Automatic continuity of biorthogonality preservers between weakly compact $JB^*$-triples and atomic $JBW^*$-triples

by

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Abstract. We prove that every biorthogonality preserving linear surjection from a weakly compact $JB^*$-triple containing no infinite-dimensional rank-one summands onto another $JB^*$-triple is automatically continuous. We also show that every biorthogonality preserving linear surjection between atomic $JBW^*$-triples containing no infinite-dimensional rank-one summands is automatically continuous. Consequently, two atomic $JBW^*$-triples containing no rank-one summands are isomorphic if and only if there exists a (not necessarily continuous) biorthogonality preserving linear surjection between them.

1. Introduction and preliminaries. Studies on the automatic continuity of linear surjections between $C^*$-algebras and von Neumann algebras preserving orthogonality relations in both directions constitute the latest variant of a problem initiated by W. Arendt in the early eighties.

We recall that two complex-valued continuous functions $f$ and $g$ are said to be orthogonal whenever they have disjoint supports. A mapping $T$ between $C(K)$-spaces is called orthogonality preserving if it maps orthogonal functions to orthogonal functions. The main result established by Arendt states that every orthogonality preserving bounded linear mapping $T : C(K) \to C(K)$ is of the form

$$T(f)(t) = h(t)f(\varphi(t)) \quad (f \in C(K), t \in K),$$

where $h \in C(K)$ and $\varphi : K \to K$ is a mapping which is continuous on $\{t \in K : h(t) \neq 0\}$.

The hypothesis of $T$ being continuous was relaxed by K. Jarosz in [21]. In fact, Jarosz obtained a complete description of all orthogonality preserving (not necessarily continuous) linear mappings between $C(K)$-spaces.

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A consequence of his description is that an orthogonality preserving linear surjection between $C(K)$-spaces is automatically continuous.

Two elements $a, b$ in a general $C^*$-algebra $A$ are said to be orthogonal (denoted by $a \perp b$) if $ab^* = b^*a = 0$. When $a = a^*$ and $b = b^*$, we have $a \perp b$ if and only if $ab = 0$. A mapping $T$ between two $C^*$-algebras $A, B$ is called orthogonality preserving if $T(a) \perp T(b)$ for every $a \perp b$ in $A$. When $T(a) \perp T(b)$ in $B$ if and only if $a \perp b$ in $A$, we say that $T$ is biorthogonality preserving. Under continuity assumptions, orthogonality preserving bounded linear operators between $C^*$-algebras are completely described in [10, §4]. This last paper is a culmination of the studies developed by W. Arendt [2], K. Jarosz [24], M. Wolff [34], and N.-C. Wong [35], among others, on bounded orthogonality preserving linear maps between $C^*$-algebras.

$C^*$-algebras belong to a wider class of complex Banach spaces in which orthogonality also makes sense. We refer to the class of (complex) $JB^*$-triples (see §2 for definitions). Two elements $a, b$ in a $JB^*$-triple $E$ are said to be orthogonal (denoted by $a \perp b$) if $L(a, b) = 0$, where $L(a, b)$ is the linear operator in $E$ given by $L(a, b)x = \{a, b, x\}$. A linear mapping $T : E \rightarrow F$ between two $JB^*$-triples is called orthogonality preserving if $T(x) \perp T(y)$ whenever $x \perp y$. The mapping $T$ is biorthogonality preserving whenever the equivalence $x \perp y \iff T(x) \perp T(y)$ holds for all $x, y$ in $E$.

Most of the novelties introduced in [10] consist in studying orthogonality preserving bounded linear operators from a $C^*$-algebra or a $JB^*$-algebra to a $JB^*$-triple to take advantage of the techniques developed in $JB^*$-triple theory. These techniques were successfully applied in the subsequent paper [11] to obtain a description of such operators (see §2 for a detailed explanation).

Despite the vast literature on orthogonality preserving bounded linear operators between $C^*$-algebras and $JB^*$-triples, just a few papers have considered the problem of automatic continuity of biorthogonality preserving linear surjections between $C^*$-algebras. Besides Jarosz [24], mentioned above, M. A. Chebotar, W.-F. Ke, P.-H. Lee, and N.-C. Wong proved in [13, Theorem 4.2] that every zero products preserving linear bijection from a properly infinite von Neumann algebra into a unital ring is a ring homomorphism followed by left multiplication by the image of the identity. J. Araujo and K. Jarosz showed that every linear bijection between algebras $L(X)$, of continuous linear maps on a Banach space $X$, which preserves zero products in both directions is automatically continuous and a multiple of an algebra isomorphism [1]. These authors also conjectured that every linear bijection between two $C^*$-algebras preserving zero products in both directions is automatically continuous (see [1, Conjecture 1]).

The authors of this note proved in [12] that every biorthogonality preserving linear surjection between two compact $C^*$-algebras or between two von Neumann algebras is automatically continuous. One of the consequences
of this result is a partial answer to [1, Conjecture 1]. Concretely, every surjective and symmetric linear mapping between von Neumann algebras (or compact $C^*$-algebras) which preserves zero products in both directions is continuous.

In this paper we study the problem of automatic continuity of biorthogonality preserving linear surjections between $JB^*$-triples, extending some of the results obtained in [12]. Section 2 contains the basic definitions and results used in the paper. Section 3 is devoted to the structure and properties of the (orthogonal) annihilator of a subset $M$ in a $JB^*$-triple, focusing on the annihilators of single elements. In Section 4 we prove that every biorthogonality preserving linear surjection from a weakly compact $JB^*$-triple containing no infinite-dimensional rank-one summands to a $JB^*$-triple is automatically continuous. In Section 5 we show that two atomic $JB^*$-triples containing no rank-one summands are isomorphic if and only if there exists a biorthogonality preserving linear surjection between them, a result which follows from the automatic continuity of every biorthogonality preserving linear surjection between atomic $JB^*$-triples containing no infinite-dimensional rank-one summands.

2. Notation and preliminaries. Given Banach spaces $X$ and $Y$, $L(X,Y)$ will denote the space of all bounded linear mappings from $X$ to $Y$. The symbol $L(X)$ will stand for the space $L(X,X)$. Throughout the paper the word “operator” will always mean bounded linear mapping. The dual space of a Banach space $X$ is denoted by $X^*$.

$JB^*$-triples were introduced by W. Kaup in [26]. A $JB^*$-triple is a complex Banach space $E$ together with a continuous triple product $\{\cdot,\cdot,\cdot\} : E \times E \times E \to E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables, and satisfies:

(a) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, where $L(a,b)$ is the operator on $E$ given by $L(a,b)x = \{a,b,x\}$;
(b) $L(a,a)$ is an hermitian operator with nonnegative spectrum;
(c) $\|L(a,a)\| = \|a\|^2$.

For each $x$ in a $JB^*$-triple $E$, $Q(x)$ will stand for the conjugate linear operator on $E$ defined by the assignment $y \mapsto Q(x)y = \{x,y,x\}$.

Every $C^*$-algebra is a $JB^*$-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x,$$

and every $JB^*$-algebra is a $JB^*$-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*. \tag{2.1}$$

The so-called Kaup–Banach–Stone theorem for $JB^*$-triples states that a bounded linear surjection between $JB^*$-triples is an isometry if and only
if it is a triple isomorphism (cf. [26 Proposition 5.5], [5 Corollary 3.4] or [18 Theorem 2.2]). It follows, among many other consequences, that when a \( JB^* \)-algebra is a \( JB^* \)-triple for a suitable triple product, then the latter coincides with the one defined in (2.1).

A \( JBW^* \)-triple is a \( JB^* \)-triple which is also a dual Banach space (with a unique isometric predual [3]). It is known that the triple product of a \( JBW^* \)-triple is separately weak* continuous [3]. The second dual of a \( JB^* \)-triple \( E \) is a \( JBW^* \)-triple with a product extending the product of \( E [15] \).

An element \( e \) in a \( JB^* \)-triple \( E \) is said to be a **tripotent** if \( \{ e, e, e \} = e \).

Each tripotent \( e \) in \( E \) gives rise to the decomposition

\[
E = E_2(e) \oplus E_1(e) \oplus E_0(e),
\]

where for \( i = 0, 1, 2 \), \( E_i(e) \) is the \( i/2 \)-eigenspace of \( L(e, e) \) (cf. [28 Theorem 25]). The natural projection of \( E \) onto \( E_i(e) \) will be denoted by \( P_i(e) \). This decomposition is termed the **Peirce decomposition** of \( E \) with respect to the tripotent \( e \). The Peirce decomposition satisfies certain rules known as **Peirce arithmetic**:

\[
\{ E_i(e), E_j(e), E_k(e) \} \subseteq E_{i-j+k}(e)
\]

if \( i - j + k \in \{ 0, 1, 2 \} \) and is zero otherwise. In addition,

\[
\{ E_2(e), E_0(e), E \} = \{ E_0(e), E_2(e), E \} = 0.
\]

The Peirce space \( E_2(e) \) is a \( JB^* \)-algebra with product \( x \circ_1 y := \{ x, e, y \} \) and involution \( x^e := \{ e, x, e \} \).

A tripotent \( e \) in \( E \) is called **complete** (resp., **unitary**) if \( E_0(e) = 0 \) (resp., \( E_2(e) = E \)). When \( E_2(e) = \mathbb{C}e \neq \{ 0 \} \), we say that \( e \) is **minimal**.

For each element \( x \) in a \( JB^* \)-triple \( E \), we shall denote \( x^{[1]} := x \), \( x^{[3]} := \{ x, x, x \} \), and \( x^{[2n+1]} := \{ x, x, x^{[2n-1]} \} \) \((n \in \mathbb{N})\). The symbol \( E_x \) will stand for the \( JB^* \)-subtriple generated by \( x \). It is known that \( E_x \) is \( JB^* \)-triple isomorphic (and hence isometric) to \( C_0(\Omega) \) for some locally compact Hausdorff space \( \Omega \) contained in \( (0, \| x \|) \) such that \( \Omega \cup \{ 0 \} \) is compact, where \( C_0(\Omega) \) denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that there exists a triple isomorphism \( \Psi \) from \( E_x \) onto \( C_0(\Omega) \) satisfying \( \Psi(x)(t) = t \ (t \in \Omega) \) (cf. [25 Corollary 4.8], [26 Corollary 1.15] and [20]). The set \( \Omega = \text{Sp}(x) \) is called the **triple spectrum** of \( x \). Note that \( C_0(\text{Sp}(x)) = C(\text{Sp}(x)) \) whenever \( 0 \notin \text{Sp}(x) \).

Therefore, for each \( x \in E \), there exists a unique element \( y \in E_x \) such that \( \{ y, y, y \} = x \). The element \( y \), denoted by \( x^{[1/3]} \), is termed the **cubic root** of \( x \). We can inductively define \( x^{[1/3^n]} = (x^{[1/3^{n-1}]})^{[1/3]} \), \( n \in \mathbb{N} \). The sequence \( (x^{[1/3^n]}) \) converges in the weak* topology of \( E^{**} \) to a tripotent denoted by \( r(x) \) and called the **range tripotent** of \( x \). The tripotent \( r(x) \) is the smallest tripotent \( e \in E^{**} \) such that \( x \) is positive in the \( JBW^* \)-algebra \( E_2^{**}(e) \) (cf. [16 Lemma 3.3]).
A subspace $I$ of a $JB^*$-triple $E$ is a triple ideal if $\{E, E, I\} + \{E, I, E\} \subseteq I$. By Proposition 1.3 in [7], $I$ is a triple ideal if and only if $\{I, E, I\} \subseteq I$. We shall say that $I$ is an inner ideal of $E$ if $\{I, E, I\} \subseteq I$. Given an $x$ in $E$, let $E(x)$ denote the norm closed inner ideal of $E$ generated by $x$. It is known that $E(x)$ coincides with the norm closure of the set $Q(x)(E)$. Moreover $E(x)$ is a $JB^*$-subalgebra of $E_2^*(r(x))$ and contains $x$ as a positive element (cf. [3]). Every triple ideal is, in particular, an inner ideal.

We recall that two elements $a, b$ in a $JB^*$-triple $E$ are said to be orthogonal (written $a \perp b$) if $L(a, b) = 0$. Lemma 1 in [10] shows that $a \perp b$ if and only if one of the following nine statements holds:

$$\{a, a, b\} = 0; \quad a \perp r(b); \quad r(a) \perp r(b);$$

(2.2) $$E_2^*(r(a)) \perp E_2^*(r(b)); \quad r(a) \in E_0^*(r(b)); \quad a \in E_0^*(r(b));$$

$$b \in E_0^*(r(a)); \quad E_a \perp E_b; \quad \{b, b, a\} = 0.$$

The Jordan identity and the above reformulations ensure that

$$a \perp \{x, y, z\} \quad \text{whenever} \quad a \perp x, y, z.$$  

An important class of $JB^*$-triples is given by the Cartan factors. A $JBW^*$-triple $E$ is called a factor if it contains no proper weak* closed ideals. The Cartan factors are precisely the $JBW^*$-triple factors containing a minimal tripotent [27]. These can be classified in six different types (see [21] or [27]).

A Cartan factor of type 1, denoted by $I_{n,m}$, is a $JB^*$-triple of the form $L(H, H')$, where $L(H, H')$ denotes the space of bounded linear operators between two complex Hilbert spaces $H$ and $H'$ of dimensions $n, m$ respectively, with the triple product defined by $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$.

We recall that given a conjugation $j$ on a complex Hilbert space $H$, we can define the linear involution $x \mapsto x^j := jx^*j$ on $L(H)$. A Cartan factor of type 2 (respectively, type 3), denoted by $II_n$ (respectively, $III_n$), is the subtriple of $L(H)$ formed by the $t$-skew-symmetric (respectively, $t$-symmetric) operators, where $H$ is an $n$-dimensional complex Hilbert space. Moreover, $II_n$ and $III_n$ are, up to isomorphism, independent of the conjugation $j$ on $H$.

A Cartan factor of type 4, $IV_n$ (also called a complex spin factor), is an $n$-dimensional complex Hilbert space provided with a conjugation $x \mapsto \bar{x}$, where the triple product and norm are given by

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|z)y$$

and $\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |(x|x)|^2}$, respectively.

The Cartan factor of type 6 is the 27-dimensional exceptional $JB^*$-algebra $VI = H_3(\mathbb{O}_C)$ of all symmetric $3 \times 3$ matrices with entries in the complex octonions $\mathbb{O}_C$, while the Cartan factor of type 5, $V = M_{1,2}(\mathbb{O}_C)$, is the subtriple of $H_3(\mathbb{O}_C)$ consisting of all $1 \times 2$ matrices with entries in $\mathbb{O}_C$. 

\begin{align*}
\{x, y, z\} &= (x|y)z + (z|y)x - (x|z)y \\
\|x\|^2 &= (x|x) + \sqrt{(x|x)^2 - |(x|x)|^2},
\end{align*}
Remark 2.1. Let $E$ be a spin factor with inner product $(\cdot|\cdot)$ and conjugation $x \mapsto \overline{x}$. It is not hard to check (and part of the folklore of $JB^*$-triple theory) that an element $w$ in $E$ is a minimal tripotent if and only if $(w|\overline{w}) = 0$ and $(w|w) = 1/2$. For every minimal tripotent $w$ in $E$ we have $E_2(w) = Cw$, $E_0(w) = C\overline{w}$ and $E_1(w) = \{x \in E : (x|w) = (x|\overline{w}) = 0\}$. Therefore, every minimal tripotent $w_2 \in E$ satisfying $w \perp w_2$ can be written in the form $w_2 = \lambda \overline{w}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

3. Biorthogonality preservers. Let $M$ be a subset of a $JB^*$-triple $E$. We write $M_M$ for the (orthogonal) annihilator of $M$ defined by

$$M_M := \{y \in E : y \perp x, \forall x \in M\}.$$ 

When no confusion can arise, we shall write $M_M$ instead of $M_M_E$.

The next result summarises some basic properties of the annihilator. The reader is referred to [17, Lemma 3.2] for a detailed proof.

**Lemma 3.1.** Let $M$ a nonempty subset of a $JB^*$-triple $E$.

(a) $M_M$ is a norm closed inner ideal of $E$.

(b) $M \cap M_M = \{0\}$.

(c) $M \subseteq M_M$.

(d) If $B \subseteq C$ then $C_M \subseteq B_M$.

(e) $M_M$ is weak* closed whenever $E$ is a $JBW^*$-triple.

As illustration of the main identity (axiom (a) in the definition of a $JB^*$-triple) we shall prove statement (a). For $a, a'$ in $M_M$, $b$ in $M$, and $c, d$ in $E$ we have $\{c, a, \{d, a', b\}\} = \{\{c, a, d\}, a', b\} - \{d, \{a, c, a'\}, b\} + \{d, a', \{c, a, b\}\}$, which shows that $\{a, c, a'\} \perp b$.

Let $e$ be a tripotent in a $JB^*$-triple $E$. Clearly, $\{e\} \subseteq E_2(e)$. Therefore, by Peirce arithmetic and Lemma 3.1

$$E_2(e)_{\perp} \subseteq \{e\}_{\perp} = E_0(e) \subseteq E_2(e)_{\perp},$$

and hence

$$E_2(e)_{\perp} = \{e\}_{\perp} = E_0(e).$$

The next lemma describes the annihilator of an element in an arbitrary $JB^*$-triple. Its proof follows directly from the reformulations of orthogonality in [2.2] (see also [10, Lemma 1]).

**Lemma 3.2.** Let $x$ be an element in a $JB^*$-triple $E$. Then

$$\{x\}_{E_M} = E_0^*(r(x)) \cap E.$$

Moreover, when $E$ is a $JBW^*$-triple we have

$$\{x\}_{E_M} = E_0(r(x)).$$
PROPOSITION 3.3. Let $e$ be a tripotent in a $JB^*$-triple $E$. Then

$$E_2(e) \oplus E_1(e) \supseteq \{e\}^\perp_{E} \supseteq E_0(e)^\perp \supseteq E_2(e).$$

Proof. It follows from (3.1) that $\{e\}^\perp_{E} = \{e\} = (E_0(e))^\perp \supseteq E_2(e)$. Now select $x \in (E_0(e))^\perp$. For each $i \in \{0, 1, 2\}$ we write $x_i = P_i(e)(x)$, where $P_i(e)$ denotes the Peirce $i$-projection with respect to $e$. Since $x \in (E_0(e))^\perp$, $x$ must be orthogonal to $x_0$ and so $\{x_0, x_0, x\} = 0$. This equality, together with Peirce arithmetic, shows that $\{x_0, x_0, x_0\} + \{x_0, x_0, x_1\} = 0$, which implies that $\|x_0\|^3 = \|x_0, x_0, x_0\| = 0$. \]

REMARK 3.4. For a tripotent $e$ in a $JB^*$-triple $E$, the equality $\{e\}^\perp_{E} = E_0(e)^\perp = E_2(e)$ does not hold in general. Let $H_1$ and $H_2$ be two infinite-dimensional complex Hilbert spaces and let $p$ be a minimal projection in $L(H_1)$. We define $E$ as the orthogonal sum $pL(H_1) \oplus \infty L(H_2)$. In this example $\{p\}^\perp_{E} = L(H_2)$ and $\{p\}^\perp_{E} = pL(H_1) \neq \mathbb{C}p = E_2(p)$.

However, if $E$ is a Cartan factor and $e$ is a noncomplete tripotent in $E$, then the equality $\{e\}^\perp_{E} = E_0(e)^\perp = E_2(e)$ always holds (cf. Lemma 5.6 in [27]).

COROLLARY 3.5. Let $x$ be an element in a $JB^*$-triple $E$. Then

$$E(x) \subseteq E_2^{**}(r(x)) \cap E \subseteq \{x\}^\perp_{E}.$$

Proof. Clearly, $E(x) = \overline{Q(x)(E)} \subseteq E_2^{**}(r(x)) \cap E$. Pick $y$ in $E_2^{**}(r(x)) \cap E$. Then $y \in E_2^{**}(r(x)) \subseteq \{x\}^\perp_{E^**}$. Since $\{x\}^\perp_{E^**} \subseteq \{x\}^\perp_{E^**}$, we conclude that $y \in \{x\}^\perp_{E^**} \cap E \subseteq \{x\}^\perp_{E^**} \cap E = \{x\}^\perp_{E}$. \]

In the setting of $C^*$-algebras the following conditions describing the first and second annihilator of a projection were established in [12] Lemma 3].

LEMMA 3.6. Let $p$ be a projection in a (not necessarily unital) $C^*$-algebra $A$. The following assertions hold:

(a) $\{p\}^\perp_{A} = (1 - p)A(1 - p)$, where $1$ denotes the unit of $A^{**}$;
(b) $\{p\}^\perp_{A} = pAp$. \]

Let $x$ be an element in a $JB^*$-triple $E$. We say that $x$ is weakly compact (respectively, compact) if the operator $Q(x) : E \rightarrow E$ is weakly compact (respectively, compact). A $JB^*$-triple is weakly compact (respectively, compact) if every element in $E$ is weakly compact (respectively, compact).

Let $E$ be a $JB^*$-triple. If we denote by $K(E)$ the Banach subspace of $E$ generated by its minimal tripotents, then $K(E)$ is a (norm closed) triple ideal of $E$ and it coincides with the set of weakly compact elements of $E$ (see Proposition 4.7 in [7]). For a Cartan factor $C$ we define the elementary $JB^*$-triple of the corresponding type to be $K(C)$. Consequently, the elementary $JB^*$-triples $K_i \ (i = 1, \ldots, 6)$ are defined as follows: $K_1 = K(H, H')$ (the
compact operators between complex Hilbert spaces $H$ and $H'$; $K_i = C_i \cap \overline{K(H)}$ for $i = 2, 3$, and $K_i = C_i$ for $i = 4, 5, 6$.

It follows from \cite[Lemma 3.3 and Theorem 3.4]{res} that a $JB^*$-triple $E$ is weakly compact if and only if one of the following statement holds:

(a) $K(E^{**}) = K(E)$.
(b) $K(E) = E$.
(c) $E$ is a $c_0$-sum of elementary $JB^*$-triples.

Let $E$ be a $JB^*$-triple. A subset $S \subseteq E$ is said to be orthogonal if $0 \notin S$ and $x \perp y$ for every $x \neq y$ in $S$. The minimal cardinal number $r$ satisfying $\text{card}(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the rank of $E$ (and will be denoted by $r(E)$).

For every orthogonal family $(e_i)_{i \in I}$ of minimal tripotents in a $JBW^*$-triple $E$ the weak* convergent sum $e := \sum_ie_i$ is a tripotent, and we call $(e_i)_{i \in I}$ a frame in $E$ if $e$ is a maximal tripotent in $E$ (i.e., $e$ is a complete tripotent and $\dim(E_1(e)) \leq \dim(E_1(\tilde{e}))$ for every complete tripotent $\tilde{e}$ in $E$).

Every frame is a maximal orthogonal family of minimal tripotents; the converse is not true in general (see \cite[\S 3]{res} for more details).

**Proposition 3.7.** Let $e$ be a minimal tripotent in a $JB^*$-triple $E$. Then $\{e\}^\perp_\perp$ is a rank-one norm closed inner ideal of $E$.

**Proof.** Let $F$ denote $\{e\}_{E}^{\perp\perp}$. Since $e$ is a minimal tripotent (i.e. $E_2(e) = \mathbb{C}e$), the set of states on $E_2(e)$, $\{\varphi \in E^* : \varphi(e) = 1 = \|\varphi\|\}$, reduces to one point $\varphi_0$ in $E^*$. Proposition 2.4 and Corollary 2.5 in \cite{res} imply that the norm of $E$ restricted to $E_1(e)$ is equivalent to a Hilbertian norm. More precisely, in the terminology of \cite{res}, the norm $\|\cdot\|_e$ coincides with the Hilbertian norm $\|\cdot\|_{\varphi_0}$ and is equivalent to the norm of $E_1(e)$.

Proposition 3.3 guarantees that $F$ is a norm closed subspace of $E_2(e) \oplus E_1(e) = \mathbb{C}e \oplus E_1(e)$, and hence $F$ is isomorphic to a Hilbert space.

We deduce, by Proposition 4.5(iii) in \cite{res} (and its proof), that $F$ is a finite orthogonal sum of Cartan factors $C_1, \ldots, C_m$ which are finite-dimensional, or infinite-dimensional spin factors, or of the form $L(H, H')$ for suitable complex Hilbert spaces $H$ and $H'$ with $\dim(H') < \infty$. Since $F$ is an inner ideal of $E$ (and hence a $JB^*$-subtriple of $E$) and $e$ is a minimal tripotent in $E$, we can easily check that $e$ is a minimal tripotent in $F = \bigoplus_{j=1,\ldots,m}^\ell C_j$. If we write $e = e_1 + \cdots + e_m$, where each $e_j$ is a tripotent in $C_j$ and $e_j \perp e_k$ whenever $j \neq k$, then since $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_1 \subseteq F_2(e) = \mathbb{C}e$, we deduce that there exists a unique $j_0 \in \{1, \ldots, m\}$ satisfying $e_j = 0$ for all $j \neq j_0$ and $e = e_{j_0} \in C_{j_0}$.

For each $j \neq j_0$, we have $C_j \subseteq \{e\}_{E}^{\perp\perp}$, and hence

$$\bigoplus_{j=1,\ldots,m}^\ell C_j = F = \{e\}_{E}^{\perp\perp} \subseteq C_{j_0}^{\perp\perp}.$$
This implies that \( C_j \perp C_i \) (or equivalently \( C_j = 0 \)) for every \( j \neq j_0 \). We consequently have \( F = \{ e \}_{E}^{\perp} = C_{j_0} \).

Finally, if \( r(F) \geq 2 \), then we deduce, via Proposition 5.8 in [27], that there exist minimal tripotents \( e_2, \ldots, e_r \) in \( F \) such that \( e, e_2, \ldots, e_r \) is a frame in \( F \). For each \( i \in \{2, \ldots, r \} \), \( e_i \) is orthogonal to \( e \) and lies in \( F = \{ e \}_{E}^{\perp} \), which is impossible. \( \blacksquare \)

Let \( T : E \to F \) be a linear map between two \( JB^* \)-triples. We shall say that \( T \) is orthogonality preserving if \( T(x) \perp T(y) \) whenever \( x \perp y \). The mapping \( T \) is said to be biorthogonality preserving whenever the equivalence

\[
x \perp y \iff T(x) \perp T(y)
\]

holds for all \( x, y \) in \( E \).

It can be easily seen that every biorthogonality preserving linear mapping \( T : E \to F \) between \( JB^* \)-triples is injective. Indeed, for each \( x \in E \), the condition \( T(x) = 0 \) implies that \( T(x) \perp T(x) \), and hence \( x \perp x \), which gives \( x = 0 \).

Orthogonality preserving bounded linear maps from a \( JB^* \)-algebra to a \( JB^* \)-triple were completely described in [11].

Before stating the result, let us recall some basic definitions. Two elements \( a \) and \( b \) in a \( JB^* \)-algebra \( J \) are said to operator commute in \( J \) if the multiplication operators \( M_a \) and \( M_b \) commute, where \( M_a \) is defined by \( M_a(x) := a \circ x \). That is, \( a \) and \( b \) operator commute if and only if \( (a \circ x) \circ b = a \circ (x \circ b) \) for all \( x \) in \( J \). Self-adjoint elements \( a \) and \( b \) in \( J \) generate a \( JB^* \)-subalgebra that can be realised as a \( JC^* \)-subalgebra of some \( B(H) \) [36], and, in this realisation, \( a \) and \( b \) commute in the usual sense whenever they operator commute in \( J \) [33, Proposition 1]. Similarly, two self-adjoint elements \( a \) and \( b \) in \( J \) operator commute if and only if \( a^2 \circ b = \{ a, a, b \} = \{ a, b, a \} \) (i.e., \( a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b \)). If \( b \in J \) we use \( \{ b \}' \) to denote the set of elements in \( J \) that operator commute with \( b \). We shall write \( Z(J) := J' \) for the center of \( J \) (this agrees with the usual notation in von Neumann algebras).

**Theorem 3.8** ([11, Theorem 4.1]). Let \( T : J \to E \) be a bounded linear mapping from a \( JB^* \)-algebra to a \( JB^* \)-triple. For \( h = T^{**}(1) \) and \( r = r(h) \) the following assertions are equivalent:

(a) \( T \) is orthogonality preserving.

(b) There exists a unique Jordan \( * \)-homomorphism \( S : J \to E^{**}_{2}(r) \) such that \( S^{**}(1) = r, S(J) \) and \( h \) operator commute, and \( T(z) = h \circ_t S(z) \) for all \( z \in J \).

(c) \( T \) preserves zero triple products, that is, \( \{ T(x), T(y), T(z) \} = 0 \) whenever \( \{ x, y, z \} = 0 \). \( \blacksquare \)
The above characterisation proves that the bitranspose of an orthogonality preserving bounded linear mapping from a $JB^*$-algebra onto a $JB^*$-triple is also orthogonality preserving.

The following theorem was essentially proved in [11]. We include here a sketch of proof for completeness.

**Theorem 3.9.** Let $T : J \to E$ be a surjective linear operator from a $JBW^*$-algebra onto a $JBW^*$-triple and let $h$ denote $T(1)$. Then $T$ is biorthogonality preserving if and only if $r(h)$ is a unitary tripotent in $E$, $h$ is an invertible element in the $JB^*$-algebra $E = E_2(r(h))$, and there exists a Jordan $^*$-isomorphism $S : J \to E = E_2(r(h))$ such that $S(J) \subseteq \{h\}'$ and $T = h \circ r(h) S$. Further, if $J$ is a factor (i.e., $Z(J) = \mathbb{C}1$) then $T$ is a scalar multiple of a triple isomorphism.

*Proof.* The sufficiency is clear. We shall prove the necessity. To this end let $T : J \to E$ be a surjective linear operator from a $JBW^*$-algebra onto a $JBW^*$-triple and let $h = T(1) \in E$. We have already seen that every biorthogonality preserving linear mapping between $JB^*$-triples is injective. Therefore $T$ is a linear bijection.

From Corollary 4.1(b) in [11] and its proof, we deduce that

$$T(J_{sa}) \subseteq E_2(r(h))_{sa}, \quad \text{and hence} \quad E = T(J) \subseteq E_2(r(h)) \subseteq E.$$ 

This implies that $E = E_2(r(h))$, which ensures that $r(h)$ is a unitary tripotent in $E$. Since the range tripotent of $h$, $r(h)$, is the unit of $E_2(r(h))$, and $h$ is a positive element in the $JBW^*$-algebra $E_2(r(h))$, we can easily check that $h$ is invertible in $E_2(r(h))$. Furthermore, $h^{1/2}$ is invertible in $E_2(r(h))$ with inverse $h^{-1/2}$.

The proof of [11] Theorem 4.1 can be literally applied here to show the existence of a Jordan $^*$-homomorphism $S : J \to E = E_2(r(h))$ such that $S(J) \subseteq \{h\}'$ and $T = h \circ r(h) S$. Since, for each $x \in J$, $h$ and $S(x)$ operator commute and $h^{1/2}$ lies in the $JB^*$-subalgebra of $E_2(r(h))$ generated by $h$, we can easily check that $S(x)$ and $h^{1/2}$ operator commute. Thus,

$$T = h \circ r(h) S = U_{h^{1/2}} S,$$

where $U_{h^{1/2}} : E_2(r(h)) \to E_2(r(h))$ is the linear mapping defined by

$$U_{h^{1/2}}(x) = 2(h^{1/2} \circ r(h) x) \circ r(h) h^{1/2} - (h^{1/2} \circ r(h) h^{1/2}) \circ r(h) x.$$ 

It is well known that $h^{1/2}$ is invertible if and only if $U_{h^{1/2}}$ is an invertible operator and, in this case, $U_{h^{1/2}}^{-1} = U_{h^{-1/2}}$ (cf. [22] Lemma 3.2.10). Therefore, $S = U_{h^{-1/2}} T$. It follows from the bijectivity of $T$ that $S$ is a Jordan $^*$-isomorphism.

Finally, when $Z(J) = \mathbb{C}1$, the center of $E_2(r(h))$ also reduces to $\mathbb{C}r(h)$, and since $h$ is an invertible element in the center of $E_2(r(h))$, we deduce that $T$ is a scalar multiple of a triple isomorphism.
Proposition 3.10. Let $E_1, E_2$ and $F$ be three $JB^*$-triples (respectively, $JBW^*$-triples). Let $T: E_1 \oplus^\infty E_2 \to F$ be a biorthogonality preserving linear surjection. Then $T(E_1)$ and $T(E_2)$ are norm closed (respectively, weak* closed) inner ideals of $F$, $B = T(A_1) \oplus^\infty T(A_2)$, and for $j = 1, 2$, $T|_{A_j}: A_j \to T(A_j)$ is a biorthogonality preserving linear surjection.

Proof. Fix $j \in \{1, 2\}$. Since $E_j = E_j^\perp_{\perp}$ and $T$ is a biorthogonality preserving linear surjection, we deduce that $T(E_j) = T(E_j^\perp_{\perp}) = T(E_j)^\perp_{\perp}$. Lemma 3.1 guarantees that $T(E_j)$ is a norm closed inner ideal of $F$ (respectively, a weak* closed inner ideal of $F$ whenever $E_1, E_2$ and $F$ are $JBW^*$-triples). The rest of the assertion follows from Lemma 3.1 and the fact that $F$ coincides with the orthogonal sum of $T(E_1)$ and $T(E_2)$. ■

4. Biorthogonality preservers between weakly compact $JB^*$-triples. The following theorem generalises [12, Theorem 5] by proving that biorthogonality preserving linear surjections between $JB^*$-triples send minimal tripotents to scalar multiples of minimal tripotents.

Theorem 4.1. Let $T: E \to F$ be a biorthogonality preserving linear surjection between two $JB^*$-triples and let $e$ be a minimal tripotent in $E$. Then $\|T(e)\|^{-1}T(e) = f_e$ is a minimal tripotent in $F$. Further, $T(E_2(e)) = F_2(f_e)$ and $T(E_0(e)) = F_0(f_e)$.

Proof. Since $T$ is a biorthogonality preserving surjection, the equality

$$T(S_{E_1}^E) = T(S_{F_1}^F)$$

holds for every subset $S$ of $E$. Lemma 3.1 ensures that for each minimal tripotent $e$ in $E$, $\{T(e)\}^\perp_{\perp_E} = T(\{e\}^\perp_{\perp_E})$ is a norm closed inner ideal in $F$. By Proposition 3.7, $\{e\}^\perp_{\perp_E}$ is a rank-one $JB^*$-triple, and hence $\{T(e)\}^\perp_{\perp_F}$ cannot contain two nonzero orthogonal elements. Thus, $\{T(e)\}^\perp_{\perp_F}$ is a rank-one $JB^*$-triple.

The arguments given in the proof of Proposition 3.7 above (see also Proposition 4.5.(iii) in [7] and its proof or [11, §3]) show that the inner ideal $\{T(e)\}^\perp_{\perp_F}$ is a rank-one Cartan factor, and hence a type 1 Cartan factor of the form $L(H, \mathbb{C})$, where $H$ is a complex Hilbert space, or a type 2 Cartan factor $II_3$ (it is known that $II_3$ is a $JB^*$-triple isomorphic to a 3-dimensional complex Hilbert space). This implies that $\|T(e)\|^{-1}T(e) = f_e$ is a minimal tripotent in $F$ and $T(e) = \lambda_e f_e$ for a suitable $\lambda_e \in \mathbb{C} \setminus \{0\}$.

The equality $T(E_2(e)) = F_2(f_e)$ has been proved. Concerning the Peirce zero subspace we have

$$T(E_0(e)) = T(E_2(e))^{\perp_{\perp_E}} = T(E_2(e))^{\perp_{\perp_F}} = F_2(f_e)^{\perp_{\perp_F}} = F_0(f_e).$$

Let $H$ and $H'$ be complex Hilbert spaces. Given $k \in H'$ and $h \in H$, we define $k \otimes h$ in $L(H, H')$ by $k \otimes h(\xi) := (\xi|h)k$. Then every minimal tripotent
in $L(H, H')$ can be written in the form $k \otimes h$, where $h$ and $k$ are norm-one elements in $H$ and $H'$, respectively. It can be easily seen that two minimal tripotents $k_1 \otimes h_1$ and $k_2 \otimes h_2$ are orthogonal if and only if $h_1 \perp h_2$ and $k_1 \perp k_2$.

**Theorem 4.2.** Let $T : E \rightarrow F$ be a biorthogonality preserving linear surjection between two $JB^*$-triples, where $E$ is a type $I_{n,m}$ Cartan factor with $n, m \geq 2$. Then there exists a positive real number $\lambda$ such that $\|T(e)\| = \lambda$ for every minimal tripotent $e$ in $E$.

*Proof.* Let $H, H'$ be complex Hilbert spaces such that $E = L(H, H')$. Let $e_1 := k_1 \otimes h_1$ and $e_2 := k_2 \otimes h_2$ be two minimal tripotents in $E$. We write $H_1 = \text{span}(\{h_1, h_2\})$ and $H'_1 = \text{span}(\{k_1, k_2\})$. The tripotents $k_1 \otimes h_1$ and $k_2 \otimes h_2$ can be identified with elements in $L(H_1, H'_1)$. By Theorem 4.1, $T(e_1) = \alpha_1 f_1$ and $T(e_2) = \alpha_2 f_2$, where $f_1$ and $f_2$ are two minimal tripotents in $F$.

If $\dim(H_1) = \dim(H'_1) = 2$, then the norm closed inner ideal $E_{e_1,e_2}$ of $E$ generated by $e_1$ and $e_2$ identifies with $L(H_1, H'_1)$, which is $JB^*$-isomorphic to $M_2(\mathbb{C})$ and coincides with the inner ideal generated by the orthogonal minimal tripotents $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, where $g_1 + g_2$ is the unit element in $E_{e_1,e_2} \cong M_2(\mathbb{C})$.

By Theorem 4.1, $w_1 := \frac{1}{\|T(g_1)\|} T(g_1)$ and $w_2 := \frac{1}{\|T(g_2)\|} T(g_2)$ are orthogonal minimal tripotents in $F$. The element $w = w_1 + w_2$ is a rank-2 tripotent in $F$ and coincides with the range tripotent of the element $h = T(g_1 + g_2) = \|T(g_1)\| w_1 + \|T(g_2)\| w_2$. By Theorem 3.8 (see also [11, Corollary 4.1(b)]), $T(E_{e_1,e_2}) \subseteq F_2(w)$. It is not hard to see that $h$ is invertible in $F_2(w)$ with inverse $h^{-1} = \frac{1}{\|T(g_1)\|} w_1 + \frac{1}{\|T(g_2)\|} w_2$.

The inner ideal $E_{e_1,e_2}$ is finite-dimensional, $T(E_{e_1,e_2})$ is norm closed and $T|_{E_{e_1,e_2}} : E_{e_1,e_2} \rightarrow F$ is a continuous biorthogonality preserving linear operator. Theorem 3.8 guarantees the existence of a Jordan $^*$-homomorphism $S : E_{e_1,e_2} \cong M_2(\mathbb{C}) \rightarrow F_2(w)$ such that $S(g_1 + g_2) = w$, $S(E_{e_1,e_2})$ and $h$ operator commute and

$$T(z) = h \circ_w S(z) \quad \text{for all } z \in E_{e_1,e_2}.$$  

It follows from the operator commutativity of $h^{-1}$ and $S(E_{e_1,e_2})$ that $S(z) = h^{-1} \circ_w T(z)$ for all $z \in E_{e_1,e_2}$. The injectivity of $T$ implies that $S$ is a Jordan $^*$-monomorphism.

Lemma 2.7 in [19] shows that $F_2(w) = F_2(\bar{w}_1 + w_2)$ coincides with $\mathbb{C} \oplus \ell_\infty \mathbb{C}$ or with a spin factor. Since $4 = \dim(T(E_{e_1,e_2})) \leq \dim(F_2(w))$, we deduce that $F_2(w)$ is a spin factor with inner product $(\cdot | \cdot)$ and conjugation $x \mapsto \bar{x}$. From Remark 2.1 we may assume, without loss of generality, that $(w_1 | w_1) = 1/2$, $(w_1 | \bar{w}_1) = 0$, and $w_2 = \bar{w}_1$. 


Now, we take \( g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( g_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) in \( E_{e_1,e_2} \). The elements \( w_3 := S(g_3) \) and \( w_4 := S(g_4) \) are orthogonal minimal tripotents in \( F_2(w) \) with \( \{w_i, w_i, w_j\} = \frac{1}{2} w_j \) for every \( (i, j), (j, i) \in \{1, 2\} \times \{3, 4\} \). Applying again Remark 2.1, we may assume that \( (w_3|w_3) = 1/2, (w_3|\wbar_3) = 0, w_4 = \wbar_3, \) and \( (w_3|w_1) = (w_3|w_2) = 0. \) Applying the definition of the triple product in a spin factor given in (2.4) we can check that \( (w_1, w_3, w_2 = \wbar_1, w_4 = \wbar_3) \) are four minimal tripotents in \( F_2(w) \) with \( w_1 \perp w_2, w_3 \perp w_4, \{w_i, w_i, w_j\} = \frac{1}{2} w_j \) for every \( (i, j), (j, i) \in \{1, 2\} \times \{3, 4\}, \) \( \{w_1, w_3, w_2\} = -\frac{1}{2} w_4, \{w_3, w_2, -w_4\} = \frac{1}{2} w_1, \{w_2, -w_4, w_1\} = \frac{1}{2} w_3, \) and \( \{-w_4, w_1, w_3\} = \frac{1}{2} w_2. \) Thus, denoting by \( M \) the \( JB^* \)-subtriple of \( F_2(w) \) generated by \( w_1, w_3, w_2, \) and \( w_4, \) we have shown that \( M \) is a \( JB^* \)-triple isomorphic to \( M_2(\mathbb{C}). \)

Combining (4.1) and (2.4) we get
\[
T(g_3) = h \circ_w S(g_3) = \{h, w, w_3\} = \frac{\|T(g_1)\| + \|T(g_2)\|}{2} w_3,
\]
\[
T(g_4) = h \circ_w S(g_4) = \{h, w, w_4\} = \frac{\|T(g_1)\| + \|T(g_2)\|}{2} w_4.
\]

Since \( T(g_1) = \|T(g_1)\||w_1, T(g_2) = \|T(g_2)\||w_2, \) and \( E_{e_1,e_2} \) is linearly generated by \( g_1, g_2, g_3 \) and \( g_4, \) we deduce that \( T(E_{e_1,e_2}) \subseteq M \) with \( 4 = \dim(T(E_{e_1,e_2})) \leq \dim(M) = 4. \) Thus, \( T(E_{e_1,e_2}) = M \) is a \( JB^* \)-subtriple of \( F. \)

The mapping \( T|_{E_{e_1,e_2}} : E_{e_1,e_2} \cong M_2(\mathbb{C}) \rightarrow T(E_{e_1,e_2}) \) is a continuous biorthogonality preserving linear bijection. Theorem 3.9 implies that \( T|_{E_{e_1,e_2}} \) is a (nonzero) scalar multiple of a triple isomorphism, and hence \( \|T(e_1)\| = \|T(e_2)\|. \)

If \( \dim(H'_1) = 1, \) then \( L(H_1, H'_1) \) is a rank-one \( JB^* \)-triple. Since \( n, m \geq 2, \) we can find a minimal tripotent \( e \in E \) such that the norm closed inner ideals of \( E \) generated by \( \{e, e_1\} \) and \( \{e, e_2\} \) both coincide with \( M_2(\mathbb{C}). \) The arguments in the above paragraph show that \( \|T(e_1)\| = \|T(e)\| = \|T(e_2)\|. \)

Finally, the case \( \dim(H_1) = 1 \) follows from the same arguments. ■

**Remark 4.3.** Given a sequence \( (\mu_n) \subset c_0 \) and a bounded sequence \( (x_n) \) in a Banach space \( X, \) the series \( \sum_k \mu_k x_k \) need not be, in general, convergent in \( X. \) However, when \( (x_n) \) is a bounded sequence of mutually orthogonal elements in a \( JB^* \)-triple \( E, \) the equality
\[
\left\| \sum_{k=1}^{n} \mu_k x_k - \sum_{k=1}^{m} \mu_k x_k \right\| = \max\{|\mu_{n+1}|, \ldots, |\mu_m|\} \sup\{\|x_n\|\}
\]
holds for every \( n < m \) in \( \mathbb{N}. \) It follows that \( (\sum_{k=1}^{n} \mu_k x_k) \) is a Cauchy sequence and hence converges in \( E. \)

The following three results generalise [12, Lemmas 8, 9 and Proposition 10] to the setting of \( JB^* \)-triples.
Lemma 4.4. Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB*-triples and let $(e_n)$ be a sequence of mutually orthogonal minimal tripotents in $E$. Then there exist positive constants $m \leq M$ satisfying $m \leq \|T(e_n)\| \leq M$ for all $n \in \mathbb{N}$.

Proof. We deduce from Theorem 4.1 that, for each natural $n$, there exist a minimal tripotent $f_n$ and a scalar $\lambda_n \in \mathbb{C} \setminus \{0\}$ such that $T(e_n) = \lambda_n f_n$, where $\|T(e_n)\| = \lambda_n$. Note that $T$ being biorthogonality preserving implies $(f_n)$ is a sequence of mutually orthogonal minimal tripotents in $F$.

Let $(\mu_n)$ be any sequence in $c_0$. Since the $e_n$'s are mutually orthogonal the series $\sum_{k \geq 1} \mu_k e_k$ converges to an element in $E$ (cf. Remark 4.3). For each natural $n$, $\sum_{k \geq 1} \mu_k e_k$ decomposes as the orthogonal sum of $\mu_n e_n$ and $\sum_{k \neq n} \mu_k e_k$, therefore

$$T\left(\sum_{k \geq 1} \mu_k e_k\right) = \mu_n \lambda_n f_n + T\left(\sum_{k \neq n} \mu_k e_k\right)$$

with $\mu_n \lambda_n f_n \perp T\left(\sum_{k \neq n} \mu_k e_k\right)$, which in particular implies

$$\|T\left(\sum_{k \geq 1} \mu_k e_k\right)\| = \max \left\{ |\mu_n| \lambda_n, \|T\left(\sum_{k \neq n} \mu_k e_k\right)\| \right\} \geq |\mu_n| |\lambda_n|.$$  

This establishes that, for each $(\mu_n)$ in $c_0$, $(\mu_n \lambda_n)$ is a bounded sequence, which in particular implies that $(\lambda_n)$ is bounded.

Finally, since $T$ is a biorthogonality preserving linear surjection and $T^{-1}(f_n) = \lambda_n^{-1} e_n$, we can similarly show that $(\lambda_n^{-1})$ is also bounded. ■

Lemma 4.5. Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB*-triples, $(\mu_n)$ a sequence in $c_0$, and $(e_n)$ a sequence of mutually orthogonal minimal tripotents in $E$. Then the sequence $(T(\sum_{k \geq n} \mu_k e_k))_{n \in \mathbb{N}}$ is well defined and converges in norm to zero.

Proof. From Theorem 4.1 and Lemma 4.4 it follows that $(T(e_n))$ is a bounded sequence of mutually orthogonal elements in $F$. Let $M$ denote a bound of the above sequence. For each natural $n$, Remark 4.3 ensures that the series $\sum_{k \geq n} \mu_k e_k$ converges.

Define $y_n := T(\sum_{k \geq n} \mu_k e_k)$. We claim that $(y_n)$ is a Cauchy sequence in $F$. Indeed, given $n < m$ in $\mathbb{N}$, we have

$$(4.2) \quad \|y_n - y_m\| = \left\|T\left(\sum_{k \geq n}^{m-1} \mu_k e_k\right)\right\| = \left\|\sum_{k \geq n}^{m-1} \mu_k T(e_k)\right\|$$

$$\leq M \max\{|\mu_n|, \ldots, |\mu_{m-1}|\},$$

where in the last inequality we have used the fact that $(T(e_n))$ is a sequence of mutually orthogonal elements. Consequently, $(y_n)$ converges in norm to some element $y_0$ in $F$. Let $z_0$ denote $T^{-1}(y_0)$. 


Fix a natural \( m \). By hypothesis, for each \( n > m \), \( e_m \) is orthogonal to \( \sum_{k \geq n} \mu_k e_k \). This implies that \( T(e_m) \perp y_n \) for every \( n > m \), which in particular implies \( \{T(e_m), T(e_m), y_n\} = 0 \) for every \( n > m \). Letting \( n \) tend to \( \infty \) we have \( \{T(e_m), T(e_m), y_0\} = 0 \). This shows that \( y_0 = T(z_0) \) is orthogonal to \( T(e_m) \), and hence \( e_m \perp z_0 \). Since \( m \) was arbitrary, we deduce that \( z_0 \) is orthogonal to \( \sum_{k \geq n} \mu_k e_k \) for every \( n \). Therefore, \( \langle y_n \rangle \subset \{y_0\}^\perp \), and hence \( y_0 \) belongs to the norm closure of \( \{y_0\}^\perp \), which implies \( y_0 = 0 \). 

**Proposition 4.6.** Let \( T : E \to F \) be a biorthogonality preserving linear surjection between two \( JB^* \)-triples, where \( E \) is weakly compact. Then \( T \) is continuous if and only if the set \( \mathcal{T} := \{\|T(e)\| : e \text{ a minimal tripotent in } E\} \) is bounded. Moreover, in that case \( \|T\| = \sup(\mathcal{T}) \).

**Proof.** The necessity being obvious, suppose that

\[
M = \sup\{\|T(e)\| : e \text{ a minimal tripotent in } E\} < \infty.
\]

Since \( E \) is weakly compact, each nonzero element \( x \) of \( E \) can be written as a norm convergent (possibly finite) sum \( x = \sum_n \lambda_n u_n \), where \( u_n \) are mutually orthogonal minimal tripotents of \( E \), and \( \|x\| = \sup\{|\lambda_n| : n \geq 1\} \) (cf. Remark 4.6 in [7]). If the series \( x = \sum_n \lambda_n u_n \) is finite then

\[
\|T(x)\| = \left\| \sum_{n=1}^m \lambda_n T(u_n) \right\| \overset{(*)}{=} \max\{\|\lambda_n T(u_n)\| : n = 1, \ldots, m\} \leq M \|x\|,
\]

where at \((*)\) we apply the fact that \( (T(u_n)) \) is a finite set of mutually orthogonal tripotents in \( F \). When the series \( x = \sum_n \lambda_n u_n \) is infinite we may assume that \( (\lambda_n) \in c_0 \).

It follows from Lemma 4.5 that the sequence \( (T(\sum_{k \geq m} \lambda_k u_k))_n \) is well defined and converges in norm to zero. We can find a natural \( m \) such that \( \|T(\sum_{k \geq m} \lambda_k u_k)\| < M \|x\| \). Since the elements \( \lambda_1 u_1, \ldots, \lambda_{m-1} u_{m-1}, \sum_{k \geq m} \lambda_k u_k \) are mutually orthogonal, we have

\[
\|T(x)\| = \max\left\{\|T(\lambda_1 u_1)\|, \ldots, \|T(\lambda_{m-1} u_{m-1})\|, \left\|T\left(\sum_{k \geq m} \lambda_k u_k\right)\right\|\right\}
\]

\[
\leq M \|x\|.
\]

Let \( E \) be an elementary \( JB^* \)-triple of type 1 (that is, an elementary \( JB^* \)-triple such that \( E^{**} \) is a type 1 Cartan factor), and let \( T : E \to F \) be a biorthogonality preserving linear surjection from \( E \) onto another \( JB^* \)-triple. Then by Theorem 4.2 and Proposition 4.6, \( T \) is continuous. Further, we claim that \( T \) is a scalar multiple of a triple isomorphism. Indeed, let us see that \( S = (1/\lambda)T \) is a triple isomorphism, where \( \lambda = \|T(e)\| = \|T\| \) for some (and hence any) minimal tripotent \( e \) in \( E \) (cf. Theorem 4.2). Let \( x \in E \). Then \( x = \sum_n \lambda_n e_n \) for a suitable \( (\lambda_n) \in c_0 \) and a family of mutually orthogonal minimal tripotents \( (e_n) \) in \( E \) [7, Remark 4.6]. Then by observing that \( T \) is
continuous we have
\[ \|S(x)\| = \frac{1}{\lambda} \|T(x)\| = \frac{1}{\lambda} \left\| T\left( \sum_n \lambda_n e_n \right) \right\| = \frac{1}{\lambda} \left\| \sum_n \lambda_n T(e_n) \right\| = \frac{1}{\lambda} \sup_n |\lambda_n| \|T(e_n)\| = \frac{1}{\lambda} \sup_n |\lambda_n| = \sup_n |\lambda_n| = \|x\|. \]
This proves that $S$ is a surjective linear isometry between $JB^*$-triples, and hence a triple isomorphism (see [26, Proposition 5.5], [5, Corollary 3.4], [18, Theorem 2.2]). We have thus proved the following result:

**Corollary 4.7.** Let $T : E \to F$ a biorthogonality preserving linear surjection from a type 1 elementary $JB^*$-triple of rank greater than one onto another $JB^*$-triple. Then $T$ is a scalar multiple of a triple isomorphism. 

Let $p$ and $q$ be two minimal projections in a $C^*$-algebra $A$ with $q \neq p$. It is known that the $C^*$-subalgebra of $A$ generated by $p$ and $q$ is isometrically isomorphic to $\mathbb{C} \oplus \mathbb{C}$ when $p$ and $q$ are orthogonal, and isomorphic to $M_2(\mathbb{C})$ otherwise. More concretely, by [31, Theorem 1.3] (see also [29, §3]), denoting by $C_{p,q}$ the $C^*$-subalgebra of $A$ generated by $p$ and $q$, we have the following statements:

(a) If $p \perp q$ then there exists an isometric $C^*$-isomorphism $\Phi : C_{p,q} \to \mathbb{C} \oplus \mathbb{C}$ such that $\Phi(p) = (1, 0)$ and $\Phi(q) = (0, 1)$.

(b) If $p$ and $q$ are not orthogonal then there exist $0 < t < 1$ and an isometric $C^*$-isomorphism $\Phi : C_{p,q} \to M_2(\mathbb{C})$ such that
\[
\Phi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(q) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.
\]

In the setting of $JB^*$-algebras we have:

**Lemma 4.8.** Let $p$ and $q$ be two minimal projections in a $JB^*$-algebra $J$ with $q \neq p$ and let $J_{p,q}$ denote the $JB^*$-subalgebra of $J$ generated by $p$ and $q$.

(a) If $p \perp q$ then there exists an isometric $JB^*$-isomorphism $\Phi : J_{p,q} \to \mathbb{C} \oplus \mathbb{C}$ such that $\Phi(p) = (1, 0)$ and $\Phi(q) = (0, 1)$.

(b) If $p$ and $q$ are not orthogonal then there exist $0 < t < 1$ and an isometric $JB^*$-isomorphism $\Phi : C \to S_2(\mathbb{C})$ such that
\[
\Phi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(q) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix},
\]

where $S_2(\mathbb{C})$ denotes the type 3 Cartan factor of all symmetric operators on a two-dimensional complex Hilbert space.

Moreover, the $JB^*$-subtriple of $J$ generated by $p$ and $q$ coincides with $J_{p,q}$. 

Proof. Statement (a) is clear. Now assume that \( p \) and \( q \) are not orthogonal. The Shirshov–Cohn theorem (see [22, Theorem 7.2.5]) ensures that \( J_{p,q} \) is a JC*-algebra, that is, a Jordan *-subalgebra of some C*-algebra \( A \). The symbol \( C_{p,q} \) will stand for the (associative) C*-subalgebra of \( A \) generated by \( p \) and \( q \). Set

\[
P := \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
Q := \begin{pmatrix}
t & \sqrt{t(1-t)} \\
\sqrt{t(1-t)} & 1 - t
\end{pmatrix}.
\]

We have already mentioned that there exist \( 0 < t < 1 \) and an isometric C*-isomorphism \( \Phi : C_{p,q} \to M_2(\mathbb{C}) \) such that \( \Phi(p) = P \) and \( \Phi(q) = Q \).

Since \( J_{p,q} \) is a Jordan *-subalgebra of \( C_{p,q} \), \( J_{p,q} \) can be identified with the Jordan *-subalgebra of \( M_2(\mathbb{C}) \) generated by the matrices \( P \) and \( Q \). It can be easily checked that

\[
P \circ Q = \begin{pmatrix}
t & \frac{1}{2} \sqrt{t(1-t)} \\
\frac{1}{2} \sqrt{t(1-t)} & 0
\end{pmatrix},
\]

\[
2P \circ Q - 2tP = \begin{pmatrix}
0 & \frac{1}{2} \sqrt{t(1-t)} \\
\frac{1}{2} \sqrt{t(1-t)} & 0
\end{pmatrix},
\]

\[
Q - (2P \circ Q - 2tP) - tP = \begin{pmatrix}
0 & 0 \\
0 & 1 - t
\end{pmatrix}.
\]

These identities show that \( J_{p,q} \) contains the generators of the JB*-algebra \( S_2(\mathbb{C}) \), and hence identifies with \( S_2(\mathbb{C}) \).

In order to prove the last assertion, let \( E_{p,q} \) denote the JB*-subtriple of \( J \) generated by \( p \) and \( q \). As \( J_{p,q} \) is itself a subtriple containing \( p \) and \( q \), we have \( E_{p,q} \subseteq J_{p,q} \). If \( p \perp q \) then it can easily be seen that \( E_{p,q} \cong \mathbb{C} \oplus \mathbb{C} \cong J_{p,q} \).

Now assume that \( p \) and \( q \) are not orthogonal.

From Proposition 5 in [20], \( E_{p,q} \) is a JB*-triple isometrically isomorphic to \( M_{1,2}(\mathbb{C}) \) or \( S_2(\mathbb{C}) \). If \( E_{p,q} \) is a rank-one JB*-triple, that is, \( E \cong M_{1,2}(\mathbb{C}) \), then \( P_0(p)(q) \) must be zero. Thus, according to the above representation, we have \( 1 - t = 0 \), which is impossible.

A JB*-algebra which is a weakly compact JB*-triple will be called weakly compact or dual (see [6]). Every positive element \( x \) in a weakly compact JB*-algebra \( J \) can be written in the form \( x = \sum \lambda_n p_n \) for a suitable \( (\lambda_n) \in c_0 \) and a family \( (p_n) \) of mutually orthogonal minimal projections in \( J \) (see Theorem 3.3 in [6]).

Our next theorem extends [12, Theorem 11].

**Theorem 4.9.** Let \( T : J \to E \) be a biorthogonality preserving linear surjection from a weakly compact JB*-algebra onto a JB*-triple. Then \( T \) is continuous and \( \|T\| \leq 2 \sup \{\|T(p)\| : p \text{ a minimal projection in } J\} \).
Proof. Since $J$ is a $JB^*$-algebra, it is enough to show that $T$ is bounded on positive norm-one elements. In this case, it suffices to prove that the set

$$\mathcal{P} = \{\|T(p)\| : p \text{ a minimal projection in } J\}$$

is bounded (cf. the proof of Proposition 4.6).

Suppose, on the contrary, that $\mathcal{P}$ is unbounded. We shall show by induction that there exists a sequence $(p_n)$ of mutually orthogonal minimal projections in $J$ such that $\|T(p_n)\| > n$.

The case $n = 1$ is clear. The induction hypothesis guarantees the existence of mutually orthogonal minimal projections $p_1, \ldots, p_n$ in $J$ with $\|T(p_k)\| > k$ for all $k \in \{1, \ldots, n\}$.

By assumption, there exists a minimal projection $q \in J$ satisfying

$$\|T(q)\| > \max\{\|T(p_1)\|, \ldots, \|T(p_n)\|, n + 1\}.$$

We claim that $q$ must be orthogonal to each $p_j$. If that is not the case, there exists $j$ such that $p_j$ and $q$ are not orthogonal. Let $C$ denote the $JB^*$-subtriple of $J$ generated by $q$ and $p_j$. We conclude from Lemma 4.8 that $C$ is isomorphic to the $JB^*$-algebra $S_2(\mathbb{C})$.

Let $g_1 = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$ and $g_2 = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})$. Then $g_1 + g_2$ is the unit element in $C \cong S_2(\mathbb{C})$. By Theorem 4.1, $w_1 := \frac{1}{\|T(g_1)\|}T(g_1)$ and $w_2 := \frac{1}{\|T(g_2)\|}T(g_2)$ are two orthogonal minimal tripotents in $E$. The element $w = w_1 + w_2$ is a rank-2 tripotent in $E$ and coincides with the range tripotent of the element $h = T(g_1 + g_2) = \|T(g_1)\|w_1 + \|T(g_2)\|w_2$. Furthermore, $h$ is invertible in $E_2(w)$, and by Theorem 3.8 (see also [11, Corollary 4.1(b)]), $T(C) \subseteq E_2(w)$.

The rest of the argument is parallel to the argument in the proof of Theorem 4.2.

The finite-dimensionality of the $JB^*$-subtriple $C$ ensures that $T(C)$ is norm closed and $T|_C : C \cong S_2(\mathbb{C}) \to E$ is a continuous biorthogonality preserving linear operator. Theorem 3.8 guarantees the existence of a Jordan $*$-homomorphism $S : C \to E_2(w)$ such that $S(g_1 + g_2) = w$, $S(C)$ and $h$ operator commute and

$$T(z) = h \circ_w S(z) \quad \text{for all } z \in C.$$  

(4.3)

It follows from the operator commutativity of $h^{-1}$ and $S(C)$ that $S(z) = h^{-1} \circ_w T(z)$ for all $z \in C$. The injectivity of $T$ implies that $S$ is a Jordan $*$-monomorphism.

Lemma 2.7 in [19] shows that $E_2(w) = E_2(w_1 + w_2)$ coincides with $\mathbb{C} \oplus \ell^\infty \mathbb{C}$ or with a spin factor. Since $3 = \dim(T(C)) \leq \dim(E_2(w))$, we deduce that $E_2(w)$ is a spin factor with inner product $(\cdot|\cdot)$ and conjugation $x \mapsto \overline{x}$. We may assume, by Remark 2.1 that $(w_1|w_1) = 1/2$, $(w_1|\overline{w_1}) = 0$, and $w_2 = \overline{w_1}$.
Now, taking \( g_3 = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \in C \cong S_2(\mathbb{C}) \), the element \( w_3 := S(g_3) \) is a tripotent in \( E_2(w) \) with \( \{w_i, w_i, w_3\} = \frac{1}{2} w_3 \) for every \( i \in \{1, 2\} \). Remark 2.1 implies that \( (w_3|w_1) = (w_3|w_2) = 0 \). Let \( M \) denote the \( JB^* \)-subtriple of \( E_2(w) \) generated by \( w_1, w_2, \) and \( w_3 \). The mapping \( S: C \cong S_2(\mathbb{C}) \rightarrow M \) is a Jordan *-isomorphism.

Combining (4.3) and (2.4) we get

\[
T(g_3) = h \circ_w S(g_3) = \{h, w, w_3\} = \frac{||T(g_1)|| + ||T(g_2)||}{2} w_3.
\]

Since \( T(g_1) = ||T(g_1)|| w_1, T(g_2) = ||T(g_2)|| w_2, \) and \( C \) is linearly generated by \( g_1, g_2 \) and \( g_3 \), we deduce that \( T(C) \subseteq M \) with \( 3 = \dim(T(C)) \leq \dim(M) = 3 \). Thus, \( T(C) = M \) is a \( JB^* \)-subtriple of \( E \).

The mapping \( T|_C : C \cong S_2(\mathbb{C}) \rightarrow T(C) \) is a continuous biorthogonality preserving linear bijection. Theorem 3.9 guarantees the existence of a scalar \( \lambda \in \mathbb{C} \setminus \{0\} \) and a triple isomorphism \( \Psi : C \rightarrow T(C) \) such that \( T(x) = \lambda \Psi(x) \) for all \( x \in C \). Since \( p_j \) and \( q \) are projections, \( ||\Psi(q)|| = ||\Psi(p_j)|| = 1 \). Hence \( ||T(p_j)|| = |\lambda| \) and \( ||T(q)|| = |\lambda| \), contradicting the induction hypothesis. Therefore \( q \perp p_j \) for every \( j = 1, \ldots, n \).

It follows by induction that there exists a sequence \( (p_n) \) of mutually orthogonal minimal projections in \( J \) such that \( ||T(p_n)|| > n \). The series \( \sum_{n=1}^{\infty} (1/\sqrt{n})p_n \) defines an element \( a \) in \( J \) (cf. Remark 4.3). For each natural \( m, a \) decomposes as the orthogonal sum of \( (1/\sqrt{m})p_m \) and \( \sum_{n \not= m} (1/\sqrt{n})p_n \), therefore

\[
T(a) = \frac{1}{\sqrt{m}} T(p_m) + T \left( \sum_{n \not= m} \frac{1}{\sqrt{n}} p_n \right),
\]

with orthogonal summands. This argument implies that

\[
||T(a)|| = \max \left\{ \frac{1}{\sqrt{m}} ||T(p_m)||, \left|| T \left( \sum_{n \not= m} \frac{1}{\sqrt{n}} p_n \right) \right|| \right\} > \sqrt{m}.
\]

Since \( m \) was arbitrary, we have arrived at the desired contradiction. \( \blacksquare \)

By Proposition 2 in [23], every Cartan factor of type 1 with \( \dim(H) = \dim(H') \), every Cartan factor of type 2 with \( \dim(H) \) even or infinite, and every Cartan factor of type 3 is a \( JBW^* \)-algebra factor for a suitable Jordan product and involution. In the case of \( C \) being a Cartan factor which is also a \( JBW^* \)-algebra, the corresponding elementary \( JB^* \)-triple \( K(C) \) is a weakly compact \( JB^* \)-algebra.

**Corollary 4.10.** Let \( K \) be an elementary \( JB^* \)-triple of type 1 with \( \dim(H) = \dim(H') \), or of type 2 with \( \dim(H) \) even or infinite, or of type 3. Suppose that \( T : K \rightarrow E \) is a biorthogonality preserving linear surjection from \( K \) onto a \( JB^* \)-triple. Then \( T \) is continuous. Further, since \( K^{**} \) is a
JBW*-algebra factor, Theorem 3.9 ensures that $T$ is a scalar multiple of a triple isomorphism. ■

**Theorem 4.11.** Let $T : E \to F$ be a biorthogonality preserving linear surjection between JB*-triples, where $E$ is weakly compact containing no infinite-dimensional rank-one summands. Then $T$ is continuous.

**Proof.** Since $E$ is a weakly compact JB*-triple, the statement follows from Proposition 4.6 as soon as we prove that the set

$$T := \{ \|T(e)\| : e \text{ a minimal tripotent in } E \}$$

is bounded.

We know that $E = \bigoplus_{\alpha \in \Gamma} K_\alpha$, where $\{K_\alpha : \alpha \in \Gamma\}$ is a family of elementary JB*-triples (see Lemma 3.3 in [7]). Now, Lemma 3.1 guarantees that $T(K_\alpha) = T(K_\alpha^{\perp \perp}) = T(K_\alpha)^{\perp \perp}$ is a norm closed inner ideal for every $\alpha \in \Gamma$.

For each $\alpha \in \Gamma$, $K_\alpha$ is finite-dimensional, or a type 1 elementary JB*-triple of rank greater than one, or a JB*-algebra. It follows, by Corollary 4.7 and Theorem 4.9 that $T|_{K_\alpha} : K_\alpha \to T(K_\alpha)$ is continuous.

Suppose that $T$ is unbounded. Having in mind that every minimal tripotent in $E$ belongs to a unique factor $K_\alpha$, by Proposition 4.6 there exists a sequence $(e_n)$ of mutually orthogonal minimal tripotents in $E$ such that $\|T(e_n)\|$ diverges to $+\infty$. The element $z := \sum_{n=1}^{\infty} \|T(e_n)\|^{-1/2} e_n$ lies in $E$ and hence $\|T(z)\| < \infty$. We fix an arbitrary natural $m$. Since $z - \|T(e_m)\|^{-1/2} e_m$ and $\|T(e_m)\|^{-1/2} e_m$ are orthogonal, we have

$$T(z - \|T(e_m)\|^{-1/2} e_m) \perp T(\|T(e_m)\|^{-1/2} e_m),$$

and hence

$$\|T(z)\| = \|T(z - \|T(e_m)\|^{-1/2} e_m)\| + T(\|T(e_m)\|^{-1/2} e_m)\|
= \max\{\|T(z - \|T(e_m)\|^{-1/2} e_m)\|, \|T(e_m)\|^{-1/2} \|T(e_m)\|\} \geq \sqrt{\|T(e_m)\|},$$

which contradicts that $\|T(e_m)\|^{1/2} \to +\infty$. Therefore $T$ is bounded. ■

**Corollary 4.12.** Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB*-triples, where $K(E)$ contains no infinite-dimensional rank-one summands. Then $T|_{K(E)} : K(E) \to K(F)$ is continuous.

**Proof.** Pick $x \in K(E)$. It can be written in the form $x = \sum_n \lambda_n u_n$, where $u_n$ are mutually orthogonal minimal tripotents of $E$, and $\|x\| = \sup\{|\lambda_n| : n \geq 1\}$ (cf. Remark 4.6 in [7]). For each natural $m$ we define $y_m := T(\sum_{n \geq m+1} \lambda_n u_n)$. Theorem 4.1 guarantees that $T(x_m) = T(\sum_{n=1}^{m} \lambda_n u_n)$ defines a sequence in $K(F)$.
Since, by Lemma 4.5, \( y_m \to 0 \) in norm, we deduce that \( T(x_m) = T(x) - y_m \) tends to \( T(x) \) in norm. Therefore \( T(K(E)) = K(F) \) and \( T|_{K(E)} : K(E) \to K(F) \) is a biorthogonality preserving linear surjection between weakly compact \( JB^* \)-triples. The result now follows from Theorem 4.11.

**Remark 4.13.** In Remark 15 of [10] it was already pointed out that the conclusion of Theorem 4.11 is no longer true if we allow \( E \) to have infinite-dimensional rank-one summands. Indeed, let \( E = L(H) \oplus^\infty L(H, \mathbb{C}) \), where \( H \) is an infinite-dimensional complex Hilbert space. We can always find an unbounded bijection \( S : L(H, \mathbb{C}) \to L(H, \mathbb{C}) \). Since \( L(H, \mathbb{C}) \) is a rank-one \( JB^* \)-triple, \( S \) is a biorthogonality preserving linear bijection and the mapping \( T : E \to E \) given by \( x + y \mapsto x + S(y) \) has the same properties.

**Corollary 4.14.** Two weakly compact \( JB^* \)-triples containing no rank-one summands are isomorphic if and only if there exists a biorthogonality preserving linear surjection between them.

5. **Biorthogonality preservers between atomic \( JBW^* \)-triples.**

A \( JBW^* \)-triple \( E \) is said to be atomic if it coincides with the weak* closed ideal generated by its minimal tripotents. Every atomic \( JBW^* \)-triple can be written as an \( \ell_\infty \)-sum of Cartan factors [21].

The aim of this section is to study when the existence of a biorthogonality preserving linear surjection between two atomic \( JBW^* \)-triples implies that they are isomorphic (note that continuity is not assumed). We shall establish an automatic continuity result for biorthogonality preserving linear surjections between atomic \( JBW^* \)-triples containing no rank-one factors.

Before dealing with the main result, we survey some results describing the elements in the predual of a Cartan factor. We make use of the description of the predual of \( L(H) \) in terms of the trace class operators (cf. [32, §II.1]). The results, included here for completeness, are direct consequences of this description but we do not know an explicit reference.

Let \( C = L(H, H') \) be a type 1 Cartan factor. Lemma 2.6 in [30] ensures that each \( \varphi \) in \( C_* \) can be written in the form \( \varphi := \sum_{n=1}^\infty \lambda_n \varphi_n \), where \( (\lambda_n) \) is a sequence in \( \ell_1^+ \) and each \( \varphi_n \) is an extreme point of the closed unit ball of \( C_* \). More concretely, for each natural \( n \) there exist norm-one elements \( h_n \in H \) and \( k_n \in H' \) such that \( \varphi_n(x) = (x(h_n)|k_n) \) for every \( x \in C \), that is, for each natural \( n \) there exists a minimal tripotent \( e_n \) in \( C \) such that \( P_2(e_n)(x) = \varphi_n(x)e_n \) for every \( x \in C \) (cf. [20, Proposition 4]).

We now consider (infinite-dimensional) type 2 and type 3 Cartan factors. Let \( j \) be a conjugation on a complex Hilbert space \( H \), and consider the linear involution on \( L(H) \) defined by \( x \mapsto x^j := jx^*j \). Let \( C_2 = \{ x \in L(H) : \)
and $C_3 = \{ x \in L(H) : x^t = x \}$ be Cartan factors of type 2 and 3, respectively.

Noticing that $L(H) = C_2 \oplus C_3$, it is easy to see that every element $\varphi$ in $(C_2)_*$ (respectively, $(C_3)_*$) admits an extension of the form $\tilde{\varphi} = \varphi \pi$, where $\pi$ denotes the canonical projection of $L(H)$ onto $C_2$ (respectively, $C_3$). Making use of [32, Lemma 1.5], we can find an element $x_{\tilde{\varphi}} \in K(H)$ satisfying
\begin{equation}
(x_{\tilde{\varphi}}(h)|k) = \tilde{\varphi}(h \otimes k) \quad (h, k \in H).
\end{equation}

Since, for each $x \in L(H)$, $\tilde{\varphi}(x) = \frac{1}{2} \varphi(x-x^t)$, we can easily check, via (5.1), that $x_{\tilde{\varphi}} = -x_{\varphi}$. Therefore $x_{\tilde{\varphi}} \in K_2 = K(C_2)$. From [7, Remark 4.6] it may be deduced that $x_{\tilde{\varphi}}$ can be (uniquely) written as a norm convergent (possibly finite) sum $x_{\tilde{\varphi}} = \sum_n \lambda_n u_n$, where $u_n$ are mutually orthogonal minimal tripotents in $K_2$ and $(\lambda_n) \in c_0$ (notice that $u_n$ is a minimal tripotent in $C_2$ but it need not be minimal in $L(H)$; in any case, either $u_n$ is minimal in $L(H)$ or it can be written as a convex combination of two minimal tripotents in $L(H)$). For each $(\beta_n) \in c_0$, $z := \sum_n \beta_n u_n \in K_2$ and, by (5.1), $\sum_n \lambda_n \beta_n = \tilde{\varphi}(z) = \varphi(z) < \infty$. Thus, $(\lambda_n) \in \ell_1$, and another application of (5.1) shows that $\varphi(x) = \sum_n \lambda_n \varphi_n(x)$ for all $x \in C_2$, where $\varphi_n$ lies in $(C_2)_*$ and satisfies $P_2(u_n)(x) = \varphi_n(x)u_n$. A similar reasoning remains true for $C_3$.

We have thus proved:

**Proposition 5.1.** Let $C$ be an infinite-dimensional Cartan factor of type 1, 2 or 3. For each $\varphi$ in $C_*$, there exist a sequence $(\lambda_n) \in \ell_1$ and a sequence $(u_n)$ of mutually orthogonal minimal tripotents in $C$ such that
\[
\|\varphi\| = \sum_{n=1}^{\infty} |\lambda_n| \quad \text{and} \quad \varphi(x) = \sum_n \lambda_n \varphi_n(x) \quad (x \in C),
\]
where for each $n \in \mathbb{N}$, $\varphi_n(x)u_n = P_2(u_n)(x) \quad (x \in C)$. $lacksquare$

Let $T : E \to F$ be a biorthogonality preserving linear surjection between atomic $JBW^*$-triples, where $E$ contains no rank-one Cartan factors. In this case $K(E)$ and $K(F)$ are weakly compact $JB^*$-triples with $K(E)^** = E$ and $K(F)^** = F$. Corollary 4.12 ensures that $T|_{K(E)} : K(E) \to K(F)$ is continuous. This is not, a priori, enough to guarantee that $T$ is continuous. In fact, for each nonreflexive Banach space $X$ there exists an unbounded linear operator $S : X^{**} \to X^{**}$ such that $S|_X : X \to X$ is continuous. The main result of this section establishes that a mapping $T$ as above is automatically continuous.

**Theorem 5.2.** Let $T : E \to F$ be a biorthogonality preserving linear surjection between atomic $JBW^*$-triples, where $E$ contains no rank-one Cartan factors. Then $T$ is continuous.
Proof. Corollary 4.12 ensures that $T|_{K(E)} : K(E) \to K(F)$ is continuous. By Lemma 3.3 in [7], $K(E)$ decomposes as a $c_0$-sum of all elementary triple ideals of $E$, that is, if $E = \bigoplus_{\ell}^{\infty} C_{\alpha}$, where each $C_{\alpha}$ is a Cartan factor, then $K(E) = \bigoplus_{\ell}^{c_0} K(C_{\alpha})$. By Proposition 3.10, for each $\alpha$, $T(K_{\alpha})$ (respectively, $T(C_{\alpha})$) is a norm closed (respectively, weak* closed) inner ideal of $K(F)$ (respectively, $F$) and $K(F) = \bigoplus_{\ell}^{c_0} T(K(C_{\alpha}))$ (respectively, $F = \bigoplus_{\ell}^{c_0} T(C_{\alpha}))$.

For each $\alpha$, $C_{\alpha}$ is either finite-dimensional, or an infinite-dimensional Cartan factor of type 1, 2 or 3. Corollaries 4.7 and 4.10 prove that the operator $T|_{K(C_{\alpha})} : K(C_{\alpha}) \to T(K(C_{\alpha}))$ is a scalar multiple of a triple isomorphism. We claim that, for each $\alpha$ and each $\varphi_{\alpha}$ in the predual of $T(C_{\alpha})$, $\varphi_{\alpha} T$ is weak* continuous. There is no loss of generality in assuming that $C_{\alpha}$ is infinite-dimensional.

Each minimal tripotent $f$ in $F$ lies in a unique elementary $JB^*$-triple $T(K(C_{\alpha})_i)$. Since $T|_{K(C_{\alpha})} : K(C_{\alpha}) \to T(K(C_{\alpha}))$ is a scalar multiple of a triple isomorphism, there exist a nonzero scalar $\lambda_{\alpha}$ and a minimal tripotent $e$ satisfying $T^{-1}(f) = \lambda_{\alpha} e$, $|\lambda_{\alpha}| \leq \|T|_{K(C_{\alpha})}^{-1}\| \leq \|T|_{K(E)}^{-1}\|$, and

$$T(K(C_{\alpha})_i(e)) = T(K(C_{\alpha}))_i(f)$$

for every $i = 0, 1, 2$. Theorem 4.1 shows that $T((C_{\alpha})_i(e)) = T(C_{\alpha})_i(f)$ for every $i = 0, 2$. Since $K(E)$ is an ideal of $E$ and $e$ is a minimal tripotent, $(C_{\alpha})_1(e) = E_1(e) = K(E)_1(e) = K(C_{\alpha})_1(e)$. It follows from (5.2) that

$$T((C_{\alpha})_i(e)) = T((C_{\alpha}))_i(f)$$

for every $i = 0, 1, 2$. Consequently, $P_2(f)T = \lambda_{\alpha}^{-1} P_2(e) \in (C_{\alpha})_*$, and $|\lambda_{\alpha}^{-1}| \leq \|T|_{K(C_{\alpha})}\| \leq \|T|_{K(E)}\|$. Since $f$ was an arbitrary minimal tripotent in $F$ (equivalently, in $T(K(C_{\alpha}))$), Proposition 5.1 ensures that $\varphi_{\alpha} T \in E_*$ with $\|\varphi_{\alpha} T\| \leq \|T|_{K(E)}\|$ for every $\varphi_{\alpha} \in (T(C_{\alpha}))_*$. Therefore, $T$ is bounded with

$$\|T\| \leq \|T|_{K(E)}\| \leq \|T\|. \quad \blacksquare$$

**Corollary 5.3.** Two atomic $JBW^*$-triples containing no rank-one summands are isomorphic if and only if there is a biorthogonality preserving linear surjection between them. \(\blacksquare\)

The conclusion of Theorem 5.2 does not hold for atomic $JBW^*$-triples containing rank-one summands.

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