2} = -(-\Delta)^{\alpha/2}$. At the end of the paper we give references and discuss these two important examples in some detail.

The primary goal of the paper is, however, to construct and estimate analogous time-dependent, or nonautonomous, Schrödinger perturbations for more general transition densities. We work under the appropriate assumption of *relative smallness*, or *relative Kato condition* on q . We give explicit upper bounds for the resulting transition density, which are new even in the autonomous Gaussian case. Our development is motivated by the role of the celebrated 3G Theorem in studying Schrödinger-type perturbations of Green functions [19, 20, 21] (see also [12], [11], [9], [3], [4], [10]). Another motivation comes from a recent estimate, the 3P Theorem of [5] for the fundamental solution of $\Delta^{\alpha/2}$. The estimate was used in [5] to construct the transition density of autonomous *gradient* perturbations of $\Delta^{\alpha/2}$, in a way resembling the above mentioned study of Schrödinger perturbations of Green functions by means of the 3G Theorem (see also [23]). [5] and the present paper show that a perturbation technique similar to that of [20] applies even more naturally to the *parabolic* Green function (that is, the fundamental solution, or transition density). We propose an explicit construction of the perturbed transition density under a minimum of assumptions, corresponding to the generality of [19, 20, 21, 10]. We refer the reader to [13, Theorem 3.5, p. 71],

[24], and [14, Theorem 19, Ch.VIII] for a selection of results in the perturbation theory of linear operators and semigroups on Banach spaces. For recent developments in Schrödinger perturbations of time-nonhomogeneous transition probabilities we refer the reader to [17], [27], [16]. We point out that our main estimate, Theorem 3 below, is more precise and explicit than those mentioned above. Accordingly, it also strengthens in the present context the celebrated Khas'minski's lemma, one of the main tools in the probabilistic theory of Schrödinger perturbations of generators of Markov processes (see, e.g., [11], [16]). This strengthening is of independent interest—the estimate is valid in the full range of times, rather than only in small time intervals, and the proof gives a deeper insight into the interplay between individual terms of the series involved.

To explain the connection of Theorem 1 to our general results we note that \tilde{g} satisfies the following equation for all $x, y \in \mathbb{R}^d$ and $s, t \in \mathbb{R}$:

$$\tilde{g}(s, x, t, y) = g(s, x, t, y) + \int_{\mathbb{R}^d} \int_X g(s, x, u, z) q(u, z) \tilde{g}(u, z, t, y) dz du$$

(see the proof of Theorem 2). This is called *Duhamel's formula* or *perturbation formula*. The equation implicitly defines the perturbed transition density in this situation (and in more general ones), which we will now discuss.

Consider an arbitrary set X with a σ -algebra \mathcal{M} and a (nonnegative) measure m defined on \mathcal{M} . To simplify the notation we will write dz for $m(dz)$ in what follows. Consider the σ -algebra \mathcal{B} of Borel subsets of \mathbb{R} , and the Lebesgue measure, du , defined on \mathcal{B} . The *space-time*, $\mathbb{R} \times X$, will be equipped with the σ -algebra $\mathcal{B} \times \mathcal{M}$ and product measure $du dz = du m(dz)$.

Let p be a $\mathcal{B} \times \mathcal{M} \times \mathcal{B} \times \mathcal{M}$ -measurable function defined (everywhere) on $\mathbb{R} \times X \times \mathbb{R} \times X$. We will call p a *transition density* on X if

$$(3) \quad p(s, x, t, y) = 0 \quad \text{whenever } s \geq t,$$

$$(4) \quad 0 < p(s, x, t, y) < \infty \quad \text{when } s < t, x, y \in X,$$

and the following *Chapman-Kolmogorov equation* holds: if $s < u < t$ then

$$(5) \quad \int_X p(s, x, u, z) p(u, z, t, y) dz = p(s, x, t, y), \quad x, y \in X.$$

Note that we require strict positivity in (4), while (3) is merely a convention. By (4) and (5), m is necessarily σ -finite on X . The reader may regard s and x in $p(s, x, t, y)$ as the starting time and position of a *variable* mass spreading in X , and t, y as the ending time and position. Thus, $\int_X p(s, x, t, y) dy$ is the total mass at time t . We will say that transition densities p' and p'' on X are *comparable locally in time* if for every $h > 0$ there is a (finite) constant $c = c(h)$ such that $c^{-1}p''(s, x, t, y) \leq p'(s, x, t, y) \leq cp''(s, x, t, y)$ for all $x, y \in X$ provided $s < t < s + h$. We will say that they are *asymptotically equal* if $c(h)$ may be chosen in such a way that $c(h) \rightarrow 1$ as $h \rightarrow 0^+$.

All the functions discussed below will be assumed measurable with respect to the relevant σ -algebras, usually with respect to $\mathcal{B} \times \mathcal{M}$.

If q is a *nonnegative* function defined on $\mathbb{R} \times \mathcal{M}$, then we let

$$(6) \quad \eta^*(q) = \inf \eta,$$

where the infimum is taken over all $\eta > 0$ with the property that there exists $h > 0$ such that for all $x, y \in \mathbb{R}^d$ and $s < t \leq s + h$,

$$(7) \quad \int_s^t \int_X p(s, x, u, z)q(u, z)p(u, z, t, y) dz du \leq \eta p(s, x, t, y).$$

We will say that a (signed) function $q : \mathbb{R} \times X \rightarrow \mathbb{R}$ is *relatively bounded* (at small times, with respect to p and m) if $0 \leq \eta^*(|q|) < \infty$. We will say that q is *relatively small* if $0 \leq \eta^*(|q|) < 1$, and we will say that q is *relatively Kato* ⁽¹⁾ if $\eta^*(|q|) = 0$.

THEOREM 2. *If $q : \mathbb{R} \times X \rightarrow \mathbb{R}$ is relatively small, then there is a unique transition density \tilde{p} on X locally in time comparable with p , such that for all $s, t \in \mathbb{R}$ and $x, y \in X$ we have*

$$(8) \quad \tilde{p}(s, x, t, y) = p(s, x, t, y) + \int_{\mathbb{R} \times X} p(s, x, u, z)q(u, z)\tilde{p}(u, z, t, y) dz du.$$

If q is relatively Kato, then \tilde{p} and p are asymptotically equal.

We note that explicit upper and lower bounds for \tilde{p} exist expressed in terms of $\eta^*(q_+)$ and $\eta^*(q_-)$ (see Theorem 3 and (27), and also (41)).

The paper is organized as follows. In Section 2 we describe the basic formalism of the *perturbation series* in the context of the Chapman–Kolmogorov equation. In Section 3 we reformulate the relative boundedness and smallness of q . By a combinatorial argument, we prove our main estimate, Theorem 3, for the perturbation series for relatively small $q \geq 0$. In Section 4 we consider signed relatively small q and we prove Theorem 2. In Section 5 we discuss in more detail Schrödinger perturbations of the transition densities of Laplacian and fractional Laplacian, and we give the proof of Theorem 1. In view of the fact that the transition density of the fractional Laplacian (but not that of the Laplacian) satisfies the 3P Theorem, in Section 5 we characterize the relative Kato condition by means of the parabolic Kato condition studied in [30].

Our main goal is to give applications motivating the use of *relative smallness* in perturbation theory of transition densities, along with a self-contained exposition of some of the relevant techniques. We do not aim at full generality here. Possible and forthcoming generalizations are mentioned in Section 6, where we also give a probabilistic interpretation of our results.

⁽¹⁾ A *different* “relative Kato condition” is used in [7, (4)] and [8, (2)].

2. Algebra of perturbation series. Let $q : \mathbb{R} \times X \rightarrow \mathbb{R}$. The identities we intend to prove below rely merely on changing the order of integration, which is justified if the integrals involved are absolutely convergent or *non-negative*. We shall first consider the latter situation and we will *assume that* $q \geq 0$.

Duhamel’s formula (8) suggests the following definitions. For $s, t \in \mathbb{R}$ and $x, y \in X$, we let $p_0(s, x, t, y) = p(s, x, t, y)$ and

$$(9) \quad p_n(s, x, t, y) = \int_s^t \int_X p_{n-1}(s, x, u, z) p(u, z, t, y) q(u, z) dz du$$

for $n \geq 1$, and we define the *perturbation of p by q* ,

$$(10) \quad \tilde{p}_q(s, x, t, y) = \sum_{n=0}^{\infty} p_n(s, x, t, y), \quad x, y \in X, s, t \in \mathbb{R}.$$

If $s \geq t$, then $p_n(s, x, t, y) = 0$ for every $n \geq 0$ and hence $\tilde{p}_q(s, x, t, y) = 0$.

Since $p(s, x, t, y) = 0$ for $s \geq t$, we could write (9) as

$$p_n(s, x, t, y) = \int_{\mathbb{R}} \int_X p_{n-1}(s, x, u, z) p(u, z, t, y) q(u, z) dz du$$

for *all* $s, t \in \mathbb{R}$ and $x, y \in X$, so the reader should not be alarmed if we occasionally simplify our notation in this way.

LEMMA 1. *For all $s < u < t$, $x, y \in X$, and $n = 0, 1, \dots$,*

$$(11) \quad \sum_{m=0}^n \int_X p_m(s, x, u, z) p_{n-m}(u, z, t, y) dz = p_n(s, x, t, y).$$

Proof. We note that (11) is true for $n = 0$ by (5). Assume that $n \geq 1$ and (11) holds for $n - 1$. The sum of the first n terms in (11) can be dealt with by induction:

$$(12) \quad \begin{aligned} & \sum_{m=0}^{n-1} \int_X p_m(s, x, u, z) p_{n-m}(u, z, t, y) dz \\ &= \sum_{m=0}^{n-1} \int_X p_m(s, x, u, z) \int_u^t \int_X p_{n-1-m}(u, z, r, w) p(r, w, t, y) q(r, w) dw dr dz \\ &= \int_u^t \int_X \left(\sum_{m=0}^{n-1} \int_X p_m(s, x, u, z) p_{n-1-m}(u, z, r, w) dz \right) p(r, w, t, y) q(r, w) dw dr \\ &= \int_u^t \int_X p_{n-1}(s, x, r, w) p(r, w, t, y) q(r, w) dw dr. \end{aligned}$$

By (9), the $(n + 1)$ -st term is

$$\begin{aligned}
 (13) \quad & \int_X p_n(s, x, u, z) p_0(u, z, t, y) dz \\
 &= \int_X \int_s^u \int_X p_{n-1}(s, x, r, w) p(r, w, u, z) q(r, w) dw dr p(u, z, t, y) dz \\
 &= \int_s^u \int_X p_{n-1}(s, x, r, w) p(r, w, t, y) q(r, w) dw dr.
 \end{aligned}$$

and (11) follows on adding (12) and (13). ■

We next prove the Chapman–Kolmogorov equation for $\tilde{p}_q = \sum_{n=0}^\infty p_n$.

LEMMA 2. *For all $s < u < t$ and $x, y \in X$,*

$$\int_X \tilde{p}_q(s, x, u, z) \tilde{p}_q(u, z, t, y) dz = \tilde{p}_q(s, x, t, y).$$

Proof. By Lemma 1,

$$\begin{aligned}
 \int_X \tilde{p}_q(s, x, u, z) \tilde{p}_q(u, z, t, y) dz &= \int_X \sum_{i=0}^\infty p_i(s, x, u, z) \sum_{j=0}^\infty p_j(u, z, t, y) dz \\
 &= \sum_{n=0}^\infty \sum_{m=0}^n \int_X p_m(s, x, u, z) p_{n-m}(u, z, t, y) dz \\
 &= \sum_{n=0}^\infty p_n(s, x, t, y) = \tilde{p}_q(s, x, t, y). \quad \blacksquare
 \end{aligned}$$

We will need the following extension of (9).

LEMMA 3. *For all $n = 1, 2, \dots, m = 0, 1, \dots, n - 1, s, t \in \mathbb{R}$, and $x, y \in X$,*

$$(14) \quad p_n(s, x, t, y) = \int_s^t \int_X p_{n-1-m}(s, x, u, z) p_m(u, z, t, y) q(u, z) dz du.$$

Proof. For $m = 0$, equality (14) holds by definition of p_n . In particular, this proves our claim for $n = 1$. If the claim is proved for some $n \geq 1$, then, for every $m = 1, \dots, n$,

$$\begin{aligned}
 p_{n+1}(s, x, t, y) &= \int_{\mathbb{R}} \int_X p_n(s, x, u, z) p(u, z, t, y) q(u, z) dz du \\
 &= \int_{\mathbb{R}} \int_X \int_{\mathbb{R}} \int_X p_{n-1-(m-1)}(s, x, w, v) p_{m-1}(w, v, u, z) q(v, w) dw dv \\
 &\quad \times p(u, z, t, y) q(u, z) dz du \\
 &= \int_{\mathbb{R}} \int_X p_{n-m}(s, x, w, v) p_m(w, v, t, y) q(v, w) dw dv. \quad \blacksquare
 \end{aligned}$$

3. Estimate from above. In this section we will only consider *relatively bounded* $q \geq 0$. Given $s < t$, we let $I(s, t)$ be the smallest number such that for all $x, y \in X$,

$$(15) \quad \int_s^t \int_X p(s, x, u, z)p(u, z, t, y)q(u, z) dz du \leq I(s, t) p(s, x, t, y).$$

Relative boundedness of q implies that $I(s, t)$ is finite if $t - s$ is small. The following lemma implies that $I(s, t)$ is then finite for *all* $s < t$.

LEMMA 4. $I(s, v) \leq I(s, t) + I(t, v)$ whenever $s < t < v$.

Proof. Let $s < t < v$ and $x, y \in X$. We have

$$\begin{aligned} \int_s^v \int_X p(s, x, u, z)p(u, z, v, y) q(u, z) dz du &= \int_s^t \int_X p(s, x, u, z)p(u, z, t, w)p(t, w, v, y) dw q(u, z) dz du \\ &\quad + \int_t^v \int_X p(s, x, t, w)p(t, w, u, z)p(u, z, v, y) dw q(u, z) dz du \\ &\leq [I(s, t) + I(t, v)] \int_X p(s, x, t, w)p(t, w, v, y) dw \\ &= [I(s, t) + I(t, v)]p(s, x, v, y). \blacksquare \end{aligned}$$

This *subadditivity* and the relative boundedness yield the following.

LEMMA 5. If $\eta > \eta^*(q)$, then there is $\beta \geq 0$ such that

$$(16) \quad \int_s^t \int_X p(s, x, u, z)p(u, z, t, y)q(u, z) dz du \leq [\eta + \beta(t - s)]p(s, x, t, y)$$

whenever $s < t \in \mathbb{R}$ and $x, y \in X$.

Proof. For $\eta > \eta^*(q)$ let $h > 0$ be as in the line preceding (7). If k is a natural number and $s + (k - 1)h < t \leq s + kh$, then $k < 1 + (t - s)/h$, and, by Lemma 4, $I(s, t) \leq k\eta \leq \eta + \eta(t - s)/h$. We can take

$$(17) \quad \beta = \eta/h. \blacksquare$$

Conversely, if (16) holds with some finite η and β , then q is relatively bounded, and $\eta^*(q) \leq \eta$. Also, (16) with $0 \leq \eta < 1$ (and some finite β) characterizes relative smallness, and (16) being true for *every* $\eta \geq 0$ (with some finite β) is equivalent to the relative Kato condition. Thus, our focus in (16) is on the value of η ; the term $\beta(t - s)$ is merely a technically convenient replacement of h .

In the remainder of the section we will assume that (16) holds with $0 \leq \eta < 1$ and (finite) $\beta \geq 0$. For instance, every bounded (nonnegative) q satisfies that assumption with $\eta = 0$ and $\beta = \sup_{\mathbb{R} \times X} q(u, z)$. Indeed,

$$\int_s^t \int_X p(s, x, u, z) p(u, z, t, y) q(u, z) dz du \leq [\sup q](t - s) p(s, x, t, y).$$

We shall need the following identity.

LEMMA 6. *For all $n = 0, 1, \dots$ and $\xi, \nu, \eta \in \mathbb{R}$,*

$$(18) \quad \sum_{m=0}^n \sum_{k=0}^{n-m} \sum_{j=0}^m \binom{n-m}{k} \binom{m}{j} \frac{\xi^k \nu^j}{k! j!} \eta^{n-k-j} = \sum_{r=0}^n \binom{n+1}{r+1} \frac{(\xi + \nu)^r}{r!} \eta^{n-r}.$$

Proof. Let j, r be integers with $0 \leq j \leq r \leq n$. There are $\binom{n+1}{r+1}$ subsets of $\{0, 1, \dots, n\}$ having $r+1$ elements. Such a subset, $\{i_1, \dots, i_{r+1}\}$ with $i_1 < \dots < i_{r+1}$, may be chosen by first fixing $m := i_{j+1} \in \{j, j+1, \dots, n-r+j\}$ and then taking j integers $0 \leq i_1 < \dots < i_j < m$, and $r-j$ integers $m < i_{j+2} < \dots < i_{r+1} \leq n$. This shows that (for every such j)

$$\sum_{j \leq m \leq n-r+j} \binom{m}{j} \binom{n-m}{r-j} = \binom{n+1}{r+1}$$

(cf. [15, (5.26)]). Considering $k = r - j$ we see that the conditions $0 \leq r \leq n$, $0 \leq j \leq r$, $j \leq m \leq n - r + j$ are equivalent to $0 \leq m \leq n$, $0 \leq k \leq n - m$, $0 \leq j \leq m$. Therefore

$$\begin{aligned} \sum_{m=0}^n \sum_{k=0}^{n-m} \sum_{j=0}^m \binom{n-m}{k} \binom{m}{j} \frac{\xi^k \nu^j}{k! j!} \eta^{n-k-j} &= \sum_{r=0}^n \sum_{j=0}^r \sum_{m=j}^{n+j-r} \binom{m}{j} \binom{n-m}{r-j} \frac{\nu^j \xi^{r-j}}{j! (r-j)!} \eta^{n-r} \\ &= \sum_{r=0}^n \binom{n+1}{r+1} \frac{(\nu + \xi)^r}{r!} \eta^{n-r}. \quad \blacksquare \end{aligned}$$

The following result is our main technical observation, justifying the assumption (16).

LEMMA 7. *For all $n = 0, 1, 2, \dots$, $x, y \in X$, and $s, t \in \mathbb{R}$,*

$$(19) \quad p_n(s, x, t, y) \leq p(s, x, t, y) \sum_{k=0}^n \binom{n}{k} \frac{(\beta(t-s))^k}{k!} \eta^{n-k}.$$

Proof. Of course, (19) holds for $n = 0$. By Lemma 3, induction, Lemma 6, and (16),

$$\begin{aligned}
 & (n + 1)p_{n+1}(s, x, t, y) \\
 &= \sum_{m=0}^n \int \int_{s X}^t p_{n-m}(s, x, u, z) p_m(u, z, t, y) q(u, z) dz du \\
 &\leq \int \int_{s X}^t p(s, x, u, z) p(u, z, t, y) q(u, z) dz du \\
 &\quad \times \sum_{m=0}^n \sum_{k=0}^{n-m} \sum_{j=0}^m \binom{n-m}{k} \binom{m}{j} \frac{(\beta(u-s))^k}{k!} \eta^{n-m-k} \frac{(\beta(t-u))^j}{j!} \eta^{m-j} \\
 &= \int \int_{s X}^t p(s, x, u, z) p(u, z, t, y) q(u, z) dz du \sum_{r=0}^n \binom{n+1}{r+1} \frac{(\beta(t-s))^r}{r!} \eta^{n-r} \\
 &\leq p(s, x, t, y) [\eta + \beta(t-s)] \sum_{r=0}^n \binom{n+1}{r+1} \frac{(\beta(t-s))^r}{r!} \eta^{n-r} \\
 &= p(s, x, t, y) \left[\sum_{r=0}^n \binom{n+1}{r} \frac{n+1-r}{r+1} \frac{(\beta(t-s))^r}{r!} \eta^{n+1-r} \right. \\
 &\quad \left. + \sum_{r=1}^{n+1} \binom{n+1}{r} \frac{(\beta(t-s))^r}{r!} \eta^{n+1-r} \right] \\
 &\leq (n+1)p(s, x, t, y) \sum_{r=0}^{n+1} \binom{n+1}{r} \frac{(\beta(t-s))^r}{r!} \eta^{n+1-r}. \blacksquare
 \end{aligned}$$

THEOREM 3. *If q satisfies (16) with $\eta < 1$, then, for all $s < t$ and $x, y \in X$,*

$$(20) \quad \tilde{p}_q(s, x, t, y) \leq \frac{1}{1-\eta} \exp\left(\frac{\beta}{1-\eta}(t-s)\right) p(s, x, t, y).$$

Proof. By (10) and Lemma 7,

$$\tilde{p}_q(s, x, t, y) \leq p(s, x, t, y) \sum_{k=0}^{\infty} \frac{(\beta(t-s))^k}{k!} \sum_{n=k}^{\infty} \binom{n}{k} \eta^{n-k},$$

where

$$\begin{aligned}
 \sum_{n=k}^{\infty} \binom{n}{k} \eta^{n-k} &= \frac{1}{k!} \frac{d^k}{d\eta^k} \sum_{n=0}^{\infty} \eta^n = \frac{1}{k!} \frac{d^k}{d\eta^k} \frac{1}{1-\eta} = \frac{1}{k!} \frac{k!}{(1-\eta)^{k+1}} \\
 &= \frac{1}{(1-\eta)^{k+1}}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 (21) \quad \tilde{p}_q(s, x, t, y) &\leq p(s, x, t, y) \frac{1}{1-\eta} \sum_{k=0}^{\infty} \frac{(\beta(t-s)/(1-\eta))^k}{k!} \\
 &= \frac{1}{1-\eta} \exp\left(\frac{\beta}{1-\eta}(t-s)\right) p(s, x, t, y). \blacksquare
 \end{aligned}$$

4. Small signed perturbations. We will present some immediate consequences of Theorem 3 for *signed* q . Let $q_+ = \max(q, 0)$ and $q_- = \max(-q, 0)$, so that $q = q_+ - q_-$. It will be convenient to consider the following integral kernels over space-time $\mathbb{R} \times X$:

$$\begin{aligned}
 Pf(s, x) &:= P_p f(s, x) := \int \int_{\mathbb{R} \times X} p(s, x, u, z) f(u, z) dz du, \\
 (22) \quad P^q f(s, x) &:= P_p^q f(s, x) := \int \int_{\mathbb{R} \times X} p(s, x, u, z) f(u, z) q(u, z) dz du.
 \end{aligned}$$

For $t \in \mathbb{R}$ and $y \in X$ we have

$$\begin{aligned}
 (P^q p(\cdot, \cdot, t, y))(s, x) &= \int \int_{\mathbb{R} \times X} p(s, x, u, z) p(u, z, t, y) q(u, z) dz du \\
 &= p_1(s, x, t, y), \quad s \in \mathbb{R}, x \in X.
 \end{aligned}$$

Using Lemma 3 we find by induction that, for every natural n ,

$$((P^q)^n p(\cdot, \cdot, t, y))(s, x) = P^q(p_{n-1}(\cdot, \cdot, t, y))(s, x) = p_n(s, x, t, y).$$

For ease of notation, we can write $p_n(s, x, t, y) = (P^q)^n p(s, x, t, y)$, or even $p_n = (P^q)^n p$. In view of (10) we define (for signed q)

$$(23) \quad \tilde{p}_q = \sum_{n=0}^{\infty} (P^q)^n p,$$

whenever the integrals and the sum are nonnegative or absolutely convergent. Given (signed) q_1 and q_2 we consider the effect of two consecutive perturbations,

$$\widetilde{(\tilde{p}_{q_1})}_{q_2} := \sum_{n=0}^{\infty} (P_{\tilde{p}_{q_1}}^{q_2})^n \tilde{p}_{q_1}.$$

Here $P_{\tilde{p}_{q_1}}^{q_2} f(s, x) = \int_{\mathbb{R}} \int_X \tilde{p}_{q_1}(s, x, u, z) f(u, z) q_2(u, z) dz du$ (cf. (22)). The following is a special case of a general result on perturbations of integral kernels (see, e.g., [28, Problem 1.13]).

LEMMA 8. *Under the assumptions of nonnegativity or absolute convergence,*

$$(24) \quad \widetilde{(\tilde{p}_{q_1})}_{q_2} = \tilde{p}_{q_1+q_2}.$$

Proof. It is not hard to verify that

$$P_{\widetilde{p}_{q_1}}^{q_2} = \sum_{k=0}^{\infty} (P^{q_1})^k P^{q_2}.$$

It follows that

$$\begin{aligned} \widetilde{(\widetilde{p}_{q_1})}_{q_2} &= \sum_{n=0}^{\infty} (P_{\widetilde{p}_{q_1}}^{q_2})^n \widetilde{p}_{q_1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} (P^{q_1})^k P^{q_2} \right)^n \sum_{l=0}^{\infty} (P^{q_1})^l p \\ &= \sum_{n=0}^{\infty} \sum_{\sigma \in \{1,2\}^n} P^{q_{\sigma(1)}} \dots P^{q_{\sigma(n)}} p = \widetilde{p}_{q_1+q_2}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 2. We assume that $\eta^*(|q|) < 1$, in particular $\eta^*(q_-) < 1$ and $\eta^*(q_+) < 1$. By Theorem 3, $\widetilde{p}_{q_-} = \sum_{n=0}^{\infty} (P^{q_-})^n p$ is convergent, hence

$$\widetilde{p}_{-q_-} = \sum_{n=0}^{\infty} (-1)^n (P^{q_-})^n p$$

is well defined and $|\widetilde{p}_{-q_-}| \leq \widetilde{p}_{q_-}$. Therefore the arguments of Section 2 apply, in particular \widetilde{p}_{-q_-} satisfies the Chapman–Kolmogorov equation (see Lemma 2). We will prove that $\widetilde{p}_{-q_-} \geq 0$. Indeed, let $\eta^*(q_-) < \eta < 1$ and let $h > 0$ be as required between (6) and (7), that is, for all $s < t \leq s + h$,

$$P^{q_-} p(s, x, t, y) \leq \eta p(s, x, t, y) \leq p(s, x, t, y), \quad x, y \in X.$$

Then $(P^{q_-})^{n+1} p(s, x, t, y) \leq \eta (P^{q_-})^n p(s, x, t, y)$ for $n = 1, 2, \dots$, and hence

$$(25) \quad \widetilde{p}_{-q_-} = (p - P^{q_-} p) + ((P^{q_-})^2 p - (P^{q_-})^3 p) + \dots \geq (1 - \eta)p.$$

In particular, $\widetilde{p}_{-q_-}(s, x, t, y) \geq 0$ provided $s < t < s + h$. By the Chapman–Kolmogorov equation, $\widetilde{p}_{-q_-}(s, x, t, y)$ is nonnegative for all times s, t . Also,

$$\widetilde{p}_{-q_-} = p - (P^{q_-} p - (P^{q_-})^2 p) - ((P^{q_-})^3 p - (P^{q_-})^4 p) + \dots \leq p$$

provided $s < t < s + h$. Therefore,

$$(26) \quad \widetilde{p}_{-q_-}(s, x, t, y) \leq p(s, x, t, y) \quad \text{for all } s, t \in \mathbb{R}, x, y \in X.$$

This further shows that the perturbation of \widetilde{p}_{-q_-} by q_+ is well defined, and $\widetilde{(\widetilde{p}_{-q_-})}_{q_+} \leq \widetilde{p}_{q_+}$ (cf. (9)). On the other hand, the (nonnegative) series defining $\widetilde{p}_{|q|}$ is convergent. Thus, by Lemma 8 and the considerations above,

$$\widetilde{p}_{-q_-} \leq \widetilde{(\widetilde{p}_{-q_-})}_{q_+} = \widetilde{p}_q \leq \widetilde{p}_{q_+}.$$

We note that if n is a natural number and $s < t < s + nh$, then Chapman–Kolmogorov, (25), Theorem 3, and (17) imply that, for all $x, y \in X$,

$$(27) \quad (1 - \eta)^n \leq \frac{\widetilde{p}_q(s, x, t, y)}{p(s, x, t, y)} \leq \frac{1}{1 - \eta} \exp \frac{n\eta}{1 - \eta}.$$

By Theorem 3 (applied to $|q|$), the series $\sum_{n=0}^{\infty} (P^q)^n p$ is absolutely convergent and hence

$$(I - P^q)\tilde{p}_q = \sum_{n=0}^{\infty} (P^q)^n p - \sum_{n=0}^{\infty} (P^q)^{n+1} p = p.$$

Thus $\tilde{p} = \tilde{p}_q$ solves (8).

To prove the uniqueness of the solution, let \tilde{p} be any transition density which locally in time is comparable with p . Then the integral in (8) is absolutely convergent. This follows from Lemma 5 (note that the domain of integration in (8) is merely $(s, t) \times X$). Therefore $(I - P^q)\tilde{p} = p$ and

$$\tilde{p} = \sum_{n=0}^{\infty} (P^q)^n (I - P^q)\tilde{p} = \sum_{n=0}^{\infty} (P^q)^n p = \tilde{p}_q.$$

If $\eta^*(|q|) = 0$ then we can have $0 < \eta < 1$ arbitrarily small in the above discussion, therefore \tilde{p}_q and p are asymptotically equal by (27). ■

Let $\tilde{P} = P_{\tilde{p}}$, that is, $\tilde{P}f(s, x) = \int_{\mathbb{R}} \int_X \tilde{p}(s, x, u, z) f(u, z) dz du$, where $\tilde{p} = \tilde{p}_q$. Note that

$$(28) \quad \tilde{P} = \sum_{n=0}^{\infty} (P^q)^n P.$$

Following [30], [16], we will say that $q : \mathbb{R} \times X \rightarrow \mathbb{R}$ belongs to the parabolic (space-time) *Kato class* for p if

$$(29) \quad \lim_{h \rightarrow 0^+} \sup_{s \in \mathbb{R}, x \in X} \int_X \int_s^{s+h} p(s, x, u, z) |q(u, z)| du dz = 0$$

and

$$(30) \quad \lim_{h \rightarrow 0^+} \sup_{t \in \mathbb{R}, y \in X} \int_X \int_{t-h}^t p(u, z, t, y) |q(u, z)| du dz = 0.$$

We say that p is *probabilistic* if

$$(31) \quad \int_X p(s, x, t, y) dy = 1 \quad \text{for } s < t, x \in X.$$

LEMMA 9. *If p is probabilistic, $p(s, x, t, y) = p(s, y, t, x)$ for $s, t \in \mathbb{R}$, $x, y \in X$, and q is relatively Kato for p , then q is in the parabolic Kato class.*

Proof. Let $h, \eta \geq 0$, and assume that for all $x, y \in \mathbb{R}^d$ and $s < t \leq s + h$,

$$\int_s^t \int_X p(s, x, u, z) p(u, z, t, y) |q(u, z)| dz du \leq \eta p(s, x, t, y).$$

Integrating this with respect to dy we obtain

$$\int_s^t \int_X p(s, x, u, z) |q(u, z)| dz du \leq \eta.$$

Thus, $\eta^*(|q|) = 0$ yields (29). Integrating with respect to dx instead, we obtain (30). ■

We note that Corollary 11 below gives a special converse to Lemma 9.

In view of Theorem 3, the *relative Kato condition* seems more intrinsic to Schrödinger perturbations than the parabolic Kato condition (see [16, Lemma 5.2] and [27, Theorem 3.10]), but the former may be cumbersome to verify in specific situations. For instance, the relative Kato condition for the Gaussian kernel is rather difficult to explicitly characterize (as opposed to that for the transition density of $\Delta^{\alpha/2}$, $0 < \alpha < 2$, see below). It should be noted that each transition density p determines a specific class of relatively Kato functions q , and a detailed analysis is required to exhibit interesting (unbounded) q . We also remark that the relative Kato condition may be interpreted as a Kato condition for *bridges* (see Section 6).

5. Examples. In this section we assume that $X = \mathbb{R}^d$, $d \geq 1$, \mathcal{M} is the σ -algebra of Borel subsets of \mathbb{R}^d , and dz is the Lebesgue measure on \mathbb{R}^d . Let $\alpha \in (0, 2)$. Our aim is to discuss the transition density, $\tilde{p}(s, x, t, y)$, of $\Delta^{\alpha/2} + q$, where q is relatively small. To state our estimates of \tilde{p} , we specialize to

$$(32) \quad p(s, x, t, y) = p_{t-s}(y - x),$$

where $x, y \in \mathbb{R}^d$, $s < t$, and p_t is the smooth real-valued function on \mathbb{R}^d determined by

$$(33) \quad \int_{\mathbb{R}^d} p_t(z) e^{iz \cdot \xi} dz = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d, t > 0.$$

In particular, for $\alpha = 1$ we have

$$(34) \quad p_t(z) = \Gamma((d + 1)/2) \pi^{-(d+1)/2} \frac{t}{(|z|^2 + t^2)^{(d+1)/2}},$$

the Cauchy convolution semigroup. Note that for every $\alpha \in (0, 2)$,

$$(35) \quad p_t(z) = t^{-d/\alpha} p_1(t^{-1/\alpha} z), \quad t > 0, z \in \mathbb{R}^d.$$

This follows from (33). We let $p(s, x, t, y) = 0$ if $s \geq t$. By the definitions, p is time- and space-homogeneous: for all $s, t, h \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^d$ we have

$$p(s, x, t, y) = p(s + h, x + z, t + h, y + z).$$

The semigroup $P_t f(x) = \int_{\mathbb{R}^d} f(y) p_t(y - x) dy$ has $\Delta^{\alpha/2}$ as infinitesimal generator ([1], [29], [3], [22]). Referring to the abstract, $p(s, x, t, y)$ is the funda-

mental solution of $\partial_t - \Delta_y^{\alpha/2}$ in the sense of distributions ([3]):

$$(36) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^d} p(s, x, t, y) [\partial_t + \Delta_y^{\alpha/2}] \phi(t, y) dy dt = -\phi(s, x),$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$. Here $C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ is the class of all infinitely differentiable compactly supported functions on $\mathbb{R} \times \mathbb{R}^d$, and

$$\begin{aligned} \Delta^{\alpha/2} \varphi(y) &= \lim_{t \downarrow 0} \frac{P_t \varphi(y) - \varphi(y)}{t} \\ &= \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|} \lim_{\varepsilon \downarrow 0} \int_{\{|z| > \varepsilon\}} \frac{\varphi(y + z) - \varphi(y)}{|z|^{d+\alpha}} dz, \quad y \in \mathbb{R}^d. \end{aligned}$$

A simple proof of (36) can be given by using Fourier transform in the space variable, and (33) (we omit the details).

We will assume that q is relatively small for p . Let $(L\phi)(t, y) = \partial_t \phi(t, y) + \Delta_y^{\alpha/2} \phi(t, y)$. We also introduce $(Q\phi)(t, y) = q(t, y)\phi(t, y)$, the operation of multiplication by q . Referring to our previous notation we have $P^q = PQ$, and (28) now reads

$$(37) \quad \tilde{P} = \sum_{n=0}^{\infty} (PQ)^n P.$$

We can interpret (36) as

$$(38) \quad PL\phi = -\phi \quad (\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)).$$

This implies that

$$(39) \quad \tilde{P}(L + Q)\phi = -\phi \quad (\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)).$$

Indeed, by (38),

$$\tilde{P}(L + Q)\phi = PL\phi + \sum_{n=1}^{\infty} (PQ)^n PL\phi + \sum_{n=0}^{\infty} (PQ)^{n+1} \phi = -\phi.$$

The associativity of the operations involved, which we have used freely above, follows from Fubini's theorem. Indeed, $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ and $L\phi$ are bounded by a constant multiple of $p(\cdot, \cdot, t_0, y_0)$ for some $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^d$ (see the remarks on (8) in the proof of Theorem 2). This proves (39):

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \tilde{p}(s, x, t, y) [\partial_t \phi(t, y) + \Delta_y^{\alpha/2} \phi(t, y) + q(t, y)\phi(t, y)] dy dt = -\phi(s, x),$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$. In fact, $(s, x, t, y) \mapsto \tilde{p}(s, x, t, y)$ is continuous, except when $s = t$. Indeed, continuity is first proved inductively for each p_n in (10), by using an argument of uniform integrability. We omit the details of the proof, and refer the reader to a similar argument in the proof of [5, Lemma 14] (see also Lemma 9 and (19) above). The continuity

of $\tilde{p} = \tilde{p}_q$ then follows from the locally (in time) uniform convergence of the series in (21).

Proof of Theorem 1. We consider the Gaussian transition density g introduced in Section 1. This corresponds to $\alpha = 2$ in (32), and an analogue of (36) holds for g and the Laplacian with a similar proof. The above discussion of the fractional Laplacian also applies to the Laplacian, provided q is relatively small with respect to g . ■

Apart from obvious similarities, there exist important differences between p ($0 < \alpha < 2$) and g ($\alpha = 2$). For instance the global decay of p in space is qualitatively different from that of g . In fact, we have the following estimate of p (cf. (34) and see, e.g., [6] for a proof).

LEMMA 10. *There exists $c = c(d, \alpha)$ such that, for all $z \in \mathbb{R}^d$ and $t > 0$,*

$$c^{-1} \left(\frac{t}{|z|^{d+\alpha}} \wedge t^{-d/\alpha} \right) \leq p_t(z) \leq c \left(\frac{t}{|z|^{d+\alpha}} \wedge t^{-d/\alpha} \right).$$

This *power-type* asymptotics yields the following 3P Theorem ([5]).

THEOREM 4. *There exists a constant $c = c(d, \alpha)$ such that*

$$p(s, x, u, z) \wedge p(u, z, t, y) \leq cp(s, x, t, y) \quad \text{for } x, z, y \in X, s, u, t \in \mathbb{R}.$$

For numbers $a, b \geq 0$ we have $ab = (a \vee b)(a \wedge b)$ and $a \vee b \leq a + b$. Therefore Theorem 4 yields the following variant form with five occurrences of p :

$$(40) \quad p(s, x, u, z)p(u, z, t, y) \leq cp(s, x, t, y)[p(s, x, u, z) + p(u, z, t, y)].$$

From this and Lemma 9 we immediately obtain the following consequence.

COROLLARY 11. *For the transition density of the fractional Laplacian $\Delta^{\alpha/2}$ with $0 < \alpha < 2$, the parabolic Kato class equals the relative Kato class.*

For instance, if $\alpha < d$, then Lemma 10 (see also (35)) yields

$$\int_0^h p_t(z) dt \approx |z|^{\alpha-d} \wedge [h^2|z|^{-d-\alpha}], \quad z \in \mathbb{R}^d, h > 0.$$

If $0 < \varepsilon \leq \alpha$, $|q(u, z)| \leq |z|^{-\alpha+\varepsilon}$, $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and we let $h \rightarrow 0$, then

$$\int_s^{s+h} \int_{\mathbb{R}^d} p(s, x, u, z)|q(u, z)| dz du \leq \int_{\mathbb{R}^d} |z|^{-\alpha+\varepsilon} \int_0^h p_t(z) dt dz \rightarrow 0.$$

Therefore each such q belongs to the parabolic Kato class, and so it is relatively Kato. Let us note that the local (in time) comparability of the Schrödinger semigroups of the fractional Laplacian for q in the (parabolic) Kato class is a new result even in the autonomous case of [3]. We also refer the reader to [31] for recent Gaussian results, with a definition [31, (1.2)] not

unrelated to our relative Kato condition (see also [10, Definition 3.1] and [16, Lemma 5.2], [27, Theorem 3.10]).

For a study of other consequences of the Kato condition for autonomous additive perturbations we refer the reader to [32], [4], [5].

6. Further discussion. For the sake of clarity, let us add a comment on the lower bound in (27). If q is relatively Kato for g (see Section 1) then for $\tilde{g} = \tilde{g}_q$ we have

$$(41) \quad \frac{g_q(s, x, t, y)}{g(s, x, t, y)} \geq \exp\left(-\int_s^t \int_X \frac{g(s, x, u, z)g(u, z, t, y)}{g(s, x, t, y)} q_-(u, z) dz du\right).$$

An analogous estimate holds for the transition density p of the fractional Laplacian $\Delta^{\alpha/2}$. These results follow from the fact that

$$R_\lambda := (I + \lambda P^{q-})^{-1} P^{q-}, \quad \lambda > 0,$$

is a sub-Markov resolvent (of P^{q-}), a unique kernel satisfying

$$R_\lambda + \lambda P^{q-} R_\lambda = P^{q-}.$$

For further background, we refer the reader to [2, 7.2–7.7]. By an argument of log-convexity (see, e.g., [20] or [18, p. 429], and also [2, 8.1–8.2]), we obtain (41). It is noteworthy that one only needs relative boundedness of q_- to obtain satisfactory lower bounds for \tilde{p} , as defined by (8).

We omit the (standard) details for two reasons. Firstly, our emphasis in this paper is on upper bounds, or non-explosion results. Secondly, in view of possible generalizations mentioned below it seems economical to postpone the full discussion to a future paper. We remark that, in principle, the lower bound (41) is well known (see, e.g., [11], [3]).

We now comment on possible and forthcoming generalizations of our results. It is possible to extend the present results to more general integral kernels or to measures (rather than functions q ; see [20], [26], [16] for a related study). In fact, considering $q(dudz) = \eta \delta_{u_0}(du)dz$, where $\eta \geq 1$ and δ_{u_0} is the probability measure concentrated at u_0 , shows that \tilde{p}_q may explode in finite time u_0 .

The technique based on Theorem 3 applies to more general *additive* perturbations (of the generator). In studying these, one should attempt a natural and general description (in terms of p) of a class of perturbations which lead to comparability theorems. In this connection we refer to [25] for a discussion of *nonlocal perturbations* of the fractional Laplacian, and to [5] for a study of *gradient perturbations* of $\Delta^{\alpha/2}$ under the assumption that $1 < \alpha < 2$.

There is a deep well-known connection of Schrödinger operators to the theory of multiplicative functionals of Markov processes (see, e.g., [16]). We now discuss this connection in the case of the Wiener process Y in \mathbb{R}^d ,

defined by the transition density g of Section 1 (note that $Y_t = B_{2t}$, where B is the standard Brownian motion). Let $\mathbb{E}_{s,x}$ and $\mathbb{P}_{s,x}$ be respectively the expectation and distribution of the process starting at the point $x \in \mathbb{R}^d$ at time $s \in \mathbb{R}$, so that $\mathbb{P}_{s,x}[Y_t \in A] = \int_A g(s, x, t, y) dy$, where Y_t is the canonical continuous coordinate process evaluated at time $t > s$. For $y \in \mathbb{R}^d$ we let $\mathbb{E}_{s,x}^{t,y}$ and $\mathbb{P}_{s,x}^{t,y}$ denote, respectively, the expectation and distribution of the process starting at x at time s and conditioned to reach y at time t (Brownian bridge). The process is defined by the transition probability density function $r(u_1, z_1, u_2, z_2) = g(u_1, z_1, u_2, z_2)g(u_2, z_2, t, y)/g(u_1, z_1, t, y)$, where $s \leq u_1 < u_2 < t$ and $z_1, z_2 \in \mathbb{R}^d$. Thus, the finite-dimensional distributions are given by the density functions

$$(42) \quad \frac{g(s, x, u_1, z_1)g(u_1, z_1, u_2, z_2) \dots g(u_n, z_n, t, y)}{g(s, x, t, y)},$$

and we have the following *disintegration* of $\mathbb{P}_{s,x}$:

$$(43) \quad \mathbb{P}_{s,x}(Y_{u_1} \in A_1, \dots, Y_{u_n} \in A_n; Y_t \in B) = \int_B \mathbb{P}_{s,x}^{t,y}(Y_{u_1} \in A_1, \dots, Y_{u_n} \in A_n)g(s, x, t, y) dy.$$

Here $x, z_1, \dots, z_n, y \in \mathbb{R}^d$, $s \leq u_1 < \dots < u_n < t$, and $A_1, \dots, A_n \subset \mathbb{R}^d$ are Borel sets. Consider the *multiplicative functional* ([11]) $e_q(s, t) = \exp(\int_s^t q(u, Y_u) du)$. We have

$$\mathbb{E}_{s,x}^{t,y} e_q(s, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_{s,x}^{t,y} \left(\int_s^t q(u, Y_u) du \right)^n.$$

According to (9) and (42),

$$(44) \quad \mathbb{E}_{s,x}^{t,y} \int_s^t q(u, Y_u) du = \int_s^t \int_{\mathbb{R}^d} \frac{g(s, x, u, z)q(u, z)g(u, z, t, y)}{g(s, x, t, y)} dz du = \frac{g_1(s, x, t, y)}{g(s, x, t, y)}.$$

Furthermore,

$$\begin{aligned} \mathbb{E}_{s,x}^{t,y} \frac{1}{2} \left(\int_s^t q(u, Y_u) du \right)^2 &= \mathbb{E}_{s,x}^{t,y} \int_s^t \int_s^t q(u, Y_u)q(v, Y_v) dv du \\ &= \int_s^t \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(s, x, u, z)g(u, z, v, w)g(v, w, t, y)}{g(s, x, t, y)} q(u, z)q(v, w) dw dz dv du \\ &= \int_s^t \int_{\mathbb{R}^d} \frac{g(s, x, u, z)g_1(u, z, t, y)}{g(s, x, t, y)} q(u, z) dz du = \frac{g_2(s, x, t, y)}{g(s, x, t, y)}. \end{aligned}$$

By induction, for every $n = 0, 1, \dots$,

$$\frac{1}{n!} \mathbb{E}_{s,x}^{t,y} \left(\int_s^t q(u, Y_u) du \right)^n = \frac{g_n(s, x, t, y)}{g(s, x, t, y)},$$

hence

$$\mathbb{E}_{s,x}^{t,y} e_q(s, t) = \frac{\tilde{g}_q(s, x, t, y)}{g(s, x, t, y)}.$$

We may interpret $\tilde{g}_q(s, x, t, y)/g(s, x, t, y)$ as the eventual inflation of mass of the Brownian particle moving from (s, x) to (t, y) . The mass grows multiplicatively where $q > 0$, and decreases where $q < 0$. Thus we may consider the results of this paper as uniform bounds for this mass.

The following example illustrates inequality (20). For a general transition density p we consider a function $q(u, z) = q(u)$, depending only on time and locally integrable in time. It easily follows from (9) that

$$\frac{\tilde{p}_q(s, x, t, y)}{p(s, x, t, y)} = \exp\left(\int_s^t q(u) du\right).$$

Note that $\eta^*(|q|) = 0$ in this example, whilst the emphasis in Theorem 3 is on $\eta^*(|q|) > 0$.

In view of (44) and (7), the relative Kato class may be considered as a Kato class for the *conditional* processes (bridges); see (29, 30).

Acknowledgments. We thank Panki Kim and the referees of the paper for discussion, corrections and insightful suggestions. The results of this paper were presented at the 2nd International Conference on Stochastic Analysis and Its Applications, May 28–31, 2008 at Seoul National University.

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Received September 4, 2007
Revised version September 22, 2008

(6210)