The path space of a higher-rank graph

by

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Abstract. We construct a locally compact Hausdorff topology on the path space of a finitely aligned \( k \)-graph \( \Lambda \). We identify the boundary-path space \( \partial \Lambda \) as the spectrum of a commutative \( C^* \)-subalgebra \( D_\Lambda \) of \( C^*(\Lambda) \). Then, using a construction similar to that of Farthing, we construct a finitely aligned \( k \)-graph \( \tilde{\Lambda} \) with no sources in which \( \Lambda \) is embedded, and show that \( \partial \Lambda \) is homeomorphic to a subset of \( \partial \tilde{\Lambda} \). We show that when \( \Lambda \) is row-finite, we can identify \( C^*(\Lambda) \) with a full corner of \( C^*(\tilde{\Lambda}) \), and deduce that \( D_\Lambda \) is isomorphic to a corner of \( D_{\tilde{\Lambda}} \). Lastly, we show that this isomorphism implements the homeomorphism between the boundary-path spaces.

1. Introduction. Cuntz and Krieger’s work \cite{2} on \( C^* \)-algebras associated to \((0,1)\)-matrices and its subsequent interpretation by Enomoto and Watatani \cite{4} were the foundation of the field we now call graph algebras. Directed graphs and their higher-rank analogues provide an intuitive framework for the analysis of this broad class of \( C^* \)-algebras; there is an explicit relationship between the dynamics of a graph and various properties of its associated \( C^* \)-algebra. Kumjian and Pask in \cite{7} introduced higher-rank graphs (or \( k \)-graphs) as analogues of directed graphs in order to study Robertson and Steger’s higher-rank Cuntz–Krieger algebras \cite{18} using the techniques previously developed for directed graphs. Higher-rank graph \( C^* \)-algebras have received a great deal of attention in recent years, not least because they extend the already rich and tractable class of graph \( C^* \)-algebras to include all tensor products of graph \( C^* \)-algebras (and thus many Kirchberg algebras whose \( K_1 \) contains torsion elements \cite{7}), as well as (up to Morita equivalence) the irrational rotation algebras and many other examples of simple \( \text{AT} \)-algebras with real rank zero \cite{8}.

Although the definition of a \( k \)-graph (see Definition 2.1) is not quite as straightforward as that of a directed graph, \( k \)-graphs are a natural generalisation of directed graphs: Kumjian and Pask show in \cite{7} Example 1.3 that 1-graphs are precisely the path-categories of directed graphs. Like directed

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graph $C^*$-algebras, higher-rank graph $C^*$-algebras were first studied using
groupoid techniques. Kumjian and Pask defined the $k$-graph $C^*$-algebra $C^*(\Lambda)$
to be the universal $C^*$-algebra for a set of Cuntz–Krieger relations
among partial isometries associated to paths of the $k$-graph $\Lambda$. Using direct
analysis, they proved a version of the gauge-invariant uniqueness theorem
for $k$-graph algebras. They then constructed a groupoid $G_\Lambda$ from each $k$-
graph $\Lambda$, and used the gauge-invariant uniqueness theorem to prove that
the groupoid $C^*$-algebra $C^*(G_\Lambda)$ is isomorphic to $C^*(\Lambda)$. This allowed them
to make use of Renault’s theory of groupoid $C^*$-algebras to analyse $C^*(\Lambda)$.
The unit space $G_\Lambda^{(0)}$ of $G_\Lambda$, which must be locally compact and Hausdorff,
is a collection of paths in the graph: for a row-finite graph with no sources,$G_\Lambda^{(0)}$ is the collection of infinite paths in $\Lambda$ (the definition of an infinite path
in a $k$-graph is not straightforward, see Remark 2.4). For more complicated
graphs, the infinite paths are replaced with boundary paths (Definition 2.9).

In [12], Raeburn, Sims and Yeend developed a “bare-hands” analysis
of $k$-graph $C^*$-algebras. They found a slightly weaker alternative to the no-
sources hypothesis from Kumjian and Pask’s theorems, called local convexity
(Definition 2.7). The same authors later introduced finitely aligned $k$-graphs
in [13], and gave a direct analysis of their $C^*$-algebras. This remains the
most general class of $k$-graphs to which a $C^*$-algebra has been associated
and studied in detail.

Many results for row-finite directed graphs with no sources can be ex-
tended to arbitrary graphs via a process called desingularisation. Given an
arbitrary directed graph $E$, Drinen and Tomforde show in [3] how to con-
struct a row-finite directed graph $F$ with no sources by adding vertices and
edges to $E$ in such a way that the $C^*$-algebra associated to $F$ contains the
$C^*$-algebra associated to $E$ as a full corner. The modified graph $F$ is now
called a Drinen–Tomforde desingularisation of $E$. Although no analogue of
a Drinen–Tomforde desingularisation is currently available for higher-rank
graphs, Farthing provided a construction in [5] analogous to that in [1] for
removing the sources in a locally convex, row-finite higher-rank graph. The
statements of the results of [5] do not contain the local convexity hypothesis,
but Farthing alerted us to an issue in the proof of [5, Theorem 2.28] (see
Remark 6.2), which arises when the graph is not locally convex.

The goal of this paper is to explore the path spaces of higher-rank graphs
and investigate how these path spaces interact with desingularisation pro-
cedures such as Farthing’s.

In Section 2, we recall the definitions and standard notation for higher-
rank graphs. In Section 3, following the approach of [9], we build a topology
on the path space of a higher-rank graph, and show that the path space is
locally compact and Hausdorff under this topology.
In Section 4, given a finitely aligned $k$-graph $\Lambda$, we construct a $k$-graph $\tilde{\Lambda}$ with no sources which contains a subgraph isomorphic to $\Lambda$. Our construction is modelled on Farthing’s construction in [5], and the reader is directed to [5] for several proofs. The crucial difference is that our construction involves extending elements of the boundary-path space $\partial \Lambda$, whereas Farthing extends paths from a different set $\Lambda \leq \infty$ (see Remark 2.10). Interestingly, although $\partial \Lambda$ and $\Lambda \leq \infty$ are potentially different when $\Lambda$ is row-finite and not locally convex (Proposition 2.12), our construction and Farthing’s yield isomorphic $k$-graphs except in the non-row-finite case (Example 4.10 and Proposition 4.12). We follow Robertson and Sims’ notational refinement [17] of Farthing’s desourcification: we construct a new $k$-graph in which the original $k$-graph is embedded, whereas Farthing’s construction adds bits onto the existing $k$-graph. This simplifies many arguments involving $\tilde{\Lambda}$; however, the main reason for modifying Farthing’s construction is that $\Lambda \leq \infty$ is not as well-behaved topologically as $\partial \Lambda$ (see Remark 3.5), and in particular, no analogue of Theorem 5.1 holds for Farthing’s construction.

In Section 5, we prove that given a row-finite $k$-graph $\Lambda$, there is a natural homeomorphism from the boundary-path space of $\Lambda$ onto the space of infinite paths in $\tilde{\Lambda}$ with range in the embedded copy of $\Lambda$. We provide examples and discussion showing that the topological basis constructed in Section 3 is the one we want.

In Section 6, we recall the definition of the Cuntz–Krieger algebra $C^*(\Lambda)$ of a higher-rank graph $\Lambda$. We show that if $\Lambda$ is a row-finite $k$-graph and $\tilde{\Lambda}$ is the graph with no sources obtained by applying the construction of Section 4 to $\Lambda$, then the embedding of $\Lambda$ in $\tilde{\Lambda}$ induces an isomorphism $\pi$ of $C^*(\Lambda)$ onto a full corner of $C^*(\tilde{\Lambda})$.

Section 7 contains results about the diagonal $C^*$-subalgebra of a $k$-graph $C^*$-algebra: the $C^*$-algebra generated by range projections associated to paths in the $k$-graph. We identify the boundary-path space of a finitely aligned higher-rank graph with the spectrum of its diagonal $C^*$-algebra. We then show that the isomorphism $\pi$ of Section 6 restricts to an isomorphism of diagonals which implements the homeomorphism of Section 5.

2. Preliminaries

Definition 2.1. Given $k \in \mathbb{N}$, a $k$-graph is a pair $(\Lambda, d)$ consisting of a countable category $\Lambda = (\text{Obj}(\Lambda), \text{Mor}(\Lambda), r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the degree map, which satisfies the factorisation property: for every $\lambda \in \text{Mor}(\Lambda)$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \text{Mor}(\Lambda)$ such that $\lambda = \mu \nu$, $d(\mu) = m$ and $d(\nu) = n$. Elements $\lambda \in \text{Mor}(\Lambda)$ are called paths. We follow the usual abuse of notation and write $\lambda \in \Lambda$ to mean $\lambda \in \text{Mor}(\Lambda)$. For $m \in \mathbb{N}^k$ we define $\Lambda^m := \{\lambda \in \Lambda :$
For subsets \( F \subset A \) and \( V \subset \text{Obj}(A) \), we write \( VF := \{ \lambda \in F : r(\lambda) \in V \} \) and \( FV := \{ \lambda \in F : s(\lambda) \in V \} \). If \( V = \{ v \} \), we drop the braces and write \( vF \) and \( Fv \). A morphism between two \( k \)-graphs \((A_1, d_1)\) and \((A_2, d_2)\) is a functor \( f : A_1 \to A_2 \) which respects the degree maps. The factorisation property allows us to identify \( \text{Obj}(A) \) with \( A^0 \). We refer to elements of \( A^0 \) as vertices.

**Remark 2.2.** To visualise a \( k \)-graph we draw its 1-skeleton: a directed graph with vertices \( A^0 \) and edges \( \bigcup_{i=1}^k A^{ei} \). To each edge we assign a colour determined by the edge’s degree. We tend to use 2-graphs for examples, and we draw edges of degree \((1,0)\) as solid lines, and edges of degree \((0,1)\) as dashed lines.

**Example 2.3.** For \( k \in \mathbb{N} \) and \( m \in (\mathbb{N} \cup \{\infty\})^k \), we define \( k \)-graphs \( \Omega_{k,m} \) as follows. Set \( \text{Obj}(\Omega_{k,m}) = \{ p \in \mathbb{N}^k : p_i \leq m_i \text{ for all } i \leq k \} \), \( \text{Mor}(\Omega_{k,m}) = \{ (p,q) : p,q \in \text{Obj}(\Omega_{k,m}) \text{ and } p_i \leq q_i \text{ for all } i \leq k \} \), \( r(p,q) = p \), \( s(p,q) = q \) and \( d(p,q) = q - p \), with composition given by \( (p,q)(q,t) = (p,t) \). If \( m = (\infty)^k \), we drop \( m \) from the subscript and write \( \Omega_k \).

The 1-skeleton of \( \Omega_{2,2} \) is depicted in Figure 1.

![Figure 1. The 2-graph \( \Omega_{2,2} \)](image)

**Remark 2.4.** The graphs \( \Omega_{k,m} \) provide an intuitive model for paths: every path \( \lambda \) of degree \( m \) in a \( k \)-graph \( A \) determines a \( k \)-graph morphism \( x_\lambda : \Omega_{k,m} \to A \). To see this, let \( p,q \in \mathbb{N}^k \) be such that \( p \leq q \leq m \). Define \( x_\lambda(p,q) = \lambda'' \), where \( \lambda = \lambda' \lambda'' \lambda''' \), and \( d(\lambda') = p \), \( d(\lambda'') = q - p \) and \( d(\lambda''') = m - q \). In this way, paths in \( A \) are often identified with the graph morphisms \( x_\lambda : \Omega_{k,m} \to A \). We refer to the segment \( \lambda'' \) of \( \lambda \) (as factorised above) as \( \lambda(p,q) \), and for \( n \leq m \), we refer to the vertex \( r(\lambda(n,m)) = s(\lambda(0,n)) \) as \( \lambda(n) \). By analogy, for \( m \in (\mathbb{N} \cup \{\infty\})^k \) we define \( A^m := \{ x : \Omega_{k,m} \to A : x \text{ is a graph morphism} \} \). For clarity of notation, if \( m = (\infty)^k \) we write \( A^\infty \).

Define \[ W_A := \bigcup_{n \in (\mathbb{N} \cup \{\infty\})^k} A^n. \]
We call $W_{\Lambda}$ the *path space* of $\Lambda$. We drop the subscript when confusion is unlikely.

For $m, n \in \mathbb{N}^k$, we denote by $m \wedge n$ the coordinate-wise minimum, and by $m \vee n$ the coordinate-wise maximum. With no parentheses, $\lor$ and $\land$ take priority over the group operation: $a - b \land c$ means $a - (b \lor c)$.

Since finite and infinite paths are fundamentally different, that one can compose them is not immediately obvious.

**Lemma 2.5** ([19 Proposition 3.0.1.1]). Let $\Lambda$ be a $k$-graph. Suppose $\lambda \in \Lambda$ and suppose that $x \in W$ satisfies $r(x) = s(\lambda)$. Then there exists a unique $k$-graph morphism $\lambda x : \Omega_{k,d(\lambda)+d(x)} \to \Lambda$ such that $(\lambda x)(0, d(\lambda)) = \lambda$ and $(\lambda x)(d(\lambda), n + d(\lambda)) = x(0, n)$ for all $n \leq d(x)$.

**Definition 2.6.** For $\lambda, \mu \in \Lambda$, write

$$A^{\min}(\lambda, \mu) := \{ (\alpha, \beta) \in \Lambda \times \Lambda : \lambda \alpha = \mu \beta, d(\lambda \alpha) = d(\lambda) \lor d(\mu) \}$$

for the collection of pairs which give *minimal common extensions* of $\lambda$ and $\mu$, and denote the set of minimal common extensions by

$$\text{MCE}(\lambda, \mu) := \{ \lambda \alpha : (\alpha, \beta) \in A^{\min}(\lambda, \mu) \} = \{ \mu \beta : (\alpha, \beta) \in A^{\min}(\lambda, \mu) \}.$$

**Definition 2.7.** A $k$-graph $\Lambda$ is *row-finite* if for each $v \in \Lambda^0$ and $m \in \mathbb{N}^k$, the set $v\Lambda^m$ is finite; $\Lambda$ has no sources if $v\Lambda^m \neq \emptyset$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

We say that $\Lambda$ is *finitely aligned* if $A^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$.

As in [12 Definition 3.1], a $k$-graph $\Lambda$ is *locally convex* if for all $v \in \Lambda^0$, all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, all $\lambda \in v\Lambda^e_i$ and all $\mu \in v\Lambda^e_j$, the sets $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are non-empty. Roughly speaking, local convexity stipulates that $\Lambda$ contains no subgraph resembling

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   u
  \downarrow
   v

\lambda
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**Definition 2.8.** For $v \in \Lambda^0$, a subset $E \subset v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$ there exists a $\lambda \in E$ such that $A^{\min}(\lambda, \mu) \neq \emptyset$. We denote the set of all finite exhaustive subsets of $\Lambda$ by $\mathcal{FE}(\Lambda)$.

**Definition 2.9.** An element $x \in W$ is a *boundary path* if for all $n \in \mathbb{N}^k$ with $n \leq d(x)$ and for all $E \in x(n)\mathcal{FE}(\Lambda)$ there exists $m \in \mathbb{N}^k$ such that $x(n, m) \in E$. We write $\partial \Lambda$ for the set of all boundary paths.

Define the set $\Lambda^{\leq \infty}$ as follows. A $k$-graph morphism $x : \Omega_{k,m} \to \Lambda$ is an element of $\Lambda^{\leq \infty}$ if there exists $n_x \leq d(x)$ such that for $n \in \mathbb{N}^k$ satisfying $n_x \leq n \leq d(x)$ and $n_i = d(x)_i$, we have $x(n)\Lambda^{e_i} = \emptyset$. 
Remark 2.10. Raeburn, Sims and Yeend introduced $Λ≤∞$ to construct a non-zero Cuntz–Krieger $Λ$-family [13, Proposition 2.12]. Farthing, Muhly and Yeend introduced $∂Λ$ in [6]; in order to construct a groupoid to which Renault’s theory of groupoid $C^*$-algebras [15] applied, they required a path space which was locally compact and Hausdorff in an appropriate topology, and $Λ≤∞$ did not suffice. The differences between $∂Λ$ and $Λ≤∞$ can be easily seen if $Λ$ contains any infinite receivers (e.g. any path in a 1-graph $Λ$ with source an infinite receiver is an element of $∂Λ \setminus Λ≤∞$), but can even show themselves in the row-finite case if $Λ$ is not locally convex.

Example 2.11. Suppose $Λ$ is the 2-graph with the skeleton pictured below:

Consider the paths $x = x_0x_1\ldots$, and $ω^n = x_0x_1\ldots x_n-1ω_n$ for $n = 0,1,2,\ldots$. Observe that $x ∉ Λ≤∞$: for each $n ∈ \mathbb{N}$, we have $d(x)_2 = 0 = (n,0)_2$, and $x((n,0))Λ^e_2 = v_nΛ^e_2 ≠ ∅$.

We claim that $x ∈ ∂Λ$. Fix $m ∈ \mathbb{N}$ and $E ∈ v_mFE(Λ)$. Since $E$ is exhaustive, for each $n ≥ m$, there exists $λ^n ∈ E$ such that $MCE(λ^n, x_m\ldots x_{n-1}ω_n)$ is non-empty. Since $E$ is finite, it cannot contain $x_m\ldots x_{n-1}ω_n$ for every $n ≥ m$, so it must contain $x_m\ldots x_p$ for some $p ∈ \mathbb{N}$. So $x((m,0),(m+p)) = x_m\ldots x_p$ belongs to $E$.

The 2-graph of Example 2.11 first appeared in Robertson’s honours thesis [16] to illustrate a subtlety arising in Farthing’s procedure [5] for removing sources in $k$-graphs when the $k$-graphs in question are not locally convex. It was for this reason that only locally convex $k$-graphs were considered in the main results of [16, 17].

Proposition 2.12. Suppose $Λ$ is a finitely aligned $k$-graph. Then $Λ≤∞ ⊂ ∂Λ$. If $Λ$ is row-finite and locally convex, then $Λ≤∞ = ∂Λ$.

To prove this we use the following lemma.

Lemma 2.13. Let $Λ$ be a row-finite, locally convex $k$-graph, and suppose that $v ∈ Λ^0$ satisfies $vΛ^{e_i} ≠ ∅$ for some $i ≤ k$. Then $vΛ^{e_i} ∈ vFE(Λ)$.

Proof. Since $Λ$ is row-finite, $vΛ^{e_i}$ is finite. To see that it is exhaustive, let $μ ∈ vΛ$. If $d(μ)_i > 0$, then $g = μ(0,e_i) ∈ vΛ^{e_i}$ implies that $Λ^{min}(μ, g) ≠ ∅$. Suppose that $d(μ)_i = 0$. Let $μ = μ_1\ldots μ_n$ be a factorisation of $μ$ such that $|d(μ_j)| = 1$ for each $j ≤ n$. Since $Λ$ is locally convex, $s(μ)Λ^{e_i} = s(μ_n)Λ^{e_i} ≠ ∅$. 


Fix $g \in s(\mu)A^{e_i}$. Let $f := (\mu g)(0,e_i)$. Then $f \in vA^{e_i}$. Since $d(\mu_i) = 0$, we have $d(\mu g) = d(\mu) \vee d(f)$. Hence $(g, (\mu g)(e_i,d(\mu g))) \in A^{\min}(\mu,f)$ as required.

**Proof of Proposition 2.12.** Fix $x \in A^{\leq \infty}$, $m \leq d(x)$ and $E \in x(m)FE(\Lambda)$. Define $t \in \mathbb{N}^k$ by

$$t_i := \begin{cases} d(x)_i & \text{if } d(x)_i < \infty, \\ \max_{\lambda \in E}(n_x \vee (m + d(\lambda)))_i & \text{if } d(x)_i = \infty. \end{cases}$$

Then $(x(m),t) \in x(m)\Lambda$, so there exists $\lambda \in \Lambda$ such that $A^{\min}(x(m,t),\lambda)$ is non-empty. Let $(\alpha,\beta) \in A^{\min}(x(m,t),\lambda)$. We first show that $d(\alpha) = 0$. Since $x \in A^{\leq \infty}$ and $n_x \leq t \leq d(x)$, if $d(x)_i < \infty$ then $x(t)A^{e_i} = \emptyset$. So for each $i$ such that $d(x)_i < \infty$, we have $d(\alpha)_i = 0$. Now suppose that $d(x)_i = \infty$. Then $d(x(m,t))_i = t_i - m_i \geq d(\lambda)_i$. So $d(x(m,t)\alpha)_i = \max \{d(x(m,t))_i,d(\lambda)_i\} = d(x(m,t))_i$, giving $d(\alpha)_i = 0$. Hence $x(m,t) = \lambda \beta$, so $x(m,m + d(\lambda)) = \lambda$.

Now suppose that $\Lambda$ is row-finite and locally convex. We want to show $\partial \Lambda \subset A^{\leq \infty}$. Fix $x \in \partial \Lambda$, and $n \in \mathbb{N}^k$ such that $n \leq d(x)$ and $n_i = d(x)_i$. It suffices to show that $x(n)A^{e_i} = \emptyset$. Since $n_i = d(x)_i$, we have $x(n)A^{e_i} \notin x(n)FE(\Lambda)$. Lemma 2.13 then implies that $x(n)A^{e_i} = \emptyset$. ■

### 3. Path space topology.

Following the approach of Paterson and Welch in [9], we construct a locally compact Hausdorff topology on the path space $W$ of a finitely aligned $k$-graph $\Lambda$. The *cylinder set* of $\mu \in \Lambda$ is $Z(\mu) := \{\nu \in W : \nu(0,d(\mu)) = \mu\}$. Define $\alpha : W \to \{0,1\}^\Lambda$ by $\alpha(w)(y) = 1$ if $w \in Z(y)$ and 0 otherwise. For a finite subset $G \subset s(\mu)\Lambda$ we define

$$(3.1) \quad Z(\mu \setminus G) := Z(\mu) \setminus \bigcup_{\nu \in G} Z(\mu \nu).$$

Our goals for this section are the following two theorems. The basis we end up with is slightly different to that in [9, Corollary 2.4], revealing a minor oversight of the authors.

**Theorem 3.1.** Let $\Lambda$ be a finitely aligned $k$-graph. Then the collection

$$\left\{Z(\mu \setminus G) : \mu \in \Lambda \text{ and } G \subset \bigcup_{i=1}^k (s(\mu)A^{e_i}) \text{ is finite}\right\}$$

is a base for the initial topology on $W$ induced by $\{\alpha\}$.

**Theorem 3.2.** Let $\Lambda$ be a finitely aligned higher-rank graph. With the topology described in Theorem 3.1, $W$ is a locally compact Hausdorff space.

Let $F$ be a set of paths in a $k$-graph $\Lambda$. A path $\beta \in W$ is a common extension of the paths in $F$ if for each $\mu \in F$, we can write $\beta = \mu \beta_\mu$ for some $\beta_\mu \in W$. If in addition $d(\beta) = \bigvee_{\mu \in F} d(\mu)$, then $\beta$ is a minimal common extension of the paths in $F$. We denote the set of all minimal common
extensions of the paths in $F$ by MCE($F$). Since MCE($\{\mu, \nu\}$) = MCE($\mu, \nu$), this definition is consistent with Definition 2.6.

**Remark 3.3.** If $F \subset A$ is finite, then $\bigcap_{\mu \in F} \mathcal{Z}(\mu) = \bigcup_{\beta \in \text{MCE}(F)} \mathcal{Z}(\beta)$.

**Proof of Theorem 3.1.** We first describe the topology on $\{0, 1\}^A$. Given disjoint finite subsets $F, G \subset A$ and $\mu \in A$, define sets $U_{\mu}^{F, G}$ to be $\{1\}$ if $\mu \in F$, $\{0\}$ if $\mu \in G$ and $\{0, 1\}$ otherwise. Then the sets $N(F, G) := \prod_{\mu \in A} U_{\mu}^{F, G}$, where $F, G$ range over all finite disjoint pairs of subsets of $A$, form a base for the topology on $\{0, 1\}^A$.

Clearly, $\alpha$ is a homeomorphism onto its range, so the sets $\alpha^{-1}(N(F, G))$ are a base for a topology on $W$. Routine calculation shows that

$$\alpha^{-1}(N(F, G)) = \left( \bigcup_{\mu \in \text{MCE}(F)} \mathcal{Z}(\mu) \right) \setminus \left( \bigcup_{\nu \in G} \mathcal{Z}(\nu) \right),$$

so the sets $\mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\mu \nu) = \mathcal{Z}(\mu \setminus G)$ are a base for our topology.

To finish the proof, it suffices to show that for $\mu \in A$, a finite subset $G \subset s(\mu)A$ and $\lambda \in \mathcal{Z}(\mu \setminus G)$, there exist $\alpha \in A$ and a finite $F \subset \bigcup_{i=1}^k (s(\alpha)A^e)$ such that $\lambda \in \mathcal{Z}(\alpha \setminus F) \subset \mathcal{Z}(\mu \setminus G)$. Let $N := (\bigvee_{\nu \in G} d(\mu \nu)) \wedge d(\lambda)$ and $\alpha = \lambda(0, N)$. To define $F$, we first define a set $F_\nu$ associated to each $\nu \in G$, then take $F = \bigcup_{\nu \in G} F_\nu$. Fix $\nu \in G$. We consider the following cases:

1. If $N \geq d(\mu \nu)$ or MCE($\alpha, \mu \nu$) = $\emptyset$, let $F_\nu = \emptyset$.
2. If $N \not\geq d(\mu \nu)$ and MCE($\alpha, \mu \nu$) $\not= \emptyset$, define $F_\nu$ as follows: Since $N \not\geq d(\mu \nu)$, there exists $j_\nu \leq k$ such that $N_{j_\nu} < d(\mu \nu)_{j_\nu}$. Hence each $\gamma \in \text{MCE}(\alpha, \mu \nu)$ satisfies $d(\gamma)_{j_\nu} = (N \vee d(\mu \nu))_{j_\nu} > N_{j_\nu}$. Define $F_\nu = \{\gamma(N, N + e_{j_\nu}) : \gamma \in \text{MCE}(\alpha, \mu \nu)\}$. Since $A$ is finitely aligned, $F_\nu$ is finite.

We now show that $\lambda \in \mathcal{Z}(\alpha \setminus F)$. We have $\lambda \in \mathcal{Z}(\alpha)$ by choice of $\alpha$. If $F = \emptyset$ we are done. If not, then fix $\nu \in G$ such that $F_\nu \not= \emptyset$, and fix $e \in F_\nu$. Then $e = \gamma(N, N + e_{j_\nu})$ for some $\gamma \in \text{MCE}(\alpha, \mu \nu)$. Therefore $d(\lambda)_{j_\nu} = N_{j_\nu} < (N + e_{j_\nu})_{j_\nu} = d(\alpha e)_{j_\nu}$. So $\lambda \not\in \mathcal{Z}(\alpha e)$, hence $\lambda \in \mathcal{Z}(\alpha \setminus F)$.

We now show that $\mathcal{Z}(\alpha \setminus F) \subset \mathcal{Z}(\mu \setminus G)$. Fix $\beta \in \mathcal{Z}(\alpha \setminus F)$. Since $\alpha \in \mathcal{Z}(\mu)$, we have $\beta \in \mathcal{Z}(\mu)$. Fix $\nu \in G$. We show that $\beta \not\in \mathcal{Z}(\mu \nu)$ in cases:

1. Suppose that $N \geq d(\mu \nu)$. Since $\beta \in \mathcal{Z}(\alpha) = \mathcal{Z}(\lambda(0, N))$ and $\lambda \not\in \mathcal{Z}(\mu \nu)$, it follows that $\beta \not\in \mathcal{Z}(\mu \nu)$.
2. If $N \not\geq d(\mu \nu)$, then either
   a) MCE($\alpha, \mu \nu$) = $\emptyset$, in which case $\beta \in \mathcal{Z}(\alpha)$ implies that $\beta \not\in \mathcal{Z}(\mu \nu)$; or
   b) MCE($\alpha, \mu \nu$) $\not= \emptyset$. Then for each $\gamma \in \text{MCE}(\alpha, \mu \nu)$, we know $\beta(N, N + e_{j_\nu}) \not= \gamma(N, N + e_{j_\nu})$. Hence $\beta \not\in \mathcal{Z}(\mu \nu)$. ■
Lemma 3.4. Let \( \{\nu^{(n)}\} \) be a sequence of paths in \( \Lambda \) such that

(i) \( d(\nu^{(n+1)}) \geq d(\nu^{(n)}) \) for all \( n \in \mathbb{N} \),

(ii) \( \nu^{(n+1)}(0,d(\nu^{(n)})) = \nu^{(n)} \) for all \( n \in \mathbb{N} \).

Then there exists a unique \( \omega \in W \) such that \( d(\omega) = \bigvee_{n \in \mathbb{N}} d(\nu^{(n)}) \) and \( \omega(0,d(\nu^{(n)})) = \nu^{(n)} \) for all \( n \in \mathbb{N} \).

Proof. Let \( m = \bigvee_{n \in \mathbb{N}} d(\nu^{(n)}) \in (\mathbb{N} \cup \{\infty\})^k \). Then

\[ (3.2) \quad \text{For } a \in \mathbb{N}^k \text{ with } a \leq m, \text{ there exists } N_a \in \mathbb{N} \text{ such that } d(\nu^{(N_a)}) \geq a. \]

For each \( (p, q) \in \Omega_{k,m} \) apply (3.2) with \( a = q \) and define \( \omega(p,q) = \nu^{(N_q)}(p,q) \).

Routine calculations using (3.2) show that \( \omega : \Omega_{k,m} \to \Lambda \) is a well-defined graph morphism with the required properties. ■

Proof of Theorem 3.2. Fix \( v \in A^0 \). We follow the strategy of [9, Theorem 2.2] to show \( Z(v) \) is compact: since \( \alpha \) is a homeomorphism onto its range, and since \( \{0,1\}^A \) is compact, it suffices to prove that \( \alpha(Z(v)) \) is closed in \( \{0,1\}^A \). Suppose that \( (\omega^{(n)})_{n \in \mathbb{N}} \) is a sequence in \( Z(v) \) converging to \( f \in \{0,1\}^A \). We seek \( \omega \in Z(v) \) such that \( f = \alpha(\omega) \). Define \( A = \{\nu \in \Lambda : \alpha(\omega^{(n)})(\nu) \to 1 \text{ as } n \to \infty\} \). Then \( A \neq \emptyset \) since \( v \in A \). Let \( d(A) := \bigvee_{\nu \in A} d(\nu) \).

Claim 3.2.1. There exists \( \omega \in vA^{d(A)} \) such that:

\[ \bullet \ d(\omega) \geq d(\mu) \text{ for all } \mu \in A, \]
\[ \bullet \ \omega(0,n) \in A \text{ for all } n \in \mathbb{N}^k \text{ with } n \leq d(A). \]

Proof. To define \( \omega \) we construct a sequence of paths and apply Lemma 3.4. We first show that for each pair \( \mu, \nu \in A \), \( \text{MCE}(\mu, \nu) \cap A \) contains exactly one element. Fix \( \mu, \nu \in A \). Then for large enough \( n \), there exist \( \beta^n \in \text{MCE}(\mu, \nu) \) such that \( \omega^n = \beta^n(\omega^n)' \). Since \( \text{MCE}(\mu, \nu) \) is finite, there exists \( M \) such that \( \omega^n = \beta^M(\omega^n)' \) for infinitely many \( n \). Define \( \beta_{\mu,\nu} := \beta^M \).

Then \( \beta_{\mu,\nu} \in A \). For uniqueness, suppose that \( \phi \in \text{MCE}(\mu, \nu) \cap A \). Then for large \( n \) we have \( \beta_{\mu,\nu} = \omega^n(0,d(\mu) \lor d(\nu)) = \phi \).

Since \( A \) is countable, we can list \( A = \{\nu^1, \nu^2, \ldots\} \). Let \( y^1 := \nu^1 \), and iteratively define \( y^n = \beta_{y^{n-1},\nu^n} \). Then \( d(y^n) = d(y^{n-1}) \lor d(\nu^n) \geq d(y^{n-1}) \), and \( y^n(0,y^{n-1}) = y^{n-1} \). By Lemma 3.4 there exists a unique \( \omega \in W \) satisfying \( d(\omega) = d(A) \) and \( \omega(0,d(y^n)) = y^n \) for all \( n \). It then follows from (3.2) that \( \omega(0,n) \in A \) for all \( n \leq d(A) \). ■

To see \( \alpha(Z(v)) \) is closed, fix \( \lambda \in A \). We show that \( \alpha(\omega^{(n)})(\lambda) \to \alpha(\omega)(\lambda) \).

If \( \alpha(\omega)(\lambda) = 1 \), then \( \lambda = \omega(0,d(\lambda)) \in A \) by Claim 3.2.1 and thus \( \alpha(\omega^{(n)})(\lambda) \) approaches 1 as \( n \) approaches \( \infty \). Now suppose that \( \alpha(\omega)(\lambda) = 0 \). If \( d(\lambda) \neq d(\omega) \), then \( \lambda \notin A \) by Claim 3.2.1 forcing \( \alpha(\omega^{(n)})(\lambda) \to 0 \). Suppose that \( d(\lambda) \leq d(\omega) \). Since \( \omega(0,d(\lambda)) \in A \), we have \( \omega^{(n)}(0,d(\lambda)) = \omega(0,d(\lambda)) \) for
large $n$. Then $\alpha(\omega)(\lambda) = 0$ implies that $\omega(0, d(\lambda)) \neq \lambda$. So for large enough $n$ we have $\omega^{(n)}(0, d(\lambda)) \neq \lambda$, forcing $\alpha(\omega^{(n)})(\lambda) \to 0$. ■

**Remark 3.5.** It has been shown that $\partial \Lambda$ is a closed subset of $W$ [6, Lemma 5.12]. Hence $\partial \Lambda$, with the relative topology, is a locally compact Hausdorff space. Consider the 2-graph of Example 2.11. For each $m, n \in \mathbb{N}$, we have $\omega^n \in \Lambda^{\leq \infty}$. For each $n \in \mathbb{N}$, we have $\omega^n \in \Lambda^{\leq \infty}$. Notice that $\omega^n \to x \notin \Lambda^{\leq \infty}$. So $\Lambda^{\leq \infty}$ is not closed in general, and hence is not locally compact.

### 4. Removing sources

**Theorem 4.1.** Let $\Lambda$ be a finitely aligned $k$-graph. Then there exists a finitely aligned $k$-graph $\tilde{\Lambda}$ with no sources, and an embedding $\iota$ of $\Lambda$ in $\tilde{\Lambda}$. If $\Lambda$ is row-finite, then so is $\tilde{\Lambda}$.

**Definition 4.2.** Define a relation $\approx$ on $V_{\Lambda} := \{(x; m) : x \in \partial \Lambda, m \in \mathbb{N}^k\}$ by: $(x; m) \approx (y; p)$ if and only if

(V1) $x(m \wedge d(x)) = y(p \wedge d(y))$,
(V2) $m - m \wedge d(x) = p - p \wedge d(y)$.

**Definition 4.3.** Define a relation $\sim$ on $P_{\Lambda} := \{(x; (m, n)) : x \in \partial \Lambda, m \leq n \in \mathbb{N}^k\}$ by: $(x; (m, n)) \sim (y; (p, q))$ if and only if

(P1) $x(m \wedge d(x), n \wedge d(Px)) = y(p \wedge d(y), q \wedge d(y))$,
(P2) $m - m \wedge d(x) = p - p \wedge d(y)$,
(P3) $n - m = q - p$.

It is clear from the definitions that both $\approx$ and $\sim$ are equivalence relations.

**Lemma 4.4.** Suppose that $(x; (m, n)) \sim (y; (p, q))$. Then $n - n \wedge d(x) = q - q \wedge d(y)$.

**Proof.** It follows from (P1) and (P3) that

$n - n \wedge d(x) - (m - m \wedge d(x)) = q - q \wedge d(y) - (p - p \wedge d(y))$.

The result is then a consequence of (P2). ■

Let $\tilde{P}_{\Lambda} := P_{\Lambda}/\sim$ and $\tilde{V}_{\Lambda} := V_{\Lambda}/\approx$. The class in $\tilde{P}_{\Lambda}$ of $(x; (m, n)) \in P_{\Lambda}$ is denoted $[x; (m, n)]$, and similarly the class in $\tilde{V}_{\Lambda}$ of $(x; m) \in V_{\Lambda}$ is denoted $[x; m]$.

To define the range and source maps, observe that if $(x; (m, n)) \sim (y; (p, q))$, then $(x; m) \approx (y; p)$ by definition of $\sim$, and $(x; n) \approx (y; q)$ by Lemma 4.4. We define range and source maps as follows.

**Definition 4.5.** Define $\tilde{r}, \tilde{s} : \tilde{P}_{\Lambda} \to \tilde{V}_{\Lambda}$ by

$\tilde{r}([x; (m, n)]) = [x; m]$ and $\tilde{s}([x; (m, n)]) = [x, n]$. 
We now define composition. For each \( m \in \mathbb{N}^k \), we define the shift map \( \sigma^m : \bigcup_{n \geq m} \Lambda^n \to \Lambda \) by \( \sigma^m(\lambda)(p, q) = \lambda(p + m, q + m) \).

**Proposition 4.6.** Suppose that \( \Lambda \) is a \( k \)-graph, and that \([x; (m, n)]\) and \([y; (p, q)]\) are elements of \( \tilde{P}_\Lambda \) satisfying \([x; n] = [y; p]\). Let \( z := x(0, n \wedge d(x))\sigma^p\wedge d(y)y \). Then

1. \( z \in \partial \Lambda \),
2. \( m \wedge d(x) = m \wedge d(z) \) and \( n \wedge d(x) = n \wedge d(z) \),
3. \( x(m \wedge d(x), n \wedge d(x)) = z(m \wedge d(z), n \wedge d(z)) \) and \( y(p \wedge d(y), q \wedge d(y)) = z(n \wedge d(z), (n + q - p) \wedge d(z)) \).

**Proof.** Part (1) follows from [6, Lemma 5.13], and (2) and (3) can be proved as in [5, Proposition 2.11].

Fix \([x; (m, n)], [y; (p, q)] \in \tilde{P}_\Lambda \) such that \([x; n] = [y; p]\), and let \( z = x(0, n \wedge d(x))\sigma^p\wedge d(y)y \). That the formula

\[
[x; (m, n)] \circ [y; (p, q)] = [z; (m, n + q - p)]
\]

determines a well-defined composition follows from Proposition 4.6.

Define \( \text{id} : \tilde{\Lambda}_A \to \tilde{P}_\Lambda \) by \( \text{id}_{[x; m]} = [x; (m, m)] \).

**Proposition 4.7 ([5, Lemma 2.19]).** \( \tilde{\Lambda} := (\tilde{\Lambda}_A, \tilde{P}_\Lambda, \tilde{r}, \tilde{s}, \circ, \text{id}) \) is a category.

**Definition 4.8.** Define \( \tilde{d} : \tilde{\Lambda} \to \mathbb{N}^k \) by \( \tilde{d}(v) = \star \) for all \( v \in \tilde{\Lambda}_A \), and \( \tilde{d}([x; (m, n)]) = n - m \) for all \([x; (m, n)] \in \tilde{P}_\Lambda \).

**Proposition 4.9 ([5, Theorem 2.22]).** The map \( \tilde{d} \) defined above has the factorisation property. Hence with \( \tilde{\Lambda} \) as in Proposition 4.7, \( (\tilde{\Lambda}, \tilde{d}) \) is a \( k \)-graph with no sources.

**Example 4.10.** If we allow infinite receivers, our construction yields a different \( k \)-graph to Farthing’s construction in [5, §2]: consider the 1-graph \( E \) with an infinite number of loops \( f_i \) on a single vertex \( v \):

![Diagram](image)

Here we have \( E^{\leq \infty} = \emptyset \), so Farthing’s construction yields a 1-graph \( \overline{E} \cong E \). Since \( v \) belongs to every finite exhaustive set in \( E \), we have \( \partial E = E \). Furthermore \([f_j; p] = [f_i; p] = [v; p] \) for all \( i, j, p \in \mathbb{N} \), and

\([f_j; (p, q)] = [f_i; (p, q)] = [v; (p - 1, q - 1)]\)
for all \(i, j, p, q\) such that \(1 < p \leq q\). Thus there is exactly one path between any two of the added vertices, resulting in a head at \(v\), yielding the graph illustrated below:

It is intriguing that following Drinen and Tomforde’s desingularisation, a head is also added at infinite receivers like this, and then the ranges of the edges \(f_i\) are distributed along this head—we cannot help but wonder whether this might suggest an approach to a Drinen–Tomforde desingularisation for \(k\)-graphs.

4.1. Row-finite 1-graphs. While one expects this style of desourcification to agree with adding heads to a row-finite 1-graph as in [1], this appears not to have been checked anywhere.

**Proposition 4.11.** Let \(E\) be a row-finite directed graph and \(F\) be the graph obtained by adding heads to sources, as in [1, p. 4]. Let \(\Lambda\) be the 1-graph associated to \(E\). Then \(\tilde{\Lambda} \cong F^*\), where \(F^*\) is a the path-category of \(F\).

**Proof.** Define \(\eta' : P_\Lambda \to F^*\) as follows. Fix \(x \in \partial E\) and \(m, n \in \mathbb{N}\). Then either \(x \in E^\infty\), or \(x \in E^*\) and \(s(x)\) is a source in \(E\). If \(x \in E^\infty\), define \(\eta'((x; (m, n))) = x(m, n)\). For \(x \in E^*\), let \(\mu_x\) be the head added to \(s(x)\), and define \(\eta'((x; (m, n))) = (x \mu_x)(m, n)\). It is straightforward to check that \(\eta'\) respects the equivalence relation \(\sim\) on \(P_\Lambda\). Define \(\eta : \tilde{\Lambda} \to F^*\) by \(\eta([x; (m, n)]) = \eta'((x; (m, n)))\). Easy but tedious calculations show that \(\eta\) is a graph morphism.

We now construct a graph morphism \(\xi : F^* \to \tilde{\Lambda}\). Let \(\nu \in F^*\). To define \(\xi\) we first need some preliminary notation. We will define \(\xi\) casewise, broken up as follows:

(i) \(\nu \in E^*\),
(ii) \(r(\nu) \in E^*\) and \(s(\nu) \in F^* \setminus E^*\), or
(iii) \(r(\nu), s(\nu) \in F^* \setminus E^*\).

If \(\nu \in E^*\), fix \(\alpha_\nu \in s(\nu)\partial E\). If \(\nu\) has \(r(\nu) \in E^*\) and \(s(\nu) \in F^* \setminus E^*\), let \(p_\nu = \max\{p \in \mathbb{N} : \nu(0, p) \in E^*\}\). Then \(\nu(p_\nu)\) is a source in \(E^*\), and \(\nu(0, p_\nu) \in \partial E\). If \(\nu \in F^* \setminus E^*\), then \(\nu\) is a segment of a head \(\mu_\nu\) added to a source in \(E^*\), and we let \(q_\nu\) be such that \(\nu = \mu_\nu(q_\nu, q_\nu + d(\mu))\).
We then define $\xi$ by

$$\xi(\nu) = \begin{cases} [\nu; (0, d(\nu))], & \text{if } \nu \in E^*, \\ [\nu(0, p_{\nu}); (0, d(\nu))], & \text{if } r(\nu) \in E^* \text{ and } s(\nu) \notin E^*, \\ [r(\mu_{\nu}); (q_{\nu}, q_{\nu} + d(\nu))], & \text{if } r(\nu), s(\nu) \in E^* \setminus E^*. \end{cases}$$

Again, tedious but straightforward calculations show that $\xi$ is a well-defined graph morphism, and that $\xi \circ \eta = 1_{\tilde{\Lambda}}$ and $\eta \circ \xi = 1_{E^*}$. ■

When $\Lambda$ is row-finite and locally convex, Proposition 2.12 implies that $\Lambda^{\leq \infty} = \partial \Lambda$. In this case our construction is essentially the same as that of Farthing [5, §2], with notation as in [17]. If $\Lambda$ is row-finite but not locally convex, then $\Lambda^{\leq \infty} \subset \partial \Lambda$ (Example 2.11 shows that this may be a strict containment). Thus it is reasonable to suspect that our construction could result in a larger path space than Farthing’s. Interestingly, this is not the case.

**Proposition 4.12.** Let $\Lambda$ be a row-finite $k$-graph. Suppose that $x \in \partial \Lambda \setminus \Lambda^{\leq \infty}$ and $m \leq n \in \mathbb{N}^k$. Then there exists $y \in \Lambda^{\leq \infty}$ such that $(x; (m, n)) \sim (y; (m, n))$.

**Proof.** Since $x \notin \Lambda^{\leq \infty}$, there exists $q \geq n \wedge d(x)$ and $i \leq k$ such that $q \leq d(x)$, $q_i = d(x)_i$, and $x(q)\Lambda^{e_i} \neq \emptyset$. Let

$$J := \{i \leq k : q_i = d(x)_i \text{ and } x(q)\Lambda^{e_i} \neq \emptyset\}.$$

Since $x \in \partial \Lambda$, for each $E \subseteq x(q)\mathcal{FE}(\Lambda)$ there exists $t \in \mathbb{N}^k$ such that $x(q, q + t) \in E$. Since $q_i = d(x)_i$ for all $i \in J$, the set $\bigcup_{i \in J} x(q)\Lambda^{e_i}$ contains no such segments of $x$, and thus cannot be finite exhaustive. Since $\Lambda$ is row-finite, $\bigcup_{i \in J} x(q)\Lambda^{e_i}$ is finite, so $\bigcup_{i \in J} x(q)\Lambda^{e_i}$ is not exhaustive. Thus there exists $\mu \in x(q)\Lambda$ such that $\text{MCE}(\mu, \nu) = \emptyset$ for all $\nu \in \bigcup_{i \in J} x(q)\Lambda^{e_i}$. By [13] Lemma 2.11], $s(\mu)\Lambda^{\leq \infty} \neq \emptyset$. Let $z \in s(\mu)\Lambda^{\leq \infty}$, and define $y := x(0, q)\mu z$. Then $y \in \Lambda^{\leq \infty}$ by [13] Lemma 2.10].

Now we show that $(x; (m, n)) \sim (y; (m, n))$. Condition (F3) is trivially satisfied. To see that (F1) and (F2) hold, it suffices to show that $n \wedge d(x) = n \wedge d(y)$. Firstly, let $i \in J$. If $d(\mu z)_i = 0$, then $(\mu z)(0, d(\mu) + e_i) \in \text{MCE}(\mu, \nu)$ for $\nu = (\mu z)(0, e_i) \in r(\mu)\Lambda^{e_i} = x(q)\Lambda^{e_i}$, a contradiction. So for each $i \in J$, we have $d(\mu z)_i = 0$, and hence $d(y)_i = d(x)_i$. Now suppose that $i \notin J$. Then either $x(q)\Lambda^{e_i} = \emptyset$ or $q_i < d(x)_i$. If $x(q)\Lambda^{e_i} = \emptyset$ then $d(y)_i = d(x)_i$. So suppose that $q_i < d(x)_i$. Since $n \wedge d(x) \leq q$, it follows that $n_i < d(x)_i$ and $n_i \leq q_i \leq d(y)_i$, hence $(n \wedge d(x))_i = n_i = (n \wedge d(y))_i$. So $n \wedge d(x) = n \wedge d(y)$. ■

The following result allows us to identify $\Lambda$ with a subgraph of $\tilde{\Lambda}$.

**Proposition 4.13.** Suppose that $\Lambda$ is a $k$-graph, and that $\lambda \in \Lambda$. Then $s(\lambda)\partial \Lambda \neq \emptyset$. If $x, y \in s(\lambda)\partial \Lambda$, then $\lambda x, \lambda y \in \partial \Lambda$ and $(\lambda x; (0, d(\lambda))) \sim (\lambda y; (0, d(\lambda)))$. Moreover, there is an injective $k$-graph morphism $\iota : \Lambda \to \tilde{\Lambda}$.
such that for $\lambda \in \Lambda$,

$$\iota(\lambda) = [\lambda x; (0, d(\lambda))] \quad \text{for any } x \in s(\lambda)\partial \Lambda.$$  

Proof. By [6, Lemma 5.15], we have $v\partial \Lambda \neq \emptyset$ for all $v \in \Lambda^0$. In particular, $s(\lambda)\partial \Lambda \neq \emptyset$. Let $x, y \in s(\lambda)\partial \Lambda$. Then [6, Lemma 5.13(ii)] says that $\lambda x, \lambda y \in \partial \Lambda$. It follows from the definition of $\sim$ that $(\lambda x; (0, d(\lambda))) \sim (\lambda y; (0, d(\lambda)))$.

Then straightforward calculations show that $\iota$ is an injective $k$-graph morphism.

We want to extend $\iota$ to an injection of $W_\Lambda$ into $W_{\tilde{\Lambda}}$. The next proposition shows that any injective $k$-graph morphism defined on $\Lambda$ can be extended to $W_\Lambda$.

**Proposition 4.14.** Let $\Lambda, \Gamma$ be $k$-graphs and $\phi : \Lambda \to \Gamma$ be a $k$-graph morphism. Let $x \in W_\Lambda \setminus \Lambda$. Then $\phi(x) : \Omega_{k,d(x)} \to W_\Gamma$ defined by $\phi(x)(p, q) = \phi(x(p, q))$ belongs to $W_\Gamma$.

Proof. This follows from $\phi$ being a $k$-graph morphism.

In particular, we can extend $\iota$ to paths with non-finite degree. We need to know that composition works as expected for non-finite paths.

**Proposition 4.15.** Let $\Lambda, \Gamma$ be $k$-graphs and $\phi : \Lambda \to \Gamma$ be a $k$-graph morphism. Let $\lambda \in \Lambda, x \in s(\lambda)W_\Lambda$, and suppose that $n \in \mathbb{N}^k$ satisfies $n \leq d(x)$. Then

1. $\phi(\lambda)\phi(x) = \phi(\lambda x),$
2. $\sigma^n(\phi(x)) = \phi(\sigma^n(x)).$

Proof. Again this follows from $\phi$ being a $k$-graph morphism.

**Remark 4.16.** We deduce that the extension of an injective $k$-graph morphism to $W_\Lambda$ is also injective. In particular, the map $\iota : \Lambda \to \tilde{\Lambda}$ has an injective extension $\iota : W_\Lambda \to W_{\tilde{\Lambda}}$.

We need to be able to ‘project’ paths from $\tilde{\Lambda}$ onto the embedding $\iota(\Lambda)$ of $\Lambda$. For $y \in \partial \Lambda$ define

$$\pi([y; (m, n)]) = [y; (m \land d(y), n \land d(y))].$$

Straightforward calculations show that $\pi$ is a surjective functor, and is a projection in the sense that $\pi(\pi([y; (m, n)])) = \pi([y; (m, n)])$ for all $[y; (m, n)] \in \tilde{\Lambda}$. In particular, $\pi|_{\iota(\Lambda)} = \text{id}_{\iota(\Lambda)}$.

**Lemma 4.17.** Let $\Lambda$ be a $k$-graph. Suppose that $\lambda, \mu \in \tilde{\Lambda}$, and that $\lambda \in \mathcal{Z}(\mu)$. Then $\pi(\lambda) \in \mathcal{Z}(\pi(\mu))$. If $d(\pi(\lambda))_i > d(\pi(\mu))_i$ for some $i \leq k$, then $d(\mu)_i = d(\pi(\mu))_i$. 

Proof. Write \( \lambda = [x; (m, m + d(\lambda))] \). Then \( \mu = [x; (m, m + d(\mu))] \), so
\[
\pi(\lambda) = [x; (m \land d(x), (m + d(\lambda)) \land d(x))], \\
\pi(\mu) = [x; (m \land d(x), (m + d(\mu)) \land d(x))].
\]
Since \( d(\lambda) \geq d(\mu) \), it follows that \( \pi(\lambda) \in \mathcal{Z}(\pi(\mu)) \).

If \( d(\pi(\lambda))_i > d(\pi(\mu))_i \), then \( d(x)_i > m_i + d(\mu)_i \), so
\[
d(\pi(\mu))_i = m_i + d(\mu)_i - m_i = d(\mu)_i.
\]

Lemma 4.18. Let \( \Lambda \) be a \( k \)-graph and \( \mu, \nu \in \tilde{\Lambda} \). Then
\[
\pi(\text{MCE}(\mu, \nu)) \subset \text{MCE}(\pi(\mu), \pi(\nu)).
\]

Proof. Suppose that \( \lambda \in \text{MCE}(\mu, \nu) \). By Lemma 4.17, we have \( \pi(\lambda) \in \mathcal{Z}(\pi(\mu)) \cap \mathcal{Z}(\pi(\nu)) \), hence \( d(\pi(\lambda)) \geq d(\pi(\mu)) \lor d(\pi(\nu)) \).

It remains to prove that \( d(\pi(\lambda)) = d(\pi(\mu)) \lor d(\pi(\nu)) \). Suppose, for a contradiction, that \( i \leq k \) is such that \( d(\pi(\lambda))_i > \max\{d(\pi(\mu))_i, d(\pi(\nu))_i\} \).

By Lemma 4.17, we then have \( d(\pi(\mu))_i = d(\mu)_i \) and \( d(\pi(\nu))_i = d(\nu)_i \). So \( d(\lambda)_i \geq d(\pi(\lambda))_i \), contradicting \( \lambda \in \text{MCE}(\mu, \nu) \).

Lemma 4.19. Let \( \Lambda \) be a \( k \)-graph, and let \( \mu, \lambda \in \nu(\Lambda^0)\tilde{\Lambda} \) be such that \( d(\lambda) = d(\mu) \) and \( \pi(\lambda) = \pi(\mu) \). Then \( \lambda = \mu \).

Proof. Since \( \mu, \lambda \in \nu(\Lambda^0)\tilde{\Lambda} \) and \( d(\lambda) = d(\mu) \), we can write \( \lambda = [x; (0, n)] \) and \( \mu = [y; (0, n)] \) for some \( x, y \in \partial \Lambda \) and \( n \in \mathbb{N}^k \). We will show that \( (x; (0, n)) \sim (y; (0, n)) \). Conditions (H2) and (H3) are trivially satisfied. Since
\[
[x; (0, n \land d(x))] = \pi(\lambda) = \pi(\mu) = [y; (0, n \land d(y))],
\]
we have \( (x; (0, n \land d(x))) \sim (y; (0, n \land d(y))) \). Hence \( x(0, n \land d(x)) = y(0, n \land d(y)) \), and (H1) is satisfied.

Proof of Theorem 4.1. The existence of \( \tilde{\Lambda} \) follows from Proposition 4.9 and the embedding from Proposition 4.13.

To check that \( \tilde{\Lambda} \) is finitely aligned, fix \( \mu, \nu \in \tilde{\Lambda} \), and \( \alpha \in \nu(\Lambda^0)\tilde{\Lambda}r(\mu) \). Then \( |\text{MCE}(\mu, \nu)| = |\text{MCE}(\alpha\mu, \alpha\nu)| \). We know that \( |\text{MCE}(\pi(\alpha\mu), \pi(\alpha\nu))| \) is finite since \( \Lambda \) is finitely aligned. We will show that \( |\text{MCE}(\alpha\mu, \alpha\nu)| = |\text{MCE}(\pi(\alpha\mu), \pi(\alpha\nu))| \).

It follows from Lemma 4.18 that
\[
|\text{MCE}(\alpha\mu, \alpha\nu)| \geq |\text{MCE}(\pi(\alpha\mu), \pi(\alpha\nu))|.
\]

For the opposite inequality, suppose that \( \lambda = \beta \) are distinct elements of \( \text{MCE}(\alpha\mu, \alpha\nu) \). Then \( d(\lambda) = d(\beta) \). Since \( r(\alpha\mu), r(\alpha\nu) \in \nu(\Lambda^0) \), Lemma 4.19 implies that \( \pi(\lambda) \neq \pi(\beta) \). So \( |\text{MCE}(\alpha\mu, \alpha\nu)| = |\text{MCE}(\pi(\alpha\mu), \pi(\alpha\nu))| \).

For the last part of the statement, we prove the contrapositive. Suppose that \( \tilde{\Lambda} \) is not row-finite. Let \( [x; m] \in \tilde{\Lambda}^0 \) and \( i \leq k \) be such that \( |[x; m]\tilde{\Lambda}^{e_i}| = \infty \). Then for each \( [y; (n, n + e_i)] \in [x; m]\tilde{\Lambda}^{e_i} \) we have \( [y; n] = \)
Let \([x;m]\), so \([x;(m,m+e_i)] \neq [y;(n,n+e_i)]\) only if (H) fails. That is,
\[
(4.3) \quad x(m \land d(x), (m+e_i) \land d(x)) \neq y(n \land d(y), (n+e_i) \land d(y)).
\]

Since \(|[x;m]|\lambda e_i| = \infty\), there are infinitely many \([y;(n,n+e_i)] \in [x;m]\lambda e_i|\) satisfying (4.3). Hence \([x(m \land d(x))]\lambda e_i| = \infty\). \(\blacksquare\)

**Remark 4.20.** Suppose that \(\Lambda\) is a finitely aligned \(k\)-graph, that \(x \in \partial \Lambda\) and that \(E \subseteq x(0)\Lambda\). Since \(\iota : \Lambda \to \iota(\Lambda)\) is a bijective \(k\)-graph morphism, we have \(E \subseteq x(0)\mathcal{FE}(\Lambda)\) if and only if \(\iota(E) \subseteq [x;0]\mathcal{FE}(\iota(\Lambda))\).

The following results show how sets of minimal common extensions and finite exhaustive sets in a \(k\)-graph \(\Lambda\) relate to those in \(\tilde{\Lambda}\).

**Proposition 4.21** ([5 Lemma 2.25]). Suppose that \(\Lambda\) is a finitely aligned \(k\)-graph, and that \(v \in \iota(\Lambda^0)\). Then \(E \subseteq v\mathcal{FE}(\iota(\Lambda))\) implies that \(E \subseteq v\mathcal{FE}(\tilde{\Lambda})\).

**Lemma 4.22.** Let \(\Lambda\) be a finitely aligned \(k\)-graph and let \(\mu, \nu \in \iota(\Lambda)\). Then \(\text{MCE}_{\iota(\Lambda)}(\mu, \nu) = \text{MCE}_{\tilde{\Lambda}}(\mu, \nu)\).

**Proof.** Since \(\iota(\Lambda) \subseteq \tilde{\Lambda}\), we have \(\text{MCE}_{\iota(\Lambda)}(\mu, \nu) \subseteq \text{MCE}_{\tilde{\Lambda}}(\mu, \nu)\). Suppose that \(\lambda \in \text{MCE}_{\tilde{\Lambda}}(\mu, \nu)\). It suffices to show that \(\lambda \in \iota(\Lambda)\). Write \(\mu = [x;(0,n)], \nu = [y;(0,q)]\) and \(\lambda = [z;(0,n \lor q)]\). Then \(\lambda \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)\) implies that \(d(z) \geq n \lor q\), hence \(\lambda \in \iota(\Lambda)\). \(\blacksquare\)

**Remark 4.23.** Since there is a bijection from \(\Lambda^\min(\mu, \nu)\) onto \(\text{MCE}(\mu, \nu)\), it follows from Lemma 4.22 that \(\tilde{\Lambda}^\min(\mu, \nu) = \iota(\Lambda)^\min(\mu, \nu)\) for all \(\mu, \nu \in \iota(\Lambda)\).

**5. Topology of path spaces under desourcification.** We extend the projection \(\pi\) defined in (4.2) to the set of infinite paths in \(\tilde{\Lambda}\), and prove that its restriction to \(\iota(\Lambda^0)\tilde{\Lambda}^\infty\) is a homeomorphism onto \(\iota(\partial \Lambda)\). For \(x \in \iota(\Lambda^0)\tilde{\Lambda}^\infty\), let \(p_x = \bigvee\{p \in \mathbb{N}^k : (x(0,p) \in \iota(\Lambda))\}\), and define \(\pi(x)\) to be the composition of \(x\) with the inclusion of \(\Omega_{d(x)}\) in \(\partial \Lambda\). Then \(\pi(x)\) is a \(k\)-graph morphism. Our goal for this section is the following theorem.

**Theorem 5.1.** Let \(\Lambda\) be a row-finite \(k\)-graph. Then \(\pi : \iota(\Lambda^0)\tilde{\Lambda}^\infty \to \iota(\partial \Lambda)\) is a homeomorphism.

We first show that the range of \(\pi\) is a subset of \(\iota(\partial \Lambda)\).

**Proposition 5.2.** Let \(\Lambda\) be a finitely aligned \(k\)-graph. Let \(x \in \iota(\Lambda^0)\tilde{\Lambda}^\infty\). Suppose that \(\{y_n : n \in \mathbb{N}^k\} \subseteq \partial \Lambda\) satisfy \([y_n;(0,n)] = x(0,n)\). Then

1. \(\lim_{n \in \mathbb{N}^k} \iota(y_n) = \pi(x)\) in \(W_{\tilde{\Lambda}}\),
2. there exists \(y \in \partial \Lambda\) such that \(\pi(x) = \iota(y)\), and for \(m,n \in \mathbb{N}^k\) with \(m \leq n \leq p_x\) we have \(\pi(x)(m,n) = \iota(y(m,n))\).

**Proof.** For part (i), fix a basic open set \(\mathcal{Z}(\mu \setminus G) \subseteq W_{\tilde{\Lambda}}\) containing \(\pi(x)\). Fix \(n \geq N := \bigvee_{\nu \in G} d(\mu\nu)\). We first show that \(\iota(y_n) \in \mathcal{Z}(\mu)\). Since \(\pi(x) \in \mathcal{Z}(\mu)\), we have \(\mu \in \iota(\Lambda)\). Since \(n \geq d(\mu)\), we have \([y_n;(0,d(\mu))] = \mu\).
Let $\alpha = \iota^{-1}(\mu)$ and $z \in s(\alpha)\partial \Lambda$. Then $[y_n; (0, d(\mu))] = \mu = [\alpha z; (0, d(\mu))]$, and (P1) gives $\iota(y_n(0, d(\mu) \wedge d(y_n))) = \iota((\alpha z)(0, d(\mu))) = \iota(\alpha) = \mu$. So $\iota(y_n) \in \mathcal{Z}(\mu)$.

We now show that $\iota(y_n) \notin \bigcup_{\nu \in G} \mathcal{Z}(\mu \nu)$. Fix $\nu \in G$. If $d(y_n) \notin d(\mu \nu)$, then trivially we have $\iota(y_n) \notin \mathcal{Z}(\mu \nu)$. Suppose that $d(y_n) \geq d(\mu \nu)$. Since $n \geq d(\mu \nu)$, we have

$$x(0, d(\mu \nu)) = [y_n; (0, n)](0, d(\mu \nu)) = \iota(y_n)(0, d(\mu \nu)) \in \iota(\Lambda).$$

So $\iota(y_n)(0, d(\mu \nu)) = x(0, d(\mu \nu)) = \pi(x)(0, d(\mu \nu)) \neq \mu \nu$.

For part (ii), recall that $\iota$ is injective, hence we can define $y : \Omega_{k,p_x} \rightarrow \Lambda$ by $\iota(y(m, n)) = \pi(x)(m, n)$. So $\iota(y) = \pi(x)$. To see that $y \in \partial \Lambda$, fix $m \in \mathbb{N}^k$ such that $m \leq d(y)$ and fix $E \in y(m)\mathcal{F}\mathcal{E}(\Lambda)$. We seek $t \in \mathbb{N}^k$ such that $y(m, m + t) \in E$. Let $p := m + \bigvee_{\mu \in E} d(\mu)$. Then since $m \leq d(y) = p_x$, we get

$$[y_p; (0, m)] = x(0, m) = \pi(x)(0, m) = \iota(y(0, m)) = [y(0, m) y'; (0, m)]$$

for some $y' \in y(m)\partial \Lambda$. So $(y_p; (0, m)) \sim (y(0, m) y'; (0, m))$, hence

$$y_p(0, m \wedge d(y_p)) = (y(0, m) y')(0, m \wedge d(y(0, m) y')) = y(0, m)$$

by (P1). In particular, this implies that $y_p(m) = y(m)$. Since $y_p \in \partial \Lambda$, there exists $t \in \mathbb{N}^k$ such that $y_p(m, m + t) \in E$. So $m + t \leq p$, and we have

$$\iota(y_p(m, m + t)) = [y_p; (0, p)](m, m + t) = x(0, p)(m, m + t) = x(m, m + t).$$

So $x(m, m + t) \in \iota(\Lambda)$, giving

$$\iota(y_p(m, m + t)) = x(m, m + t) = \pi(x)(m, m + t) = \iota(y(m, m + t)).$$

Finally, injectivity of $\iota$ gives $y(m, m + t) = y_p(m, m + t) \in E$. □

The next few results ensure that our definition of $\pi$ on $\tilde{\Lambda}^\infty$ is compatible with (4.2) when we regard finite paths as $k$-graph morphisms. The following lemma is also crucial in showing that $\pi$ is injective on $\iota(\Lambda^0)\tilde{\Lambda}^\infty$.

**Lemma 5.3.** Let $\Lambda$ be a finitely aligned $k$-graph. Let $x \in \iota(\Lambda^0)\tilde{\Lambda}^\infty$. Suppose that $w \in \partial \Lambda$ satisfies $\pi(x) = \iota(w)$. Then $x(0, n) = [w; (0, n)]$ for all $n \in \mathbb{N}^k$.

**Proof.** Fix $n \in \mathbb{N}^k$. Let $z \in \partial \Lambda$ be such that $x(0, n) = [z; (0, n)]$. We aim to show that $(z; (0, n)) \sim (w; (0, n))$. That (P2) and (P3) hold follows immediately from their definitions. It remains to verify condition (P1):

$$z(0, n \wedge d(z)) = w(0, n \wedge d(\omega)).$$

Since $\pi(x) = \iota(w)$ we have $d(\omega) = p_x$. Thus

$$[w; (0, n \wedge p_x)] = \iota(w(0, n \wedge p_x)) = x(0, n \wedge p_x) = [z; (0, n \wedge p_x)].$$

So $(w; (0, n \wedge p_x)) \sim (z; (0, n \wedge p_x))$. It then follows from (P1) that

$$w(0, n \wedge p_x) = z(0, n \wedge p_x).$$

(5.2)
Hence \( n \wedge d(z) \geq n \wedge p_x \). Furthermore, 
\[
x(0, n \wedge d(z)) = [z; (0, n \wedge d(z))] = \iota(z(0, n \wedge d(z))) \in \iota(\Lambda)
\]
implies that \( n \wedge p_x \geq n \wedge d(z) \). So \( n \wedge d(z) = n \wedge p_x \), and (5.2) becomes (5.1), as required. 

**Remark 5.4.** Suppose that \( \Lambda \) is a finitely aligned \( k \)-graph, and that \( y \in \partial \Lambda \) and \( m, n \in \mathbb{N}^k \) satisfy \( m \leq n \leq d(y) \). Then 
\[
[y; (m, n)] = [\sigma^m(y); (0, n - m)] = \iota(\sigma^m(y)(0, n - m)) = \iota(y(m, n)),
\]
so \([y; (m, n)] = \iota(y(m, n))\).

The next proposition shows that our definitions of \( \pi \) for finite and infinite paths are compatible:

**Proposition 5.5.** Let \( \Lambda \) be a finitely aligned \( k \)-graph. Suppose that \( x \in \tilde{\Lambda}^\infty \), and \( m \leq n \in \mathbb{N}^k \). Then \( \pi(x(m, n)) = \pi(x)(m \wedge p_x, n \wedge p_x) \).

**Proof.** Fix \( y \in \partial \Lambda \) such that \( \pi(x) = \iota(y) \). Then
\[
(x(m, n)) = \pi([y; (m, n)]) = [y; (m \wedge p_x, n \wedge p_x)] \quad \text{by Lemma 5.3}
\]
\[
= \iota(y(m \wedge p_x, n \wedge p_x)) \quad \text{since } d(y) = p_x \\
= \pi(x)(m \wedge p_x, n \wedge p_x) \quad \text{by Remark 5.4}
\]
We can now show that \( \pi \) restricts to a homeomorphism of \( \iota(A^0)\tilde{\Lambda}^\infty \) onto \( \iota(\partial \Lambda) \). We first show that it is a bijection, then show it is continuous. Openness is the trickiest part, and the proof of it completes this section.

**Proposition 5.6.** Let \( \Lambda \) be a finitely aligned \( k \)-graph. Then the map \( \pi : \iota(A^0)\tilde{\Lambda}^\infty \to \iota(\partial \Lambda) \) is a bijection.

**Proof.** That \( \pi \) is injective follows from Lemma 5.3. To see that \( \pi \) is onto \( \iota(\partial \Lambda) \), let \( w \in \partial \Lambda \) and define \( x : \Omega_k \to \tilde{\Lambda} \) by \( x(p, q) = [w; (p, q)] \). Then \( p_x = d(w) \), and \( r(x) \in \iota(\Lambda) \). To see that \( \pi(x) = \iota(w) \), fix \( m, n \in \mathbb{N}^k \) with \( m \leq n \leq d(w) \). Then
\[
(x)(m, n) = x(m, n) = [w; (m, n)] \quad \text{by Proposition 5.5}
\]
\[
= \iota(w(m, n)) \quad \text{by Remark 5.4}
\]
\[
= \iota(w)(m, n) \quad \text{by Proposition 4.14}
\]

**Proposition 5.7.** Let \( \Lambda \) be a finitely aligned \( k \)-graph. Then \( \pi : \iota(A^0)\tilde{\Lambda}^\infty \to \iota(\partial \Lambda) \) is continuous.

**Proof.** Fix a basic open set \( Z(\mu \setminus G) \subset W_{\tilde{\Lambda}} \). If \( Z(\mu \setminus G) \cap \iota(\partial \Lambda) = \emptyset \), then \( \pi^{-1}(Z(\mu \setminus G) \cap \iota(\partial \Lambda)) = \emptyset \) is open. Suppose that \( Z(\mu \setminus G) \cap \iota(\partial \Lambda) \neq \emptyset \),
and fix \( y \in \mathcal{Z}(\mu \setminus G) \cap \iota(\partial \Lambda) \). Let \( F = G \cap \iota(\Lambda) \). We will show that

\[
\pi^{-1}(y) \in \mathcal{Z}(\mu \setminus F) \cap (\tilde{\Lambda}^{\infty} \cap r^{-1}(\iota(\Lambda))) \subset \pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial \Lambda)).
\]

Since \( y \in \mathcal{Z}(\mu) \), we have \( \pi^{-1}(y) \in \mathcal{Z}(\mu) \). To see that \( \pi^{-1}(y) \notin \bigcup_{\beta \in F} \mathcal{Z}(\mu \beta) \), fix \( \beta \in F \). First suppose that \( d(\mu \beta) \notin d(y) \). Then \( \pi^{-1}(y)(0,d(\mu \beta)) \notin \iota(\Lambda) \). Since \( \mu \beta \in \iota(\Lambda) \), we have \( \pi^{-1}(y)(0,d(\mu \beta)) \neq \mu \beta \). Now suppose that \( d(\mu \beta) \leq d(y) \); then

\[
\pi^{-1}(y)(0,d(\mu \beta)) = y(0,d(\mu \beta)) \neq \mu \beta.
\]

We now show that \( \mathcal{Z}(\mu \setminus F) \cap \iota(A^0) \tilde{\Lambda}^{\infty} \subset \pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial \Lambda)) \). Let \( z \in \mathcal{Z}(\mu \setminus F) \cap \iota(A^0) \tilde{\Lambda}^{\infty} \). It suffices to show that \( \pi(z) \in \mathcal{Z}(\mu \setminus G) \). Firstly, \( \pi(z)(0,d(\mu)) = z(0,d(\mu)) = \mu \in \iota(\Lambda) \). To see that \( \pi(z) \notin \bigcup_{\nu \in G} \mathcal{Z}(\mu \nu) \), fix \( \nu \in G \). If \( d(\mu \nu) \notin d(\pi(z)) \), then trivially \( \pi(z) \notin \mathcal{Z}(\mu \nu) \). Suppose that \( d(\mu \nu) \leq d(\pi(z)) \). If \( \nu \notin \iota(\Lambda) \), then \( \pi(z)(0,d(\mu \nu)) \neq \mu \nu \). Otherwise, \( \nu \in \iota(\Lambda) \), so \( \nu \in F \) and we have \( \pi(z)(0,d(\mu \nu)) = z(0,d(\mu \nu)) \neq \mu \nu \).

**Proposition 5.8.** Let \( \Lambda \) be a row-finite \( k \)-graph. Then \( \pi : \iota(A^0) \tilde{\Lambda}^{\infty} \rightarrow \iota(\partial \Lambda) \) is open.

**Proof.** Fix \( \pi(y) \in \mathcal{Z}(\mu \setminus G) \cap \iota(A^0) \tilde{\Lambda}^{\infty} \). Let \( \omega \in \partial \Lambda \) be such that \( \pi(y) = \iota(\omega) \). Define \( \lambda := y(0,\bigvee_{\nu \in G} d(\mu \nu)) \), and

\[
F := \bigcup \{ s(\pi(\lambda)) \iota(A^{e_i}) : d(\lambda)_i > d(\pi(y))_i \}.
\]

We claim that

\[
\pi(y) \in \mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial \Lambda) \subset \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(A^0) \tilde{\Lambda}^{\infty}).
\]

First we show that \( \pi(y) \in \mathcal{Z}(\pi(\lambda)) \). It follows from Lemma 5.3 that \( \pi(\lambda) = [\omega; (0, d(\lambda) \wedge d(\omega))] \). Since \( d(\omega) = d(\pi(y)) \), Proposition 5.5 implies that

\[
\pi(y)(0,d(\pi(\lambda))) = \pi(y)(0,d(\lambda) \wedge d(\omega)) = \pi(y)(0,d(\lambda)) = \pi(\lambda).
\]

Now we show that \( \pi(y) \notin \bigcup_{f \in F} \mathcal{Z}(\pi(\lambda)f) \). Fix \( f \in F \); say \( d(f) = e_i \). Then by definition of \( F \), \( d(\lambda)_i > d(\pi(y))_i \) and thus

\[
d(\pi(\lambda))_i = \min\{d(\lambda)_i,d(\omega)_i\} = d(\omega)_i = d(\pi(y))_i.
\]

So \( d(\pi(y)) \neq d(\pi(\lambda)) \), and hence \( \pi(y) \notin \mathcal{Z}(\pi(\lambda)f) \) as required.

Now we show that \( \mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial \Lambda) \subset \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(A^0) \tilde{\Lambda}^{\infty}) \). Let \( \pi(\beta) \in \mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial \Lambda) \). We aim to show that \( \beta \in \mathcal{Z}(\mu \setminus G) \). Since \( \mathcal{Z}(\lambda) \subset \mathcal{Z}(\mu \setminus G) \), it suffices to show that \( \beta \in \mathcal{Z}(\lambda) \). Clearly \( \beta \in \mathcal{Z}(\pi(\lambda) \setminus F) \). If \( d(\lambda) = d(\pi(\lambda)) \) then \( \pi(\lambda) = \lambda \) and we are done. Suppose that \( d(\lambda) > d(\pi(\lambda)) \), and let \( \tau = \beta(d(\pi(\lambda)),d(\lambda)) \). We know that \( \beta \in \mathcal{Z}(\pi(\lambda)) \). We aim to use Lemma 4.19 to show that \( \tau = \lambda(d(\pi(\lambda)),d(\lambda)) \). Fix \( i \leq k \) such that \( d(\lambda)_i > d(\pi(\lambda))_i \). Then since \( d(\pi(\lambda)) = d(\lambda) \wedge d(\omega) \), we have \( d(\lambda)_i > d(\omega)_i = d(\pi(y))_i \). Now \( \beta \in \mathcal{Z}(\pi(\lambda) \setminus F) \) implies that \( \tau(0,e_i) \notin F \). In particular,
\[ \tau(0, e_i) \notin \nu(A). \] We claim that \( d(\pi(\tau)) = 0. \) Suppose, for a contradiction, that \( d(\pi(\tau))_j > 0 \) for some \( j \leq k. \) Then \( \pi(\tau)(0, e_j) = \tau(0, e_j) \notin \nu(A). \) But \( \pi(\tau) \in \nu(A) \) by definition of \( \pi. \) So we must have \( d(\pi(\tau)) = 0, \) which implies that
\[
\pi(\tau) = r(\tau) = s(\pi(\lambda)) = \pi(\lambda(d(\pi(\lambda)), d(\lambda))).
\]
Now Lemma \[4.19\] implies that \( \tau = \lambda(d(\pi(\lambda)), d(\lambda)). \) Then
\[
\beta(0, \lambda) = \beta(0, d(\pi(\lambda))) = \pi(\lambda)\lambda(d(\pi(\lambda)), d(\lambda)) = \lambda. \]

**Example 5.9.** We can see that \( \pi \) is not open for non-row-finite graphs by considering the 1-graph \( E \) from Example \[4.10\] with ‘desourcification’ \( \tilde{E}. \) Observe that \( Z(\mu_1) \cap \nu(E^0)^\infty = \{\mu_1\mu_2\cdots\} \) is open in \( \tilde{E}, \) and \( \pi(Z(\mu_1) \cap \nu(E^0)^\infty) = \{v\}. \) Since \( \partial E = E, \) any basic open set in \( \partial E \) containing \( v \) is of the form \( Z(v \setminus G) \) for some finite \( G \subseteq E^1. \) Since \( E^1 \) is infinite, there is no finite \( G \subseteq E^1 \) such that \( Z(v \setminus G) \subseteq \{v\}. \) Hence \( \{v\} \) is not open in \( E, \) and \( \pi \) is not an open map.

**Proof of Theorem 5.1.** Propositions \[5.6 \ 5.7 \ 5.8\] say precisely that \( \pi \) is a bijection, is continuous, and is open.

**Remark 5.10.** Although \( \pi|_{\nu(A)^\infty} \) is open for all row-finite \( k \)-graphs, it behaves particularly well with respect to cylinder sets for locally convex \( k \)-graphs. The following discussion and example arose in preliminary work on a proof that \( \pi \) is open when \( A \) is row-finite and locally convex. We have retained this example since it helps illustrate some of the issues surrounding the map \( \pi. \)

Denote our standard topology for a finitely \( k \)-graph by \( \tau_1. \) The collection \( \{Z(\mu) : \mu \in \Lambda\} \) of cylinder sets also forms a base for a topology: they cover \( W_\Lambda, \) and if \( x \in Z(\lambda) \cap Z(\nu), \) then \( x \in Z(x(0, d(\lambda) \vee d(\nu))) \subseteq Z(\lambda) \cap Z(\nu). \) This topology, denoted \( \tau_2, \) is not necessarily Hausdorff: we cannot separate any edge from its range: if \( r(f) \in Z(\mu) \) then \( \mu = r(f), \) and thus \( f \in Z(\mu). \)

It may seem reasonable to expect that \( \{Z(\mu) \cap \partial \Lambda : \mu \in \Lambda\} \) is a base for the restriction of \( \tau_1 \) to \( \partial \Lambda. \) However, this is not so. To see why, consider the 2-graph of Example \[2.11\] Let \( y \) be the boundary path beginning with \( f_0. \) So \( x, y \in \partial \Lambda. \) Let \( \mu \) be such that \( x \in Z(\mu). \) Then \( \mu = x_0\ldots x_n \) for some \( n \in \mathbb{N}, \) so \( y \in Z(\mu) \) as well. So the topology \( \tau_1 \) is not Hausdorff even when restricted to \( \partial \Lambda. \) In the topology \( \tau_2, \) it is easy to see how to separate these two points: \( y \in Z(f_0) \cap \partial \Lambda \) and \( x \in Z(r(x) \setminus \{f_0\}) \cap \partial \Lambda, \) and these two sets are disjoint.

If we restrict ourselves to locally convex \( k \)-graphs, \( \tau_1 \) and \( \tau_2 \) do restrict to the same topology on \( \partial \Lambda. \) certainly, for each \( \mu \in \Lambda, \) we can realise a cylinder set \( Z(\mu) \) as a set of the form \( Z(\mu \setminus G) \) by taking \( G = \emptyset. \) Now suppose that
Let \( x \in \mathcal{Z}(\mu \setminus G) \cap \partial \Lambda \). We claim that with

\[
\nu_x := x\left(0, \bigvee_{\alpha \in G} d(\mu_\alpha) \wedge d(x)\right),
\]
we have \( x \in \mathcal{Z}(\nu_x) \cap \partial \Lambda \subset \mathcal{Z}(\mu \setminus G) \cap \partial \Lambda \). Clearly \( x \in \mathcal{Z}(\nu_x) \cap \partial \Lambda \). The containment requires a little more work. Clearly \( y \in \mathcal{Z}(\mu) \). Fix \( \alpha \in G \). We will show that \( y \notin \mathcal{Z}(\mu \alpha) \). If \( d(y) \geq d(\mu_\alpha) \), then trivially \( y \notin \mathcal{Z}(\mu \alpha) \). Suppose that \( d(y) \leq d(\mu_\alpha) \). We claim that \( d(x) \geq d(\mu_\alpha) \). Suppose, for a contradiction, that \( d(x) \geq d(\mu_\alpha) \). Then there exists \( i \leq k \) such that \( d(x)_i < d(\mu_\alpha)_i \). Then \( d(x)_i = d(\nu_x)_i \). Since \( x \in \partial \Lambda \), we must have \( x(d(\nu_x)) \Lambda^{e_i} \notin x(d(\nu_x)) \mathcal{F} \mathcal{E}(\Lambda) \). Since \( \Lambda \) is locally convex, Lemma 2.13 implies that \( y(d(\nu_x)) \Lambda^{e_i} = x(d(\nu_x)) \Lambda^{e_i} \). So \( d(y)_i = d(\nu_x)_i = d(x)_i < d(\mu_\alpha)_i \), a contradiction. Hence \( d(x) \geq d(\mu_\alpha) \). This implies that \( d(\nu_x) \geq d(\mu_\alpha) \). So

\[
y(0, d(\mu_\alpha)) = v_x(0, d(\mu_\alpha)) = x(0, d(\mu_\alpha)) \neq \mu_\alpha.
\]

**Proposition 5.11.** Suppose that \( \Lambda \) is a row-finite, locally convex \( k \)-graph, and let \( \mu \in \iota(\Lambda^0, \Lambda) \). Then \( \pi(\mathcal{Z}(\mu) \cap \iota(\Lambda^0) \Lambda^\infty) = \mathcal{Z}(\pi(\mu)) \cap \iota(\partial \Lambda) \). In particular, \( \pi \) is open.

**Proof.** We first show that \( \pi(\mathcal{Z}(\mu) \cap \iota(\Lambda^0) \Lambda^\infty) \subset \mathcal{Z}(\pi(\mu)) \cap \iota(\partial \Lambda) \). Suppose that \( \pi(y) \in \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0) \Lambda^\infty) \). Trivially \( \pi(y) \in \iota(\partial \Lambda) \). We will show that \( \pi(y) \in \mathcal{Z}(\pi(\mu) \setminus \pi(G)) \). Since \( y(0, d(\mu)) = \mu \), we have

\[
\pi(\mu) = \pi(y(0, d(\mu))) = \pi(y)(0, d(\mu) \wedge d(\pi(y))).
\]
So \( \pi(y) \in \mathcal{Z}(\pi(\mu)) \). Furthermore, \( d(\pi(\mu)) = d(\mu) \wedge d(\pi(y)) \).

Fix \( \nu \in G \). We will show that \( \pi(y) \notin \mathcal{Z}(\pi(\mu)) \). Since \( y \in \mathcal{Z}(\mu \setminus G) \), we have \( y(0, d(\mu_\nu)) \neq \mu_\nu \). Since \( d(y(0, d(\mu_\nu))) = d(\mu_\nu) \) and \( r(y) = r(\mu_\nu) \in \iota(\Lambda^0) \), Lemma 4.19 implies that

\[
\pi(\mu_\nu) \neq \pi(y(0, d(\mu_\nu))) = \pi(y)(0, d(\mu_\nu) \wedge d(\pi(y))).
\]
So \( \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0) \Lambda^\infty) \subset \mathcal{Z}(\pi(\mu) \setminus \pi(G)) \cap \iota(\partial \Lambda) \).

Now suppose that \( \iota(\omega) \in \mathcal{Z}(\pi(\mu)) \cap \iota(\partial \Lambda) \), and let \( y = \pi^{-1}(\iota(\omega)) \). We show that \( y \in \mathcal{Z}(\mu) \). Write \( \mu = [z; (0, d(\mu))] \). Then \( \pi(\mu) = [z; (0, d(\mu) \wedge d(z))] \) and \( y(0, d(\mu)) = [\omega; (0, d(\mu))] \). We claim that \( (z; (0, d(\mu))) \sim (\omega; (0, d(\mu))) \).

That (F2) and (F3) hold follows immediately from their definition. To show that (F1) is satisfied, we must show that \( z(0, d(\mu) \wedge d(z)) = w(0, d(\mu) \wedge d(w)) \).

Since \( \pi(y) \in \mathcal{Z}(\pi(\mu)) \), we have \( y \in \mathcal{Z}(\pi(\mu)) \). Then

\[
[\omega; (0, d(\pi(\mu))) = y(0, d(\pi(\mu))) = \pi(\mu) = [z; (0, d(\mu) \wedge d(z))].
\]
So \( (\omega; (0, d(\pi(\mu)))) \sim (z; (0, d(\mu) \wedge d(z))) \). Therefore (F1) implies that

\[
\omega(0, d(\pi(\mu))) = \omega(0, d(\pi(\mu)) \wedge d(\omega)) = z(0, d(\mu) \wedge d(z)),
\]
and \( d(\pi(\mu)) = d(\mu) \wedge d(z) \). We will show \( d(\mu) \wedge d(w) = d(\pi(\mu)) \). Fix \( i \leq k \). We argue the following cases separately:
(1) If \( d(\pi(\mu))_i = d(\mu)_i \), we have \( d(w) \geq d(\pi(\mu)) = d(\mu)_i \). It follows that \( (d(\mu) \wedge d(w))_i = d(\mu)_i = d(\pi(\mu))_i \).

(2) If \( d(\pi(\mu))_i < d(\mu)_i \), it requires a little more work: Since \( d(\mu)_i > d(\pi(\mu))_i = \min \{ d(\mu)_i, d(z)_i \} \), we have \( d(\pi(\mu))_i = d(z)_i \). So \( z \in \partial \Lambda \) implies that \( z(d(\pi(\mu))) \Lambda^{e_i} \notin z(d(\pi(\mu))) \mathcal{FE}(\Lambda) \). By Lemma 2.13 we have \( z(d(\pi(\mu))) \Lambda^{e_i} = \emptyset \), and hence \( \omega(d(\pi(\mu))) \Lambda^{e_i} = \emptyset \). So \( d(\omega)_i = d(\pi(\mu))_i < d(\mu)_i \), giving \( (d(\mu) \wedge d(\omega))_i = d(\omega)_i = d(\pi(\mu))_i \). \( \blacksquare \)

6. High-rank graph \( C^* \)-algebras

Definition 6.1. Let \( \Lambda \) be a finitely aligned \( k \)-graph. A Cuntz–Krieger \( \Lambda \)-family in a \( C^* \)-algebra \( B \) is a collection \( \{ t_\lambda : \lambda \in \Lambda \} \) of partial isometries satisfying

\( \text{(CK1)} \) \( \{ s_v : v \in \Lambda^0 \} \) is a set of mutually orthogonal projections,

\( \text{(CK2)} \) \( s_\mu s_v = s_\mu v \) whenever \( s(\mu) = r(\nu) \),

\( \text{(CK3)} \) \( s_\mu s_v = \sum_{(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)} s_{\alpha} s_{\beta}^* \) for all \( \mu, \nu \in \Lambda \),

\( \text{(CK4)} \) \( \prod_{\mu \in E} (s_v - s_\mu s_\mu^*) = 0 \) for all \( v \in \Lambda^0 \) and \( E \in v \mathcal{FE}(\Lambda) \).

The \( C^* \)-algebra \( C^*(\Lambda) \) of a \( k \)-graph \( \Lambda \) is the universal \( C^* \)-algebra generated by a Cuntz–Krieger \( \Lambda \)-family \( \{ s_\lambda : \lambda \in \Lambda \} \).

Remark 6.2. The following theorem appears as [5] Theorem 2.28]. Farthing alerted us to an issue in the proof of the theorem. It contains a claim which is proved in cases, and in the proof of Case 1 of the claim (on page 189), there is an error when \( i_0 \) is such that \( m_{i_0} = d(x)_i + 1 \). Then \( a_{i_0} = d(x)_{i_0} \), and [5] equation (2.13) gives \( t_{i_0} \leq d(z)_{i_0} \); not \( t_{i_0} \geq d(z)_{i_0} \) as stated.

Theorem 6.3. Let \( \Lambda \) be a row-finite \( k \)-graph. Let \( C^*(\Lambda) \) and \( C^*(\tilde{\Lambda}) \) be generated by the Cuntz–Krieger families \( \{ s_\lambda : \lambda \in \Lambda \} \) and \( \{ t_\lambda : \lambda \in \tilde{\Lambda} \} \). Then the sum \( \sum_{v \in \iota(A^0)} t_v \) converges strictly to a full projection \( p \in M(C^*(\tilde{\Lambda})) \) such that \( pC^*(\tilde{\Lambda})p = C^*(\{ t_{\iota(\lambda)} : \lambda \in \Lambda \} ) \), and \( s_\lambda \mapsto t_{\iota(\lambda)} \) determines an isomorphism \( \varsigma : C^*(\Lambda) \cong pC^*(\tilde{\Lambda})p \).

Before proving Theorem 6.3 we need the following results.

Proposition 6.4 ([5] Theorem 2.26]). Let \( \Lambda \) be a finitely aligned \( k \)-graph. If \( \{ t_\lambda : \lambda \in \tilde{\Lambda} \} \) is a Cuntz–Krieger \( \tilde{\Lambda} \)-family, then \( \{ t_\lambda : \lambda \in \iota(\Lambda) \} \) is a Cuntz–Krieger \( \iota(\Lambda) \)-family.

Remark 6.5. Let \( \Lambda \) be a finitely aligned \( k \)-graph. It follows from the universal properties of \( C^*(\Lambda) \) and \( C^*(\iota(\Lambda)) \) that \( C^*(\Lambda) \cong C^*(\iota(\Lambda)) \).

Proposition 6.6 ([5] Theorem 2.27]). Let \( \Lambda \) be a finitely aligned \( k \)-graph, and let \( \{ t_\lambda : \lambda \in \tilde{\Lambda} \} \) be the universal Cuntz–Krieger \( \tilde{\Lambda} \)-family which
generates \( C^*(\tilde{\Lambda}) \). Then \( C^*(\Lambda) \) is isomorphic to the subalgebra of \( C^*(\tilde{\Lambda}) \) generated by \( \{ t_\lambda : \lambda \in \iota(\Lambda) \} \).

**Lemma 6.7.** Suppose that \( \Lambda \) is a finitely aligned \( k \)-graph. Let \( \lambda \in \tilde{\Lambda} \), and let \( \lambda' = \lambda(d(\pi(\lambda)), d(\lambda)) \). Suppose that \( x \in \partial \Lambda \) satisfies \( \iota(r(x)) = r(\lambda') \) and \( d(x) \land d(\lambda') = 0 \). Then \( \lambda' = [x; (0, d(\lambda'))] \).

**Proof.** Write \( \lambda = [y; (0, d(\lambda))] \) then \( \lambda' = [y; (d(\lambda) \land d(y), d(\lambda))] \). We must show that \( (y; (d(\lambda) \land d(y), d(\lambda))) \sim (x; (0, d(\lambda'))) \). That conditions \( (P2) \) and \( (P3) \) hold follows immediately from their definitions. It remains to show that \( (P1) \) is satisfied. Since \( d(x) \land d(\lambda') = 0 \), it suffices to show that \( y(d(\lambda) \land d(y)) = x(0) \). We have

\[
\iota(x(0)) = \iota(r(x)) = r(\lambda') = [y; d(\lambda) \land d(y)] = \iota(y(d(\lambda) \land d(y))).
\]

Injectivity of \( \iota \) then gives \( y(d(\lambda) \land d(y)) = x(0) \).

**Lemma 6.8.** Let \( \lambda \in \tilde{\Lambda} \). Let \( \lambda' = \lambda(d(\pi(\lambda)), d(\lambda)) \) and define

\[
G_\lambda := \bigcup_{i=1}^{k} \{ \alpha \in s(\pi(\lambda))\iota(\Lambda)^{e_i} : \text{MCE}(\alpha, \lambda') = \emptyset \}.
\]

Then \( G_\lambda \cup \{ \lambda' \} \in s(\pi(\lambda))\mathcal{FE}(\tilde{\Lambda}) \).

**Proof.** Fix \( \mu \in s(\pi(\lambda))\tilde{\Lambda} \), and suppose that \( \text{MCE}(\mu, \alpha) = \emptyset \) for all \( \alpha \in G_\lambda \). We will show that \( \text{MCE}(\mu, \lambda') \neq \emptyset \). Fix \( \nu \in s(\mu)\tilde{\Lambda}^{d(\mu) \land d(\lambda') - d(\mu)} \). Then \( d(\mu \nu) \geq d(\lambda') \). It suffices to show that \( \text{MCE}(\mu \nu, \lambda') \neq \emptyset \). Write \( \mu \nu = [z; (0, d(\mu \nu))] \).

We first show that \( d(\lambda') \land d(\pi(\mu \nu)) = 0 \). Suppose for a contradiction that \( d(\lambda') \land d(\pi(\mu \nu)) > 0 \). So we have \( d(\lambda') \land d(\mu \nu) \land d(z) > 0 \), hence there exists \( i \leq k \) such that \( d(\lambda')_i, d(\mu \nu)_i, \) and \( d(z)_i \) are all greater than zero. Then \( (\mu \nu)(0, e_i) = [z; (0, e_i)] = \iota(z)(0, e_i) \in \iota(\Lambda) \). Since \( \pi|_{\iota(\Lambda)} = id_{\iota(\Lambda)} \) and \( \pi(\lambda') = s(\pi(\lambda)) \neq \lambda' \), we have \( \lambda' \notin \iota(\Lambda) \). This implies that \( (\mu \nu)(0, e_i) \neq \lambda'(0, e_i) \). So \( \text{MCE}((\mu \nu)(0, e_i), \lambda') = \emptyset \), and thus \( (\mu \nu)(0, e_i) \notin G_\lambda \). But \( \text{MCE}(\mu \nu(0, e_i), \mu \nu) \neq \emptyset \), which implies that \( \text{MCE}(\mu, \mu \nu(0, e_i)) \neq \emptyset \). This contradicts our supposition that \( \text{MCE}(\mu, \alpha) = \emptyset \) for all \( \alpha \in G_\lambda \). So \( d(\lambda') \land d(\pi(\mu \nu)) = 0 \).

Since \( d(\mu \nu) \geq d(\lambda') \), we have

\[
d(z) \land d(\lambda') = d(z) \land d(\mu \nu) \land d(\lambda') = d(\pi(\mu \nu)) \land d(\lambda') = 0.
\]

Since \( r(\lambda') = r(\mu \nu) = \iota(r(z)) \), it follows from Lemma 6.7 that \( \lambda' = [z; (0, \lambda')] \). Thus \( \mu \nu = [z; (0, \mu \nu)] \in \text{MCE}(\mu \nu, \lambda') \).

**Proof of Theorem 6.3.** Let \( A := C^*(\{ t_\lambda : \lambda \in \iota(\Lambda) \}) \). Then \( A \cong C^*(\tilde{\Lambda}) \) by Proposition 6.6. We will show that \( A \) is a full corner of \( C^*(\tilde{\Lambda}) \).
Following the argument of [10] Lemma 2.10, the sum \( \sum_{v \in \iota(A^0)} t_v \) converges strictly in \( M(C^*(\tilde{A})) \) to a projection \( p \) satisfying
\[
ptx^*_\mu p = \begin{cases} 
  t_\lambda t_\mu^* & \text{if } \overline{\tau}(\lambda), \overline{\tau}(\mu) \in \iota(A^0), \\
  0 & \text{otherwise.}
\end{cases}
\]

(6.1)

The standard argument shows that \( p \) is a full projection in \( M(C^*(\tilde{A})) \). It follows from (6.1) that \( A \subset pC^*(\tilde{A})p \). Now suppose that \( \lambda, \mu \in \iota(A^0) \tilde{A} \).

We will show that \( ptx^*_\mu p = A \). If \( \tilde{s}(\lambda) \neq \tilde{s}(\mu) \), then (CKI) implies that \( ptx^*_\mu p = 0 \in A \). Suppose that \( \tilde{s}(\lambda) = \tilde{s}(\mu) \). We first show
\[
(6.2) \quad \lambda(d(\pi(\lambda)), d(\lambda)) = \mu(d(\pi(\mu)), d(\mu)).
\]

Let \( x, y \in \partial A \) be such that \( \lambda = [x; (0, d(\lambda))] \) and \( \mu = [y; (0, d(\mu))] \). Let
\[
\lambda' = \lambda(d(\pi(\lambda)), d(\lambda)) = [x; (d(\lambda) \wedge d(x), d(\lambda)], \\
\mu' = \mu(d(\pi(\mu)), d(\mu)) = [y; (d(\mu) \wedge d(y), d(\mu))].
\]

We claim that \( \lambda' = \mu' \). Condition (F2) is trivially satisfied, and (F1) and (F3) follow from the vertex equivalence \([x; d(\lambda)] = \tilde{s}(\lambda) = \tilde{s}(\mu) = [y; d(\mu)]\).

Hence \( \lambda' = \mu' \).

Claim 6.3.1. Let \( G_\lambda := \bigcup_{i=1}^k \{ \alpha \in s(\pi(\lambda))\iota(A)^{e_i} : \text{MCE}(\alpha, \lambda') = \emptyset \} \).

Then
\[
t_{\lambda'}t_{\lambda'}^* = \prod_{\alpha \in G_\lambda} (t_{s(\pi(\lambda))} - t_{\alpha}t_{\alpha}^*).
\]

Proof. Lemma 6.8 implies that \( G_\lambda \cup \{ \lambda' \} \) is finite exhaustive, so (CK4) implies
\[
\prod_{\beta \in G_\lambda \cup \{ \lambda' \}} (t_{s(\pi(\lambda))} - t_{\beta}t_{\beta}^*) = 0.
\]

Furthermore,
\[
\prod_{\beta \in G_\lambda \cup \{ \lambda' \}} (t_{s(\pi(\lambda))} - t_{\beta}t_{\beta}^*) = \left( \prod_{\alpha \in G_\lambda} (t_{s(\pi(\lambda))} - t_{\alpha}t_{\alpha}^*) \right) (t_{s(\pi(\lambda))} - t_{\lambda'}t_{\lambda'}^*)
\]
\[
= \left( \prod_{\alpha \in G_\lambda} (t_{s(\pi(\lambda))} - t_{\alpha}t_{\alpha}^*) \right) - \left( t_{\lambda'}t_{\lambda'}^* \prod_{\alpha \in G_\lambda} (t_{s(\pi(\lambda))} - t_{\alpha}t_{\alpha}^*) \right).
\]

Fix \( \alpha \in G_\lambda \). By [13] Lemma 2.7(i),
\[
t_{\lambda'}t_{\lambda'}^*(t_{s(\pi(\lambda))} - t_{\alpha}t_{\alpha}^*) = t_{\lambda'}t_{\lambda'}^* - \sum_{\gamma \in \text{MCE}(\lambda', \alpha)} t_{\gamma}t_{\gamma}^* = t_{\lambda'}t_{\lambda'}^*.
\]

So
\[
0 = \prod_{\beta \in G_\lambda \cup \{ \lambda' \}} (t_{s(\pi(\lambda))} - t_{\beta}t_{\beta}^*) = \prod_{\alpha \in G_\lambda} (t_{s(\pi(\lambda))} - t_{\alpha}t_{\alpha}^*) - t_{\lambda'}t_{\lambda'}^*. \quad \blacksquare
Now we put the pieces together:

\[ pt^*_\lambda t^*_\mu = t^*_\lambda t^*_\mu \]

\[ = t^*_\pi(\lambda) t^*_\lambda t^*_\mu \]

\[ = t^*_\pi(\lambda) \prod_{\alpha \in G_\lambda} \left( ts(\pi(\lambda)) - t^*_\alpha t^*_\mu \right) \] by Claim 6.3.1.

This is an element of \( A \) since \( \pi(\lambda), \pi(\mu), \alpha \in \iota(\Lambda) \) for all \( \alpha \in G_\lambda \). So \( A = pC^*(\tilde{\Lambda})p \).

**The diagonal and the spectrum.** For a \( k \)-graph \( \Lambda \), we call \( C^* \{ s^*_\mu s^*_\mu : \mu \in \Lambda \} \subset C^*(\Lambda) \) the diagonal \( C^* \)-algebra of \( \Lambda \) and denote it by \( D_\Lambda \), dropping the subscript when confusion is unlikely. For a commutative \( C^* \)-algebra \( A \), denote by \( \Delta(A) \) the spectrum of \( A \). Given a homomorphism \( \pi : A \to B \) of commutative \( C^* \)-algebras, denote by \( \pi^* \) the induced map from \( \Delta(B) \) to \( \Delta(A) \) such that \( \pi^*(f)(y) = f(\pi(y)) \) for all \( f \in \Delta(B) \) and \( y \in A \).

**Theorem 7.1.** Let \( \Lambda \) be a row-finite higher-rank graph. Let \( p \in M(C^*(\tilde{\Lambda})) \) and \( \varsigma : C^*(\Lambda) \cong pC^*(\tilde{\Lambda})p \) be from Theorem 6.3. Then the restriction \( \varsigma|_{D_\Lambda} = : \rho \) is an isomorphism of \( D_\Lambda \) onto \( pD_e\Lambda p \). Let \( \pi : \iota(\Lambda^0)\tilde{\Lambda}^\infty \to \iota(\partial\Lambda) \) be the homeomorphism from Theorem 5.1. Then there exist homeomorphisms \( h_\Lambda : \partial\Lambda \to \Delta(D_\Lambda) \) and \( \eta : \iota(\Lambda^0)\tilde{\Lambda}^\infty \to \Delta(pD_e\Lambda p) \) such that the following diagram commutes:

\[ \begin{array}{ccc} 
\iota(\Lambda^0)\tilde{\Lambda}^\infty & \xrightarrow{\pi} & \iota(\partial\Lambda) \\
\eta \downarrow & & \downarrow h_\Lambda \circ \iota^{-1} \\
\Delta(pD_e\Lambda p) & \xrightarrow{\rho^*} & \Delta(D_\Lambda) 
\end{array} \]

As in \[ \Pi \], for a finite subset \( F \subset \Lambda \), define

\[ \vee F := \bigcup_{G \subset F} \text{MCE}(G) = \bigcup_{G \subset F} \left\{ \lambda \in \bigcap_{\mu \in G} \mu \Lambda : d(\lambda) = \bigvee_{\mu \in G} d(\mu) \right\}. \]

**Lemma 7.2.** Let \( \Lambda \) be a finitely aligned \( k \)-graph and let \( F \) be a finite subset of \( \Lambda \). Suppose that \( r(\lambda) \in F \) for each \( \lambda \in F \). For \( \mu \in F \), define

\[ q^*_\mu := s^*_\mu s^*_\mu \prod_{\mu' \in \vee F \setminus \{\mu\}} (s^*_\mu s^*_\mu - s^*_\mu s^*_\mu'). \]

Then the \( q^*_\mu \) are mutually orthogonal projections in \( \text{span}\{ s^*_\mu : \mu \in \vee F \} \), and for each \( \nu \in \vee F \),

\[ s^*_\nu s^*_\nu = \sum_{\nu' \in \vee F} q^*_\nu \nu'. \]
Proof. Since
\[ s_\mu s^*_\mu \prod_{\mu' \in \mathcal{V} \setminus \{\mu\}} \left( s_\mu s^*_\mu - s_\mu s^*_\mu s^*_\mu' \right) = s_\mu s^*_\mu \prod_{\mu' \in \mathcal{V}, d(\mu') \neq 0} \left( s_\mu - s_\mu s^*_\mu s^*_\mu' \right), \]
[11, Proposition 8.6] says precisely that the \( q_{\mu}^{\mathcal{V} \mathcal{F}} \) are mutually orthogonal projections. That
\[ q_{\mu}^{\mathcal{V} \mathcal{F}} = \sum_{\nu' \in \mathcal{V} \mathcal{F}} q_{\nu' \mu}^{\mathcal{V} \mathcal{F}} \]
is established in the proof of [11, Proposition 8.6] on page 421.

Remark 7.3. We have
\[ q_{\mu}^{\mathcal{V} \mathcal{F}} = s_\mu \left( \prod_{\mu' \in \mathcal{V} \mathcal{F} \setminus \{s(\mu)\}} (s_{s(\mu)} - s_\mu s^*_\mu s^*_\mu') \right) s^*_\mu. \]
This follows from a straightforward induction on \(|\mathcal{V} \mathcal{F}|\).

The following lemma can be verified through routine calculation. The reader is referred to the author’s PhD thesis for details.

Lemma 7.4 ([19, Lemma A.0.7]). Let \( \Lambda \) be a \( C^* \)-algebra, let \( p \) be a projection in \( A \), let \( Q \) be a finite set of commuting subprojections of \( p \) and let \( q_0 \) be a non-zero subprojection of \( p \). Then \( \prod_{q \in Q} (p - q) \) is a projection. If \( q_0 \) is orthogonal to each \( q \in Q \), then \( q_0 \prod_{q \in Q} (p - q) = q_0 \), so in particular, \( \prod_{q \in Q} (p - q) \neq 0 \).

Proposition 7.5. Let \( \Lambda \) be a finitely aligned \( k \)-graph. Then \( D = \text{span} \{ s_\mu s^*_\mu : \mu \in \Lambda \} \), and for each \( x \in \partial \Lambda \) there exists a unique \( h(x) \in \Delta(D) \) such that
\[ h(x)(s_\mu s^*_\mu) = \begin{cases} 1 & \text{if } x = \mu' \mu, \\ 0 & \text{otherwise.} \end{cases} \]
Moreover, \( x \mapsto h(x) \) is a homeomorphism \( h : \partial \Lambda \to \Delta(D) \).

Proof. Let \( \mu, \nu \in \Lambda \). It follows from (CK3) that
\[ (s_\mu s^*_\mu)(s_\nu s^*_\nu) = \sum_{\lambda \in \text{MCE}(\mu, \nu)} s_\lambda s^*_\lambda, \]
and hence \( D = \text{span} \{ s_\mu s^*_\mu : \mu \in \Lambda \} \).

Fix \( x \in \partial \Lambda \) and \( \sum_{\mu \in \mathcal{F}} b_\mu s_\mu s^*_\mu \in \text{span} \{ s_\mu s^*_\mu : \mu \in \Lambda \} \). By setting extra coefficients to zero we can assume that each path in \( \mathcal{F} \) has its range in \( \mathcal{F} \), and write
\[ \sum_{\mu \in \mathcal{F}} b_\mu s_\mu s^*_\mu = \sum_{\mu \in \mathcal{V} \mathcal{F}} b_\mu s_\mu s^*_\mu. \]
Let \( n = \sqrt{\{ p \in \mathbb{N}^k : x(0, p) \in \mathcal{V} \mathcal{F} \}} \). Since \( \mathcal{V} \mathcal{F} \) is a finite set of finite paths, \( n \) is finite. Since \( \mathcal{V} \mathcal{F} \) is closed under minimal common extensions, \( x(0, n) \in \mathcal{V} \mathcal{F} \).
Furthermore, since $x \in \partial A$, we have

$$F_x := \{ \mu' \in x(n)A \setminus \{ x(n) \} : x(0, n)\mu' \in \forall F \} \notin x(n)\mathcal{FE}(A).$$

So there exists $\nu \in x(n)A$ such that for each $\mu' \in F_x$, $\text{MCE}(\nu, \mu') = \emptyset$. Then $s_{\nu s_{\mu'}^*}^* s_{\mu'}^* = 0$ for all $\mu' \in F_x$. Applying Lemma 7.4 with $p = s_{x(n)}$, $q_0 = s_{\nu}^*$ and $Q = \{ s_{\mu'}^* : \mu' \in F_x \}$, we have $\prod_{\mu' \in F_x} (s_{x(n)} - s_{\mu'}^*) \neq 0$. So

$$q_{F_x(0, n)}^* = s_{x(0, n)} \prod_{\mu' \in F_x} (s_{x(n)} - s_{\mu'}^*)^* s_{x(0, n)}^* \neq 0.$$

We have

$$\left\| \sum_{\nu \in \forall F} b_{\mu} s_{\mu}^* s_{\mu}^* \right\| = \left\| \sum_{\nu \in \forall F} \left( \sum_{\mu \in \forall F} b_{\mu} \right) q_{\nu F}^* \right\| \quad \text{by (7.1)}$$

$$= \max_{\{\nu \in \forall F : q_{\nu F}^* \neq 0\}} \left( \sum_{\mu \in \forall F} b_{\mu} \right)$$

$$\geq \left( \sum_{\mu \in \forall F} b_{\mu} \right)_{x(0, n) \in \mathcal{Z}(\mu)}$$

$$= \left( \sum_{\mu \in \forall F} b_{\mu} \right)_{x(0, n) \in \mathcal{Z}(\mu)}$$

Hence the formula

$$h(x) \left( \sum_{\mu \in \forall F} b_{\mu} s_{\mu}^* s_{\mu}^* \right) = \sum_{\mu \in \forall F} b_{\mu}$$

determines a norm-decreasing linear map on span$\{ s_{\mu}^* : \mu \in A \}$.

To see that $h(x)$ is a homomorphism, it suffices to show that

$$h(x)(s_{\mu}^* s_{\lambda}^*) = h(x)(s_{\lambda}^*) h(x)(s_{\mu}^*).$$

Calculating the right-hand side of (7.3) yields

$$h(x)(s_{\mu}^*) h(x)(s_{\lambda}^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\lambda), \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the left-hand side of (7.3) gives

$$h(x)(s_{\mu}^* s_{\lambda}^*) = h(x) \left( \sum_{\lambda \in \text{MCE}(\mu, \alpha)} s_{\lambda}^* \right).$$

There exists at most one $\lambda \in \text{MCE}(\mu, \alpha)$ such that $x \in \mathcal{Z}(\lambda)$. Such a $\lambda$ exists if and only if $x \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\alpha)$, so

$$h(x)(s_{\mu}^* s_{\lambda}^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\alpha) \cap \mathcal{Z}(\mu), \\ 0 & \text{otherwise.} \end{cases}$$
Thus we have established (7.3), hence \( h(x) \) is a homomorphism, and thus extends uniquely to a non-zero homomorphism \( h(x) : D \rightarrow C \).

We claim the map \( h : \partial \Lambda \rightarrow \Delta(D) \) is a homeomorphism. The trickiest part is to show \( h \) is onto:

**Claim 7.5.1.** The map \( h \) is surjective.

**Proof.** Fix \( \phi \in \Delta(D) \). We seek \( x \in \partial \Lambda \) such that \( h(x) = \phi \). For each \( n \in \mathbb{N}^k \), the \( \{ s_\mu s_\mu^* : d(\mu) = n \} \) are mutually orthogonal projections. It follows that for each \( n \in \mathbb{N}^k \) there exists at most one \( \nu^m \in \Lambda^n \) such that \( \phi(s_{\nu^m} s_{\nu^m}^*) = 1 \). Let \( S \) denote the set of \( n \) for which such \( \nu^m \) exist. If \( \nu = \mu \nu' \) and \( \phi(s_{\nu} s_{\nu}^*) = 1 \), then

\[
1 = \phi(s_{\nu} s_{\nu}^*) = \phi(s_\nu s_{\nu}^* s_\mu s_\mu^*) = \phi(s_\nu s_{\nu}^*)\phi(s_\mu s_\mu^*) = \phi(s_\mu s_\mu^*).
\]

This implies that if \( n \in S \) and \( m \leq n \), then \( m \in S \) and \( \nu^m = \nu^n(0, m) \). Set \( N := \vee S \), and define \( x : \Omega_{k, N} \rightarrow \Lambda \) by \( x(p, q) = \nu^q(p, q) \). Then since each \( \nu^q \) is a \( k \)-graph morphism, so is \( x \).

We now show that \( x \in \partial \Lambda \). Fix \( n \in \mathbb{N}^k \) such that \( n \leq d(x) \), and \( E \in x(n)\mathcal{FE}(\Lambda) \). We seek \( m \in \mathbb{N}^k \) such that \( (x(n), n + m) \in E \). Since \( E \) is finite exhaustive, (CK4) implies that \( \prod_{\lambda \in E} (s_{x(n)} - s_{\lambda}s_{\lambda}^*) = 0 \). Multiplying on the left by \( s_{x(0,n)} \) and on the right by \( s_{x(0,n)}^* \) yields

\[
\prod_{\lambda \in E} (s_{x(0,n)} s_{x(0,n)}^* - s_{x(0,n)} s_{x(0,n)}^*) = 0.
\]

Thus, since \( \phi \) is a homomorphism, there exists \( \lambda \in E \) such that

\[
0 = \phi(s_{x(0,n)} s_{x(0,n)}^*) - \phi(s_{x(0,n)} s_{x(0,n)}^*) = \phi(s_{\nu^n} s_{\nu^n}^*) - \phi(s_{x(0,n)} s_{x(0,n)}^*)
\]

\[
= 1 - \phi(s_{x(0,n)} s_{x(0,n)}^*).
\]

So \( \phi(s_{x(0,n)} s_{x(0,n)}^*) = 1 \). Thus \( x(0,n)\lambda = \nu^n + d(\lambda) = x(0,n + d(\lambda)) \), and hence \( x \in \partial \Lambda \).

Now we must show that \( h(x) = \phi \). For each \( \mu \in \Lambda \) we have

\[
\phi(s_\mu s_\mu^*) = 1 \iff d(\mu) \in S \text{ and } \nu^{d(\mu)} = \mu
\]

\[
\iff x(0,d(\mu)) = \mu
\]

\[
\iff h(x)(s_\mu s_\mu^*) = 1.
\]

As \( \phi(s_\mu s_\mu^*) \) and \( h(x)(s_\mu s_\mu^*) \) take values in \( \{0, 1\} \), we have \( h(x) = \phi \). \( \blacksquare \)

To see that \( h \) is injective, suppose that \( h(x) = h(y) \). Then for each \( n \in \mathbb{N}^k \), we have

\[
h(y)(s_{x(0,n \land d(x))} s_{x(0,n \land d(x))}^*) = h(x)(s_{x(0,n \land d(x))} s_{x(0,n \land d(x))}^*) = 1.
\]

Hence \( y(0, n \land d(x)) = x(0, n \land d(x)) \). By symmetry, we also have \( y(0, n \land d(y)) = x(0, n \land d(y)) \) for all \( n \). In particular, \( d(x) = d(y) \) and \( y(0, n) = x(0, n) \) for all \( n \leq d(x) \). Thus \( x = y \).
Recall that $\Delta(D)$ carries the topology of pointwise convergence. For openness, it suffices to check that $h^{-1}$ is continuous. Suppose that $h(x^n) \to h(x)$. Fix a basic open set $\mathcal{Z}(\mu)$ containing $x$, so $h(x)(s_\mu s_\mu^*) = 1$. Since $h(x^n)(s_\mu s_\mu^*) \in \{0, 1\}$ for all $n$, for large enough $n$ we have $h(x^n)(s_\mu s_\mu^*) = 1$. So $x^n \in \mathcal{Z}(\mu)$. For continuity, a similarly straightforward argument shows that if $x^n \to x$, then $h(x^n)(s_\mu s_\mu^*) \to h(x)(s_\mu s_\mu^*)$. This convergence extends to $\text{span}\{s_\mu s_\mu^* : \mu \in A\}$ by linearity, and to $D$ by an $\varepsilon/3$ argument.

We can now prove our main result.

Proof of Theorem 7.1 Let $A$ be a row-finite $k$-graph, and $\tilde{A}$ be the desourcification described in Proposition 4.9. Let $\{s_\lambda : \lambda \in A\}$ and $\{t_\lambda : \lambda \in \tilde{A}\}$ be universal Cuntz–Krieger families in $C^*(A)$ and $C^*(\tilde{A})$. Let $A$ be the $C^*$-subalgebra of $C^*(\tilde{A})$ generated by $\{t_\lambda : \lambda \in \iota(A)\}$, and define the diagonal subalgebra of $A$ by $D_A := \text{span}\{tt_\lambda^* : \lambda \in \iota(A)\}$. Replacing $tt_\lambda^*$ with $tt_\lambda^*$ in the proof Theorem 6.3 yields $D_A \cong pD_{\tilde{A}}$. Since $A \cong C^*(A)$, it follows that $D_A \cong D_{\tilde{A}}$. Thus $D_A \cong pD_{\tilde{A}}$ as required.

We now construct $\eta$ and show that it is a homeomorphism. That $p$ commutes with $D_{\tilde{A}}$ implies that $pD_{\tilde{A}}$ is an ideal in $D_{\tilde{A}}$. Then Propositions A26(a) and A27(b) imply that the map $k : \phi \mapsto \phi|_{pD_{\tilde{A}}}$ is a homeomorphism of $\{\phi \in \Delta(D_{\tilde{A}}) : \phi|_{pD_{\tilde{A}}} \neq 0\}$ onto $\Delta(pD_{\tilde{A}})$. Since $\tilde{A}$ is row-finite with no sources, $\partial\tilde{A} = \tilde{A}^\infty$. Let $h_{\tilde{A}} : \tilde{A}^\infty \to \Delta(D_{\tilde{A}})$ be the homeomorphism obtained from Proposition 7.5. Then $h_{\tilde{A}}(x) \in \text{dom}(k)$ for all $x \in \iota(A^0)\tilde{A}^\infty$. Define $\eta := k \circ h_{\tilde{A}}\iota(A^0)\tilde{A}^\infty : \iota(A^0)\tilde{A}^\infty \to \Delta(pD_{\tilde{A}})$.

We now show that $h_A \circ \iota^{-1} \circ \pi = \rho^* \circ \eta$. Since $\rho$ is an isomorphism, it suffices to fix $x \in \iota(A^0)\tilde{A}^\infty$ and $\mu \in A$ and show that

$$(7.4) \quad (h_A \circ \iota^{-1} \circ \pi)(x)(s_\mu s_\mu^*) = (\rho^* \circ \eta)(x)(s_\mu s_\mu^*).$$

Let $\omega \in \partial A$ be such that $\pi(x) = \iota(\omega)$. Then the left-hand side of $(7.4)$ becomes

$$h_A \circ \iota^{-1} \circ \pi)(x)(s_\mu s_\mu^*) = h_A(w)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } \omega \in \mathcal{Z}(\mu), \\ 0 & \text{otherwise}. \end{cases}$$

Since $r(x) \in \iota(A^0)$, the right-hand side of $(7.4)$ simplifies to

$$(\rho^* \circ \eta)(x)(s_\mu s_\mu^*) = \eta(x)(\rho(s_\mu s_\mu^*)) = h_{\tilde{A}}(x)(tt_\mu t_\mu^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\iota(\mu)), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $x \in \mathcal{Z}(\iota(\mu))$ if and only if $\omega \in \mathcal{Z}(\mu)$. Suppose that $x \in \mathcal{Z}(\iota(\mu))$. Since $\mu \in A$ and $\pi(x) = \iota(\omega)$, we have $\pi(x(0,d(\mu))) = \pi(\iota(\mu)) = \iota(\mu)$. So $d(\pi(x(0,d(\mu)))) = d(\mu)$, and thus $d(x) \wedge d(w) \geq d(\mu)$. So $d(\omega)$

≥ \(d(\mu)\). Then we have
\[
x \in \mathcal{Z}(\nu(\mu)) \iff x(0, d(\mu)) = \nu(\mu)
\]
since \(\nu\) preserves degree
\[
\iff [\omega; (0, d(\mu))] = \nu(\mu)
\]
by Lemma 5.3
\[
\iff \nu(\omega(0, d(\mu))) = \nu(\mu)
\]
by Remark 5.4
\[
\iff \omega(0, d(\mu)) = \mu
\]
since \(\nu\) is a injective
\[
\iff \omega \in \mathcal{Z}(\mu).
\]
So equation (7.4) holds, and we are done. ■

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References

The path space of a higher-rank graph


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