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Automatic continuity of biorthogonality preservers between weakly compact JB^*-triples and atomic JBW^*-triples

by

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Abstract. We prove that every biorthogonality preserving linear surjection from a weakly compact JB^*-triple containing no infinite-dimensional rank-one summands onto another JB^*-triple is automatically continuous. We also show that every biorthogonality preserving linear surjection between atomic JBW^*-triples containing no infinite-dimensional rank-one summands is automatically continuous. Consequently, two atomic JBW^*-triples containing no rank-one summands are isomorphic if and only if there exists a (not necessarily continuous) biorthogonality preserving linear surjection between them.

1. Introduction and preliminaries. Studies on the automatic continuity of linear surjections between C^*-algebras and von Neumann algebras preserving orthogonality relations in both directions constitute the latest variant of a problem initiated by W. Arendt in the early eighties.

We recall that two complex-valued continuous functions f and g are said to be *orthogonal* whenever they have disjoint supports. A mapping T between $C(K)$-spaces is called *orthogonality preserving* if it maps orthogonal functions to orthogonal functions. The main result established by Arendt states that every orthogonality preserving bounded linear mapping $T : C(K) \to C(K)$ is of the form

$$T(f)(t) = h(t)f(\varphi(t)) \quad (f \in C(K),\, t \in K),$$

where $h \in C(K)$ and $\varphi : K \to K$ is a mapping which is continuous on $\{t \in K : h(t) \neq 0\}$.

The hypothesis of T being continuous was relaxed by K. Jarosz in [24]. In fact, Jarosz obtained a complete description of all orthogonality preserving (not necessarily continuous) linear mappings between $C(K)$-spaces.

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A consequence of his description is that an orthogonality preserving linear surjection between $C(K)$-spaces is automatically continuous.

Two elements a, b in a general C^*-algebra A are said to be *orthogonal* (denoted by $a \perp b$) if $ab^* = b^*a = 0$. When $a = a^*$ and $b = b^*$, we have $a \perp b$ if and only if $ab = 0$. A mapping T between two C^*-algebras A, B is called *orthogonality preserving* if $T(a) \perp T(b)$ for every $a \perp b$ in A. When $T(a) \perp T(b)$ in B if and only if $a \perp b$ in A, we say that T is *biorthogonality preserving*. Under continuity assumptions, orthogonality preserving bounded linear operators between C^*-algebras are completely described in [10, §4]. This last paper is a culmination of the studies developed by W. Arendt [2], K. Jarosz [24], M. Wolff [34], and N.-C. Wong [35], among others, on bounded orthogonality preserving linear maps between C^*-algebras.

C^*-algebras belong to a wider class of complex Banach spaces in which orthogonality also makes sense. We refer to the class of (complex) JB^*-triples (see §2 for definitions). Two elements a, b in a JB^*-triple E are said to be *orthogonal* (denoted by $a \perp b$) if $L(a, b) = 0$, where $L(a, b)$ is the linear operator in E given by $L(a, b)x = \{a, b, x\}$. A linear mapping $T : E \to F$ between two JB^*-triples is called *orthogonality preserving* if $T(x) \perp T(y)$ whenever $x \perp y$. The mapping T is *biorthogonality preserving* whenever the equivalence $x \perp y \Leftrightarrow T(x) \perp T(y)$ holds for all x, y in E.

Most of the novelties introduced in [10] consist in studying orthogonality preserving bounded linear operators from a C^*-algebra or a JB^*-algebra to a JB^*-triple to take advantage of the techniques developed in JB^*-triple theory. These techniques were successfully applied in the subsequent paper [11] to obtain a description of such operators (see §2 for a detailed explanation).

Despite the vast literature on orthogonality preserving bounded linear operators between C^*-algebras and JB^*-triples, just a few papers have considered the problem of automatic continuity of biorthogonality preserving linear surjections between C^*-algebras. Besides Jarosz [24], mentioned above, M. A. Chebotar, W.-F. Ke, P.-H. Lee, and N.-C. Wong proved in [13, Theorem 4.2] that every zero products preserving linear bijection from a properly infinite von Neumann algebra into a unital ring is a ring homomorphism followed by left multiplication by the image of the identity. J. Araujo and K. Jarosz showed that every linear bijection between algebras $L(X)$, of continuous linear maps on a Banach space X, which preserves zero products in both directions is automatically continuous and a multiple of an algebra isomorphism [1]. These authors also conjectured that every linear bijection between two C^*-algebras preserving zero products in both directions is automatically continuous (see [1, Conjecture 1]).

The authors of this note proved in [12] that every biorthogonality preserving linear surjection between two compact C^*-algebras or between two von Neumann algebras is automatically continuous. One of the consequences of this result is a partial answer to [1, Conjecture 1]. Concretely, every surjective and symmetric linear mapping between von Neumann algebras (or compact C^*-algebras) which preserves zero products in both directions is continuous.

In this paper we study the problem of automatic continuity of biorthogonality preserving linear surjections between JB^*-triples, extending some of the results obtained in [12]. Section 2 contains the basic definitions and results used in the paper. Section 3 is devoted to the structure and properties of the (orthogonal) annihilator of a subset M in a JB^*-triple, focusing on the annihilators of single elements. In Section 4 we prove that every biorthogonality preserving linear surjection from a weakly compact JB^*-triple containing no infinite-dimensional rank-one summands to a JB^*-triple is automatically continuous. In Section 5 we show that two atomic JB^*-triples containing no rank-one summands are isomorphic if and only if there exists a biorthogonality preserving linear surjection between them, a result which follows from the automatic continuity of every biorthogonality preserving linear surjection between atomic JB^*-triples containing no infinite-dimensional rank-one summands.

2. Notation and preliminaries. Given Banach spaces X and Y, $L(X,Y)$ will denote the space of all bounded linear mappings from X to Y. The symbol $L(X)$ will stand for the space $L(X,X)$. Throughout the paper the word "operator" will always mean bounded linear mapping. The dual space of a Banach space X is denoted by X^*.

JB^*-triples were introduced by W. Kaup in [26]. A JB^*-*triple* is a complex Banach space E together with a continuous triple product $\{\cdot,\cdot,\cdot\}$: $E\times E\times E\to E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables, and satisfies:

(a) $L(a,b)L(x,y)=L(x,y)L(a,b)+L(L(a,b)x,y)-L(x,L(b,a)y)$, where $L(a,b)$ is the operator on E given by $L(a,b)x=\{a,b,x\}$;
(b) $L(a,a)$ is an hermitian operator with nonnegative spectrum;
(c) $\|L(a,a)\|=\|a\|^2$.

For each x in a JB^*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by the assignment $y\mapsto Q(x)y=\{x,y,x\}$.

Every C^*-algebra is a JB^*-triple via the triple product given by

$$2\{x,y,z\}=xy^*z+zy^*x,$$

and every JB^*-algebra is a JB^*-triple under the triple product

$$\{x,y,z\}=(x\circ y^*)\circ z+(z\circ y^*)\circ x-(x\circ z)\circ y^*. \tag{2.1}$$

The so-called *Kaup–Banach–Stone theorem* for JB^*-triples states that a bounded linear surjection between JB^*-triples is an isometry if and only if it is a triple isomorphism (cf. [26, Proposition 5.5], [5, Corollary 3.4] or [18, Theorem 2.2]). It follows, among many other consequences, that when a JB^*-algebra is a JB^*-triple for a suitable triple product, then the latter coincides with the one defined in (2.1).

A JBW^*-*triple* is a JB^*-triple which is also a dual Banach space (with a unique isometric predual [3]). It is known that the triple product of a JBW^*-triple is separately weak* continuous [3]. The second dual of a JB^*-triple E is a JBW^*-triple with a product extending the product of E [15].

An element e in a JB^*-triple E is said to be a *tripotent* if $\{e, e, e\} = e$. Each tripotent e in E gives rise to the decomposition

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e)$ is the $i/2$-eigenspace of $L(e, e)$ (cf. [28, Theorem 25]). The natural projection of E onto $E_i(e)$ will be denoted by $P_i(e)$. This decomposition is termed the *Peirce decomposition* of E with respect to the tripotent e. The Peirce decomposition satisfies certain rules known as *Peirce arithmetic*:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e)$$

if $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The Peirce space $E_2(e)$ is a JB^*-algebra with product $x \circ_e y := \{x, e, y\}$ and involution $x^{\sharp_e} := \{e, x, e\}$.

A tripotent e in E is called *complete* (resp., *unitary*) if $E_0(e) = 0$ (resp., $E_2(e) = E$). When $E_2(e) = \mathbb{C}e \neq \{0\}$, we say that e is *minimal*.

For each element x in a JB^*-triple E, we shall denote $x^{[1]} := x$, $x^{[3]} := \{x, x, x\}$, and $x^{[2n+1]} := \{x, x, x^{[2n-1]}\}$ $(n \in \mathbb{N})$. The symbol E_x will stand for the JB^*-subtriple generated by x. It is known that E_x is JB^*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$ such that $\Omega \cup \{0\}$ is compact, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that there exists a triple isomorphism Ψ from E_x onto $C_0(\Omega)$ satisfying $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. [25, Corollary 4.8], [26, Corollary 1.15] and [20]). The set $\overline{\Omega} = \mathrm{Sp}(x)$ is called the *triple spectrum* of x. Note that $C_0(\mathrm{Sp}(x)) = C(\mathrm{Sp}(x))$ whenever $0 \notin \mathrm{Sp}(x)$.

Therefore, for each $x \in E$, there exists a unique element $y \in E_x$ such that $\{y, y, y\} = x$. The element y, denoted by $x^{[1/3]}$, is termed the *cubic root* of x. We can inductively define $x^{[1/3^n]} = (x^{[1/3^{n-1}]})^{[1/3]}$, $n \in \mathbb{N}$. The sequence $(x^{[1/3^n]})$ converges in the weak* topology of E^{**} to a tripotent denoted by $r(x)$ and called the *range tripotent* of x. The tripotent $r(x)$ is the smallest tripotent $e \in E^{**}$ such that x is positive in the JBW^*-algebra $E_2^{**}(e)$ (cf. [16, Lemma 3.3]).

A subspace I of a JB^*-triple E is a *triple ideal* if $\{E,E,I\}+\{E,I,E\}\subseteq I$. By Proposition 1.3 in [7], I is a triple ideal if and only if $\{E,E,I\}\subseteq I$. We shall say that I is an *inner ideal* of E if $\{I,E,I\}\subseteq I$. Given an x in E, let $E(x)$ denote the norm closed inner ideal of E generated by x. It is known that $E(x)$ coincides with the norm closure of the set $Q(x)(E)$. Moreover $E(x)$ is a JB^*-subalgebra of $E_2^{**}(r(x))$ and contains x as a positive element (cf. [8]). Every triple ideal is, in particular, an inner ideal.

We recall that two elements a, b in a JB^*-triple E are said to be *orthogonal* (written $a\perp b$) if $L(a,b)=0$. Lemma 1 in [10] shows that $a\perp b$ if and only if one of the following nine statements holds:

$$\begin{aligned}&\{a,a,b\}=0;\quad a\perp r(b);\quad r(a)\perp r(b);\\ (2.2)\qquad &E_2^{**}(r(a))\perp E_2^{**}(r(b));\quad r(a)\in E_0^{**}(r(b));\quad a\in E_0^{**}(r(b));\\ &b\in E_0^{**}(r(a));\quad E_a\perp E_b;\quad \{b,b,a\}=0.\end{aligned}$$

The Jordan identity and the above reformulations ensure that

$$a\perp\{x,y,z\}\quad\text{whenever}\quad a\perp x,y,z. \tag{2.3}$$

An important class of JB^*-triples is given by the Cartan factors. A JBW^*-triple E is called a *factor* if it contains no proper weak* closed ideals. The *Cartan factors* are precisely the JBW^*-triple factors containing a minimal tripotent [27]. These can be classified in six different types (see [21] or [27]).

A Cartan factor *of type 1*, denoted by $I_{n,m}$, is a JB^*-triple of the form $L(H,H')$, where $L(H,H')$ denotes the space of bounded linear operators between two complex Hilbert spaces H and H' of dimensions n,m respectively, with the triple product defined by $\{x,y,z\}=\frac{1}{2}(xy^*z+zy^*x)$.

We recall that given a conjugation j on a complex Hilbert space H, we can define the linear involution $x\mapsto x^t:=jx^*j$ on $L(H)$. A Cartan factor *of type 2* (respectively, *type 3*), denoted by II_n (respectively, III_n), is the subtriple of $L(H)$ formed by the t-skew-symmetric (respectively, t-symmetric) operators, where H is an n-dimensional complex Hilbert space. Moreover, II_n and III_n are, up to isomorphism, independent of the conjugation j on H.

A Cartan factor *of type 4*, IV_n (also called a *complex spin factor*), is an n-dimensional complex Hilbert space provided with a conjugation $x\mapsto\overline{x}$, where the triple product and norm are given by

$$\{x,y,z\}=(x|y)z+(z|y)x-(x|\overline{z})\overline{y} \tag{2.4}$$

and $\|x\|^2=(x|x)+\sqrt{(x|x)^2-|(x|\overline{x})|^2}$, respectively.

The Cartan factor *of type 6* is the 27-dimensional exceptional JB^*-algebra $VI=H_3(\mathbb{O}^{\mathbb{C}})$ of all symmetric 3×3 matrices with entries in the complex octonions $\mathbb{O}^{\mathbb{C}}$, while the Cartan factor *of type 5*, $V=M_{1,2}(\mathbb{O}^{\mathbb{C}})$, is the subtriple of $H_3(\mathbb{O}^{\mathbb{C}})$ consisting of all 1×2 matrices with entries in $\mathbb{O}^{\mathbb{C}}$.

REMARK 2.1. Let E be a spin factor with inner product $(\cdot|\cdot)$ and conjugation $x \mapsto \overline{x}$. It is not hard to check (and part of the folklore of JB^*-triple theory) that an element w in E is a minimal tripotent if and only if $(w|\overline{w}) = 0$ and $(w|w) = 1/2$. For every minimal tripotent w in E we have $E_2(w) = \mathbb{C}w$, $E_0(w) = \mathbb{C}\overline{w}$ and $E_1(w) = \{x \in E : (x|w) = (x|\overline{w}) = 0\}$. Therefore, every minimal tripotent $w_2 \in E$ satisfying $w \perp w_2$ can be written in the form $w_2 = \lambda\overline{w}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

3. Biorthogonality preservers. Let M be a subset of a JB^*-triple E. We write $M_E^\perp$ for the *(orthogonal) annihilator* of M defined by

$$M_E^\perp := \{y \in E : y \perp x, \forall x \in M\}.$$

When no confusion can arise, we shall write $M^\perp$ instead of $M_E^\perp$.

The next result summarises some basic properties of the annihilator. The reader is referred to [17, Lemma 3.2] for a detailed proof.

LEMMA 3.1. *Let M a nonempty subset of a JB^*-triple E.*

(a) *$M^\perp$ is a norm closed inner ideal of E.*
(b) *$M \cap M^\perp = \{0\}$.*
(c) *$M \subseteq M^{\perp\perp}$.*
(d) *If $B \subseteq C$ then $C^\perp \subseteq B^\perp$.*
(e) *$M^\perp$ is weak* closed whenever E is a JBW^*-triple.* ∎

As illustration of the main identity (axiom (a) in the definition of a JB^*-triple) we shall prove statement (a). For a, a' in $M^\perp$, b in M, and c, d in E we have $\{c, a, \{d, a', b\}\} = \{\{c, a, d\}, a', b\} - \{d, \{a, c, a'\}, b\} + \{d, a', \{c, a, b\}\}$, which shows that $\{a, c, a'\} \perp b$.

Let e be a tripotent in a JB^*-triple E. Clearly, $\{e\} \subseteq E_2(e)$. Therefore, by Peirce arithmetic and Lemma 3.1,

$$E_2(e)^\perp \subseteq \{e\}^\perp = E_0(e) \subseteq E_2(e)^\perp,$$

and hence

$$E_2(e)^\perp = \{e\}^\perp = E_0(e). \tag{3.1}$$

The next lemma describes the annihilator of an element in an arbitrary JB^*-triple. Its proof follows directly from the reformulations of orthogonality in (2.2) (see also [10, Lemma 1]).

LEMMA 3.2. *Let x be an element in a JB^*-triple E. Then*

$$\{x\}_E^\perp = E_0^{**}(r(x)) \cap E.$$

Moreover, when E is a JBW^*-triple we have

$$\{x\}_E^\perp = E_0(r(x)). \quad ∎$$

Proposition 3.3. *Let e be a tripotent in a JB^*-triple E. Then*

$$E_2(e) \oplus E_1(e) \supseteq \{e\}_E^{\perp\perp} = E_0(e)^\perp \supseteq E_2(e).$$

Proof. It follows from (3.1) that $\{e\}^{\perp\perp} = \{e\}_E^{\perp\perp} = (E_0(e))^\perp \supseteq E_2(e)$. Now select $x \in (E_0(e))^\perp$. For each $i \in \{0,1,2\}$ we write $x_i = P_i(e)(x)$, where $P_i(e)$ denotes the Peirce i-projection with respect to e. Since $x \in (E_0(e))^\perp$, x must be orthogonal to x_0 and so $\{x_0, x_0, x\} = 0$. This equality, together with Peirce arithmetic, shows that $\{x_0, x_0, x_0\} + \{x_0, x_0, x_1\} = 0$, which implies that $\|x_0\|^3 = \|\{x_0, x_0, x_0\}\| = 0$. ∎

Remark 3.4. For a tripotent e in a JB^*-triple E, the equality $\{e\}_E^{\perp\perp} = E_0(e)^\perp = E_2(e)$ does not hold in general. Let H_1 and H_2 be two infinite-dimensional complex Hilbert spaces and let p be a minimal projection in $L(H_1)$. We define E as the orthogonal sum $pL(H_1) \oplus^\infty L(H_2)$. In this example $\{p\}_E^\perp = L(H_2)$ and $\{p\}_E^{\perp\perp} = pL(H_1) \neq \mathbb{C}p = E_2(p)$.

However, if E is a Cartan factor and e is a noncomplete tripotent in E, then the equality $\{e\}^{\perp\perp} = E_0(e)^\perp = E_2(e)$ always holds (cf. Lemma 5.6 in [27]).

Corollary 3.5. *Let x be an element in a JB^*-triple E. Then*

$$E(x) \subseteq E_2^{**}(r(x)) \cap E \subseteq \{x\}_E^{\perp\perp}.$$

Proof. Clearly, $E(x) = \overline{Q(x)(E)} \subseteq E_2^{**}(r(x)) \cap E$. Pick y in $E_2^{**}(r(x)) \cap E$. Then $y \in E_2^{**}(r(x)) \subseteq \{x\}_{E^{**}}^{\perp\perp}$. Since $\{x\}_E^\perp \subset \{x\}_{E^{**}}^\perp$, we conclude that $y \in \{x\}_{E^{**}}^{\perp\perp} \cap E \subseteq (\{x\}_E^\perp)_{E^{**}}^\perp \cap E = \{x\}_E^{\perp\perp}$. ∎

In the setting of C^*-algebras the following conditions describing the first and second annihilator of a projection were established in [12, Lemma 3].

Lemma 3.6. *Let p be a projection in a (not necessarily unital) C^*-algebra A. The following assertions hold:*

(a) $\{p\}_A^\perp = (1-p)A(1-p)$, *where* 1 *denotes the unit of* A^{**}*;*
(b) $\{p\}_A^{\perp\perp} = pAp$. ∎

Let x be an element in a JB^*-triple E. We say that x is *weakly compact* (respectively, *compact*) if the operator $Q(x) : E \to E$ is weakly compact (respectively, compact). A JB^*-triple is *weakly compact* (respectively, *compact*) if every element in E is weakly compact (respectively, compact).

Let E be a JB^*-triple. If we denote by $K(E)$ the Banach subspace of E generated by its minimal tripotents, then $K(E)$ is a (norm closed) triple ideal of E and it coincides with the set of weakly compact elements of E (see Proposition 4.7 in [7]). For a Cartan factor C we define the *elementary JB^*-triple* of the corresponding type to be $K(C)$. Consequently, the elementary JB^*-triples K_i $(i = 1, \ldots, 6)$ are defined as follows: $K_1 = K(H, H')$ (the compact operators between complex Hilbert spaces H and H'); $K_i = C_i \cap K(H)$ for $i = 2, 3$, and $K_i = C_i$ for $i = 4, 5, 6$.

It follows from [7, Lemma 3.3 and Theorem 3.4] that a JB^*-triple E is weakly compact if and only if one of the following statement holds:

(a) $K(E^{**}) = K(E)$.
(b) $K(E) = E$.
(c) E is a c_0-sum of elementary JB^*-triples.

Let E be a JB^*-triple. A subset $S \subseteq E$ is said to be *orthogonal* if $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. The minimal cardinal number r satisfying $\mathrm{card}(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the *rank* of E (and will be denoted by $r(E)$).

For every orthogonal family $(e_i)_{i\in I}$ of minimal tripotents in a JBW^*-triple E the weak* convergent sum $e := \sum_i e_i$ is a tripotent, and we call $(e_i)_{i\in I}$ a *frame* in E if e is a *maximal tripotent* in E (i.e., e is a complete tripotent and $\dim(E_1(e)) \leq \dim(E_1(\widetilde{e}))$ for every complete tripotent $\widetilde{e}$ in E). Every frame is a maximal orthogonal family of minimal tripotents; the converse is not true in general (see [4, §3] for more details).

Proposition 3.7. *Let e be a minimal tripotent in a JB^*-triple E. Then $\{e\}_E^{\perp\perp}$ is a rank-one norm closed inner ideal of E.*

Proof. Let F denote $\{e\}_E^{\perp\perp}$. Since e is a minimal tripotent (i.e. $E_2(e) = \mathbb{C}e$), the set of states on $E_2(e)$, $\{\varphi \in E^* : \varphi(e) = 1 = \|\varphi\|\}$, reduces to one point φ_0 in E^*. Proposition 2.4 and Corollary 2.5 in [9] imply that the norm of E restricted to $E_1(e)$ is equivalent to a Hilbertian norm. More precisely, in the terminology of [9], the norm $\|\cdot\|_e$ coincides with the Hilbertian norm $\|\cdot\|_{\varphi_0}$ and is equivalent to the norm of $E_1(e)$.

Proposition 3.3 guarantees that F is a norm closed subspace of $E_2(e) \oplus E_1(e) = \mathbb{C}e \oplus E_1(e)$, and hence F is isomorphic to a Hilbert space.

We deduce, by Proposition 4.5(iii) in [7] (and its proof), that F is a finite orthogonal sum of Cartan factors $C_1, \ldots, C_m$ which are finite-dimensional, or infinite-dimensional spin factors, or of the form $L(H, H')$ for suitable complex Hilbert spaces H and H' with $\dim(H') < \infty$. Since F is an inner ideal of E (and hence a JB^*-subtriple of E) and e is a minimal tripotent in E, we can easily check that e is a minimal tripotent in $F = \bigoplus_{j=1,\ldots,m}^{\ell_\infty} C_j$. If we write $e = e_1 + \cdots + e_m$, where each e_j is a tripotent in C_j and $e_j \perp e_k$ whenever $j \neq k$, then since $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_1 \subseteq F_2(e) = \mathbb{C}e$, we deduce that there exists a unique $j_0 \in \{1, \ldots, m\}$ satisfying $e_j = 0$ for all $j \neq j_0$ and $e = e_{j_0} \in C_{j_0}$.

For each $j \neq j_0$, we have $C_j \subseteq \{e\}_E^{\perp}$, and hence

$$\bigoplus_{j=1,\ldots,m}^{\ell_\infty} C_j = F = \{e\}^{\perp\perp} \subseteq C_j^{\perp}.$$

This implies that $C_j \perp C_j$ (or equivalently $C_j = 0$) for every $j \neq j_0$. We consequently have $F = \{e\}_E^{\perp\perp} = C_{j_0}$.

Finally, if $r(F) \geq 2$, then we deduce, via Proposition 5.8 in [27], that there exist minimal tripotents $e_2, \ldots, e_r$ in F such that $e, e_2, \ldots, e_r$ is a frame in F. For each $i \in \{2, \ldots, r\}$, e_i is orthogonal to e and lies in $F = \{e\}_E^{\perp\perp}$, which is impossible. ∎

Let $T : E \to F$ be a linear map between two JB^*-triples. We shall say that T is *orthogonality preserving* if $T(x) \perp T(y)$ whenever $x \perp y$. The mapping T is said to be *biorthogonality preserving* whenever the equivalence

$$x \perp y \;\Leftrightarrow\; T(x) \perp T(y)$$

holds for all x, y in E.

It can be easily seen that every biorthogonality preserving linear mapping $T : E \to F$ between JB^*-triples is injective. Indeed, for each $x \in E$, the condition $T(x) = 0$ implies that $T(x) \perp T(x)$, and hence $x \perp x$, which gives $x = 0$.

Orthogonality preserving bounded linear maps from a JB^*-algebra to a JB^*-triple were completely described in [11].

Before stating the result, let us recall some basic definitions. Two elements a and b in a JB^*-algebra J are said to *operator commute* in J if the multiplication operators M_a and M_b commute, where M_a is defined by $M_a(x) := a \circ x$. That is, a and b operator commute if and only if $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in J. Self-adjoint elements a and b in J generate a JB^*-subalgebra that can be realised as a JC^*-subalgebra of some $B(H)$ [36], and, in this realisation, a and b commute in the usual sense whenever they operator commute in J [33, Proposition 1]. Similarly, two self-adjoint elements a and b in J operator commute if and only if $a^2 \circ b = \{a, a, b\} = \{a, b, a\}$ (i.e., $a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b$). If $b \in J$ we use $\{b\}'$ to denote the set of elements in J that operator commute with b. We shall write $Z(J) := J'$ for the center of J (this agrees with the usual notation in von Neumann algebras).

THEOREM 3.8 ([11, Theorem 4.1]). *Let $T : J \to E$ be a bounded linear mapping from a JB^*-algebra to a JB^*-triple. For $h = T^{**}(1)$ and $r = r(h)$ the following assertions are equivalent:*

(a) *T is orthogonality preserving.*
(b) *There exists a unique Jordan *-homomorphism $S : J \to E_2^{**}(r)$ such that $S^{**}(1) = r$, $S(J)$ and h operator commute, and $T(z) = h \circ_r S(z)$ for all $z \in J$.*
(c) *T preserves zero triple products, that is, $\{T(x), T(y), T(z)\} = 0$ whenever $\{x, y, z\} = 0$.* ∎

The above characterisation proves that the bitranspose of an orthogonality preserving bounded linear mapping from a JB^*-algebra onto a JB^*-triple is also orthogonality preserving.

The following theorem was essentially proved in [11]. We include here a sketch of proof for completeness.

THEOREM 3.9. *Let $T : J \to E$ be a surjective linear operator from a JBW^*-algebra onto a JBW^*-triple and let h denote $T(1)$. Then T is biorthogonality preserving if and only if $r(h)$ is a unitary tripotent in E, h is an invertible element in the JB^*-algebra $E = E_2(r(h))$, and there exists a Jordan *-isomorphism $S : J \to E = E_2(r(h))$ such that $S(J) \subseteq \{h\}'$ and $T = h \circ_{r(h)} S$. Further, if J is a factor (i.e. $Z(J) = \mathbb{C}1$) then T is a scalar multiple of a triple isomorphism.*

Proof. The sufficiency is clear. We shall prove the necessity. To this end let $T : J \to E$ be a surjective linear operator from a JBW^*-algebra onto a JBW^*-triple and let $h = T(1) \in E$. We have already seen that every biorthogonality preserving linear mapping between JB^*-triples is injective. Therefore T is a linear bijection.

From Corollary 4.1(b) in [11] and its proof, we deduce that

$$T(J_{\mathrm{sa}}) \subseteq E_2(r(h))_{\mathrm{sa}}, \quad \text{and hence} \quad E = T(J) \subseteq E_2(r(h)) \subseteq E.$$

This implies that $E = E_2(r(h))$, which ensures that $r(h)$ is a unitary tripotent in E. Since the range tripotent of h, $r(h)$, is the unit of $E_2(r(h))$, and h is a positive element in the JBW^*-algebra $E_2(r(h))$, we can easily check that h is invertible in $E_2(r(h))$. Furthermore, $h^{1/2}$ is invertible in $E_2(r(h))$ with inverse $h^{-1/2}$.

The proof of [11, Theorem 4.1] can be literally applied here to show the existence of a Jordan *-homomorphism $S : J \to E = E_2(r(h))$ such that $S(J) \subseteq \{h\}'$ and $T = h \circ_{r(h)} S$. Since, for each $x \in J$, h and $S(x)$ operator commute and $h^{1/2}$ lies in the JB^*-subalgebra of $E_2(r(h))$ generated by h, we can easily check that $S(x)$ and $h^{1/2}$ operator commute. Thus,

$$T = h \circ_{r(h)} S = U_{h^{1/2}} S,$$

where $U_{h^{1/2}} : E_2(r(h)) \to E_2(r(h))$ is the linear mapping defined by

$$U_{h^{1/2}}(x) = 2(h^{1/2} \circ_{r(h)} x) \circ_{r(h)} h^{1/2} - (h^{1/2} \circ_{r(h)} h^{1/2}) \circ_{r(h)} x.$$

It is well known that $h^{1/2}$ is invertible if and only if $U_{h^{1/2}}$ is an invertible operator and, in this case, $U_{h^{1/2}}^{-1} = U_{h^{-1/2}}$ (cf. [22, Lemma 3.2.10]). Therefore, $S = U_{h^{-1/2}} T$. It follows from the bijectivity of T that S is a Jordan *-isomorphism.

Finally, when $Z(J) = \mathbb{C}1$, the center of $E_2(r(h))$ also reduces to $\mathbb{C}r(h)$, and since h is an invertible element in the center of $E_2(r(h))$, we deduce that T is a scalar multiple of a triple isomorphism. ■

PROPOSITION 3.10. *Let E_1, E_2 and F be three JB^*-triples (respectively, JBW^*-triples). Let $T : E_1 \oplus^\infty E_2 \to F$ be a biorthogonality preserving linear surjection. Then $T(E_1)$ and $T(E_2)$ are norm closed (respectively, weak* closed) inner ideals of F, $B = T(A_1) \oplus^\infty T(A_2)$, and for $j = 1, 2$, $T|_{A_j} : A_j \to T(A_j)$ is a biorthogonality preserving linear surjection.*

Proof. Fix $j \in \{1, 2\}$. Since $E_j = E_j^{\perp\perp}$ and T is a biorthogonality preserving linear surjection, we deduce that $T(E_j) = T(E_j^{\perp\perp}) = T(E_j)^{\perp\perp}$. Lemma 3.1 guarantees that $T(E_j)$ is a norm closed inner ideal of F (respectively, a weak* closed inner ideal of F whenever E_1, E_2 and F are JBW^*-triples). The rest of the assertion follows from Lemma 3.1 and the fact that F coincides with the orthogonal sum of $T(E_1)$ and $T(E_2)$. ■

4. Biorthogonality preservers between weakly compact JB^*-triples. The following theorem generalises [12, Theorem 5] by proving that biorthogonality preserving linear surjections between JB^*-triples send minimal tripotents to scalar multiples of minimal tripotents.

THEOREM 4.1. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples and let e be a minimal tripotent in E. Then $\|T(e)\|^{-1}T(e) = f_e$ is a minimal tripotent in F. Further, $T(E_2(e)) = F_2(f_e)$ and $T(E_0(e)) = F_0(f_e)$.*

Proof. Since T is a biorthogonality preserving surjection, the equality

$$T(S_E^\perp) = T(S)_F^\perp$$

holds for every subset S of E. Lemma 3.1 ensures that for each minimal tripotent e in E, $\{T(e)\}_F^{\perp\perp} = T(\{e\}_E^{\perp\perp})$ is a norm closed inner ideal in F. By Proposition 3.7, $\{e\}_E^{\perp\perp}$ is a rank-one JB^*-triple, and hence $\{T(e)\}_F^{\perp\perp}$ cannot contain two nonzero orthogonal elements. Thus, $\{T(e)\}_F^{\perp\perp}$ is a rank-one JB^*-triple.

The arguments given in the proof of Proposition 3.7 above (see also Proposition 4.5.(iii) in [7] and its proof or [4, §3]) show that the inner ideal $\{T(e)\}_F^{\perp\perp}$ is a rank-one Cartan factor, and hence a type 1 Cartan factor of the form $L(H, \mathbb{C})$, where H is a complex Hilbert space, or a type 2 Cartan factor II_3 (it is known that II_3 is a JB^*-triple isomorphic to a 3-dimensional complex Hilbert space). This implies that $\|T(e)\|^{-1}\, T(e) = f_e$ is a minimal tripotent in F and $T(e) = \lambda_e f_e$ for a suitable $\lambda_e \in \mathbb{C} \setminus \{0\}$.

The equality $T(E_2(e)) = F_2(f_e)$ has been proved. Concerning the Peirce zero subspace we have

$$T(E_0(e)) = T(E_2(e)_E^\perp) = T(E_2(e))_F^\perp = F_2(f_e)_F^\perp = F_0(f_e). \ ■$$

Let H and H' be complex Hilbert spaces. Given $k \in H'$ and $h \in H$, we define $k \otimes h$ in $L(H, H')$ by $k \otimes h(\xi) := (\xi|h)k$. Then every minimal tripotent in $L(H,H')$ can be written in the form $k\otimes h$, where h and k are norm-one elements in H and H', respectively. It can be easily seen that two minimal tripotents $k_1\otimes h_1$ and $k_2\otimes h_2$ are orthogonal if and only if $h_1\perp h_2$ and $k_1\perp k_2$.

THEOREM 4.2. *Let $T:E\to F$ be a biorthogonality preserving linear surjection between two JB^*-triples, where E is a type $I_{n,m}$ Cartan factor with $n,m\geq 2$. Then there exists a positive real number λ such that $\|T(e)\| = \lambda$ for every minimal tripotent e in E.*

Proof. Let H,H' be complex Hilbert spaces such that $E=L(H,H')$. Let $e_1:=k_1\otimes h_1$ and $e_2:=k_2\otimes h_2$ be two minimal tripotents in E. We write $H_1=\operatorname{span}(\{h_1,h_2\})$ and $H_1'=\operatorname{span}(\{k_1,k_2\})$. The tripotents $k_1\otimes h_1$ and $k_2\otimes h_2$ can be identified with elements in $L(H_1,H_1')$. By Theorem 4.1, $T(e_1)=\alpha_1 f_1$ and $T(e_2)=\alpha_2 f_2$, where f_1 and f_2 are two minimal tripotents in F.

If $\dim(H_1)=\dim(H_1')=2$, then the norm closed inner ideal E_{e_1,e_2} of E generated by e_1 and e_2 identifies with $L(H_1,H_1')$, which is JB^*-isomorphic to $M_2(\mathbb{C})$ and coincides with the inner ideal generated by the orthogonal minimal tripotents $g_1=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $g_2=\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, where g_1+g_2 is the unit element in $E_{e_1,e_2}\cong M_2(\mathbb{C})$.

By Theorem 4.1, $w_1:=\frac{1}{\|T(g_1)\|}T(g_1)$ and $w_2:=\frac{1}{\|T(g_2)\|}T(g_2)$ are orthogonal minimal tripotents in F. The element $w=w_1+w_2$ is a rank-2 tripotent in F and coincides with the range tripotent of the element $h=T(g_1+g_2)=\|T(g_1)\|w_1+\|T(g_2)\|w_2$. By Theorem 3.8 (see also [11, Corollary 4.1(b)]), $T(E_{e_1,e_2})\subseteq F_2(w)$. It is not hard to see that h is invertible in $F_2(w)$ with inverse $h^{-1}=\frac{1}{\|T(g_1)\|}w_1+\frac{1}{\|T(g_2)\|}w_2$.

The inner ideal E_{e_1,e_2} is finite-dimensional, $T(E_{e_1,e_2})$ is norm closed and $T|_{E_{e_1,e_2}}:E_{e_1,e_2}\to F$ is a continuous biorthogonality preserving linear operator. Theorem 3.8 guarantees the existence of a Jordan *-homomorphism $S:E_{e_1,e_2}\cong M_2(\mathbb{C})\to F_2(w)$ such that $S(g_1+g_2)=w$, $S(E_{e_1,e_2})$ and h operator commute and

$$T(z)=h\circ_w S(z)\quad\text{for all } z\in E_{e_1,e_2}. \tag{4.1}$$

It follows from the operator commutativity of h^{-1} and $S(E_{e_1,e_2})$ that $S(z)=h^{-1}\circ_w T(z)$ for all $z\in E_{e_1,e_2}$. The injectivity of T implies that S is a Jordan *-monomorphism.

Lemma 2.7 in [19] shows that $F_2(w)=F_2(w_1+w_2)$ coincides with $\mathbb{C}\oplus^{\ell_\infty}\mathbb{C}$ or with a spin factor. Since $4=\dim(T(E_{e_1,e_2}))\leq\dim(F_2(w))$, we deduce that $F_2(w)$ is a spin factor with inner product $(\cdot|\cdot)$ and conjugation $x\mapsto\overline{x}$. From Remark 2.1, we may assume, without loss of generality, that $(w_1|w_1)=1/2$, $(w_1|\overline{w}_1)=0$, and $w_2=\overline{w}_1$.

Now, we take $g_3 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $g_4 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in E_{e_1,e_2}. The elements $w_3 := S(g_3)$ and $w_4 := S(g_4)$ are orthogonal minimal tripotents in $F_2(w)$ with $\{w_i, w_i, w_j\} = \frac{1}{2}w_j$ for every $(i,j),(j,i) \in \{1,2\} \times \{3,4\}$. Applying again Remark 2.1, we may assume that $(w_3|w_3) = 1/2$, $(w_3|\overline{w}_3) = 0$, $w_4 = \overline{w}_3$, and $(w_3|w_1) = (w_3|w_2) = 0$. Applying the definition of the triple product in a spin factor given in (2.4) we can check that $(w_1, w_3, w_2 = \overline{w}_1, w_4 = \overline{w}_3)$ are four minimal tripotents in $F_2(w)$ with $w_1 \perp w_2$, $w_3 \perp w_4$, $\{w_i, w_i, w_j\} = \frac{1}{2}w_j$ for every $(i,j),(j,i) \in \{1,2\}\times\{3,4\}$, $\{w_1, w_3, w_2\} = -\frac{1}{2}w_4$, $\{w_3, w_2, -w_4\} = \frac{1}{2}w_1$, $\{w_2, -w_4, w_1\} = \frac{1}{2}w_3$, and $\{-w_4, w_1, w_3\} = \frac{1}{2}w_2$. Thus, denoting by M the JB^*-subtriple of $F_2(w)$ generated by w_1, w_3, w_2, and w_4, we have shown that M is a JB^*-triple isomorphic to $M_2(\mathbb{C})$.

Combining (4.1) and (2.4) we get

$$T(g_3) = h \circ_w S(g_3) = \{h, w, w_3\} = \frac{\|T(g_1)\| + \|T(g_2)\|}{2} w_3,$$
$$T(g_4) = h \circ_w S(g_4) = \{h, w, w_4\} = \frac{\|T(g_1)\| + \|T(g_2)\|}{2} w_4.$$

Since $T(g_1) = \|T(g_1)\|w_1$, $T(g_2) = \|T(g_2)\|w_2$, and E_{e_1,e_2} is linearly generated by g_1, g_2, g_3 and g_4, we deduce that $T(E_{e_1,e_2}) \subseteq M$ with $4 = \dim(T(E_{e_1,e_2})) \leq \dim(M) = 4$. Thus, $T(E_{e_1,e_2}) = M$ is a JB^*-subtriple of F.

The mapping $T|_{E_{e_1,e_2}} : E_{e_1,e_2} \cong M_2(\mathbb{C}) \to T(E_{e_1,e_2})$ is a continuous biorthogonality preserving linear bijection. Theorem 3.9 implies that $T|_{E_{e_1,e_2}}$ is a (nonzero) scalar multiple of a triple isomorphism, and hence $\|T(e_1)\| = \|T(e_2)\|$.

If $\dim(H_1') = 1$, then $L(H_1, H_1')$ is a rank-one JB^*-triple. Since $n, m \geq 2$, we can find a minimal tripotent e in E such that the norm closed inner ideals of E generated by $\{e, e_1\}$ and $\{e, e_2\}$ both coincide with $M_2(\mathbb{C})$. The arguments in the above paragraph show that $\|T(e_1)\| = \|T(e)\| = \|T(e_2)\|$.

Finally, the case $\dim(H_1) = 1$ follows from the same arguments. ∎

Remark 4.3. Given a sequence $(\mu_n) \subset c_0$ and a bounded sequence (x_n) in a Banach space X, the series $\sum_k \mu_k x_k$ need not be, in general, convergent in X. However, when (x_n) is a bounded sequence of mutually orthogonal elements in a JB^*-triple E, the equality

$$\Big\| \sum_{k=1}^{n} \mu_k x_k - \sum_{k=1}^{m} \mu_k x_k \Big\| = \max\{|\mu_{n+1}|, \ldots, |\mu_m|\} \sup\{\|x_n\|\}$$

holds for every $n < m$ in $\mathbb{N}$. It follows that $(\sum_{k=1}^{n} \mu_k x_k)$ is a Cauchy sequence and hence converges in E.

The following three results generalise [12, Lemmas 8, 9 and Proposition 10] to the setting of JB^*-triples.

Lemma 4.4. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples and let (e_n) be a sequence of mutually orthogonal minimal tripotents in E. Then there exist positive constants $m \leq M$ satisfying $m \leq \|T(e_n)\| \leq M$ for all $n \in \mathbb{N}$.*

Proof. We deduce from Theorem 4.1 that, for each natural n, there exist a minimal tripotent f_n and a scalar $\lambda_n \in \mathbb{C} \setminus \{0\}$ such that $T(e_n) = \lambda_n f_n$, where $\|T(e_n)\| = \lambda_n$. Note that T being biorthogonality preserving implies (f_n) is a sequence of mutually orthogonal minimal tripotents in F.

Let (μ_n) be any sequence in c_0. Since the e_n's are mutually orthogonal the series $\sum_{k\geq 1} \mu_k e_k$ converges to an element in E (cf. Remark 4.3). For each natural n, $\sum_{k\geq 1} \mu_k e_k$ decomposes as the orthogonal sum of $\mu_n e_n$ and $\sum_{k\neq n} \mu_k e_k$, therefore

$$T\Big(\sum_{k\geq 1} \mu_k e_k\Big) = \mu_n \lambda_n f_n + T\Big(\sum_{k\neq n} \mu_k e_k\Big)$$

with $\mu_n \lambda_n f_n \perp T\big(\sum_{k\neq n}^{\infty} \mu_k e_k\big)$, which in particular implies

$$\Big\|T\Big(\sum_{k\geq 1} \mu_k e_k\Big)\Big\| = \max\Big\{|\mu_n|\,|\lambda_n|, \Big\|T\Big(\sum_{k\neq n} \mu_k e_k\Big)\Big\|\Big\} \geq |\mu_n|\,|\lambda_n|.$$

This establishes that, for each (μ_n) in c_0, $(\mu_n \lambda_n)$ is a bounded sequence, which in particular implies that (λ_n) is bounded.

Finally, since T is a biorthogonality preserving linear surjection and $T^{-1}(f_n) = \lambda_n^{-1} e_n$, we can similarly show that (λ_n^{-1}) is also bounded. ∎

Lemma 4.5. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples, (μ_n) a sequence in c_0, and (e_n) a sequence of mutually orthogonal minimal tripotents in E. Then the sequence $(T(\sum_{k\geq n} \mu_k e_k))_n$ is well defined and converges in norm to zero.*

Proof. From Theorem 4.1 and Lemma 4.4 it follows that $(T(e_n))$ is a bounded sequence of mutually orthogonal elements in F. Let M denote a bound of the above sequence. For each natural n, Remark 4.3 ensures that the series $\sum_{k\geq n} \mu_k e_k$ converges.

Define $y_n := T(\sum_{k\geq n} \mu_k e_k)$. We claim that (y_n) is a Cauchy sequence in F. Indeed, given $n < m$ in $\mathbb{N}$, we have

$$\|y_n - y_m\| = \Big\|T\Big(\sum_{k\geq n}^{m-1} \mu_k e_k\Big)\Big\| = \Big\|\sum_{k\geq n}^{m-1} \mu_k T(e_k)\Big\| \leq M \max\{|\mu_n|, \dots, |\mu_{m-1}|\}, \tag{4.2}$$

where in the last inequality we have used the fact that $(T(e_n))$ is a sequence of mutually orthogonal elements. Consequently, (y_n) converges in norm to some element y_0 in F. Let z_0 denote $T^{-1}(y_0)$.

Fix a natural m. By hypothesis, for each $n > m$, e_m is orthogonal to $\sum_{k\geq n} \mu_k e_k$. This implies that $T(e_m) \perp y_n$ for every $n > m$, which in particular implies $\{T(e_m), T(e_m), y_n\} = 0$ for every $n > m$. Letting n tend to ∞ we have $\{T(e_m), T(e_m), y_0\} = 0$. This shows that $y_0 = T(z_0)$ is orthogonal to $T(e_m)$, and hence $e_m \perp z_0$. Since m was arbitrary, we deduce that z_0 is orthogonal to $\sum_{k\geq n} \mu_k e_k$ for every n. Therefore, $(y_n) \subset \{y_0\}^\perp$, and hence y_0 belongs to the norm closure of $\{y_0\}^\perp$, which implies $y_0 = 0$. ∎

PROPOSITION 4.6. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples, where E is weakly compact. Then T is continuous if and only if the set $\mathcal{T} := \{\|T(e)\| : e$ a minimal tripotent in $E\}$ is bounded. Moreover, in that case $\|T\| = \sup(\mathcal{T})$.*

Proof. The necessity being obvious, suppose that

$$M = \sup\{\|T(e)\| : e \text{ a minimal tripotent in } E\} < \infty.$$

Since E is weakly compact, each nonzero element x of E can be written as a norm convergent (possibly finite) sum $x = \sum_n \lambda_n u_n$, where u_n are mutually orthogonal minimal tripotents of E, and $\|x\| = \sup\{|\lambda_n| : n \geq 1\}$ (cf. Remark 4.6 in [7]). If the series $x = \sum_n \lambda_n u_n$ is finite then

$$\|T(x)\| = \Big\|\sum_{n=1}^{m} \lambda_n T(u_n)\Big\| \overset{(*)}{=} \max\{\|\lambda_n T(u_n)\| : n = 1, \ldots, m\} \leq M\|x\|,$$

where at $(*)$ we apply the fact that $(T(u_n))$ is a finite set of mutually orthogonal tripotents in F. When the series $x = \sum_n \lambda_n u_n$ is infinite we may assume that $(\lambda_n) \in c_0$.

It follows from Lemma 4.5 that the sequence $(T(\sum_{k\geq n} \lambda_k u_k))_n$ is well defined and converges in norm to zero. We can find a natural m such that $\|T(\sum_{k\geq m} \lambda_k u_k)\| < M\|x\|$. Since the elements $\lambda_1 u_1, \ldots, \lambda_{m-1} u_{m-1}$, $\sum_{k\geq m} \lambda_k u_k$ are mutually orthogonal, we have

$$\begin{aligned}\|T(x)\| &= \max\Big\{\|T(\lambda_1 u_1)\|, \ldots, \|T(\lambda_{m-1} u_{m-1})\|, \Big\|T\Big(\sum_{k\geq m} \lambda_k u_k\Big)\Big\|\Big\}\\ &\leq M\|x\|. \ \blacksquare\end{aligned}$$

Let E be an elementary JB^*-triple of type 1 (that is, an elementary JB^*-triple such that E^{**} is a type 1 Cartan factor), and let $T : E \to F$ be a biorthogonality preserving linear surjection from E onto another JB^*-triple. Then by Theorem 4.2 and Proposition 4.6, T is continuous. Further, we claim that T is a scalar multiple of a triple isomorphism. Indeed, let us see that $S = (1/\lambda)T$ is a triple isomorphism, where $\lambda = \|T(e)\| = \|T\|$ for some (and hence any) minimal tripotent e in E (cf. Theorem 4.2). Let $x \in E$. Then $x = \sum_n \lambda_n e_n$ for a suitable $(\lambda_n) \in c_0$ and a family of mutually orthogonal minimal tripotents (e_n) in E [7, Remark 4.6]. Then by observing that T is continuous we have

$$\|S(x)\| = \frac{1}{\lambda}\|T(x)\| = \frac{1}{\lambda}\Big\|T\Big(\sum_n \lambda_n e_n\Big)\Big\| = \frac{1}{\lambda}\Big\|\sum_n \lambda_n T(e_n)\Big\|$$
$$= \frac{1}{\lambda}\sup_n |\lambda_n|\,\|T(e_n)\| = \frac{1}{\lambda}\sup_n |\lambda_n|\lambda = \sup_n |\lambda_n| = \|x\|.$$

This proves that S is a surjective linear isometry between JB^*-triples, and hence a triple isomorphism (see [26, Proposition 5.5], [5, Corollary 3.4], [18, Theorem 2.2]). We have thus proved the following result:

COROLLARY 4.7. *Let $T : E \to F$ a biorthogonality preserving linear surjection from a type 1 elementary JB^*-triple of rank greater than one onto another JB^*-triple. Then T is a scalar multiple of a triple isomorphism.* ∎

Let p and q be two minimal projections in a C^*-algebra A with $q \neq p$. It is known that the C^*-subalgebra of A generated by p and q is isometrically isomorphic to $\mathbb{C} \oplus^\infty \mathbb{C}$ when p and q are orthogonal, and isomorphic to $M_2(\mathbb{C})$ otherwise. More concretely, by [31, Theorem 1.3] (see also [29, §3]), denoting by $C_{p,q}$ the C^*-subalgebra of A generated by p and q, we have the following statements:

(a) If $p \perp q$ then there exists an isometric C^*-isomorphism $\Phi : C_{p,q} \to \mathbb{C} \oplus^\infty \mathbb{C}$ such that $\Phi(p) = (1,0)$ and $\Phi(q) = (0,1)$.

(b) If p and q are not orthogonal then there exist $0 < t < 1$ and an isometric C^*-isomorphism $\Phi : C_{p,q} \to M_2(\mathbb{C})$ such that

$$\Phi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(q) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

In the setting of JB^*-algebras we have:

LEMMA 4.8. *Let p and q be two minimal projections in a JB^*-algebra J with $q \neq p$ and let $J_{p,q}$ denote the JB^*-subalgebra of J generated by p and q.*

(a) *If $p \perp q$ then there exists an isometric JB^*-isomorphism $\Phi : J_{p,q} \to \mathbb{C} \oplus^\infty \mathbb{C}$ such that $\Phi(p) = (1,0)$ and $\Phi(q) = (0,1)$.*

(b) *If p and q are not orthogonal then there exist $0 < t < 1$ and an isometric JB^*-isomorphism $\Phi : C \to S_2(\mathbb{C})$ such that*

$$\Phi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \Phi(q) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix},$$

where $S_2(\mathbb{C})$ denotes the type 3 Cartan factor of all symmetric operators on a two-dimensional complex Hilbert space.

Moreover, the JB^*-subtriple of J generated by p and q coincides with $J_{p,q}$.

Proof. Statement (a) is clear. Now assume that p and q are not orthogonal. The Shirshov–Cohn theorem (see [22, Theorem 7.2.5]) ensures that $J_{p,q}$ is a JC^*-algebra, that is, a Jordan *-subalgebra of some C^*-algebra A. The symbol $C_{p,q}$ will stand for the (associative) C^*-subalgebra of A generated by p and q. Set

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

We have already mentioned that there exist $0 < t < 1$ and an isometric C^*-isomorphism $\Phi : C_{p,q} \to M_2(\mathbb{C})$ such that $\Phi(p) = P$ and $\Phi(q) = Q$.

Since $J_{p,q}$ is a Jordan *-subalgebra of $C_{p,q}$, $J_{p,q}$ can be identified with the Jordan *-subalgebra of $M_2(\mathbb{C})$ generated by the matrices P and Q. It can be easily checked that

$$P \circ Q = \begin{pmatrix} t & \frac{1}{2}\sqrt{t(1-t)} \\ \frac{1}{2}\sqrt{t(1-t)} & 0 \end{pmatrix},$$

$$2P \circ Q - 2tP = \begin{pmatrix} 0 & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 0 \end{pmatrix},$$

$$Q - (2P \circ Q - 2tP) - tP = \begin{pmatrix} 0 & 0 \\ 0 & 1-t \end{pmatrix}.$$

These identities show that $J_{p,q}$ contains the generators of the JB^*-algebra $S_2(\mathbb{C})$, and hence identifies with $S_2(\mathbb{C})$.

In order to prove the last assertion, let $E_{p,q}$ denote the JB^*-subtriple of J generated by p and q. As $J_{p,q}$ is itself a subtriple containing p and q, we have $E_{p,q} \subseteq J_{p,q}$. If $p \perp q$ then it can easily be seen that $E_{p,q} \cong \mathbb{C} \oplus^\infty \mathbb{C} \cong J_{p,q}$. Now assume that p and q are not orthogonal.

From Proposition 5 in [20], $E_{p,q}$ is a JB^*-triple isometrically isomorphic to $M_{1,2}(\mathbb{C})$ or $S_2(\mathbb{C})$. If $E_{p,q}$ is a rank-one JB^*-triple, that is, $E \cong M_{1,2}(\mathbb{C})$, then $P_0(p)(q)$ must be zero. Thus, according to the above representation, we have $1 - t = 0$, which is impossible. ■

A JB^*-algebra which is a weakly compact JB^*-triple will be called *weakly compact* or *dual* (see [6]). Every positive element x in a weakly compact JB^*-algebra J can be written in the form $x = \sum_n \lambda_n p_n$ for a suitable $(\lambda_n) \in c_0$ and a family (p_n) of mutually orthogonal minimal projections in J (see Theorem 3.3 in [6]).

Our next theorem extends [12, Theorem 11].

THEOREM 4.9. *Let $T : J \to E$ be a biorthogonality preserving linear surjection from a weakly compact JB^*-algebra onto a JB^*-triple. Then T is continuous and $\|T\| \leq 2 \sup\{\|T(p)\| : p$ a minimal projection in $J\}$.*

Proof. Since J is a JB^*-algebra, it is enough to show that T is bounded on positive norm-one elements. In this case, it suffices to prove that the set

$$\mathcal{P} = \{\|T(p)\| : p \text{ a minimal projection in } J\}$$

is bounded (cf. the proof of Proposition 4.6).

Suppose, on the contrary, that $\mathcal{P}$ is unbounded. We shall show by induction that there exists a sequence (p_n) of mutually orthogonal minimal projections in J such that $\|T(p_n)\| > n$.

The case $n = 1$ is clear. The induction hypothesis guarantees the existence of mutually orthogonal minimal projections $p_1, \dots, p_n$ in J with $\|T(p_k)\| > k$ for all $k \in \{1, \dots, n\}$.

By assumption, there exists a minimal projection $q \in J$ satisfying

$$\|T(q)\| > \max\{\|T(p_1)\|, \dots, \|T(p_n)\|, n+1\}.$$

We claim that q must be orthogonal to each p_j. If that is not the case, there exists j such that p_j and q are not orthogonal. Let C denote the JB^*-subtriple of J generated by q and p_j. We conclude from Lemma 4.8 that C is isomorphic to the JB^*-algebra $S_2(\mathbb{C})$.

Let $g_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $g_1 + g_2$ is the unit element in $C \cong S_2(\mathbb{C})$. By Theorem 4.1, $w_1 := \frac{1}{\|T(g_1)\|} T(g_1)$ and $w_2 := \frac{1}{\|T(g_2)\|} T(g_2)$ are two orthogonal minimal tripotents in E. The element $w = w_1 + w_2$ is a rank-2 tripotent in E and coincides with the range tripotent of the element $h = T(g_1 + g_2) = \|T(g_1)\| w_1 + \|T(g_2)\| w_2$. Furthermore, h is invertible in $E_2(w)$, and by Theorem 3.8 (see also [11, Corollary 4.1(b)]), $T(C) \subseteq E_2(w)$.

The rest of the argument is parallel to the argument in the proof of Theorem 4.2.

The finite-dimensionality of the JB^*-subtriple C ensures that $T(C)$ is norm closed and $T|_C : C \cong S_2(\mathbb{C}) \to E$ is a continuous biorthogonality preserving linear operator. Theorem 3.8 guarantees the existence of a Jordan *-homomorphism $S : C \to E_2(w)$ such that $S(g_1 + g_2) = w$, $S(C)$ and h operator commute and

$$T(z) = h \circ_w S(z) \quad \text{for all } z \in C. \tag{4.3}$$

It follows from the operator commutativity of h^{-1} and $S(C)$ that $S(z) = h^{-1} \circ_w T(z)$ for all $z \in C$. The injectivity of T implies that S is a Jordan *-monomorphism.

Lemma 2.7 in [19] shows that $E_2(w) = E_2(w_1 + w_2)$ coincides with $\mathbb{C} \oplus^{\ell_\infty} \mathbb{C}$ or with a spin factor. Since $3 = \dim(T(C)) \le \dim(E_2(w))$, we deduce that $E_2(w)$ is a spin factor with inner product $(\cdot|\cdot)$ and conjugation $x \mapsto \overline{x}$. We may assume, by Remark 2.1, that $(w_1|w_1) = 1/2$, $(w_1|\overline{w}_1) = 0$, and $w_2 = \overline{w}_1$.

Now, taking $g_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in C \cong S_2(\mathbb{C})$, the element $w_3 := S(g_3)$ is a tripotent in $E_2(w)$ with $\{w_i, w_i, w_3\} = \frac{1}{2} w_3$ for every $i \in \{1,2\}$. Remark 2.1 implies that $(w_3|w_1) = (w_3|w_2) = 0$. Let M denote the JB^*-subtriple of $E_2(w)$ generated by w_1, w_2, and w_3. The mapping $S : C \cong S_2(\mathbb{C}) \to M$ is a Jordan *-isomorphism.

Combining (4.3) and (2.4) we get

$$T(g_3) = h \circ_w S(g_3) = \{h, w, w_3\} = \frac{\|T(g_1)\| + \|T(g_2)\|}{2} w_3.$$

Since $T(g_1) = \|T(g_1)\| w_1$, $T(g_2) = \|T(g_2)\| w_2$, and C is linearly generated by g_1, g_2 and g_3, we deduce that $T(C) \subseteq M$ with $3 = \dim(T(C)) \leq \dim(M) = 3$. Thus, $T(C) = M$ is a JB^*-subtriple of E.

The mapping $T|_C : C \cong S_2(\mathbb{C}) \to T(C)$ is a continuous biorthogonality preserving linear bijection. Theorem 3.9 guarantees the existence of a scalar $\lambda \in \mathbb{C}\backslash\{0\}$ and a triple isomorphism $\Psi : C \to T(C)$ such that $T(x) = \lambda \Psi(x)$ for all $x \in C$. Since p_j and q are projections, $\|\Psi(q)\| = \|\Psi(p_j)\| = 1$. Hence $\|T(p_j)\| = |\lambda|$ and $\|T(q)\| = |\lambda|$, contradicting the induction hypothesis. Therefore $q \perp p_j$ for every $j = 1, \ldots, n$.

It follows by induction that there exists a sequence (p_n) of mutually orthogonal minimal projections in J such that $\|T(p_n)\| > n$. The series $\sum_{n=1}^{\infty} (1/\sqrt{n}) p_n$ defines an element a in J (cf. Remark 4.3). For each natural m, a decomposes as the orthogonal sum of $(1/\sqrt{m}) p_m$ and $\sum_{n \neq m} (1/\sqrt{n}) p_n$, therefore

$$T(a) = \frac{1}{\sqrt{m}} T(p_m) + T\bigg(\sum_{n \neq m} \frac{1}{\sqrt{n}} p_n\bigg),$$

with orthogonal summands. This argument implies that

$$\|T(a)\| = \max\bigg\{\frac{1}{\sqrt{m}} \|T(p_m)\|, \ \bigg\|T\bigg(\sum_{n \neq m} \frac{1}{\sqrt{n}} p_n\bigg)\bigg\|\bigg\} > \sqrt{m}.$$

Since m was arbitrary, we have arrived at the desired contradiction. ∎

By Proposition 2 in [23], every Cartan factor of type 1 with $\dim(H) = \dim(H')$, every Cartan factor of type 2 with $\dim(H)$ even or infinite, and every Cartan factor of type 3 is a JBW^*-algebra factor for a suitable Jordan product and involution. In the case of C being a Cartan factor which is also a JBW^*-algebra, the corresponding elementary JB^*-triple $K(C)$ is a weakly compact JB^*-algebra.

Corollary 4.10. *Let K be an elementary JB^*-triple of type 1 with $\dim(H) = \dim(H')$, or of type 2 with $\dim(H)$ even or infinite, or of type 3. Suppose that $T : K \to E$ is a biorthogonality preserving linear surjection from K onto a JB^*-triple. Then T is continuous. Further, since K^{**} is a*

JBW^*-algebra factor, Theorem 3.9 *ensures that T is a scalar multiple of a triple isomorphism.* ∎

THEOREM 4.11. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between JB^*-triples, where E is weakly compact containing no infinite-dimensional rank-one summands. Then T is continuous.*

Proof. Since E is a weakly compact JB^*-triple, the statement follows from Proposition 4.6 as soon as we prove that the set

$$\mathcal{T} := \{\|T(e)\| : e \text{ a minimal tripotent in } E\}$$

is bounded.

We know that $E = \bigoplus_{\alpha\in\Gamma}^{c_0} K_\alpha$, where $\{K_\alpha : \alpha \in \Gamma\}$ is a family of elementary JB^*-triples (see Lemma 3.3 in [7]). Now, Lemma 3.1 guarantees that $T(K_\alpha) = T(K_\alpha^{\perp\perp}) = T(K_\alpha)^{\perp\perp}$ is a norm closed inner ideal for every $\alpha \in \Gamma$.

For each $\alpha \in \Gamma$, K_α is finite-dimensional, or a type 1 elementary JB^*-triple of rank greater than one, or a JB^*-algebra. It follows, by Corollary 4.7 and Theorem 4.9, that $T|_{K_\alpha} : K_\alpha \to T(K_\alpha)$ is continuous.

Suppose that $\mathcal{T}$ is unbounded. Having in mind that every minimal tripotent in E belongs to a unique factor K_α, by Proposition 4.6, there exists a sequence (e_n) of mutually orthogonal minimal tripotents in E such that $\|T(e_n)\|$ diverges to $+\infty$. The element $z := \sum_{n=1}^{\infty} \|T(e_n)\|^{-1/2} e_n$ lies in E and hence $\|T(z)\| < \infty$. We fix an arbitrary natural m. Since $z - \|T(e_m)\|^{-1/2} e_m$ and $\|T(e_m)\|^{-1/2} e_m$ are orthogonal, we have

$$T(z - \|T(e_m)\|^{-1/2} e_m) \perp T(\|T(e_m)\|^{-1/2} e_m),$$

and hence

$$\begin{aligned}\|T(z)\| &= \|T(z - \|T(e_m)\|^{-1/2} e_m)) + T(\|T(e_m)\|^{-1/2} e_m)\| \\ &= \max\{\|T(z - \|T(e_m)\|^{-1/2} e_m)\|, \|T(e_m)\|^{-1/2}\|T(e_m)\|\} \geq \sqrt{\|T(e_m)\|},\end{aligned}$$

which contradicts that $\|T(e_m)\|^{1/2} \to +\infty$. Therefore $\mathcal{T}$ is bounded. ∎

COROLLARY 4.12. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples, where $K(E)$ contains no infinite-dimensional rank-one summands. Then $T|_{K(E)} : K(E) \to K(F)$ is continuous.*

Proof. Pick $x \in K(E)$. It can be written in the form $x = \sum_n \lambda_n u_n$, where u_n are mutually orthogonal minimal tripotents of E, and $\|x\| = \sup\{|\lambda_n| : n \geq 1\}$ (cf. Remark 4.6 in [7]). For each natural m we define $y_m := T(\sum_{n\geq m+1} \lambda_n u_n)$. Theorem 4.1 guarantees that $T(x_m) = T(\sum_{n=1}^{m} \lambda_n u_n)$ defines a sequence in $K(F)$.

Since, by Lemma 4.5, $y_m \to 0$ in norm, we deduce that $T(x_m) = T(x) - y_m$ tends to $T(x)$ in norm. Therefore $T(K(E)) = K(F)$ and $T|_{K(E)} : K(E) \to K(F)$ is a biorthogonality preserving linear surjection between weakly compact JB^*-triples. The result now follows from Theorem 4.11. ∎

REMARK 4.13. In Remark 15 of [10] it was already pointed out that the conclusion of Theorem 4.11 is no longer true if we allow E to have infinite-dimensional rank-one summands. Indeed, let $E = L(H) \oplus^\infty L(H, \mathbb{C})$, where H is an infinite-dimensional complex Hilbert space. We can always find an unbounded bijection $S : L(H, \mathbb{C}) \to L(H, \mathbb{C})$. Since $L(H, \mathbb{C})$ is a rank-one JB^*-triple, S is a biorthogonality preserving linear bijection and the mapping $T : E \to E$ given by $x + y \mapsto x + S(y)$ has the same properties.

COROLLARY 4.14. *Two weakly compact JB^*-triples containing no rank-one summands are isomorphic if and only if there exists a biorthogonality preserving linear surjection between them.*

5. Biorthogonality preservers between atomic JBW^*-triples. A JBW^*-triple E is said to be *atomic* if it coincides with the weak* closed ideal generated by its minimal tripotents. Every atomic JBW^*-triple can be written as an ℓ_∞-sum of Cartan factors [21].

The aim of this section is to study when the existence of a biorthogonality preserving linear surjection between two atomic JBW^*-triples implies that they are isomorphic (note that continuity is not assumed). We shall establish an automatic continuity result for biorthogonality preserving linear surjections between atomic JBW^*-triples containing no rank-one factors.

Before dealing with the main result, we survey some results describing the elements in the predual of a Cartan factor. We make use of the description of the predual of $L(H)$ in terms of the *trace class* operators (cf. [32, §II.1]). The results, included here for completeness, are direct consequences of this description but we do not know an explicit reference.

Let $C = L(H, H')$ be a type 1 Cartan factor. Lemma 2.6 in [30] ensures that each φ in C_* can be written in the form $\varphi := \sum_{n=1}^{\infty} \lambda_n \varphi_n$, where (λ_n) is a sequence in ℓ_1^+ and each φ_n is an extreme point of the closed unit ball of C_*. More concretely, for each natural n there exist norm-one elements $h_n \in H$ and $k_n \in H'$ such that $\varphi_n(x) = (x(h_n)|k_n)$ for every $x \in C$, that is, for each natural n there exists a minimal tripotent e_n in C such that $P_2(e_n)(x) = \varphi_n(x) e_n$ for every $x \in C$ (cf. [20, Proposition 4]).

We now consider (infinite-dimensional) type 2 and type 3 Cartan factors. Let j be a conjugation on a complex Hilbert space H, and consider the linear involution on $L(H)$ defined by $x \mapsto x^t := jx^*j$. Let $C_2 = \{x \in L(H) :$

$x^t = -x\}$ and $C_3 = \{x \in L(H) : x^t = x\}$ be Cartan factors of type 2 and 3, respectively.

Noticing that $L(H) = C_2 \oplus C_3$, it is easy to see that every element φ in $(C_2)_*$ (respectively, $(C_3)_*$) admits an extension of the form $\widetilde{\varphi} = \varphi\pi$, where π denotes the canonical projection of $L(H)$ onto C_2 (respectively, C_3). Making use of [32, Lemma 1.5], we can find an element $x_{\widetilde{\varphi}} \in K(H)$ satisfying

$$(x_{\widetilde{\varphi}}(h)|k) = \widetilde{\varphi}(h \otimes k) \quad (h, k \in H). \tag{5.1}$$

Since, for each $x \in L(H)$, $\widetilde{\varphi}(x) = \frac{1}{2}\widetilde{\varphi}(x - x^t)$, we can easily check, via (5.1), that $x^t_{\widetilde{\varphi}} = -x_{\widetilde{\varphi}}$. Therefore $x_{\widetilde{\varphi}} \in K_2 = K(C_2)$. From [7, Remark 4.6] it may be deduced that $x_{\widetilde{\varphi}}$ can be (uniquely) written as a norm convergent (possibly finite) sum $x_{\widetilde{\varphi}} = \sum_n \lambda_n u_n$, where u_n are mutually orthogonal minimal tripotents in K_2 and $(\lambda_n) \in c_0$ (notice that u_n is a minimal tripotent in C_2 but it need not be minimal in $L(H)$; in any case, either u_n is minimal in $L(H)$ or it can be written as a convex combination of two minimal tripotents in $L(H)$). For each $(\beta_n) \in c_0$, $z := \sum_n \beta_n u_n \in K_2$ and, by (5.1), $\sum_n \lambda_n \beta_n = \widetilde{\varphi}(z) = \varphi(z) < \infty$. Thus, $(\lambda_n) \in \ell_1$, and another application of (5.1) shows that $\varphi(x) = \sum_n \lambda_n \varphi_n(x)$ for all $x \in C_2$, where φ_n lies in $(C_2)_*$ and satisfies $P_2(u_n)(x) = \varphi_n(x) u_n$. A similar reasoning remains true for C_3.

We have thus proved:

Proposition 5.1. *Let C be an infinite-dimensional Cartan factor of type 1, 2 or 3. For each φ in C_*, there exist a sequence $(\lambda_n) \in \ell_1$ and a sequence (u_n) of mutually orthogonal minimal tripotents in C such that*

$$\|\varphi\| = \sum_{n=1}^{\infty} |\lambda_n| \quad \textit{and} \quad \varphi(x) = \sum_n \lambda_n \varphi_n(x) \quad (x \in C),$$

where for each $n \in \mathbb{N}$, $\varphi_n(x) u_n = P_2(u_n)(x)$ $(x \in C)$. ■

Let $T : E \to F$ be a biorthogonality preserving linear surjection between atomic JBW^*-triples, where E contains no rank-one Cartan factors. In this case $K(E)$ and $K(F)$ are weakly compact JB^*-triples with $K(E)^{**} = E$ and $K(F)^{**} = F$. Corollary 4.12 ensures that $T|_{K(E)} : K(E) \to K(F)$ is continuous. This is not, a priori, enough to guarantee that T is continuous. In fact, for each nonreflexive Banach space X there exists an unbounded linear operator $S : X^{**} \to X^{**}$ such that $S|_X : X \to X$ is continuous. The main result of this section establishes that a mapping T as above is automatically continuous.

Theorem 5.2. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between atomic JBW^*-triples, where E contains no rank-one Cartan factors. Then T is continuous.*

Proof. Corollary 4.12 ensures that $T|_{K(E)} : K(E) \to K(F)$ is continuous. By Lemma 3.3 in [7], $K(E)$ decomposes as a c_0-sum of all elementary triple ideals of E, that is, if $E = \bigoplus^{\ell_\infty} C_\alpha$, where each C_α is a Cartan factor, then $K(E) = \bigoplus^{c_0} K(C_\alpha)$. By Proposition 3.10, for each α, $T(K_\alpha)$ (respectively, $T(C_\alpha)$ is a norm closed (respectively, weak* closed) inner ideal of $K(F)$ (respectively, F) and $K(F) = \bigoplus^{c_0} T(K(C_\alpha))$ (respectively, $F = \bigoplus^{c_0} T(C_\alpha)$).

For each α, C_α is either finite-dimensional, or an infinite-dimensional Cartan factor of type 1, 2 or 3. Corollaries 4.7 and 4.10 prove that the operator $T|_{K(C_\alpha)} : K(C_\alpha) \to T(K(C_\alpha))$ is a scalar multiple of a triple isomorphism. We claim that, for each α and each φ_α in the predual of $T(C_\alpha)$, $\varphi_\alpha T$ is weak* continuous. There is no loss of generality in assuming that C_α is infinite-dimensional.

Each minimal tripotent f in F lies in a unique elementary JB^*-triple $T(K(C_\alpha))$. Since $T|_{K(C_\alpha)} : K(C_\alpha) \to T(K(C_\alpha))$ is a scalar multiple of a triple isomorphism, there exist a nonzero scalar λ_α and a minimal tripotent e satisfying $T^{-1}(f) = \lambda_\alpha e$, $|\lambda_\alpha| \le \|(T|_{K(C_\alpha)})^{-1}\| \le \|(T|_{K(E)})^{-1}\|$, and

$$T(K(C_\alpha)_i(e)) = T(K(C_\alpha))_i(f) \tag{5.2}$$

for every $i = 0, 1, 2$. Theorem 4.1 shows that $T((C_\alpha)_i(e)) = T(C_\alpha)_i(f)$ for every $i = 0, 2$. Since $K(E)$ is an ideal of E and e is a minimal tripotent, $(C_\alpha)_1(e) = E_1(e) = K(E)_1(e) = K(C_\alpha)_1(e)$. It follows from (5.2) that

$$T((C_\alpha)_i(e)) = T((C_\alpha))_i(f)$$

for every $i = 0, 1, 2$. Consequently, $P_2(f)T = \lambda_\alpha^{-1} P_2(e) \in (C_\alpha)_*$, and $|\lambda_\alpha^{-1}| \le \|T|_{K(C_\alpha)}\| \le \|T|_{K(E)}\|$.

Since f was an arbitrary minimal tripotent in F (equivalently, in $T(K(C_\alpha))$), Proposition 5.1 ensures that $\varphi_\alpha T \in E_*$ with $\|\varphi_\alpha T\| \le \|T|_{K(E)}\|$ for every $\varphi_\alpha \in (T(C_\alpha))_*$. Therefore, T is bounded with

$$\|T\| \le \|T|_{K(E)}\| \le \|T\|. \ \blacksquare$$

COROLLARY 5.3. *Two atomic JBW^*-triples containing no rank-one summands are isomorphic if and only if there is a biorthogonality preserving linear surjection between them.* ■

The conclusion of Theorem 5.2 does not hold for atomic JBW^*-triples containing rank-one summands.

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