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Anisotropic classes of homogeneous pseudodifferential symbols

by

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Abstract. We define homogeneous classes of x-dependent anisotropic symbols $\dot{S}^m_{\gamma,\delta}(A)$ in the framework determined by an expansive dilation A, thus extending the existing theory for diagonal dilations. We revisit anisotropic analogues of Hörmander–Mikhlin multipliers introduced by Rivière [Ark. Mat. 9 (1971)] and provide direct proofs of their boundedness on Lebesgue and Hardy spaces by making use of the well-established Calderón–Zygmund theory on spaces of homogeneous type. We then show that x-dependent symbols in $\dot{S}^0_{1,1}(A)$ yield Calderón–Zygmund kernels, yet their L^2 boundedness fails. Finally, we prove boundedness results for the class $\dot{S}^m_{1,1}(A)$ on weighted anisotropic Besov and Triebel–Lizorkin spaces extending isotropic results of Grafakos and Torres [Michigan Math. J. 46 (1999)].

1. Introduction: definitions, examples, notation. Multiplier operators, and more generally pseudodifferential operators, continue to attract attention due to their wide applications in the study of partial differential equations and signal analysis. Several classes of isotropic pseudodifferential symbols attached to such operators, in both linear and multilinear setting, are nowadays well understood. Among them we highlight the prominent role played by the classical Hörmander–Mikhlin multipliers [16], [21], their space dependent counterparts—the Coifman–Meyer symbols [8], or more generally the so-called classical classes of symbols $S^m_{\gamma,\delta}$ or their homogeneous counterparts $\dot{S}^m_{\gamma,\delta}$.

We start by recalling the definition of the isotropic classes of homogeneous symbols $\dot{S}^m_{\gamma,\delta}$; see for example the work of Grafakos and Torres [15]. We say that a symbol σ belongs to the class $\dot{S}^m_{\gamma,\delta}$ if

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha\beta}|\xi|^{m+\delta|\alpha|-\gamma|\beta|} \tag{1.1}$$

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for all multi-indices α, β, all $\xi \in \mathbb{R}^n$, and some positive constants $C_{\alpha\beta}$. In particular, if the symbol σ is x-independent, we refer to it as a *multiplier*. A multiplier $\sigma(\xi)$ belongs to the class $\dot{S}^m_{\gamma,0}$, or simply $\dot{S}^m_\gamma$, if

$$|\partial_\xi^\beta \sigma(\xi)| \leq C_\beta |\xi|^{m-\gamma|\beta|} \tag{1.2}$$

for all multi-indices β, all $\xi \in \mathbb{R}^n$, and some positive constants C_β. The nonhomogeneous version of these classes is obtained by replacing the quantity $|\xi|$ with $1 + |\xi|$. For the remainder of this paper, the absence of the dot will refer to the nonhomogeneous version of a given class of symbols.

In the early 1970s Rivière extended the theory of singular integrals to operators with kernels that satisfy a homogeneity given by a one-parameter group of transformations. His work [22] anticipated future developments surrounding what is nowadays known as the Calderón–Zygmund theory on spaces of homogeneous type. Some of the motivation for the study of such spaces and operators acting on them comes from partial differential equations where several differential operators, such as the heat operator, satisfy an anisotropic homogeneity. Of particular interest was therefore the study of the boundedness properties of homogeneous multiplier operators; see, for example, [22], and the works of Madych and Rivière [20] and Seeger [23]. In the context of operators with x-dependent nonhomogeneous anisotropic symbols, several boundedness results are known, for instance, from the works of Garello [13], Lascar [18], Leopold [19], and Yamazaki [29, 30]. As we will indicate below, the setting used by the latter authors involves diagonal dilations. However, the study of pseudodifferential operators with x-dependent anisotropic symbols associated with more general expansive dilations has not been previously explored.

In this paper we introduce and investigate the appropriate notion of *anisotropic class of multipliers* $\dot{S}^m_\gamma(A)$, and more generally of *anisotropic class of symbols* $\dot{S}^m_{\gamma,\delta}(A)$, associated to an expansive matrix A. We search for a definition analogous to the isotropic one stated above. We need to set up first some of the standard notation, which we borrow from Bownik's monograph [3]; see also [4], [6]. Given an *expansive matrix* A, that is, a matrix all of whose eigenvalues λ satisfy $|\lambda| > 1$, we can first define a canonical quasi-norm ρ_A associated to it. Specifically, if we let P be some nondegenerate $n \times n$ matrix, and $|\cdot|$ the standard norm of $\mathbb{R}^n$, there exists an *ellipsoid* $\varDelta = \{x \in \mathbb{R}^n : |Px| < 1\}$ such that $|\varDelta| = 1$ and $\varDelta \subset r\varDelta \subset A\varDelta$ for some $r > 1$. Then we can define a family of dilated balls around the origin $B_k = A^k\varDelta$, $k \in \mathbb{Z}$, that satisfy

$$B_k \subset rB_k \subset B_{k+1} \quad \text{and} \quad |B_k| = b^k,$$

where $b = |\det A|$. The *step homogeneous quasi-norm* induced by A is defined by

$$\rho(x) = b^j \quad \text{for } x \in B_{j+1} \setminus B_j, \quad \text{and} \quad \rho(0) = 0.$$

It is straightforward to verify that ρ satisfies a triangle inequality up to a constant and the homogeneity condition $\rho(Ax) = b\rho(x)$, $x \in \mathbb{R}^n$. It is known that any two homogeneous quasi-norms associated to a dilation A are equivalent; therefore, we can talk about a *canonical quasi-norm associated to* A, which we denote by ρ_A. Moreover, endowed with the quasi-norm ρ_A and the Lebesgue measure, $\mathbb{R}^n$ becomes a space of homogeneous type. Similarly we shall consider a family of dilated balls B_k^*, $k \in \mathbb{Z}$, and a canonical quasi-norm ρ_{A^*} associated with the transposed dilation A^*.

DEFINITION 1.1. We say that a bounded symbol $\sigma(x,\xi)$ belongs to the *homogeneous anisotropic class* $\dot{S}^m_{\gamma,\delta}(A)$ if it satisfies the estimates

$$|\partial_x^\alpha \partial_\xi^\beta [\sigma(A^{-k_1}\cdot, (A^*)^{k_2}\cdot)](A^{k_1}x, (A^*)^{-k_2}\xi)| \le C_{\alpha,\beta}\rho_{A^*}(\xi)^m \tag{1.3}$$

for all multi-indices α, β and $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Here, $k_1, k_2 \in \mathbb{Z}$ are given by

$$k_1 = \lfloor k\delta \rfloor, \qquad k_2 = \lfloor k\gamma \rfloor, \tag{1.4}$$

where $k \in \mathbb{Z}$ is such that $\rho_{A^*}(\xi) \sim |\det A|^k$.

The derivatives above should be interpreted as

$$\partial_x^\alpha \partial_\xi^\beta \tilde{\sigma}(A^{k_1}x, (A^*)^{-k_2}\xi),$$

where

$$\tilde{\sigma}(x,\xi) = \sigma(A^{-k_1}x, (A^*)^{k_2}\xi),$$

and $k_1, k_2 \in \mathbb{Z}$ are as in the previous definition.

The notation $\sim$ has the following interpretation: we pick k to be the unique integer such that the frequency variable ξ belongs to the annulus $B_{k+1}^* \setminus B_k^*$. We would like to point out that, for a general expansive matrix A, we need to require estimates that hold uniformly after rescaling to scale zero. This is intuitively clear, due to the definition of the quasi-norm induced by the adjoint matrix. As we shall soon see, however, this apparently small detail will translate into certain technical difficulties in our proofs.

When our symbol is x-independent, we will again refer to it as multiplier and simply write $\dot{S}^m_\gamma(A)$ for the corresponding class. At a first glance, Definition 1.1 might seem rather obscure. Nevertheless, it is not hard to see that in the isotropic case, that is, when $A = 2I_n$ (I_n is the $n \times n$ identity matrix), Definition 1.1 yields the isotropic class $\dot{S}^m_{\gamma,\delta}$. Indeed, in this case we have $\rho_{A^*}(\xi) = |\xi|^n$ and we simply need to observe that our uniform estimates (1.3) are those in (1.1) written for $|\xi| \sim 2^k$. That is, the isotropic class $\dot{S}^m_{\gamma,\delta}$ coincides with $\dot{S}^{m/n}_{\gamma,\delta}(2I_n)$. Note that the rescaling of parameter m by a factor of $1/n$ is an artifact of our definition of a quasi-norm. To be consistent with the isotropic definition of this class of symbols, one must require that the quasi-norm associated to A^* satisfies the homogeneity condition $\rho_{A^*}(A^*\xi) = |\det A|^{1/n}\rho_{A^*}(\xi)$ instead. When $A = 2I_n$ this leads to the quasi-norm $\rho_{A^*}(\xi) = |\xi|$ and the Definition 1.1 yields the isotropic class $\dot{S}^m_{\gamma,\delta}$.

More generally, suppose that the dilation A is diagonal,

$$A = \begin{pmatrix} \lambda^{a_1} & 0 & \dots & 0 \\ 0 & \lambda^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{a_n} \end{pmatrix},$$

where $\lambda > 1$, $a_1, \dots, a_n > 0$ and $a_1 + \dots + a_n = na$. Consider

$$\rho_A(x_1, \dots, x_n) = (|x_1|^{2/a_1} + \dots + |x_n|^{2/a_n})^{a/2}.$$

It is easy to check that ρ_A is a quasi-norm associated with the dilation A. In particular, we have the homogeneity condition

$$\rho_A(Ax) = \rho_A(\lambda^{a_1}x_1, \dots, \lambda^{a_n}x_n) = \lambda^a\rho_A(x) = |\det A|^{1/n}\rho_A(x).$$

Alternatively, we could have chosen

$$\rho_A(x_1, \dots, x_n) = \max_{1\le j\le n} |x_j|^{a/a_j}.$$

Pick now $\xi \in B^*_{k+1} \setminus B^*_k$ for some $k \in \mathbb{Z}$, that is, $\rho_{A^*}(\xi) \sim \lambda^{ak}$. Then

$$|\partial_x^\alpha\partial_\xi^\beta[\sigma(A^{-k_1}\cdot, (A^*)^{k_2}\cdot)](A^{k_1}x, (A^*)^{-k_2}\xi)| \\ = \lambda^{-k_1\sum_{j=1}^n a_j\alpha_j + k_2\sum_{j=1}^n a_j\beta_j}|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)|.$$

Therefore, using (1.4), we see that estimates (1.3) take the more familiar form

$$|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)| \lesssim C_{\alpha,\beta}[\rho_{A^*}(\xi)]^{m+\delta\|\alpha\|-\gamma\|\beta\|}, \tag{1.5}$$

where we denoted

$$\|\alpha\| = \frac{1}{a}\sum_{j=1}^n a_j\alpha_j, \qquad \|\beta\| = \frac{1}{a}\sum_{j=1}^n a_j\beta_j.$$

Estimates (1.5) define the so-called *homogeneous class* $\dot{S}^m_{a;\gamma,\delta}$; the corresponding nonhomogeneous version of this class, $S^m_{a;\gamma,\delta}$, was previously investigated in [13, 18, 19, 29, 30]. Our definition has the following advantage: for a general matrix A, say one that has some nontrivial Jordan blocks, the action of A on $\mathbb{R}^n$ could be rather complex, and the diagonal version employed by these authors does not capture the anisotropy of all directions. We also recover the nonhomogeneous class introduced by these authors with a straightforward adaptation of the previous definition. At least in the diagonal case, this definition is powerful enough to recover known properties of boundedness, symbolic calculus and microlocal analysis of classical Hörmander classes of symbols; see again [13, 18, 19, 29, 30] and the references therein. The relevant boundedness properties of certain nonhomogeneous classes of symbols associated to a general expansive matrix A are investigated in our complementary work [1]. The anisotropic nonhomogeneous class of symbols is simply the smoothed out version at $\xi = 0$ of the homogeneous one.

Definition 1.2. We say that a bounded symbol $\sigma(x,\xi)$ belongs to the *nonhomogeneous anisotropic class* $S^m_{\gamma,\delta}(A)$ if it satisfies the estimates

$$|\partial_x^\alpha \partial_\xi^\beta [\sigma(A^{-k_1}\cdot,(A^*)^{k_2}\cdot)](A^{k_1}x,(A^*)^{-k_2}\xi)| \le C_{\alpha,\beta}(1+\rho_{A^*}(\xi))^m \tag{1.6}$$

for all multi-indices α,β and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Here, $k_1,k_2 \in \mathbb{Z}$ are given by (1.4), where $k \in \mathbb{N}$ is such that $1+\rho_{A^*}(\xi) \sim |\det A|^k$.

Associated to any symbol $\sigma(x,\xi)$ we have a pseudodifferential operator

$$(\sigma(x,D)f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi; \tag{1.7}$$

here, $\widehat{f} = \mathcal{F}f$ denotes the Fourier transform of f. When the symbol is x-independent, we simply write $\sigma(D)$ and refer to it as a *multiplier operator*. Because of the $\xi = 0$ singularity of the symbol, it is natural to consider the operator $\sigma(x,D)$ initially defined on the subspace $\mathcal{S}_0(\mathbb{R}^n)$ of the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$, consisting of all functions whose Fourier transform vanishes to infinite order at zero. Moreover, we can show that for any $\sigma \in \dot{S}^m_{\gamma,\delta}(A)$, $\sigma(x,D)$ maps $\mathcal{S}_0$ continuously to $\mathcal{S}$. We postpone the proof of this fact, which requires some additional notation specific to the anisotropic setting, until Section 4; see Lemma 4.9. It is also well known that $\mathcal{S}$ is dense in L^p, $1 \le p < \infty$, and $\mathcal{S} \cap H^p$ is dense in H^p, $0 < p \le 1$; a similar statement holds for $\mathcal{S}_0$. For the appropriate definitions of these spaces and further properties, we refer again to the monograph [3].

The remainder of this paper will be concerned with *homogeneous* multipliers or pseudodifferential symbols, therefore allowing for a singularity at $\xi = 0$; we reiterate that the definition we provided is appropriate for any expansive matrix, not just for a diagonal one. We investigate the relevant properties of certain homogeneous classes of multipliers or symbols in this anisotropic setting, with the main goal of extending the classical isotropic results.

Our paper is organized as follows. In Sections 2 and 3 we revisit the anisotropic Mikhlin and Hörmander multipliers, and provide alternative proofs of their boundedness on Lebesgue and Hardy spaces. Our approach is simpler than the one in [22], mainly because we can appeal now to the well established Calderón–Zygmund theory on spaces of homogeneous type. In particular, the continuity results for these multipliers follow immediately once we show that their Schwartz kernels satisfy anisotropic Calderón–Zygmund estimates. In Section 4, we prove that pseudodifferential operators with symbols in $\dot{S}^0_{1,1}(A)$ have anisotropic Calderón–Zygmund kernels. Then, by making use of wavelet techniques, we show that operators with symbols in the exotic classes $\dot{S}^m_{1,1}(A)$ are bounded on weighted anisotropic Besov and Triebel–Lizorkin spaces. This extends the corresponding isotropic results of Grafakos and Torres [15].

2. Anisotropic Mikhlin multipliers. In analogy with its isotropic counterpart, we will define the *anisotropic Mikhlin class* $\dot{S}^0_1(A)$. A multiplier $\sigma \in \dot{S}^0_1(A)$ satisfies

$$|\partial^\beta_\xi[\sigma((A^*)^k\cdot)]((A^*)^{-k}\xi)| \le C_\beta \tag{2.1}$$

for all multi-indices β, all $\xi \in \mathbb{R}^n \setminus \{0\}$, and $k \in \mathbb{Z}$ such that $\rho_{A^*}(\xi) \sim b^k = |\det A|^k$. In particular, (2.1) implies that σ is a bounded function. Note that in the isotropic case $A = 2I_n$, the condition (2.1) takes the familiar form

$$|\partial^\beta_\xi \sigma(\xi)| \le C_\beta |\xi|^{-|\beta|} \quad \text{for all } \beta.$$

EXAMPLE 2.1. Consider the following simple partial differential equation in the variable function $u(x_1, x_2)$ on $\mathbb{R}^2$:

$$P(\partial)u = Q(\partial)f,$$

where $f(x_1, x_2)$ is some given Schwartz function on $\mathbb{R}^2$, and

$$P(\partial) = \partial^6_{x_1} + \partial^2_{x_2} + \partial^6_{x_2}, \qquad Q(\partial) = \partial^6_{x_1} + \partial^6_{x_2}.$$

By taking the Fourier transform on both sides of this equation, we obtain

$$(\xi_1^6 + \xi_2^2 + \xi_2^6)\widehat{u} = (\xi_1^6 + \xi_2^6)\widehat{f},$$

which gives (by taking the inverse Fourier transform)

$$u = \mathcal{F}^{-1}(\sigma\widehat{f}).$$

Here,

$$\sigma(\xi_1, \xi_2) = \frac{\xi_1^6 + \xi_2^6}{\xi_1^6 + \xi_2^2 + \xi_2^6}$$

is a typical example of homogeneous multiplier to which the anisotropic setting seems to be more appropriate than the isotropic one. This is despite the fact that σ does not have any obvious scaling property.

Indeed, a straightforward exercise verifies that $\sigma \in \dot{S}^0_1(A)$, where

$$A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix}.$$

More precisely, we can check that estimates (1.5) hold with $a_1 = 1/2$, $a_2 = 3/2$, $a = 1$, $n = 2$, i.e.,

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\sigma(\xi_1,\xi_2)| \lesssim \rho_{A^*}(\xi_1,\xi_2)^{-\|(\alpha_1,\alpha_2)\|},$$

where

$$\rho_{A^*}(\xi_1,\xi_2) = \max_{i=1,2}(|\xi_1|^2, |\xi_2|^{2/3}).$$

For example,

$$|\partial_{\xi_1}\sigma(\xi_1,\xi_2)| = \frac{6|\xi_1|^5|\xi_2|^2}{(\xi_1^6+\xi_2^2+\xi_2^6)^2} \lesssim \min_{i=1,2}(|\xi_1|^{-1}, |\xi_2|^{-1/3}) \lesssim \rho_{A^*}(\xi_1,\xi_2)^{-\|(1,0)\|}.$$

Let us briefly indicate how the estimate above was obtained. If $\rho_{A^*}(\xi_1,\xi_2) = |\xi_1|^2$, i.e., $|\xi_1|^3 \geq |\xi_2|$, then

$$\frac{|\xi_1|^5|\xi_2|^2}{(\xi_1^6+\xi_2^2+\xi_2^6)^2} \leq \frac{|\xi_1|^6|\xi_2|^2|\xi_1|^{-1}}{|\xi_1|^{12}} \leq \left(\frac{|\xi_2|}{|\xi_1|^3}\right)^2 |\xi_1|^{-1} \leq |\xi_1|^{-1}.$$

If $\rho_{A^*}(\xi_1,\xi_2) = |\xi_2|^{2/3}$, i.e., $|\xi_1|^3 \leq |\xi_2|$, then

$$\frac{|\xi_1|^5|\xi_2|^2}{(\xi_1^6+\xi_2^2+\xi_2^6)^2} \leq \frac{|\xi_2|^{5/3}|\xi_2|^2}{|\xi_2|^4} \leq |\xi_2|^{-1/3}.$$

Similar estimates hold for all multi-indices $|\alpha| \leq 2$. The results of Sections 2 and 3 will show that the L^p boundedness of the given function f is propagated to the solution u.

Consider then a multiplier operator $\sigma(D)$ with anisotropic Mikhlin symbol $\sigma(\xi)$, initially defined on $\mathcal{S}_0$. The main result of this section is the following.

THEOREM 2.2. *If $\sigma \in \dot{S}_1^0(A)$, then $\sigma(D)$ extends as a bounded operator*

(i) $\sigma(D) : L^p \to L^p$, $p > 1$,
(ii) $\sigma(D) : L^1 \to L^{1,\infty}$,
(iii) $\sigma(D) : H^p \to H^p$, $0 < p \leq 1$,
(iv) $\sigma(D) : H^p \to L^p$, $0 < p \leq 1$.

In particular, in the isotropic case $A = 2I_n$, we recover the well known result about the Mikhlin class that if $\sigma \in \dot{S}_1^0$, then $\sigma(D)$ is a bounded operator on all spaces L^p, $p > 1$.

Our proof will follow the classical approach. We refine the nice argument in Grafakos' book [14, Chapter 5], and show first that $\sigma(D)$ is a Calderón–Zygmund operator with respect to the dilation A and the canonical quasi-norm ρ_A.

PROPOSITION 2.3. *Suppose that $\sigma \in \dot{S}_1^0(A)$. Then $K = \mathcal{F}^{-1}\sigma$ is a Calderón–Zygmund kernel, that is,*

$$|\partial^\alpha[K(A^k\cdot)](A^{-k}x)| \leq C_\alpha/\rho_A(x) \tag{2.2}$$

for some $C_\alpha > 0$*, all multi-indices* α*, and all* $x \in \mathbb{R}^n \setminus \{0\}$ *such that* $\rho_A(x) \sim |\det A|^k = b^k$.

With this fact in hand, the proof of our theorem is immediate.

Proof of Theorem 2.2. Parts (i) and (ii) follow from the general Calderón–Zygmund theory on spaces of homogeneous type as explained, for example, in Stein's book [24, Chapter 1]. This is due to the fact that condition (2.2) implies the Hörmander condition (3.2); see Proposition 3.3.

Part (iii) is implied by [3, Theorem 9.8]. We only need to note that the multiplier operator is a Calderón–Zygmund singular integral of convolution type,

$$\sigma(D)f = K * f,$$

which is L^2 bounded, because $K \in L^\infty$. Moreover, as a convolution operator, $\sigma(D)$ preserves vanishing moments, i.e., $(\sigma(D))^*(x^\alpha) = 0$ for all α. Finally, part (iv) is a consequence of [3, Theorem 9.9]. ∎

To prove the proposition we will need the following elementary lemma (see [3, 4, 6]).

Lemma 2.4. *Suppose* A *is an expansive matrix, and* λ_- *and* λ_+ *are any positive real numbers such that* $1 < \lambda_- < \min_{\lambda \in \sigma(A)} |\lambda|$ *and* $\max_{\lambda \in \sigma(A)} |\lambda| < \lambda_+ < b = |\det A|$. *Then there exists* $c > 0$ *such that*

$$(1/c)(\lambda_-)^j |x| \le |A^j x| \le c(\lambda_+)^j |x| \quad \text{for } j \ge 0, \tag{2.3}$$

$$(1/c)(\lambda_+)^j |x| \le |A^j x| \le c(\lambda_-)^j |x| \quad \text{for } j \le 0. \tag{2.4}$$

Furthermore, if A *is diagonalizable over* $\mathbb{C}$*, then we may take* $\lambda_- = \min_{\lambda \in \sigma(A)} |\lambda|$ *and* $\lambda_+ = \max_{\lambda \in \sigma(A)} |\lambda|$.

We are ready to prove our proposition.

Proof of Proposition 2.3. We start by considering the Littlewood–Paley decomposition

$$\sum_{j \in \mathbb{Z}} \varphi((A^*)^j \xi) = 1, \qquad \xi \neq 0,$$

where φ is a C^∞ function compactly supported away from the origin. Define

$$\sigma_j(\xi) = \sigma(\xi) \varphi((A^*)^j \xi),$$

and note that $\sigma_j(\xi) \neq 0$ if and only $\rho_{A^*}(\xi) \sim b^{-j}$. Clearly $\sum_{j \in \mathbb{Z}} \sigma_j(\xi)$ converges boundedly to $\sigma(\xi)$ for $\xi \neq 0$; see, for example, Frazier, Jawerth and Weiss' monograph [12]. If we let

$$K_j(x) = \int_{\mathbb{R}^n} \sigma_j(\xi) e^{ix \cdot \xi} \, d\xi,$$

then $\sum_{j \in \mathbb{Z}} K_j$ converges to K in $\mathcal{S}'$.

Claim 1. *For all multi-indices α and for all $x \in \mathbb{R}^n$ such that $\rho_A(x) \sim 1$, we have*

$$\sum_{j\in\mathbb{Z}} |\partial^\alpha K_j(x)| \le C_\alpha \tag{2.5}$$

for some positive constant C_α.

Notice again that we continue with our paradigm of "rescaling to scale zero". It turns out that it suffices to have the estimate (2.5) on the kernel K, since, as we will see later, we can perform a "rescaling back to scale k" argument and thus obtain estimate (2.2).

Let us first prove Claim 1. We denote by $D_{A^k} f(\cdot) = f(A^k \cdot)$ the dilation by A^k for all $k \in \mathbb{Z}$ of some function f. We can write

$$K_j = \mathcal{F}^{-1}\sigma_j = \mathcal{F}^{-1}(D_{A^{*j}} D_{(A^*)^{-j}}\sigma_j) = b^{-j} D_{A^{-j}} \mathcal{F}^{-1}(D_{(A^*)^{-j}}\sigma_j).$$

Let

$$f_j = D_{(A^*)^{-j}}\sigma_j \quad \text{and} \quad g_j = \mathcal{F}^{-1} f_j.$$

It is easy to check that $\{f_j\}_{j\in\mathbb{Z}}$ is a subset of the *normal class*

$$\mathcal{N}_R = \{\phi \in C^\infty : \operatorname{supp}(\phi) \subset \{\xi : R^{-1} < |\xi| < R\}, \|\partial^\alpha \phi\|_{L^\infty} \le c_\alpha\},$$

where $R, c_\alpha > 0$ is some fixed collection of parameters depending only on φ. Consequently, $\{g_j\}_{j\in\mathbb{Z}}$ is a subset of

$$\mathcal{M} = \{\Phi \in C^\infty : \|\Phi\|_m = \sup_{|\alpha|\le m} (1+|x|)^m |\partial^\alpha \Phi(x)| \le c_m\},$$

where $(c_m)_{m\in\mathbb{N}}$ is some fixed sequence depending on R and c_α.

Therefore, we have

$$K = \sum_{j\in\mathbb{Z}} K_j = \sum_{j\in\mathbb{Z}} b^{-j} D_{A^{-j}}(g_j), \tag{2.6}$$

where $\{g_j\}_{j\in\mathbb{Z}} \subset \mathcal{M}$. Recall that we are trying to achieve the scale zero estimate (2.5) of K. We fix $s \in \mathbb{N}$ and try to find summable estimates on $|\partial^\alpha D_{A^{-j}}(g_j)(x)|$, $|\alpha| = s$, for $\rho_A(x) \sim 1$. We distinguish two cases.

Case 1: *"Estimates of flat functions": $j \ge 0$.* We use the chain rule and the fact that $\|g_j\|_s \le c_s$ to conclude that

$$|\partial^\alpha D_{A^{-j}}(g_j)(x)| \le C_s \|A^{-j}\|^s \sup_{|\beta|=s} |\partial^\beta g_j(A^{-j}x)| \le C_s \|A^{-j}\|^s.$$

Clearly, the series $\sum_{j\ge 0} b^{-j}\|A^{-j}\|^s$ converges as a geometric series with ratio strictly less than 1.

Case 2: *"Decay estimates at infinity": $j < 0$.* We use again the chain rule and Lemma 2.4. For some convenient $N \in \mathbb{N}$ (to be chosen later), we can write

$$|\partial^\alpha D_{A^{-j}}(g_j)(x)| \le C_{N,s} \|A^{-j}\|^s |A^{-j}x|^{-N} \le C_{N,s} \lambda_+^{-js} \lambda_-^{jN}.$$

Note that the second inequality implicitly assumes that we work at scale zero, that is, $\rho_A(x) \sim 1$. We have also used the fact that $g_j \in \mathcal{M}$. If we now recall that $1 < \lambda_- < \lambda_+ < b$, we can choose N such that

$$b^{-1}\lambda_+^{-s}\lambda_-^N < 1,$$

which guarantees the convergence of the geometric series $\sum_{j<0} b^{-j}\lambda_+^{-js}\lambda_-^{jN}$.

Hence, by combining the two summable estimates, we conclude that the scale-zero estimate (2.5) holds:

$$\sum_{j\in\mathbb{Z}} |\partial^\alpha K_j(x)| \le C_s, \qquad |\alpha| = s.$$

In order to finish the proof, we only need to prove the following

Claim 2. *If K satisfies* (2.5), *then it also satisfies* (2.2).

We use a rather natural rescaling argument. For a fixed $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$ such that $\rho_A(x) \sim b^k$, we need to estimate

$$\partial^\alpha[K(A^k\cdot)](A^{-k}x) = \partial^\alpha(D_{A^k}K)(A^{-k}x).$$

Note that

$$D_{A^k}K = \sum_{j\in\mathbb{Z}} b^{-j}D_{A^{k-j}}(g_j) = b^{-k}\sum_{j\in\mathbb{Z}} b^{-j}D_{A^{-j}}(\tilde{g}_j),$$

where $\tilde{g}_j = g_{j+k} \in \mathcal{M}$. Consequently, since $\rho_A(A^{-k}x) \sim 1$ (i.e., we are back at scale zero!), we can repeat the same argument following (2.6) and conclude that

$$|\partial^\alpha[K(A^k\cdot)](A^{-k}x)| \le C_\alpha b^{-k} \lesssim C_\alpha/\rho_A(x).$$

This shows (2.2) and the proof is complete. ■

3. Anisotropic Hörmander multipliers. In this section we revisit the class of anisotropic Hörmander multipliers introduced by Rivière [22]. We show that in the setting of expansive dilations this class corresponds to anisotropic convolution-type Calderón–Zygmund operators. As a consequence, we obtain a proof of L^p boundedness of anisotropic Hörmander multipliers alternative to the original approach of Rivière.

Definition 3.1. Let σ be a bounded function on $\mathbb{R}^n \setminus \{0\}$. We say that σ is an *anisotropic Hörmander multiplier of order M* (associated to the matrix A) if it is M-times differentiable and there exists $C_\alpha > 0$ such that

$$\int_{B_1^*\setminus B_0^*} |\partial_\xi^\alpha(\sigma(A^{*k}\cdot))(\xi)|^2\, d\xi \le C_\alpha \tag{3.1}$$

for all multi-indices α with $|\alpha| \le M$, and for all $k \in \mathbb{Z}$.

Note that in the isotropic case $A = 2I_n$, the condition (3.1) takes the more familiar form

$$\sup_{k\in\mathbb{Z}} 2^{k(-n+2|\alpha|)} \int_{2^k<|\xi|<2^{k+1}} |\partial_\xi^\alpha \sigma(\xi)|^2 d\xi \le C_\alpha \quad \text{for all } |\alpha| \le M.$$

We remark immediately that any Mikhlin multiplier is a Hörmander multiplier. Indeed, if $\sigma \in \dot{S}_1^0(A)$, then

$$\begin{aligned}\int_{B_1^*\setminus B_0^*} |\partial^\alpha(\sigma(A^{*k}\cdot))(\xi)|^2\, d\xi &= \int_{B_{k+1}^*\setminus B_k^*} |\partial^\alpha(\sigma(A^{*k}\cdot))((A^*)^{-k}\xi)|^2 b^{-k}\, d\xi \\ &\le \int_{B_{k+1}^*\setminus B_k^*} C_\alpha b^{-k}\, d\xi \le C_\alpha.\end{aligned}$$

The constant C_α is of course the one appearing in the estimates (2.1) that define the class $\dot{S}_1^0(A)$.

Conversely, suppose that σ is a Hörmander multiplier of sufficiently large order M. Then, by the Sobolev embedding theorem, $\partial^\beta(\sigma(A^{*k}\cdot))(\xi)$ are bounded on $B_1^* \setminus B_0^*$ for $|\beta| < M - n/2$. Thus, σ satisfies Mikhlin estimates (2.1) up to that order. Hence, one could easily deduce an analogue of Theorem 2.2 for Hörmander multipliers with sufficiently large order M. Instead, we will show the following more concrete result generalizing the isotropic Hörmander Multiplier Theorem (see [14, Theorem 5.2.7]), which can also be deduced from [22, Theorem II.1.2].

Theorem 3.2. *If σ is an anisotropic Hörmander multiplier of order $\lfloor n/2\rfloor + 1$, then $\sigma(D)$ extends as a bounded operator:*

(i) $\sigma(D) : L^p \to L^p$, $p > 1$,
(ii) $\sigma(D) : L^1 \to L^{1,\infty}$.

We follow the strategy outlined in the previous section. The Calderón–Zygmund theory for spaces of homogeneous type immediately yields Theorem 3.2, once we prove that $\sigma(D)$ has a Calderón–Zygmund kernel.

Proposition 3.3. *Let σ be an anisotropic Hörmander multiplier of order $\lfloor n/2\rfloor + 1$. Then $K = \mathcal{F}^{-1}\sigma$ satisfies the anisotropic Hörmander condition*

$$\sup_{y\neq 0} \int_{\rho_A(x)\ge 2c\rho_A(y)} |K(x-y) - K(x)|\, dx \le C \tag{3.2}$$

for some constants $c > 1$ and $C > 0$.

Proof. We start by observing that the annulus $B_1^* \setminus B_0^*$ can be replaced by any other annulus $B_{i+1}^* \setminus B_i^*$ in Definition 3.1. Indeed,

$$\int\limits_{B^*_{i+1}\setminus B^*_i} |\partial^\alpha(\sigma(A^{*k}\cdot))(\xi)|^2\,d\xi = b^i \int\limits_{B^*_1\setminus B^*_0} |\partial^\alpha(\sigma(A^{*k}\cdot))(A^{*i}\xi)|^2\,d\xi \tag{3.3}$$
$$\le Cb^i\|(A^*)^{-i}\|^{|\alpha|}\sum_{|\beta|=|\alpha|}\int\limits_{B^*_1\setminus B^*_0}|\partial^\beta(\sigma((A^*)^{k+i}\cdot))(\xi)|^2\,d\xi$$
$$\le Cb^i\|(A^*)^{-i}\|^{|\alpha|}\sum_{|\beta|=|\alpha|} C_\beta.$$

With the notation in the proof of Proposition 2.3, we have

$$\sigma_j(\xi) = \sigma(\xi)\varphi(A^{*j}\xi),$$

so that $\sigma_j(\xi)\neq 0$ if and only if $\rho_{A^*}(\xi)\sim b^j = |\det A|^j$. Let

$$K_j = b^{-j}D_{A^{-j}}\mathcal{F}^{-1}(D_{(A^*)^{-j}}\sigma_j).$$

By the product rule and the fact that $\operatorname{supp}\hat\varphi\subset B^*_R\setminus B^*_{-R}$ for some $R\in\mathbb{N}$, we have

$$\int\limits_{\mathbb{R}^n}|\partial^\alpha D_{(A^*)^{-j}}\sigma_j(\xi)|^2\,d\xi \lesssim \int\limits_{B^*_R\setminus B^*_{-R}}|\partial^\alpha D_{(A^*)^{-j}}\sigma(\xi)|^2\,d\xi \tag{3.4}$$
$$= \sum_{i=-R}^{R-1}\int\limits_{B^*_{i+1}\setminus B^*_i}|\partial^\alpha D_{(A^*)^{-j}}\sigma(\xi)|^2\,d\xi \lesssim \sum_{|\beta|=|\alpha|}C_\beta$$

for $|\alpha|\le M := \lfloor n/2\rfloor + 1$, where in the last step we have used (3.3). From (3.4) we infer that

$$\int\limits_{\mathbb{R}^n}|\mathcal{F}^{-1}(D_{(A^*)^{-j}}\sigma_j)(x)(1+|x|)^M|^2\,dx\le C_M,$$

which is equivalent to

$$\int\limits_{\mathbb{R}^n}|b^jK_j(A^jx)(1+|x|)^M|^2\,dx\le C_M. \tag{3.5}$$

Fix now $0<\epsilon<M-n/2$. From the Cauchy–Schwarz inequality and (3.5) we see that, for all $j\in\mathbb{Z}$, we have

$$\int\limits_{\mathbb{R}^n}|K_j(x)|(1+|A^{-j}x|)^\varepsilon\,dx = b^j\int\limits_{\mathbb{R}^n}|K_j(A^jx)|(1+|x|)^\epsilon\,dx \tag{3.6}$$
$$\le\Big(\int\limits_{\mathbb{R}^n}|b^jK_j(A^jx)(1+|x|)^M|^2\,dx\Big)^{1/2}\Big(\int\limits_{\mathbb{R}^n}(1+|x|)^{2\epsilon-2M}\,dx\Big)^{1/2}\le C_\epsilon.$$

Likewise, the estimate

$$\int\limits_{\mathbb{R}^n}|\xi^\beta\partial^\alpha(D_{(A^*)^{-j}}\sigma_j(\xi))|^2\,d\xi\le C_{\alpha,\beta}$$

yields

$$b^j \int_{\mathbb{R}^n} |\partial^\beta (D_{A^j} K_j)(x)(1+|x|)^M|^2 \, dx \le C_{M,\beta},$$

and a similar argument to the one proving (3.6) yields the following general estimates on the derivatives of the localized kernels:

$$b^j \int_{\mathbb{R}^n} |\partial^\beta (D_{A^j} K_j)(x)|(1+|x|)^\epsilon \, dx \le C_{\beta,\epsilon} \tag{3.7}$$

for all $j \in \mathbb{Z}$ and all multi-indices β. Using the estimates above, we see that $\sum_{j\in\mathbb{Z}} K_j(x)$ converges to some function $K(x)$ for all $x \neq 0$; this being the case, the function $K(x)$ coincides with the distribution $\mathcal{F}^{-1}\sigma$ for $x \neq 0$. Indeed, since for $j \ge 0$ we have $|K_j(x)| \le Cb^{-j}$, we immediately conclude that $\sum_{j\ge 0} K_j(x)$ is convergent. On the other hand, (3.6) implies that for any $\delta > 0$ we have

$$\int_{|x|\ge\delta} \sum_{j<0} |K_j(x)| \, dx < \infty,$$

and this implies that the function $\sum_{j<0} K_j(x)$ is finite almost everywhere away from the origin.

To prove now that K satisfies the anisotropic Hörmander condition (3.2), it is sufficient to show that for all $y \neq 0$, we have

$$\sum_{j\in\mathbb{Z}} \int_{\rho_A(x)\ge 2c\rho_A(y)} |K_j(x-y) - K_j(x)| \, dx \le C.$$

For a fixed $y \neq 0$, let $k \in \mathbb{Z}$ (fixed) be such that $\rho_A(y) \sim b^k$. We will break down our summation into two sums, over $j > k$ and over $j \le k$.

By appropriately choosing $c > 1$ (this choice being determined by the triangle inequality satisfied by the quasi-norm ρ_A) and using again (3.6), we can write

$$\begin{aligned} \sum_{j>k} \int_{\rho_A(x)\ge 2c\rho_A(y)} & |K_j(x-y) - K_j(x)| \, dx \\ & \le \sum_{j>k} \int_{\rho_A(x)\ge\rho_A(y)} 2|K_j(x)| \, dx \\ & = 2\sum_{j>k} \int_{\rho_A(x)\ge\rho_A(y)} |K_j(x)| \frac{(1+\rho_A(A^{-j}x))^\epsilon}{(1+\rho_A(A^{-j}x))^\epsilon} \, dx \\ & \le C_\epsilon \sum_{j>k} \sup_{x:\rho_A(x)\ge\rho_A(y)} (1+\rho_A(A^{-j}x))^{-\epsilon} \\ & \le C_\epsilon \sum_{j>k} (1+\rho_A(A^{-j}y))^{-\epsilon} \le C_\epsilon \sum_{j>k} b^{(j-k)\epsilon} \lesssim 1. \end{aligned} \tag{3.8}$$

To estimate the sum over $j \le k$, we will use the Mean Value Theorem and estimate (3.7). We have

$$\begin{aligned}
(3.9)\qquad & \int\limits_{\rho_A(x)\ge 2c\rho_A(y)} |K_j(x-y)-K_j(x)|\,dx \\
&= b^j \int\limits_{\rho_A(x)\ge 2cb^{k-j}} |D_{A^j}K_j(x-A^{-j}y)-D_{A^j}K_j(x)|\,dx \\
&\le b^j|A^{-j}y| \int\limits_{\rho_A(x)\ge 2cb^{k-j}} \int\limits_0^1 |\nabla D_{A^j}K_j(x-\theta A^{-j}y)|\,dx\,d\theta \\
&\le b^j|A^{-j}y| \int\limits_0^1 \int\limits_{\rho_A(x)\ge 2cb^{k-j}} |\nabla D_{A^j}K_j(x-\theta A^{-j}y)| \frac{(1+|x-\theta A^{-j}y|)^{\epsilon}}{(1+|x-\theta A^{-j}y|)^{\epsilon}}\,dx\,d\theta \\
&\le C_{1,\epsilon}\frac{|A^{-j}y|}{(1+|A^{-j}y|)^{-\epsilon}}.
\end{aligned}$$

Summing over $j \le k$ and taking $\epsilon < 1$, we conclude that

$$(3.10)\qquad \sum_{j\le k} \int\limits_{\rho_A(x)\ge 2c\rho_A(y)} |K_j(x-y)-K_j(x)|\,dx \lesssim C_{1,\epsilon}\sum_{j\le k} b^{(j-k)(1-\epsilon)} \lesssim 1.$$

By combining (3.8) and (3.10), we obtain (3.2). ∎

4. The class $\dot S^0_{1,1}(A)$. It is well known [24, p. 267] that in contrast with the isotropic Mikhlin class of multipliers $\dot S^0_1$, the isotropic class of x-dependent symbols $\sigma(x,\xi) \in \dot S^0_{1,0}$ does not yield L^p-bounded pseudodifferential operators

$$(\sigma(x,D)f)(x) = \int\limits_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi.$$

Recall that a symbol $\sigma \in \dot S^0_{1,0}(A)$ satisfies

$$|\partial_x^\alpha\partial_\xi^\beta[\sigma(\cdot, A^{*k}\cdot)](x,(A^*)^{-k}\xi)| \le C_{\alpha,\beta}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ and $k \in \mathbb{Z}$ such that $\rho_{A^*}(\xi) \sim b^k$. The L^p-unboundedness is propagated to the more general anisotropic class $\dot S^0_{1,0}(A)$. In fact, for each expansive matrix A, one can construct an appropriate symbol σ for which $\sigma(x,D)$ is unbounded; our examples are simple extensions of the isotropic ones appearing, for example, in [15].

EXAMPLE 4.1. Assuming that φ is a smooth bump supported away from the origin consider a sequence $(m_j(x))_{j\in\mathbb{Z}}$ of smooth functions defined on $\mathbb{R}^n$ such that

$$(4.1)\qquad \|\partial^\alpha[m_j(A^{-j}\cdot)]\|_\infty \le C_\alpha \quad \text{for all } \alpha.$$

Then a straightforward calculation using Definition 1.1 shows that the symbol

$$\sigma(x,\xi) = \sum_{j\in\mathbb{Z}} m_j(x)\varphi((A^*)^{-j}\xi) \tag{4.2}$$

belongs to the class $\dot{S}^0_{1,1}(A)$. Indeed, recall that $\sigma \in \dot{S}^0_{1,1}(A)$ if

$$|\partial_x^\alpha \partial_\xi^\beta [\sigma(A^{-k}\cdot, A^{*k}\cdot)](A^k x, (A^*)^{-k}\xi)| \le C_{\alpha,\beta} \tag{4.3}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ and $k \in \mathbb{Z}$ such that $\rho_{A^*}(\xi) \sim b^k$. The latter constraint reduces the series (4.2) to a finite sum over $|j-k| \le R$, where R depends on the size of $\operatorname{supp}\varphi$. Then (4.3) follows by the chain rule.

Example 4.2. Assume that φ is as in Example 2 and let $(v_j)_{j\in\mathbb{Z}}$ be a bounded sequence in $\mathbb{R}^n$. Define the symbol

$$\sigma_1(x,\xi) = \sum_{j\in\mathbb{Z}} e^{ix\cdot(A^*)^j v_j}\varphi((A^*)^{-j}\xi).$$

Since the functions $m_j(x) = e^{ix\cdot(A^*)^j v_j}$ satisfy (4.1) we see that $\sigma_1 \in \dot{S}^0_{1,1}(A)$. A similar argument shows that the symbol

$$\sigma_0(x,\xi) = \sum_{j=0}^{\infty} e^{ix\cdot(A^*)^j v_j}\varphi((A^*)^{-j}\xi)$$

is in the nonhomogeneous class $S^0_{1,1}(A)$. However, one can show that the operator $\sigma_0(x,D)$ is unbounded on L^2 (see [1]). Thus, the L^2-boundedness of operators with symbols in $\dot{S}^0_{1,1}(A)$ also fails in general.

4.1. Calderón–Zygmund estimates. A natural approach for proving continuity results of pseudodifferential operators, when applicable, is via their singular integral realization. It turns out that the anisotropic homogeneous forbidden symbols $\sigma \in \dot{S}^0_{1,1}(A)$ enjoy a remarkable property: the Schwartz kernels associated to $\sigma(x,D)$ satisfy Calderón–Zygmund estimates. The isotropic analogue of this fact is due to Coifman and Meyer [9]. Combined with the failure of L^2 bounds, this shows that, in general, this class yields unbounded operators on all L^p spaces.

The x-dependence of the symbol $\sigma(x,\xi)$ implies that the operator $\sigma(x,D)$ is of non-convolution type. For $f \in \mathcal{S}$, we can write

$$(\sigma(x,D)f)(x) = \int_{\mathbb{R}^n} \tilde{K}(x,y)f(y)\,dy,$$

where $\tilde{K}(x,y) = K(x,x-y)$, and (in the distribution sense)

$$K(x,z) = \int_{\mathbb{R}^n} \sigma(x,\xi)e^{iz\cdot\xi}\,d\xi = \mathcal{F}_2^{-1}\sigma(x,z).$$

THEOREM 4.3. *If $\sigma \in \dot{S}^0_{1,1}(A)$, then the corresponding kernel K satisfies the Calderón–Zygmund estimates*

$$|\partial_x^\alpha \partial_z^\beta [K(A^k\cdot, A^k\cdot)](A^{-k}x, A^{-k}z)| \le C_{\alpha,\beta}/\rho_A(z) \tag{4.4}$$

for $z \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ such that $\rho_A(z) \sim b^k$, for all multi-indices α, β, and some constants $C_{\alpha,\beta} > 0$.

As an immediate corollary, the Schwartz kernel $\tilde{K}$ satisfies the *symmetric Calderón–Zygmund condition*

$$|\partial_x^\alpha \partial_y^\beta [\tilde{K}(A^k\cdot, A^k\cdot)](A^{-k}x, A^{-k}y)| \le C_{\alpha,\beta}/\rho_A(x-y)$$

for $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ such that $\rho_A(x-y) \sim b^k$.

Proof of Theorem 4.3. Our proof is a fine tune-up of the one given for Proposition 2.3. We start again by considering the Littlewood–Paley decomposition

$$\sum_{j\in\mathbb{Z}} \varphi((A^*)^j\xi) = 1, \quad \xi \ne 0,$$

where φ is a C^∞ function compactly supported away from the origin. Define the symbols

$$\sigma_j(x,\xi) = \sigma(x,\xi)\varphi((A^*)^j\xi),$$

and note that $\sigma_j(x,\xi) \ne 0$ if and only $\rho_{A^*}(\xi) \sim b^{-j}$. Clearly $\sum_{j\in\mathbb{Z}} \sigma_j(x,\xi)$ converges boundedly to $\sigma(x,\xi)$ for $\xi \ne 0$. Let

$$K_j(x,z) = \int_{\mathbb{R}^n} \sigma_j(x,\xi)e^{iz\cdot\xi}\,d\xi.$$

We have the following

CLAIM 1. *For all multi-indices α, β and for all $z \in \mathbb{R}^n$ such that $\rho_A(z) \sim 1$, we have*

$$\sum_{j\in\mathbb{Z}} |\partial_x^\alpha \partial_z^\beta K_j(x,z)| \le C_{\alpha,\beta} \tag{4.5}$$

for some positive constant $C_{\alpha,\beta}$.

The proof of Claim 1 follows a familiar path. For a function of two variables $f(x,y)$, we denote by $D_{A,B}f(\cdot,\cdot) = f(A\cdot, B\cdot)$ the dilation by A in the first variable and by B in the second one. Thus, we can write

$$K_j = b^{-j} D_{A^{-j},A^{-j}} \mathcal{F}_2^{-1}(D_{A^j,(A^*)^{-j}}\sigma_j).$$

Let

$$f_j = D_{A^j,(A^*)^{-j}}\sigma_j \quad \text{and} \quad g_j = \mathcal{F}_2^{-1} f_j.$$

Then

$$K_j = b^{-j} D_{A^{-j},A^{-j}} g_j. \tag{4.6}$$

It is easy to check that $\{f_j\}_{j\in\mathbb{Z}}$ is a subset of the *normal class* $\mathcal{N}_R$ consisting of functions $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\mathrm{supp}(\phi) \subset \mathbb{R}^n \times \{\xi : R^{-1} < |\xi| < R\} \quad \text{and} \quad |\partial_x^\alpha \partial_\xi^\beta \phi(x,\xi)| \le c_{\alpha,\beta}, \forall x,\xi,$$

where $R, c_{\alpha,\beta} > 0$ is some fixed collection of parameters depending only on φ. Consequently, $\{g_j\}_{j\in\mathbb{Z}}$ is a subset of $\mathcal{M}$ which consists of functions $\Phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|\Phi\|_{m,\alpha} = \sup_{|\beta|\le m} \sup_{z\in\mathbb{R}^n} (1+|z|)^m |\partial_x^\alpha \partial_z^\beta \Phi(x,z)| \le c_{m,\alpha},$$

where $(c_{m,\alpha})$ is some fixed sequence depending on R and the constants $c_{\alpha,\beta}$.

Our goal is to achieve estimates (4.5). Based on the equality (4.6), if we fix $s_1 = |\alpha|$, $s_2 = |\beta| \in \mathbb{N}$, it suffices to prove summable estimates on $|\partial_x^\alpha \partial_z^\beta D_{A^{-j},A^{-j}} g_j(x,z)|$ for $\rho_A(z) \sim 1$. We split our analysis into two cases.

Case 1: *"Estimates of flat functions": $j \ge 0$.* We use the chain rule and the fact that $\|g_j\|_{s,\alpha} \le c_{s,\alpha}$ to conclude that

$$\begin{aligned}
&|\partial_x^\alpha \partial_z^\beta g_j(A^{-j}\cdot, A^{-j}\cdot)(x,z)| \\
&\qquad \le C_{s_1,s_2} \|A^{-j}\|^{s_1} \|A^{-j}\|^{s_2} \sup_{|\alpha_1|=s_1} \sup_{|\alpha_2|=s_2} |\partial_x^{\alpha_1} \partial_z^{\alpha_2} g_j(A^{-j}x, A^{-j}z)| \\
&\qquad \le C_{s_1,s_2} \|A^{-j}\|^{s_1+s_2}.
\end{aligned}$$

Clearly, the series $\sum_{j\ge 0} b^{-j} \|A^{-j}\|^{s_1+s_2}$ converges as a geometric series with ratio strictly less than 1.

Case 2: *"Decay estimates at infinity": $j < 0$.* We use again the chain rule, Lemma 2.4, and the fact that $g_j \in \mathcal{M}$ to write, for some sufficiently large N,

$$\begin{aligned}
|\partial_x^\alpha \partial_z^\beta g_j(A^{-j}\cdot, A^{-j}\cdot)(x,z)| &\le C_{N,s_1,s_2} \|A^{-j}\|^{s_1+s_2} |A^{-j}z|^{-N} \\
&\le C_{N,s_1,s_2} \lambda_+^{-j(s_1+s_2)} \lambda_-^{jN}.
\end{aligned}$$

Note that the second inequality implicitly assumes that we work at scale zero, that is, $\rho_A(z) \sim 1$. Recall also that $1 < \lambda_- < \lambda_+ < b$. Therefore, we can choose N such that $b^{-1}\lambda_+^{-s_1-s_2}\lambda_-^N < 1$, which again guarantees the convergence of the series $\sum_{j<0} b^{-j} \lambda_+^{-j(s_1+s_2)} \lambda_-^{jN}$.

Hence, both cases yield summable estimates and we conclude that

$$\sum_{j\in\mathbb{Z}} |\partial_x^\alpha \partial_z^\beta K_j(x,z)| \le C_{s_1,s_2}, \qquad |\alpha| = s_1,\ |\beta| = s_2.$$

This, in turn, clearly implies the zero-scale estimates

$$|\partial_x^\alpha \partial_z^\beta K(x,z)| \le C_{\alpha,\beta} \tag{4.7}$$

for all multi-indices α, β and all pairs (x,z) such that $\rho_A(z) \sim 1$.

Finally, we now use the standard rescaling argument to prove the following claim.

CLAIM 2. *If K satisfies* (4.7)*, then it also satisfies* (4.4)*.*

If $\rho_A(z) \sim b^k$ for some $k \in \mathbb{Z}$ fixed, we need to estimate

$$\partial_x^\alpha \partial_z^\beta [K(A^k \cdot, A^k \cdot)](A^{-k}x, A^{-k}z) = \partial^\alpha (D_{A^k, A^k} K)(A^{-k}x, A^{-k}z).$$

Note that

$$D_{A^k,A^k} K = \sum_{j\in\mathbb{Z}} b^{-j} D_{A^{k-j},A^{k-j}}(g_j) = b^{-k} \sum_{j\in\mathbb{Z}} b^{-j} D_{A^{-j},A^{-j}}(\tilde{g}_j),$$

where $\tilde{g}_j = g_{j+k} \in \mathcal{M}$. Consequently, since $\rho_A(A^{-k}z) \sim 1$ (that is, we are back at scale zero), $b^k D_{A^k,A^k} K$ satisfies (4.7). Thus,

$$b^k |\partial_x^\alpha \partial_z^\beta [K(A^k \cdot, A^k \cdot)](A^{-k}x, A^{-k}z)| \le C_{\alpha,\beta}.$$

The proof is complete. ∎

4.2. Anisotropic homogeneous Triebel–Lizorkin and Besov spaces. The study of pseudodifferential operators with exotic isotropic symbols $\dot{S}^0_{1,1}$ has received much attention. For example, using wavelet techniques, Torres [25, 26] and Grafakos–Torres [15] have studied the properties of this class of symbols on general spaces of smooth functions. These works suggest that the right setting for studying the boundedness of pseudodifferential operators with symbols in the forbidden class $\dot{S}^0_{1,1}(A)$ is provided by the anisotropic homogeneous Triebel–Lizorkin spaces $\dot{\mathbf{F}}^{\alpha,q}_p = \dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)$ or Besov spaces $\dot{\mathbf{B}}^{\alpha,q}_p = \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)$. Before stating our result, we recall the definitions and the molecular characterizations of these spaces. For an extensive treatment of these spaces, the reader is referred to [4, 5, 6]; see also [2].

DEFINITION 4.4. Let μ be a ρ_A-*doubling* measure on $\mathbb{R}^n$, i.e., there exists $\beta = \beta(\mu) > 0$ such that

$$\mu(x + B_{k+1}) \le |\det A|^\beta \mu(x + B_k) \quad \text{for all } x \in \mathbb{R}^n,\ k \in \mathbb{Z}. \tag{4.8}$$

The smallest such β is called the *doubling constant* of μ.

For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, we define the *anisotropic Triebel–Lizorkin space* $\dot{\mathbf{F}}^{\alpha,q}_p = \dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{\mathbf{F}}^{\alpha,q}_p} = \Big\| \Big(\sum_{j\in\mathbb{Z}} (|\det A|^{j\alpha} |f * \varphi_j|)^q \Big)^{1/q} \Big\|_{L^p(\mu)} < \infty, \tag{4.9}$$

where $\varphi_j(x) = |\det A|^j \varphi(A^j x)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\operatorname{supp}\hat{\varphi} := \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \neq 0\}} \subset [-\pi,\pi]^n \setminus \{0\}, \tag{4.10}$$

$$\sup_{j\in\mathbb{Z}} |\hat{\varphi}((A^*)^j \xi)| > 0 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{4.11}$$

Likewise, we define the *anisotropic Besov space* $\dot{\mathbf{B}}_p^{\alpha,q} = \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}} = \Big(\sum_{j\in\mathbb{Z}} (|\det A|^{j\alpha} \|f * \varphi_j\|_{L^p(\mu)})^q\Big)^{1/q} < \infty. \tag{4.12}$$

Recall that $\mathcal{S}'/\mathcal{P}$ can be identified with the space of all continuous functionals on $\mathcal{S}_0$. In [4, 5, 6] it is proved that the inclusion maps $\dot{\mathbf{F}}_p^{\alpha,q} \hookrightarrow \mathcal{S}'/\mathcal{P}$, $\dot{\mathbf{B}}_p^{\alpha,q} \hookrightarrow \mathcal{S}'/\mathcal{P}$ are continuous, and therefore these spaces are quasi-Banach. Moreover, the definitions of anisotropic $\dot{\mathbf{F}}_p^{\alpha,q}$ and $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces are independent of φ. Let $\mathcal{Q}$ be the collection of all *dilated cubes*

$$\mathcal{Q} = \{A^{-j}([0,1]^n + k) : j \in \mathbb{Z},\, k \in \mathbb{Z}^n\}$$

adapted to the action of the dilation A. For $Q = Q_{j,k} = A^{-j}([0,1]^n + k)$ define $\operatorname{scale}(Q) = -j$ and its "lower-left" corner $x_Q = A^{-j}k$. If φ is a function on $\mathbb{R}^n$, we define its wavelet system as

$$\varphi_Q(x) = |Q|^{1/2} \varphi(A^j x - k), \qquad Q = A^{-j}([0,1]^n + k) \in \mathcal{Q}. \tag{4.13}$$

Obviously, if $A = 2I_n$ we obtain the usual collection of *dyadic cubes*.

DEFINITION 4.5. The *discrete Triebel–Lizorkin sequence space* $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} = \Big\|\Big(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^q\Big)^{1/q}\Big\|_{L^p(\mu)} < \infty, \tag{4.14}$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q. Likewise, the *discrete Besov sequence space* $\dot{\mathbf{b}}_p^{\alpha,q} = \dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{b}}_p^{\alpha,q}} = \Big(\sum_{j\in\mathbb{Z}} \Big\| \sum_{Q\in\mathcal{Q},\,\operatorname{scale}(Q)=j} |Q|^{-\alpha} |s_Q| \tilde{\chi}_Q \Big\|_{L^p(\mu)}^q\Big)^{1/q} < \infty.$$

We will also need a definition of anisotropic molecules introduced in [4, 6] generalizing isotropic molecules of Frazier and Jawerth [11, 12]. These molecules come in two flavors depending on whether they are used in the analysis or synthesis transforms.

DEFINITION 4.6. Suppose $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure with doubling constant $\beta \ge 1$. Let $0 < \zeta_- \le 1/n \le \zeta_+ < 1$ be the parameters measuring the eccentricity of the dilation A:

$$\zeta_+ := \frac{\ln \lambda_+}{\ln |\det A|}, \qquad \zeta_- := \frac{\ln \lambda_-}{\ln |\det A|},$$

where λ_- and λ_+ are any positive real numbers such that

$$1 < \lambda_- < \min_{\lambda \in \sigma(A)} |\lambda| \le \max_{\lambda \in \sigma(A)} |\lambda| < \lambda_+ < |\det A|.$$

Let

$$J = \begin{cases} \beta \max(1, 1/p, 1/q) & \text{for } \dot{\mathbf{F}}_p^{\alpha,q} \text{ spaces,} \\ \beta/p + \max(0, 1 - 1/p) & \text{for } \dot{\mathbf{B}}_p^{\alpha,q} \text{ spaces,} \end{cases} \tag{4.15}$$
$$N = \max(\lfloor (J - \alpha - 1)/\zeta_- \rfloor, -1).$$

We say that $\Psi_Q(x)$ is a *smooth synthesis molecule* for $\dot{\mathbf{F}}_p^{\alpha,q}$ (or $\dot{\mathbf{B}}_p^{\alpha,q}$) supported near $Q \in \mathcal{Q}$ with $\operatorname{scale}(Q) = -j$ and $j \in \mathbb{Z}$ if there exist $M > J$ such that

$$|\partial^\gamma [\Psi_Q(A^{-j} \cdot)](x)| \le \frac{|\det A|^{j/2}}{(1 + \rho_A(x - A^j x_Q))^M} \quad \text{for } |\gamma| \le \lfloor \alpha/\zeta_- \rfloor + 1, \tag{4.16}$$

$$|\Psi_Q(x)| \le \frac{|\det A|^{j/2}}{(1 + \rho_A(A^j(x - x_Q)))^{\max(M, (M-\alpha)\zeta_+/\zeta_-)}}, \tag{4.17}$$

$$\int x^\gamma \Psi_Q(x)\, dx = 0 \quad \text{for } |\gamma| \le N. \tag{4.18}$$

We say that $\Phi_Q(x)$ is a *smooth analysis molecule* for $\dot{\mathbf{F}}_p^{\alpha,q}$ (or $\dot{\mathbf{B}}_p^{\alpha,q}$) supported near $Q \in \mathcal{Q}$ with $\operatorname{scale}(Q) = -j$ and $j \in \mathbb{Z}$ if there exists $M > J$ such that

$$|\partial^\gamma [\Phi_Q(A^{-j} \cdot)](x)| \le \frac{|\det A|^{j/2}}{(1 + \rho_A(x - A^j x_Q))^M} \quad \text{for } |\gamma| \le N + 1, \tag{4.19}$$

$$|\Phi_Q(x)| \le \frac{|\det A|^{j/2}}{(1 + \rho_A(A^j(x - x_Q)))^{\max(M, 1 + \alpha\zeta_+/\zeta_- + M - J)}}, \tag{4.20}$$

$$\int x^\gamma \Phi_Q(x)\, dx = 0 \quad \text{for } |\gamma| \le \lfloor \alpha/\zeta_- \rfloor. \tag{4.21}$$

We say that $\{\Phi_Q\}_{Q \in \mathcal{Q}}$ is a *family of smooth synthesis* [*analysis*] *molecules* if each Φ_Q is a smooth synthesis [analysis] molecule supported near Q.

We emphasize that in the context of smooth molecules, Ψ_Q and Φ_Q are understood as some functions indexed by $Q \in \mathcal{Q}$ which are not necessarily given by (4.13). Nevertheless, if $\varphi \in \mathcal{S}$ has sufficiently many vanishing moments, then the wavelet system $\{\varphi_Q\}_Q$ is a family of smooth molecules (both for synthesis and analysis).

The following result about smooth molecular analysis and synthesis transforms was established in the setting of anisotropic $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces [4, Theorems 5.5 and 5.7] and anisotropic $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces [5, Theorem 5.4]. Theorem 4.7

is simply a generalization of the corresponding isotropic results of Frazier and Jawerth [10, 11].

THEOREM 4.7. *Suppose that A is an expansive matrix, $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, and μ is a ρ_A-doubling measure. Then there exists a constant $C > 0$ such that:*

(i) *If $\{\Psi_Q\}_Q$ is a family of smooth synthesis molecules for $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$, then*
$$\Big\| \sum_{Q\in\mathcal{Q}} s_Q \Psi_Q \Big\|_{\dot{\mathbf{F}}_p^{\alpha,q}} \leq C\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \quad \text{for all } s = \{s_Q\}_Q \in \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu).$$

(ii) *If $\{\Phi_Q\}_Q$ is a family of smooth analysis molecules for $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$, then*
$$\|\{\langle f, \Phi_Q\rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \leq C\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} \quad \text{for all } f \in \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu).$$

Furthermore, the same result holds for Besov spaces $\dot{\mathbf{B}}_p^{\alpha,q}$.

4.3. Boundedness on anisotropic $\dot{\mathbf{F}}_p^{\alpha,q}$ and $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces. Finally, we are ready to prove our anisotropic boundedness result extending the isotropic result of Grafakos and Torres [15].

THEOREM 4.8. *Let $\sigma \in \dot{S}^m_{1,1}(A)$, $0 < p, q < \infty$, and*

$$(\sigma(x, D))^*(x^\gamma) = 0 \quad \text{for } |\gamma| \leq N = \max(\lfloor (J - \alpha - 1)/\zeta_- \rfloor, -1). \tag{4.22}$$

(Note that if $\alpha > J - 1$, then (4.22) is trivially satisfied.) Then the pseudodifferential operator $\sigma(x, D)$ (a priori defined on $\mathcal{S}_0$) extends as a bounded operator:

(i) $\sigma(x, D) : \dot{\mathbf{F}}_p^{\alpha+m,q}(\mathbb{R}^n, A, \mu) \to \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$,
(ii) $\sigma(x, D) : \dot{\mathbf{B}}_p^{\alpha+m,q}(\mathbb{R}^n, A, \mu) \to \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$.

In the statement of the theorem, we need to specify how T^* acts on polynomials, where $T = \sigma(x, D)$. Using Lemma 4.9 below and a formal duality relation $\langle T^*(x^\gamma), f\rangle = \langle T(f), x^\gamma\rangle$, the condition (4.22) means that $\langle \sigma(x, D)(f), x^\gamma\rangle = 0$ for all $f \in \mathcal{S}_0$ and $|\gamma| \leq N$. To prove Theorem 4.8, we carefully adapt the argument in [15] to the anisotropic setting. It is sufficient to prove that $\sigma(x, D)$ maps a wavelet system $\{\varphi_Q\}_Q$ into a family of smooth molecules $\{\Psi_Q\}_Q$ (modulo some scalar rescaling). As we shall see shortly, this can be achieved by some relatively easy computations. We need the following technical lemma.

LEMMA 4.9. *Let $\sigma \in \dot{S}^m_{\gamma,\delta}(A)$. Then $\sigma(x, D)$ maps $\mathcal{S}_0$ continuously into $\mathcal{S}$.*

Proof. The homogeneous anisotropic symbol estimate (1.3) combined with the chain rule and Lemma 2.4 implies that

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{|\alpha|,|\beta|}\rho_{A^*}(\xi)^m \cdot \begin{cases} (\lambda_+)^{k\delta|\alpha|}(\lambda_-)^{-k\gamma|\beta|} & \text{if } k \ge 0, \\ (\lambda_-)^{k\delta|\alpha|}(\lambda_+)^{-k\gamma|\beta|} & \text{if } k < 0, \end{cases}$$

for all multi-indices α, β, and $\xi \in \mathbb{R}^n \setminus \{0\}$, where $k \in \mathbb{Z}$ is such that $\rho_{A^*}(\xi) \sim b^k$. Thus, we can rewrite the above as

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{|\alpha|,|\beta|} \cdot \begin{cases} \rho_{A^*}(\xi)^{m+\zeta_+\delta|\alpha|-\zeta_-\gamma|\beta|} & \text{if } \rho_{A^*}(\xi) \ge 1, \\ \rho_{A^*}(\xi)^{m+\zeta_-\delta|\alpha|-\zeta_+\gamma|\beta|} & \text{if } \rho_{A^*}(\xi) < 1. \end{cases}$$

Applying [3, Lemma 3.2] now yields polynomial growth/decay estimates as $\xi \to 0$ and $\xi \to \infty$,

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{|\alpha|,|\beta|} \cdot \begin{cases} |\xi|^{(m+\zeta_+\delta|\alpha|-\zeta_-\gamma|\beta|)/\zeta_\pm} & \text{if } |\xi| \ge 1, \\ |\xi|^{(m+\zeta_-\delta|\alpha|-\zeta_+\gamma|\beta|)/\zeta_\mp} & \text{if } |\xi| < 1. \end{cases}$$

Here $\zeta_\pm$ means ζ_+ or ζ_- if the exponent is negative or positive, respectively, and similarly for $\zeta_\mp$. However, for the rest of the argument it is unimportant what the exact exponents are. That is, we shall only use the fact that

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta} \min(1,|\xi|^{-d_1}) \max(1,|\xi|^{d_2}) \tag{4.23}$$

for some exponents $d_1 = d_1(\alpha,\beta) > 0$ and $d_2 = d_2(\alpha,\beta) > 0$ depending on the multi-indices α and β. In fact, the rest of the proof follows directly the standard argument as in [15, Lemma 2.1]. One should emphasize here that we must impose a stronger topology on $\mathcal{S}_0$ than the induced topology of a closed subspace of the Schwartz class $\mathcal{S}$.

Indeed, let $f \in \mathcal{S}_0$ and $N \in \mathbb{N}$. Let Δ_ξ be the Laplacian in the ξ variable. Using the identity

$$(I - \Delta_\xi)^N (e^{ix\cdot\xi}) = (1+|x|^2)^N e^{ix\cdot\xi}$$

and the bound (4.23), integration by parts yields

$$(\sigma(x,D)f)(x) = \int e^{ix\cdot\xi} \frac{(I-\Delta_\xi)^N}{(1+|x|^2)^N} (\sigma(x,\xi)\hat{f}(\xi))\, d\xi.$$

This formula works since $f \in \mathcal{S}_0$ and thus $\hat{f}$ and all of its partial derivatives vanish to infinite order at the origin. Moreover, differentiation under the integral is allowed, resulting in

$$(\partial_x^\alpha \sigma(x,D)f)(x) = \int (I-\Delta_\xi)^N \bigg(\partial_x^\alpha \bigg(\frac{e^{ix\cdot\xi}}{(1+|x|^2)^N}\sigma(x,\xi)\bigg)\hat{f}(\xi)\bigg) d\xi$$

for any multi-index α. Applying the product rule, the bound (4.23), and the fact that $f \in \mathcal{S}_0$ yields

$$|(\partial_x^\alpha \sigma(x,D)f)(x)| \le C \frac{\|f\|_M}{(1+|x|^2)^{N-|\alpha|}}$$

for sufficiently large M. Here,

$$\|f\|_M = \sup_{|\beta|\le M} \sup_{\xi\in\mathbb{R}^n} |\partial^\beta \hat{f}(\xi)|(|\xi|^M + |\xi|^{-M}) < \infty$$

are seminorms defining the topology of the locally convex space $\mathcal{S}_0$. Since α and N are arbitrary, the above shows that $\sigma(x,D):\mathcal{S}_0\to\mathcal{S}$ is continuous. ∎

Proof of Theorem 4.8. Let $\varphi\in\mathcal{S}$ be such that

$$\operatorname{supp}\hat\varphi\subset[-\pi,\pi]^n\setminus\{0\},\qquad\qquad (4.24)$$
$$\sum_{j\in\mathbb{Z}}|\hat\varphi((A^*)^j\xi)|^2=1\quad\text{ for all }\xi\neq 0.$$

Hence, φ is a Parseval wavelet for $L^2(\mathbb{R}^n)$. By the combination of [7, Lemma 2.12] with [6, Lemma 2.8] we have

$$f=\sum_{Q\in\mathcal{Q}}\langle f,\varphi_Q\rangle\varphi_Q\quad\text{ for any } f\in\mathcal{S}_0,$$

with the unconditional convergence in $\mathcal{S}$. Therefore, by Lemma 4.9 the action of the pseudodifferential operator $\sigma(x,D)$ can be expressed as

$$\sigma(x,D)f=\sum_{Q\in\mathcal{Q}}\langle f,\varphi_Q\rangle\sigma(x,D)\varphi_Q\quad\text{ for } f\in\mathcal{S}_0.\qquad (4.25)$$

Assume for the moment that there exists a constant $C>0$ such that

$$\sigma(x,D)\varphi_Q=C|Q|^{-m}\Psi_Q\quad\text{ for all } Q\in\mathcal{Q}\qquad (4.26)$$

where $\{\Psi_Q\}_Q$ is a family of smooth synthesis molecules for $\dot{\mathbf{F}}_p^{\alpha,q}$ or $\dot{\mathbf{B}}_p^{\alpha,q}$. Then, by Theorem 4.7(i), the analysis transform $f\mapsto\{\langle f,\varphi_Q\rangle\}_Q$ is bounded as a map $\dot{\mathbf{F}}_p^{\alpha+m,q}\to\dot{\mathbf{f}}_p^{\alpha+m,q}$. Clearly, the multiplication map $\{s_Q\}_Q\mapsto\{|Q|^{-m}s_Q\}_Q$ is an isometry $\dot{\mathbf{f}}_p^{\alpha+m,q}\to\dot{\mathbf{f}}_p^{\alpha,q}$. Thus, by (4.25), (4.26), and Theorem 4.7(ii) we have $\|\sigma(x,D)f\|_{\dot{\mathbf{F}}_p^{\alpha,q}}\le\|f\|_{\dot{\mathbf{F}}_p^{\alpha+m,q}}$ for all $f\in\mathcal{S}_0$. Since $\mathcal{S}_0$ is a dense subspace of $\dot{\mathbf{F}}_p^{\alpha,q}$ if $p,q<\infty$, this yields the required conclusion. The same argument works for $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces.

Thus, it remains to establish (4.26) by computing the action of $\sigma(x,D)$ on a fixed wavelet φ_Q associated to a dilated cube $Q=Q_{j,k}$. For $x\in\mathbb{R}^n, x\neq 0$, let $\tilde x=A^jx-k$, and $b=|\det A|$. We have the following sequence of equalities:

$$\begin{aligned}(\sigma(x,D)\varphi_Q)(x)&=\int\sigma(x,\xi)\widehat{\varphi}_Q(\xi)e^{ix\cdot\xi}\,d\xi\\&=\int\sigma(x,\xi)b^{-j/2}\widehat{\varphi}((A^*)^{-j}\xi)e^{-ik\cdot(A^*)^{-j}\xi}e^{ix\cdot\xi}\,d\xi\\&=\int\sigma(x,A^{*j}\xi)b^{j/2}\widehat{\varphi}(\xi)e^{-ik\cdot\xi}e^{ix\cdot A^{*j}\xi}\,d\xi\\&=b^{j/2}\int\sigma(A^{-j}(\tilde x+k),A^{*j}\xi)\widehat{\varphi}(\xi)e^{i\tilde x\cdot\xi}\,d\xi.\end{aligned}$$

Therefore,

$$(\sigma(x,D)\varphi_Q)(x)=b^{j/2}(\sigma_Q(x,D)\varphi)(A^jx-k),\qquad (4.27)$$

where we denote

$$(\sigma_Q(x,D)f)(x)=\int\sigma(A^{-j}(x+k),A^{*j}\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi.$$

For a fixed multi-index γ, we can write

$$(4.28)\qquad (\partial^\gamma \sigma_Q(x,D)\varphi)(x) = \int e^{ix\cdot\xi}\widehat{\varphi}(\xi)\sum_{\delta\le\gamma} C_\delta (i\xi)^\delta \partial_x^{\gamma-\delta}[\sigma(A^{-j}(\cdot\,+k), A^{*j}\cdot)](x,\xi)\,d\xi.$$

Now fix β. An integration by parts in (4.28) then gives

$$(4.29)\qquad (\partial^\gamma \sigma_Q(x,D)\varphi)(x) = \sum_{\delta\le\gamma} C_\delta \int e^{ix\cdot\xi}|x|^{-|\beta|} \cdot \sum_{|\beta_1|+|\beta_2|=|\beta|} \partial_x^{\gamma-\delta}\partial_\xi^{\beta_1}[\sigma(A^{-j}(\cdot+k), A^{*j}\cdot)](x,\xi)\partial_\xi^{\beta_2}((i\xi)^\delta\widehat{\varphi}(\xi))\,d\xi.$$

By the support assumption (4.24), the above integral runs only over $\xi\in\mathbb{R}^n$ with $\rho_{A^*}(\xi)\sim 1$. Let $\xi=(A^*)^{-j}\xi'$ and $x=A^jx'$. Then $\rho_{A^*}(\xi')\sim b^j$, and since $\sigma\in\dot{S}^m_{1,1}(A)$, we have

$$|\partial_x^{\gamma-\delta}\partial_\xi^{\beta_1}[\sigma(A^{-j}\cdot, A^{*j}\cdot)](A^jx', (A^*)^{-j}\xi')| \le C_{\beta_1,\gamma,\delta} b^{jm}.$$

Therefore, by taking absolute values on both sides of (4.29) and using the triangle inequality, we get

$$|(\partial^\gamma\sigma_Q(x,D)\varphi)(x)| \le C_{\beta,\gamma}|x|^{-|\beta|}b^{jm}.$$

Since the multi-index β is arbitrary, by the quasi-norm estimate

$$(1+\rho_A(x))^{\zeta_-} \le C(1+|x|)\quad \text{for all } x\in\mathbb{R}^n,$$

we can obtain the anisotropic version of the above estimate,

$$(4.30)\qquad |(\partial^\gamma\sigma_Q(x,D)\varphi)(x)| \le C_{M,P}(1+\rho_A(x))^{-M}b^{jm}\quad \text{for } |\gamma|\le P,$$

where $M,P>0$ are some fixed integers. More precisely, we let $P=\lfloor\alpha/\zeta_-\rfloor+1$ and $M>\max(J,(J-\alpha)\zeta_+/\zeta_-)$.

Recall now that $Q=A^{-j}((0,1]^n+k)$ and define

$$(4.31)\qquad \Psi_Q(x) = (C_{M,P})^{-1}b^{j/2-jm}(\sigma_Q(x,D)\varphi)(A^jx-k),$$

so that by (4.27),

$$\sigma(x,D)(\varphi_Q) = C_{M,P}|Q|^{-m}\Psi_Q.$$

It remains to verify that Ψ_Q is a molecule. By (4.30), (4.31), and

$$\Psi_Q(A^{-j}\cdot) = (C_{M,P})^{-1}b^{j/2-jm}(\sigma_Q(x,D)\varphi)(\cdot-k)$$

we have

$$|\partial^\gamma[\Psi_Q(A^{-j}\cdot)](x)| \le b^{j/2}(1+\rho_A(x-A^jx_Q))^{-M}\quad \text{for } |\gamma|\le P.$$

By our choice of M, this simultaneously yields (4.16) and (4.17). Finally, the condition (4.18) is a direct consequence of the vanishing moment hypothesis (4.22). Thus, we conclude that Ψ_Q is a smooth synthesis molecule, which finishes the proof. ∎

References

[1] Á. Bényi and M. Bownik, *Anisotropic classes of nonhomogeneous pseudodifferential symbols*, preprint.

[2] L. Borup and M. Nielsen, *On anisotropic Triebel–Lizorkin type spaces, with applications to the study of pseudo-differential operators*, J. Funct. Spaces Appl. 6 (2008), 107–154.

[3] M. Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc. 164 (2003), no. 781, 122 pp.

[4] —, *Atomic and molecular decompositions of anisotropic Besov spaces*, Math. Z. 250 (2005), 539–571.

[5] —, *Anisotropic Triebel–Lizorkin spaces with doubling measures*, J. Geom. Anal. 17 (2007), 387–424.

[6] M. Bownik and K.-P. Ho, *Atomic and molecular decompositions of anisotropic Triebel–Lizorkin spaces*, Trans. Amer. Math. Soc. 358 (2006), 1469–1510.

[7] M. Bownik, B. Li, D. Yang, and Y. Zhou, *Weighted anisotropic product Hardy spaces and boundedness of sublinear operators*, Math. Nachr. 283 (2010), 392–442.

[8] R. R. Coifman et Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57 (1978).

[9] —, —, *Wavelets. Calderón–Zygmund and Multilinear Operators*, Cambridge Stud. Adv. Math. 48, Cambridge Univ, Press, Cambridge, 1997.

[10] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. 34 (1985), 777–799.

[11] —, —, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.

[12] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood–Paley Theory and the Study of Function Spaces*, CBMS Reg. Conf. Ser. Math. 79, Amer. Math. Soc., 1991.

[13] G. Garello, *Anisotropic pseudodifferential operators of type* $1,1$, Ann. Mat. Pura Appl. 174 (1998), 135–160.

[14] L. Grafakos, *Classical and Modern Fourier Analysis*, Prentice-Hall, 2003.

[15] L. Grafakos and R. H. Torres, *Pseudodifferential operators with homogeneous symbols*, Michigan Math. J. 46 (1999), 261–269.

[16] L. Hörmander, *Estimates for translation invariant operators in* L^p *spaces*, Acta Math. 104 (1960), 93–140.

[17] —, *Continuity of pseudo-differential operators of type* $1,1$, Comm. Partial Differential Equations 14 (1989), 231–243.

[18] R. Lascar, *Propagation des singularités des solutions d'équations pseudo-différentielles quasi homogènes*, Ann. Inst. Fourier (Grenoble) 27 (1977), no. 2, 79–123.

[19] H.-G. Leopold, *Boundedness of anisotropic pseudo-differential operators in function spaces of Besov–Hardy–Sobolev type*, Z. Anal. Anwend. 5 (1986), 409–417.

[20] W. R. Madych and N. M. Rivière, *Multipliers of the Hölder classes,* J. Funct. Anal. 21 (1976), 369–379.

[21] S. G. Mikhlin, *On the multipliers of Fourier integrals*, Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 701–703 (in Russian).

[22] N. M. Rivière, *Singular integrals and multiplier operators*, Ark. Mat. 9 (1971), 243–278.

[23] A. Seeger, *Some inequalities for singular convolution operators in* L^p*-spaces*, Trans. Amer. Math. Soc. 308 (1988), 259–272.

[24] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993.

[25] R. H. Torres, *Continuity properties of pseudodifferential operators of type* $1, 1$, Comm. Partial Differential Equations 15 (1990), 1313–1328.

[26] —, *Boundedness results for operators with singular kernels on distribution spaces*, Mem. Amer. Math. Soc. 90 (1991), no. 442, viii+172 pp.

[27] H. Triebel, *Theory of Function Spaces*, Monogr. Math. 78, Birkhäuser, 1983.

[28] —, *Theory of Function Spaces II*, Monogr. Math. 84, Birkhäuser, 1992.

[29] M. Yamazaki, *A quasihomogeneous version of paradifferential operators. I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 33 (1986), 131–174.

[30] —, *A quasihomogeneous version of paradifferential operators. II. A symbol calculus*, ibid. 33 (1986), 311–345.

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