STUDIA MATHEMATICA 204 (2) (2011)

Toeplitz operators on Bergman spaces and Hardy multipliers

by

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Abstract. We study Toeplitz operators T_a with radial symbols in weighted Bergman spaces A^p_μ, $1 < p < \infty$, on the disc. Using a decomposition of A^p_μ into finite-dimensional subspaces the operator T_a can be considered as a coefficient multiplier. This leads to new results on boundedness of T_a and also shows a connection with Hardy space multipliers. Using another method we also prove a necessary and sufficient condition for the boundedness of T_a for a satisfying an assumption on the positivity of certain indefinite integrals.

1. Introduction and notation. We study Toeplitz operators T_a with radial symbols in weighted Bergman spaces $A^p_\mu = A^p_\mu(\mathbb{D})$, $1 < p < \infty$, of the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. In the Hilbert space case $p = 2$, the article [8] (see also [22]) contains a study of T_a as the Taylor coefficient multiplier

$$T_a : \sum_{k\in\mathbb{N}} f_k z^k \mapsto \sum_{k\in\mathbb{N}} \gamma_k f_k z^k. \tag{1.1}$$

The multiplier coefficients γ_k are weighted moments γ_k of the symbol a (see (3.1)), so the boundedness of $T_a : A^2 \to A^2$ can be characterized in terms of the boundedness of the sequence $(\gamma_k)_{k=1}^\infty$. The reference [8] contains the unitary equivalence of T_a to a multiplication operator even in a more general setting, thus completely clarifiying the basic properties of T_a.

The Toeplitz operator can still be considered as a multiplier even in the case $p \neq 2$, but this is a less useful point of view, since the monomials do not form an unconditional Schauder basis in A^p, $p \neq 2$, not to speak of the case with weighted norms. However, in this paper we prove (Theorem 3.5) the fact that even with quite general weighted norms the Bergman space can still be decomposed into finite-dimensional subspaces $A^{(n)}$, $n \in \mathbb{N}$, spanned by monomials with degrees in certain subintervals $\mathbb{N}_n$ of $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2010 *Mathematics Subject Classification*: Primary 47B35; Secondary 46E15.
Key words and phrases: Toeplitz operator, Bergman space, radial symbol, radial weight.

DOI: 10.4064/sm204-2-3

The boundedness and compactness of T_a can be characterized in terms of its behaviour on these blocks (see Theorem 3.3).

In fact this leads to the following result (for the assumptions on μ, see this section below, and for γ_k and $\mathbb{N}_n$, see Section 3).

THEOREM 1.1. *Let $1 < p < \infty$. The Toeplitz operator $T_a : A^p_\mu \to A^p_\mu$ is bounded if and only if the coefficient multipliers*

$$T_a^{(n)} : \sum_{k\in\mathbb{N}} f_k e^{ik\theta} \mapsto \sum_{k\in\mathbb{N}_n} \gamma_k f_k e^{ik\theta}, \qquad \theta \in [0, 2\pi], \tag{1.2}$$

are for all $n \in \mathbb{N}$ uniformly bounded operators $H^p \to H^p$, where H^p is the Hardy space on the disc. Moreover, $T_a : A^p_\mu \to A^p_\mu$ is compact if and only if the sequence formed by the operator norms of $T_a^{(n)} : H^p \to H^p$ converges to 0.

The reason is that on every $A^{(n)}$, the Bergman norm is actually equivalent to a Hardy-type norm, a result which is contained in Theorem 3.5. We remark that a complete characterization of the boundedness of Hardy multipliers is not known. The operator norms of $T_a^{(n)} : H^p \to H^p$ will later be denoted by $\mathcal{M}_p(\tau_n)$.

As for other results, we observe in Proposition 3.1 that the boundedness of the multiplier sequence (γ_k) is still necessary for the boundedness of the operator $T_a : A^p_\mu \to A^p_\mu$ in the case $p \neq 2$, for general weights. Hence, we obtain the result that the boundedness of the operator $T_a : A^p_\mu \to A^p_\mu$, $p \neq 2$, implies the boundedness $T_a : A^2_\mu \to A^2_\mu$ (see Theorem 3.2).

The above approach to Toeplitz operators as multipliers is contained in Section 3. In Section 2 we provide another type of necessary and sufficient condition for the boundedness of T_a for special a. Namely, under a rather weak assumption on the positivity of a certain indefinite nth integral $I_a(n)$ of a, the boundedness of $T_a : A^2 \to A^2$ was characterized in [9] in terms of the boundary behaviour of $I_a^{(n)}$. In Theorem 2.1 we generalize this to the case $p \neq 2$. The proof uses some estimates of the kernel of the Berezin transform and it is considerably more complicated than in the Hilbert space case.

We recall the basic definitions and notation. An introduction to Bergman spaces on $\mathbb{D}$ and Toeplitz operators can be found in [28]. For Hardy spaces we also refer to [19]. By C, C', C_1 etc. we mean positive constants independent of given functions or indices, but which may vary from place to place. The *Toeplitz operator* T_a is defined as the product of pointwise multiplication and Bergman projection operators,

$$T_a f(z) = \int_{\mathbb{D}} \frac{a(w)f(w)}{(1 - z\bar{w})^2}\, dA(w),$$

where dA is the normalized two-dimensional Lebesgue measure on $\mathbb{D}$ and

$a : \mathbb{D} \to \mathbb{C}$ is the symbol of T_a. In the following we restrict to the case of $a \in L^1(\mathbb{D})$ radial: $a(z) = a(|z|)$. Then also the one real variable function $a(r)$ belongs to $L^1(0,1)$.

We shall work in the context of weighted Bergman spaces with rather general radial weights. Let μ be a nonatomic, bounded positive measure on $[0,1[$ such that $\mu([1-\varepsilon,1[) > 0$ for every $0 < \varepsilon < 1$. We define the space $L^p_\mu = L^p_\mu(\mathbb{D})$ using the norm

$$\|f\|_{p,\mu} := \left(\frac{1}{2\pi}\int_0^1\int_0^{2\pi} |f(re^{i\theta})|^p r\, d\theta\, d\mu(r)\right)^{1/p}. \tag{1.3}$$

The corresponding weighted Bergman space (the closed subspace of L^p_μ consisting of analytic functions) is denoted by $A^p_\mu := A^p_\mu(\mathbb{D})$. The most classical case of unweighted Bergman space $A^p := A^p(\mathbb{D})$ corresponds to the measure $d\mu = dr$; in this case we omit the index μ in the notation. Also the weighted cases with measures $d\mu = (1-r^2)dr$ are standard in the literature.

Remark 1.2. 1° It is not really important to assume that the measure is nonatomic, since, for any bounded positive measure μ on $[0,1[$ and every $\varepsilon > 0$, there exists a nonatomic (bounded positive) measure μ_0 such that the weighted Bergman spaces corresponding to μ and μ_0 are the same and

$$(1-\varepsilon)\|f\|_{p,\mu} \le \|f\|_{p,\mu_0} \le \|f\|_{p,\mu} \tag{1.4}$$

for all $f \in A_{p,\mu}$. This fact is essentially contained in Proposition 1.2 of [11].

2° With our assumptions on the measure μ, the polynomials form a dense subspace of A^p_μ. See [14, Proposition 2.1].

The basic problem of characterizing the boundedness of Toeplitz operators on Bergman spaces is still open. Well-known partial results are included in [12, 16, 25, 26, 27, 29]. Recently, quite weak sufficient conditions for the boundedness were given in [21, 17]. The latter reference also contains the definition of Toeplitz operators with distributional symbols. The recent review [18] contains some results on radial symbols, related to the present paper. Many other topics, like matrices or products of Toeplitz operators, operators on general domains, Fredholm properties and so on, have attracted a lot of attention (see e.g. [1–7, 11, 20, 23, 29]).

2. Symbols with positive integrals. The problem of characterizing the boundedness of T_a was solved for positive a in [16] (a good presentation of the topic with references to later developments and generalizations can be found in [28]): the Toeplitz operator $T_a : A^p \to A^p$ is bounded if and only if the Berezin transform of a is bounded. Our aim here is to consider the radial case for all $1 < p < \infty$ and in particular weaken the positivity assumption on the symbol a as follows. For all $n \in \mathbb{N}$, $n \ge 1$, we define the

nth indefinite integrals by

$$I_a(r) := I_a^{(1)}(r) = \int_r^1 a(s)s\,ds, \qquad I_a^{(n+1)}(r) := \int_r^1 I_a^{(n)}(s)s\,ds, \qquad r \in [0,1[.$$

It is clear that for positive a all indefinite integrals $I_a^{(n)}$ are positive functions; moreover, if $I_a^{(n)}$ is positive, then the same is true for all $I_a^{(k)}$ with $k > n$. On the other hand, $I_a^{(n)}$ may very well be positive though a is not. We shall solve the boundedness problem under the assumption that $I_a^{(n)}$ is positive for some n:

Theorem 2.1. *Assume that for some $n \geq 1$ the function I_a^n is nonnegative. Then the Toeplitz operator $T_a : A^p \to A^p$ is bounded if and only if*

$$|I_a^{(n+1)}(r)| \leq C(1-r)^{n+1} \tag{2.1}$$

for all $r \in [0,1[$.

In the case $p = 2$ this result was proven in [8, 22].

The proof requires some preparations. The normalized Bergman kernel and the kernel of the Berezin transform are denoted, respectively, by

$$k_z(w) = \frac{1-|w|^2}{(1-z\bar{w})^2}, \qquad B_z(w) = \frac{(1-|z|^2)^2}{|1-z\bar{w}|^4} = k_w(z)\overline{k_w(z)}, \tag{2.2}$$

where $z, w \in \mathbb{D}$, the Berezin transform of a function $f \in L^1(\mathbb{D})$ thus being

$$\tilde{f}(z) := \int_{\mathbb{D}} f(w)B_z(w)\,dA(w).$$

Lemma 2.2. *For every $k \in \mathbb{N}$, $k \geq 1$, there exists a constant $C_k > 0$ such that*

$$\frac{1}{C_k}\frac{r^{k-1}(1-r^2)^2}{(1-r\varrho)^{k+3}} \leq \int_0^{2\pi} \frac{\partial^k}{\partial\varrho^k}(\varrho B_z(\varrho e^{i\theta}))\,d\theta \leq C_k\frac{r^{k-1}(1-r^2)^2}{(1-r\varrho)^{k+3}}, \tag{2.3}$$

where $r := |z|$ and $\varrho \in [0,1[$.

Proof. Writing $w = \varrho e^{i\theta}$, we see that

$$\begin{aligned}\frac{1}{|1-z\bar{w}|^4} &= \frac{1}{(1-z\bar{w})^2}\frac{1}{(1-\bar{z}w)^2} \\ &= \Big(\sum_{n=0}^\infty (n+1)(z\bar{w})^n\Big)\Big(\sum_{n=0}^\infty (n+1)(\bar{z}w)^n\Big) \\ &= \sum_{n=0}^\infty\sum_{m=0}^n (n-m+1)(m+1)(z\bar{w})^{n-m}(\bar{z}w)^m \\ &= \sum_{n=0}^\infty \varrho^n \sum_{m=0}^n (n-m+1)(m+1)z^{n-m}\bar{z}^m e^{-i\theta(n-2m)}.\end{aligned} \tag{2.4}$$

Hence,

$$\frac{\partial^k}{\partial\varrho^k}\frac{\varrho}{|1-z\bar{w}|^4} = \sum_{n=k-1}^{\infty}\frac{(n+1)!}{(n-k+1)!}\varrho^{n-k+1}\sum_{m=0}^{n}(n-m+1)(m+1)z^{n-m}\bar{z}^m e^{-i\theta(n-2m)},$$

and, since only the terms with even n and $m = n/2$ are nonzero after integration, we get

$$(2.5)\qquad \int_0^{2\pi}\frac{\partial^k}{\partial\varrho^k}\frac{\varrho}{|1-z\bar{w}|^4}\,d\theta = 2\pi\sum_{\substack{n\ge k-1\\ n\text{ even}}}\frac{(n+1)!}{(n-k+1)!}\left(\frac{n}{2}+1\right)^2 z^{n/2}\bar{z}^{n/2}\varrho^{n-k+1}$$
$$= \frac{\pi}{2}r^{k-1}\sum_{\substack{n\ge 0\\ n+k\text{ odd}}}\frac{(n+k)!}{n!}(n+k+1)^2(r\varrho)^n.$$

From the identity

$$(2.6)\qquad \frac{d^{k+2}}{dx^{k+2}}\frac{1}{1-x} = \sum_{n=0}^{\infty}\frac{(n+k+2)!}{n!}x^n,\qquad k\in\mathbb{N},$$

one can develop the Taylor series for the expressions $(1-r\varrho)^{-k-3}$; the result follows from (2.5), using the estimates

$$(2.7)\qquad \frac{1}{C}\frac{(n+k+2)!}{n!} \le \frac{(n+k)!}{n!}(n+k+1)^2 \le \frac{C(n+k+2)!}{n!},$$

and the observation that the missing even, say $(n+k+1)$st, degree term in (2.5) is proportional to the existing $(n+k)$th degree term. ∎

Notice that though the ϱ-derivatives of the Berezin kernel are not positive functions, the angular integrals of the derivatives are, by the previous lemma. This will be of crucial importance for the proof of the main result.

The following lemma is well-known, and follows from the boundedness of T_a by considering the expression $\langle T_a k_\zeta, k_\zeta\rangle$, and using the definition of the operator norm and duality of Bergman spaces.

LEMMA 2.3. *If $T_a : A^p \to A^p$ is bounded, then*

$$(2.8)\qquad \Big|\int_{\mathbb{D}} a(w)B_z(w)\,dA(w)\Big| \le C.$$

Proof of Theorem 2.1. Let us consider the necessity of condition (2.1). Assuming $T_a : A^p \to A^p$ bounded, Lemma 2.3 and repeated integration by parts imply

$$C \geq \Big|\int_{\mathbb{D}} a(w)B_z(w)\,dA(w)\Big| = \Big|\int_0^{2\pi}\int_0^1 a(\varrho)B_z(\varrho e^{i\theta})\varrho\,d\varrho\,d\theta\Big| \tag{2.9}$$
$$= \Big|\int_0^{2\pi}\Big(\sum_{k=1}^{n}(-1)^{k+1}\Big[I_a^{(k)}(\varrho)\frac{\partial^{k-1}}{\partial\varrho^{k-1}}(\varrho B_z(\varrho e^{i\theta}))\Big]_{\varrho=0}^{\varrho=1}$$
$$+(-1)^n\int_0^1 I_a^{(n)}(\varrho)\frac{\partial^n}{\partial\varrho^n}(\varrho B_z(\varrho e^{i\theta}))\,d\varrho\Big)\,d\theta\Big|.$$

Since $a \in L^1(\mathbb{D}, dA)$ and it is constant with respect to θ, the function a restricted to the unit interval actually belongs to $L^1(0,1)$ with respect to the one-dimensional measure. Hence,

$$\lim_{\varrho\to 1}\int_{\varrho}^{1} a(s)\,ds = 0 \tag{2.10}$$

and by induction $\lim_{\varrho\to 1} I_a^{(k)}(\varrho) = 0$ for every k. This implies that the substitution term with $\varrho = 1$ in (2.9) is null, since for any fixed z the expression $\partial^k(\varrho B_z(\varrho))/\partial\varrho^k$ is even bounded. Moreover, the substitution term with $\varrho = 0$ is bounded by a constant times the $L^1(0,1)$-norm of a, hence the same applies to the whole term

$$\Big[I_a^{(k)}(\varrho)\frac{\partial^{k-1}}{\partial\varrho^{k-1}}(\varrho B_z(\varrho e^{i\theta}))\Big]_{\varrho=0}^{\varrho=1} \tag{2.11}$$

Concerning the last term of (2.9), its modulus can be bounded from below, using the positivity of $I_a^{(n)}$, the fact that $r \geq 1/2$, and (2.3), as follows:

$$\int_0^1 I_a^{(n)}(\varrho)\int_0^{2\pi}\frac{\partial^n}{\partial\varrho^n}(\varrho B_z(\varrho e^{i\theta}))\,d\theta\,d\varrho \geq C_n\int_0^1 I_a^{(n)}(\varrho)\frac{r^{n-1}(1-r^2)^2}{(1-r\varrho)^{n+3}}\,d\varrho \tag{2.12}$$
$$\geq C_n'(1-r)^2\int_r^1 I_a^{(n)}(\varrho)\frac{1}{(1-r\varrho)^{n+3}}\,d\varrho$$
$$\geq \frac{C_n''}{(1-r)^{n+1}}\int_r^1 I_a^{(n)}(\varrho)\,d\varrho = \frac{C_n'' I_a^{(n+1)}}{(1-r)^{n+1}}.$$

This, together with (2.9) and the boundedness of (2.11), implies that condition (2.1) is necessary for the boundedness of the Toeplitz operator.

The sufficiency part follows using the method of [17, Theorem 3.1]. Some details are however different, so we present the proof. Let $n \in \mathbb{N}$ be as in the assumption, and let f be an arbitrary polynomial and $z \in \mathbb{D}$. Repeated integration by parts yields

$$
(2.13)\qquad T_a f(z) = \int_0^{2\pi}\int_0^1 a(\varrho e^{i\theta})\frac{f(\varrho e^{i\theta})}{(1-z\varrho e^{-i\theta})^2}\,\varrho\,d\varrho\,d\theta
$$

$$
= \int_0^{2\pi}\bigg(\sum_{k=1}^{n}(-1)^{k+1}\bigg[I_a^{(k)}(\varrho)\,\frac{\partial^{k-1}}{\partial\varrho^{k-1}}\frac{\varrho f(\varrho e^{i\theta})}{(1-z\varrho e^{-i\theta})^2}\bigg]_{\varrho=0}^{\varrho=1}
$$

$$
+(-1)^n\int_0^1 I_a^{(n)}(\varrho)\,\frac{\partial^n}{\partial\varrho^n}\frac{\varrho f(\varrho e^{i\theta})}{(1-z\varrho e^{-i\theta})^2}\,d\varrho\bigg)\,d\theta.
$$

Since f is a polynomial, we deduce as around (2.10) that the substitution terms with $\varrho = 1$ vanish and those with $\varrho = 0$ are bounded by the L^1-norm of a times $\|f\|_p$. Moreover, it is plain that

$$
\bigg|\int_0^{2\pi}\int_0^{1/2} I_a^{(n)}(\varrho)\,\frac{\partial^n}{\partial\varrho^n}\frac{\varrho f(\varrho e^{i\theta})}{(1-z\varrho e^{-i\theta})^2}\,d\varrho\,d\theta\bigg|
$$

can be bounded by $C\|f\|_p$. Hence, we can add the Jacobian ϱ and bound the last term of (2.13) by $C\|f\|_p$ plus

$$
(2.14)\qquad \int_0^{2\pi}\int_{1/2}^{1}|I_a^{(n)}(\varrho)|\bigg|\frac{\partial^n}{\partial\varrho^n}\frac{f(\varrho e^{i\theta})}{(1-z\varrho e^{-i\theta})^2}\bigg|\varrho\,d\varrho\,d\theta
$$

$$
\le C\int_0^{2\pi}\int_{1/2}^{1}|I_a^{(n)}(\varrho)|\sum_{m=0}^{n}\bigg|\frac{\partial^m f(\varrho e^{i\theta})}{\partial\varrho^m}\,\frac{\partial^{n-m}}{\partial\varrho^{n-m}}\frac{1}{(1-z\varrho e^{-i\theta})^2}\bigg|\varrho\,d\varrho\,d\theta
$$

$$
\le C\int_0^{2\pi}\int_{1/2}^{1}|I_a^{(n)}(\varrho)|(1-\varrho)^{-n}\sum_{m=0}^{n}(1-\varrho)^m|f^{(m)}(\varrho e^{i\theta})|\frac{(1-\varrho)^{n-m}}{|1-z\varrho e^{-i\theta}|^{2+n-m}}\,\varrho\,d\varrho\,d\theta
$$

$$
\le C'\int_0^{2\pi}\int_0^{1}\sum_{m=0}^{n}(1-\varrho)^m|f^{(m)}(\varrho e^{i\theta})|\frac{1}{|1-z\varrho e^{-i\theta}|^2}\,\varrho\,d\varrho\,d\theta,
$$

where (2.1) and the identity $\partial^m f(\varrho e^{i\theta})/\partial\varrho^m = f^{(m)}(\varrho e^{i\theta})e^{im\theta}$ were used. The rest goes as in [21]: the last expression of (2.14) is the maximal Bergman projection of the $L^p(dA)$-function

$$
\sum_{m=0}^{n}(1-|z|)^m|f^{(m)}(z)|.
$$

As a conclusion, the $L^p(dA)$-norm of (2.14) is bounded by a constant times $\|f\|_p$, proving that $T_a : A^p \to A^p$ is bounded, since the polynomials form a dense subspace of A^p. ∎

3. Toeplitz operators as multipliers. In case $p = 2$ and for radial symbols it was proven in [8] that the Toeplitz operator T_a is unitarily equivalent to the ℓ^2-multiplier operator. We provide here a partial generalization for the case $p \neq 2$, though a complete analogue is not possible due to the fact that the monomials do not form an unconditional Schauder basis in this case. However, it is still possible to decompose the weighted Bergman space into finite-dimensional blocks such that the space is the ℓ^p-sum of the blocks, with some consequences for the boundedness and compactness properties of T_a (see Theorem 3.3). This approach has been taken by the first named author in a series of papers (see [13, 14, 15]). Moreover, our methods work in Bergman spaces with quite general radial weights: we use the weighted norms $\|\cdot\|_{p,\mu}$ as defined in (1.3).

We start with some easy observations. For all k, the quantity

$$\gamma_k = 2\int_0^1 (k+1)r^{2k+1}a(r)\,dr \tag{3.1}$$

is well-defined, since $a(r) \in L^1(0,1)$ by our basic assumptions (see Section 1).

Proposition 3.1. *Let $1 < p < \infty$. If the Toeplitz operator $T_a : A^p_\mu \to A^p_\mu$ is bounded, then it is a coefficient multiplier, i.e.,*

$$(T_a f)(z) = \sum_{n\in\mathbb{N}} \gamma_n f_n z^n, \tag{3.2}$$

and moreover

$$\sup_{k\in\mathbb{N}} |\gamma_k| \le \|T_a\| < \infty, \tag{3.3}$$

where $\|T_a\|$ is the operator norm.

Proof. We first recall that the Taylor series

$$\frac{1}{(1-z\bar{w})^2} = \sum_{n=0}^{\infty}(n+1)z^n r^n e^{-in\theta}$$

of the Bergman kernel converges uniformly for z and $w = re^{i\theta}$ in compact subsets of $\mathbb{D}$. For $f(w) = \sum_n f_n r^n e^{-in\theta} \in A^p_\mu$ we thus get

$$\begin{aligned} T_a f(z) &= \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(k+1)z^k f_n \int_0^1\int_0^{2\pi} r^{n+k}e^{-ik\theta}e^{in\theta}a(r)r\,d\theta\,dr \\ &= \sum_{n=0}^{\infty} f_n z^n \int_0^1 2(n+1)r^{2n+1}a(r)\,dr, \end{aligned}$$

i.e. (3.2) holds. Furthermore, for any k, define $h_k(z) = C_k z^k$, where $C_k > 0$ is chosen such that $\|h_k\|_{p,\mu} = 1$. Since $T_a h_k = \gamma_k h_k$, we obtain $|\gamma_k| \le \|T_a\|$ for all k. ∎

If f is a polynomial, then $T_a f$ is well-defined by the integral formula for all radial $a \in L^1(\mathbb{D})$. We remark that in this case (3.2) holds irrespective of whether T_a is a bounded operator or not.

An interesting consequence of the proposition is the following:

THEOREM 3.2. *If $T_a : A^p_\mu \to A^p_\mu$ is bounded for some $1 < p < \infty$, then T_a is bounded $A^2_\mu \to A^2_\mu$.*

This will follow from (3.8), Theorem 3.3(a) and Proposition 4.1(ii) below. In the unweighted case this already follows from [22, Corollary 6.1.2]: it was proven there that (3.3) is sufficient for the boundedness of $T_a : A^2 \to A^2$.

The main result, Theorem 3.3, is based on a natural decomposition of A^p_μ into finite-dimensional subspaces spanned by monomials of degrees in the intervals $\mathbb{N}_n := \mathbb{N} \cap]m_n, m_{n+1}]$, where $(m_n)_{n=1}^\infty$ is a positive strictly increasing sequence to be defined shortly. For analytic functions $f(z) = \sum_{k\in\mathbb{N}} f_k z^k$ on $\mathbb{D}$ we define

$$Q_n f(z) = \sum_{k\in\mathbb{N}_n} f_k z^k, \quad P_n = \sum_{k=1}^{n} Q_k, \qquad n = 1, 2, \ldots, \tag{3.4}$$

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}, \qquad r \in [0, 1[. \tag{3.5}$$

If f is analytic in a neighbourhood of the closed unit disc, say a polynomial, then the quantity $M_p(f, 1)$ can be defined by putting $r = 1$ in (3.5). Moreover, for polynomials $h = \sum_k h_k z^k$ we can define

$$\mathcal{M}_p(h) = \sup_{\substack{M_p(f,1)\le 1 \\ f \text{ polynomial}}} M_p(h * f, 1), \tag{3.6}$$

where

$$h * f = \sum_k h_k f_k z^k. \tag{3.7}$$

Finally, for all n let τ_n be the polynomial

$$\tau_n(z) = \sum_{k\in\mathbb{N}_n} \gamma_k z^k. \tag{3.8}$$

So, if we denote by $A^{(n)}$ the finite-dimensional space of the polynomials $f = \sum_{k\in\mathbb{N}_n} f_k z^k$, then the mapping Q_n is the canonical projection, say, from A^p_μ onto $A^{(n)}$. Moreover, if $f \in A^{(n)}$, then $T_a f = T_a^{(n)} f = \tau_n * f$; see (1.2) and the remark after Proposition 3.1.

THEOREM 3.3. (a) *$T_a : A^p_\mu \to A^p_\mu$ is bounded if and only if*

$$\sup_{n\in\mathbb{N}} \mathcal{M}_p(\tau_n) < \infty. \tag{3.9}$$

Moreover, if T_a is bounded, then its operator norm satisfies, for some constant $C > 0$,

$$\frac{1}{C}\|T_a\| \le \sup_{n\in\mathbb{N}} \mathcal{M}_p(\tau_n) \le C\|T_a\|.$$

(b) $T_a : A^p_\mu \to A^p_\mu$ *is compact if and only if*

$$\lim_{n\to\infty} \mathcal{M}_p(\tau_n) = 0. \tag{3.10}$$

Proof of Theorem 1.1. Working out the definitions of $\mathcal{M}_p$ and τ_n one finds that the operator norm of the multiplier $T_a^{(n)}$ on H^p is equal to $\mathcal{M}_p(\tau_n)$. So Theorem 3.3 is a reformulation of Theorem 1.1. ∎

The proof of Theorem 3.3 will be given in Section 4. The rest of the current section is devoted to the study of the finite-dimensional decomposition of the weighted Bergman space. First we need to specify the numerical sequence appearing in the above definitions.

Definition 3.4. Let $c_p > 0$ be a constant such that the bound $M_p(P_n f, r) \le c_p M_p(f, r)$ holds for all $0 < r < 1$, n and $f \in A^p_\mu$ (see [24]). We fix a number

$$b > 16 \cdot 3^{p-1}(1 + 2^p)c_p^p + 2$$

and set $m_1 = 0$, and assume that for some $n \ge 1$ the numbers $m_1 < \cdots < m_n$ and $0 =: s_0 < s_1 < \cdots < s_{n-1} < 1$ have been chosen. Let s_n be such that

$$\int_0^{s_n} r^{m_n p} r\,d\mu = b \int_{s_n}^1 r^{m_n p} r\,d\mu. \tag{3.11}$$

This is possible by what was assumed on $d\mu$ (see (1.3)); moreover, when choosing s_1, we may require by possibly increasing b that

$$s_1 \ge \max\{1/2, 1 - 1/p\}, \quad \text{hence} \quad s_1^p \ge 1/4. \tag{3.12}$$

Then find $m_{n+1} > m_n$ such that

$$\int_0^{s_n} r^{m_{n+1} p} r\,d\mu = \frac{1}{b} \int_{s_n}^1 r^{m_{n+1} p} r\,d\mu. \tag{3.13}$$

We define

$$\omega_n := \left(\int_0^{s_n} \left(\frac{r}{s_n}\right)^{m_n p} r\,d\mu + \int_{s_n}^1 \left(\frac{r}{s_n}\right)^{m_{n+1} p} r\,d\mu \right)^{1/p}. \tag{3.14}$$

The following result shows that the weighted Bergman space is an ℓ^p-sum of finite-dimensional subspaces; this is a substitute for the property that the monomials form an orthogonal basis of A^2.

THEOREM 3.5. *If every space $A^{(n)} = Q_n(A^p_\mu)$, $n \in \mathbb{N}$, is endowed with the norm*

$$\|g\|_{p,\mu,n} := \omega_n M_p(g, s_n) = \omega_n \left(\frac{1}{2\pi} \int_0^{2\pi} |g(s_n e^{i\theta})|^p \, d\theta \right)^{1/p}, \tag{3.15}$$

then A^p_μ is the ℓ^p-sum of the spaces $A^{(n)}$; more precisely, there are constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{p,\mu} \le \Big(\sum_{n=1}^{\infty} \|Q_n f\|^p_{p,\mu,n} \Big)^{1/p} \le C_2 \|f\|_{p,\mu} \tag{3.16}$$

for all $f \in A^p_\mu$.

For the proof we first provide a collection of lemmas. The symbol $[s]$ denotes the largest integer not larger than $s \in \mathbb{R}$.

LEMMA 3.6. *If $n > 0$, $0 < m < M$, $0 < r \le s$, and $g = \sum_{k=0}^{[n]} g_k z^k$ and $h = \sum_{k=[m]+1}^{M} h_k z^k$, then*

$$\left(\frac{r}{s}\right)^n M_p(g, s) \le M_p(g, r) \quad \text{and} \quad M_p(h, r) \le \left(\frac{r}{s}\right)^m M_p(h, s). \tag{3.17}$$

This follows from the monotonicity of $M_p(f, r)$ as a function of r.

LEMMA 3.7. *For all $g \in A^{(n)} = Q_n(A^p_\mu)$,*

$$\int_0^1 M^p_p(g, r) r \, d\mu \le \omega_n^p M^p_p(g, s_n) \le b \int_0^1 M^p_p(g, r) r \, d\mu. \tag{3.18}$$

Proof. One can estimate, using Lemma 3.6 and then (3.11), (3.13),

$$\begin{aligned}
\int_0^1 M^p_p(g, r) r \, d\mu &\le M^p_p(g, s_n) \int_0^{s_n} \left(\frac{r}{s_n}\right)^{m_n p} r \, d\mu + M^p_p(g, s_n) \int_{s_n}^1 \left(\frac{r}{s_n}\right)^{m_{n+1} p} r \, d\mu \\
&\le M^p_p(g, s_n) b \int_{s_n}^1 \left(\frac{r}{s_n}\right)^{m_n p} r \, d\mu + M^p_p(g, s_n) b \int_0^{s_n} \left(\frac{r}{s_n}\right)^{m_{n+1} p} r \, d\mu \\
&\le b \int_{s_n}^1 M^p_p(g, r) r \, d\mu + b \int_0^{s_n} M^p_p(g, r) r \, d\mu \le b \int_0^1 M^p_p(g, r) r \, d\mu. \ \blacksquare
\end{aligned}$$

LEMMA 3.8. *If $g \in A^{(n)}$, then*

$$\int_0^1 M^p_p(g, r) r \, d\mu \le \frac{b}{b-2} \int_{s_{n-1}}^{s_{n+1}} M^p_p(g, r) r \, d\mu. \tag{3.19}$$

Proof. Again by (3.11) and (3.13),

$$\int_0^{s_{n-1}} M_p^p(g,r)r\,d\mu \le M_p^p(g,s_{n-1}) \int_0^{s_{n-1}} \left(\frac{r}{s_{n-1}}\right)^{m_n p} r\,d\mu$$
$$\le M_p^p(g,s_{n-1})\frac{1}{b}\int_{s_{n-1}}^1 \left(\frac{r}{s_{n-1}}\right)^{m_n p} r\,d\mu \le \frac{1}{b}\int_{s_{n-1}}^1 M_p^p(g,r)r\,d\mu,$$

hence

$$\frac{b}{b+1}\int_0^1 M_p^p(g,r)r\,d\mu \le \int_{s_{n-1}}^1 M_p^p(g,r)r\,d\mu. \tag{3.20}$$

Moreover,

$$\int_{s_{n+1}}^1 M_p^p(g,r)r\,d\mu \le M_p^p(g,s_{n+1}) \int_{s_{n+1}}^1 \left(\frac{r}{s_{n+1}}\right)^{m_{n+1}p} r\,d\mu \tag{3.21}$$
$$\le \frac{1}{b}\int_0^{s_{n+1}} M_p^p(g,r)r\,d\mu \le \frac{1}{b}\int_0^1 M_p^p(g,r)r\,d\mu.$$

As a consequence,

$$\int_{s_{n-1}}^{s_{n+1}} M_p^p(g,r)r\,d\mu = \int_{s_{n-1}}^1 M_p^p(g,r)r\,d\mu - \int_{s_{n+1}}^1 M_p^p(g,r)r\,d\mu \tag{3.22}$$
$$\ge \left(\frac{b}{b+1}-\frac{1}{b}\right)\int_0^1 M_p^p(g,r)r\,d\mu \ge \frac{b-2}{b}\int_0^1 M_p^p(g,r)r\,d\mu. \quad \blacksquare$$

Corollary 3.9. *If $f \in A_\mu^p$, then*

$$\int_0^{s_n} M_p^p((\mathrm{id}-P_n)f,r)r\,d\mu \le \frac{8}{b-2}\int_{s_n}^{s_{n+2}} M_p^p((\mathrm{id}-P_n)f,r)r\,d\mu \tag{3.23}$$

for $n \ge 1$, and

$$\int_{s_{n-1}}^1 M_p^p(P_{n-2}f,r)r\,d\mu \le \frac{8}{b-2}\int_{s_{n-3}}^{s_{n-1}} M_p^p(P_{n-2}f,r)r\,d\mu \tag{3.24}$$

for $n \ge 4$.

Proof. We may assume that f is a polynomial. An application of Lemma 3.8 to the functions $g(z) = z^{[m_{n+1}]+1}$ and $h(z) = z^{[m_{n-1}]}$ yields, since $g \in A^{(n+1)}$ and $h \in A^{(n-2)}$,

$$\begin{aligned}
\int_0^{s_n} r^{(m_{n+1}+1)p} r\,d\mu &\le \int_0^{s_n} r^{([m_{n+1}]+1)p} r\,d\mu \\
&\le \int_0^1 r^{([m_{n+1}]+1)p} r\,d\mu - \int_{s_n}^{s_{n+2}} r^{([m_{n+1}]+1)p} r\,d\mu \\
&\le \left(\frac{b}{b-2}-1\right)\int_{s_n}^{s_{n+2}} r^{([m_{n+1}]+1)p} r\,d\mu \\
&\le \frac{2s_n^{-p}}{b-2}\int_{s_n}^{s_{n+2}} r^{(m_{n+1}+1)p} r\,d\mu \le \frac{8}{b-2}\int_{s_n}^{s_{n+2}} r^{(m_{n+1}+1)p} r\,d\mu
\end{aligned}$$

due to (3.12), and similarly, for $n \ge 4$,

$$\begin{aligned}
\int_{s_{n-1}}^1 r^{m_{n-1}p} r\,d\mu &\le \int_{s_{n-1}}^1 r^{[m_{n-1}]p} r\,d\mu \le \int_0^1 r^{[m_{n-1}]p} r\,d\mu - \int_{s_{n-3}}^{s_{n-1}} r^{[m_{n-1}]p} r\,d\mu \\
&\le \frac{2}{b-2}\int_{s_{n-3}}^{s_{n-1}} r^{[m_{n-1}]p} r\,d\mu \le \frac{2s_1^{-p}}{b-2}\int_{s_{n-3}}^{s_{n-1}} r^{m_{n-1}p} r\,d\mu \\
&\le \frac{8}{b-2}\int_{s_{n-3}}^{s_{n-1}} r^{m_{n-1}p} r\,d\mu.
\end{aligned}$$

(In case $n = 3$ the second to last estimate does not hold.) We obtain, using Lemma 3.6 two times in both cases,

$$\begin{aligned}
\int_0^{s_n} M_p^p((\mathrm{id}-P_n)f,r) r\,d\mu &\le M_p^p((\mathrm{id}-P_n)f,s_n)\int_0^{s_n}\left(\frac{r}{s_n}\right)^{(m_{n+1}+1)p} r\,d\mu \\
&\le \frac{8}{b-2} M_p^p((\mathrm{id}-P_n)f,s_n)\int_{s_n}^{s_{n+2}}\left(\frac{r}{s_n}\right)^{(m_{n+1}+1)p} r\,d\mu \\
&\le \frac{8}{b-2}\int_{s_n}^{s_{n+2}} M_p^p((\mathrm{id}-P_n)f,r) r\,d\mu
\end{aligned}$$

and

$$\begin{aligned}
\int_{s_{n-1}}^1 M_p^p(P_{n-2}f,r) r\,d\mu &\le M_p^p(P_{n-2}f,s_{n-1})\int_{s_{n-1}}^1\left(\frac{r}{s_{n-1}}\right)^{m_{n-1}p} r\,d\mu \\
&\le \frac{8}{b-2} M_p^p(P_{n-2}f,s_{n-1})\int_{s_{n-3}}^{s_{n-1}}\left(\frac{r}{s_{n-1}}\right)^{m_{n-1}p} r\,d\mu \\
&\le \frac{8}{b-2}\int_{s_{n-3}}^{s_{n-1}} M_p^p(P_{n-2}f,r) r\,d\mu. \blacksquare
\end{aligned}$$

Proof of Theorem 3.5. Using Lemma 3.8 and $M_p(Q_nf,r)\le 2c_pM_p(f,r)$ (see the beginning of Definition 3.4) we get

$$\sum_{n=1}^{\infty}\frac{b^2}{b-2}\int\limits_{s_{n-1}}^{s_{n+1}}M_p^p(Q_nf,r)r\,d\mu\le\sum_{n=1}^{\infty}\frac{2c_pb^2}{b-2}\int\limits_{s_{n-1}}^{s_{n+1}}M_p^p(f,r)r\,d\mu \le \frac{4c_pb^2}{b-2}\int\limits_0^1 M_p^p(f,r)r\,d\mu. \tag{3.25}$$

Use of Lemmas 3.7 and 3.8 yields

$$\int\limits_{s_{n-1}}^{s_{n+1}}M_p^p(Q_nf,r)r\,d\mu\le M_p^p(Q_nf,s_n)\omega_n^p \le\frac{b^2}{b-2}\int\limits_{s_{n-1}}^{s_{n+1}}M_p^p(Q_nf,r)r\,d\mu, \tag{3.26}$$

so combining this with (3.25) proves the second inequality of the theorem with $C_2=((4c_pb^2)/(b-2))^{1/p}$.

To prove the first inequality, consider $f=\sum_{n=1}^{\infty}Q_nf$; we may and do assume that $Q_1f=P_1f=0$, since Q_1 is a bounded finite rank projection which commutes with the other Q_k. Denoting by $1_{[a,b]}:\mathbb{R}\to\{0,1\}$ the characteristic function of the interval $[a,b]$, we put, for $z=re^{i\theta}$,

$$f_1(z)=\sum_{n=1}^{\infty}(Q_nf)(z)1_{[s_{n-1},s_{n+1}[}(r),\qquad f_2(z)=\sum_{n=1}^{\infty}(Q_nf)(z)1_{[0,s_{n-1}[}(r),$$

$$f_3(z)=\sum_{n=1}^{\infty}(Q_nf)(z)1_{[s_{n+1},1[}(r)\qquad\text{with } f=f_1+f_2+f_3.$$

We have

$$f_2=\sum_{k=1}^{\infty}\Big(\sum_{n=k+1}^{\infty}Q_nf\Big)1_{[s_{k-1},s_k[}=\sum_{k=1}^{\infty}((\mathrm{id}-P_k)f)1_{[s_{k-1},s_k[}$$

and

$$f_3=\sum_{k=3}^{\infty}\Big(\sum_{n=1}^{k-2}Q_nf\Big)1_{[s_{k-1},s_k[}=\sum_{k=3}^{\infty}(P_{k-2}f)1_{[s_{k-1},s_k[}.$$

We obtain

$$\int\limits_0^1 M_p^p(f,r)r\,d\mu \le 3^{p-1}\Big(\int\limits_0^1 M_p^p(f_1,r)r\,d\mu+\int\limits_0^1 M_p^p(f_2,r)r\,d\mu+\int\limits_0^1 M_p^p(f_3,r)r\,d\mu\Big). \tag{3.27}$$

Now $\int_0^1 M_p^p(f_1,r)r\,d\mu=\sum_{n=1}^{\infty}\int_{s_{n-1}}^{s_{n+1}}M_p^p(Q_nf,r)r\,d\mu$. Corollary 3.9 yields

$$\begin{aligned}\int_0^1 M_p^p(f_2,r)\,d\mu &= \sum_{k=1}^{\infty}\int_{s_{k-1}}^{s_k} M_p^p((\mathrm{id}-P_k)f,r)r\,d\mu\\ &\le \frac{8}{b-2}\sum_{k=1}^{\infty}\int_{s_k}^{s_{k+2}} M_p^p((\mathrm{id}-P_k)f,r)r\,d\mu\\ &\le \frac{8\cdot 2^p c_p^p}{b-2}\sum_{k=1}^{\infty}\int_{s_k}^{s_{k+2}} M_p^p(f,r)r\,d\mu \le \frac{16\cdot 2^p c_p^p}{b-2}\|f\|_{p,\mu}^p\end{aligned}$$

and (using the assumption $Q_1 f = 0$, so that (3.24) also holds for $n = 3$)

$$\begin{aligned}\int_0^1 M_p^p(f_3,r)\,d\mu &= \sum_{k=3}^{\infty}\int_{s_{k-1}}^{s_k} M_p^p(P_{k-2}f,r)r\,d\mu \le \frac{8}{b-2}\sum_{k=3}^{\infty}\int_{s_{k-3}}^{s_{k-1}} M_p^p(P_{k-2}f,r)r\,d\mu\\ &\le \frac{8c_p^p}{b-2}\sum_{k=3}^{\infty}\int_{s_{k-3}}^{s_{k-1}} M_p^p(f,r)r\,d\mu \le \frac{16c_p^p}{b-2}\|f\|_{p,\mu}^p.\end{aligned}$$

Insert this in (3.27) to get

$$\|f\|_{p,\mu}^p \le 3^{p-1}\sum_{n=1}^{\infty}\int_{s_{n-1}}^{s_{n+1}} M_p^p(Q_n f,r)r\,d\mu + 3^{p-1}\frac{16c_p^p(1+2^p)}{b-2}\|f\|_{p,\mu}^p.$$

By our choice of b we obtain $a := 16c_p^p(1+2^p)3^{p-1}/(b-2) < 1$ and hence

$$(1-a)3^{1-p}\|f\|_{p,\mu}^p \le \sum_{n=1}^{\infty}\int_{s_{n-1}}^{s_{n+1}} M_p^p(Q_n f,r)r\,d\mu.$$

In view of (3.26), the proof is complete. ■

4. Proof of Theorem 3.3. We consider the boundedness statements. Assume (3.9) holds, and let $f \in A_\mu^p$ first be a polynomial. Since T_a is a multiplier, $Q_n T_a f = T_a Q_n f = \tau_n * Q_n f$, by the remark just before Theorem 3.3.

Put $F_n(z) = (Q_n f)(s_n z)$ for $z \in \mathbb{D}$. Then $F_n \in A^{(n)}$ and we obtain

$$\begin{aligned}M_p^p(Q_n T_a f, s_n) = M_p^p(\tau_n * F_n, 1) &\le \mathcal{M}_p^p(\tau_n) M_p^p(F_n,1)\\ &= \mathcal{M}_p^p(\tau_n) M_p^p(Q_n f, s_n),\end{aligned}$$

hence

$$\|Q_n T_a f\|_{p,\mu,n} \le (\sup_l \mathcal{M}_p(\tau_l))\|Q_n f\|_{p,\mu,n}$$

for all n, so Theorem 3.5 yields

$$\|T_a f\|_{p,\mu} \le C(\sup_{n\in\mathbb{N}} \mathcal{M}_p(\tau_n))\|f\|_{p,\mu}$$

for some constant C. This bound also holds for general $f \in A_\mu^p$ due to the density of the polynomials in the Bergman space.

Assume next that the Toeplitz operator is bounded. For all n, let h_n be an analytic function with $M_p(h_n,1)=1$ and $\mathcal{M}_p(\tau_n)=M_p(\tau_n*h_n,1)$. Set

$$\tilde{h}_n=Q_nh_n\in A^{(n)}.$$

Then we have $M_p(\tau_n*h_n,1)=M_p(\tau_n*\tilde{h}_n,1)$, and $M_p(\tilde{h}_n,1)\le C$ for a constant C. Define

$$f(z)=\omega_n^{-1}\tilde{h}_n(s_n^{-1}z)\in A^{(n)},\qquad z\in\mathbb{D}.$$

Then

$$\|f\|_{p,\mu,n}=M_p(\tilde{h}_n,1)\le C,\qquad \|T_af\|_{p,\mu,n}=M_p(\tau_n*\tilde{h}_n,1)=\mathcal{M}_p(\tau_n).$$

Theorem 3.5 yields

$$\|f\|_{p,\mu}\le C',\qquad \|T_af\|_{p,\mu}\ge\mathcal{M}_p(\tau_n)/C'$$

for a constant $C'>0$, hence $\|T_a\|\ge C''\mathcal{M}_p(\tau_n)$ for a constant $C''>0$, for all n. The proof of part (a) is complete.

As for compactness, if (3.10) holds, then T_a can be approximated in the operator norm by finite rank operators (use Theorem 3.5), hence it must be compact. Conversely, if (3.10) fails, we find a constant $C>0$ and an increasing subsequence $(n_l)_{l\in\mathbb{N}}$ such that $\mathcal{M}_p(\tau_{n_l})\ge C$. For each l we thus find a polynomial $f_l\in A^{(n_l)}$ such that

$$M_p(\tau_{n_l}*f_l,1)\ge CM_p(f_l,1)/2. \tag{4.1}$$

Normalizing the polynomials so that $\|f_l\|_{p,\mu}=1$ we find that the unit ball of A^p_μ is not mapped into a precompact set by T_a, since $T_af_l=\tau_{n_l}*f_l\in A^{(n_l)}$ and $\|T_af_l\|_{p,\mu}$ is bounded from below by a positive constant, by Theorem 3.5 and (4.1). ∎

In view of the above results it is useful to consider the functional $\mathcal{M}_p$ in some more detail. We remark that statement (ii) below completes the proof of Theorem 3.2.

Proposition 4.1. *Let $1/p+1/q=1$ and g be a polynomial.*

(i) *We have $\mathcal{M}_p(g)\le M_q(g,1)$. In particular, if $\sup_{n\in\mathbb{N}}M_q(\tau_n,1)<\infty$, then $T_a:A^p_\mu\to A^p_\mu$ is bounded.*
(ii) *If $p=q=2$, then $\mathcal{M}_p(g)=\sup_k|g_k|$, where $g=\sum_k g_kz^k$.*

Proof. For (i),

$$\begin{aligned}\mathcal{M}_p^p(g)&=\sup_{M_p(f,1)\le1}\frac{1}{(2\pi)^2}\int_0^{2\pi}\Big|\int_0^{2\pi}f(e^{i(\theta-\varphi)})g(e^{i\varphi})\,d\theta\Big|^p\,d\varphi\\&\le\sup_{M_p(f,1)\le1}\frac{1}{2\pi}\int_0^{2\pi}M_p^p(f,1)M_q^p(g,1)\,d\varphi=\sup_{M_p(f,1)\le1}M_p^p(f,1)M_q^p(g,1)\\&\le M_q^p(g,1),\end{aligned}$$

and for (ii),

$$\mathcal{M}_2^2(g) = \sup_{\sum |f_n|^2 \le 1} \sum_n |g_n|^2 |f_n|^2 = \sup_{n\in\mathbb{N}} |g_n|^2, \quad \text{where} \quad g = \sum_n g_n z^n. \ \blacksquare$$

EXAMPLE 4.2. We calculate the indices s_n, m_n for $d\mu = dr$ for suitable b. Here $\int_0^s r^m r\, d\mu = s^{m+2}/(m+2)$ and $\int_s^1 r^m r\, d\mu = (1 - s^{m+2})/(m+2)$. Hence, (3.11) means

$$s_n^{m_n p+2} = b(1 - s_n^{m_n p+2}),$$

and we obtain

$$s_n = \left(\frac{b}{b+1}\right)^{1/(m_n p+2)}.$$

Also (3.13) leads to

$$b s_n^{m_{n+1} p+2} = 1 - s_n^{m_{n+1} p+2},$$

hence $s_n^{m_{n+1}p+2} = 1/(b+1)$ or

$$m_{n+1}p + 2 = \frac{\log(1/(b+1))}{\log(s_n)} = \frac{\log(1/(b+1))}{\log(b/(b+1))}(m_n p + 2). \tag{4.2}$$

From these equalities we find the indices m_n and s_n:

$$m_n = \frac{2}{p}(a^{n-1} - 1), \quad s_n = \left(\frac{b}{b+1}\right)^{a^{-n+1}/2} \quad \text{with} \quad a := \frac{\log(b+1)}{\log((b+1)/b)} > 1.$$

Acknowledgements. The second named author was partially supported by the Academy of Finland project "Functional analysis and applications". The authors thank Dr. Antti Vähäkangas (Helsinki) for some discussions on the manuscript.

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Received September 30, 2010
Revised version March 1, 2011 (6998)