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The path space of a higher-rank graph

by

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Abstract. We construct a locally compact Hausdorff topology on the path space of a finitely aligned k-graph Λ. We identify the boundary-path space $\partial\Lambda$ as the spectrum of a commutative C^*-subalgebra D_Λ of $C^*(\Lambda)$. Then, using a construction similar to that of Farthing, we construct a finitely aligned k-graph $\widetilde{\Lambda}$ with no sources in which Λ is embedded, and show that $\partial\Lambda$ is homeomorphic to a subset of $\partial\widetilde{\Lambda}$. We show that when Λ is row-finite, we can identify $C^*(\Lambda)$ with a full corner of $C^*(\widetilde{\Lambda})$, and deduce that D_Λ is isomorphic to a corner of $D_{\widetilde{\Lambda}}$. Lastly, we show that this isomorphism implements the homeomorphism between the boundary-path spaces.

1. Introduction. Cuntz and Krieger's work [2] on C^*-algebras associated to $(0,1)$-matrices and its subsequent interpretation by Enomoto and Watatani [4] were the foundation of the field we now call graph algebras. Directed graphs and their higher-rank analogues provide an intuitive framework for the analysis of this broad class of C^*-algebras; there is an explicit relationship between the dynamics of a graph and various properties of its associated C^*-algebra. Kumjian and Pask in [7] introduced *higher-rank graphs* (or *k-graphs*) as analogues of directed graphs in order to study Robertson and Steger's higher-rank Cuntz–Krieger algebras [18] using the techniques previously developed for directed graphs. Higher-rank graph C^*-algebras have received a great deal of attention in recent years, not least because they extend the already rich and tractable class of graph C^*-algebras to include all tensor products of graph C^*-algebras (and thus many Kirchberg algebras whose K_1 contains torsion elements [7]), as well as (up to Morita equivalence) the irrational rotation algebras and many other examples of simple A$\mathbb{T}$-algebras with real rank zero [8].

Although the definition of a k-graph (see Definition 2.1) is not quite as straightforward as that of a directed graph, k-graphs are a natural generalisation of directed graphs: Kumjian and Pask show in [7, Example 1.3] that 1-graphs are precisely the path-categories of directed graphs. Like directed

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graph C^*-algebras, higher-rank graph C^*-algebras were first studied using groupoid techniques. Kumjian and Pask defined the k-graph C^*-algebra $C^*(\Lambda)$ to be the universal C^*-algebra for a set of Cuntz–Krieger relations among partial isometries associated to paths of the k-graph Λ. Using direct analysis, they proved a version of the gauge-invariant uniqueness theorem for k-graph algebras. They then constructed a groupoid $\mathcal{G}_\Lambda$ from each k-graph Λ, and used the gauge-invariant uniqueness theorem to prove that the groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$ is isomorphic to $C^*(\Lambda)$. This allowed them to make use of Renault's theory of groupoid C^*-algebras to analyse $C^*(\Lambda)$. The unit space $\mathcal{G}_\Lambda^{(0)}$ of $\mathcal{G}_\Lambda$, which must be locally compact and Hausdorff, is a collection of paths in the graph: for a row-finite graph with no sources, $\mathcal{G}_\Lambda^{(0)}$ is the collection of infinite paths in Λ (the definition of an infinite path in a k-graph is not straightforward, see Remark 2.4). For more complicated graphs, the infinite paths are replaced with *boundary paths* (Definition 2.9).

In [12], Raeburn, Sims and Yeend developed a "bare-hands" analysis of k-graph C^*-algebras. They found a slightly weaker alternative to the no-sources hypothesis from Kumjian and Pask's theorems, called *local convexity* (Definition 2.7). The same authors later introduced *finitely aligned* k-graphs in [13], and gave a direct analysis of their C^*-algebras. This remains the most general class of k-graphs to which a C^*-algebra has been associated and studied in detail.

Many results for row-finite directed graphs with no sources can be extended to arbitrary graphs via a process called *desingularisation*. Given an arbitrary directed graph E, Drinen and Tomforde show in [3] how to construct a row-finite directed graph F with no sources by adding vertices and edges to E in such a way that the C^*-algebra associated to F contains the C^*-algebra associated to E as a full corner. The modified graph F is now called a *Drinen–Tomforde desingularisation* of E. Although no analogue of a Drinen–Tomforde desingularisation is currently available for higher-rank graphs, Farthing provided a construction in [5] analogous to that in [1] for removing the sources in a locally convex, row-finite higher-rank graph. The statements of the results of [5] do not contain the local convexity hypothesis, but Farthing alerted us to an issue in the proof of [5, Theorem 2.28] (see Remark 6.2), which arises when the graph is not locally convex.

The goal of this paper is to explore the path spaces of higher-rank graphs and investigate how these path spaces interact with desingularisation procedures such as Farthing's.

In Section 2, we recall the definitions and standard notation for higher-rank graphs. In Section 3, following the approach of [9], we build a topology on the path space of a higher-rank graph, and show that the path space is locally compact and Hausdorff under this topology.

In Section 4, given a finitely aligned k-graph Λ, we construct a k-graph $\widetilde{\Lambda}$ with no sources which contains a subgraph isomorphic to Λ. Our construction is modelled on Farthing's construction in [5], and the reader is directed to [5] for several proofs. The crucial difference is that our construction involves extending elements of the boundary-path space $\partial\Lambda$, whereas Farthing extends paths from a different set $\Lambda^{\leq\infty}$ (see Remark 2.10). Interestingly, although $\partial\Lambda$ and $\Lambda^{\leq\infty}$ are potentially different when Λ is row-finite and not locally convex (Proposition 2.12), our construction and Farthing's yield isomorphic k-graphs except in the non-row-finite case (Example 4.10 and Proposition 4.12). We follow Robertson and Sims' notational refinement [17] of Farthing's desourcification: we construct a new k-graph in which the original k-graph is embedded, whereas Farthing's construction adds bits onto the existing k-graph. This simplifies many arguments involving $\widetilde{\Lambda}$; however, the main reason for modifying Farthing's construction is that $\Lambda^{\leq\infty}$ is not as well-behaved topologically as $\partial\Lambda$ (see Remark 3.5), and in particular, no analogue of Theorem 5.1 holds for Farthing's construction.

In Section 5, we prove that given a row-finite k-graph Λ, there is a natural homeomorphism from the boundary-path space of Λ onto the space of infinite paths in $\widetilde{\Lambda}$ with range in the embedded copy of Λ. We provide examples and discussion showing that the topological basis constructed in Section 3 is the one we want.

In Section 6 we recall the definition of the Cuntz–Krieger algebra $C^*(\Lambda)$ of a higher-rank graph Λ. We show that if Λ is a row-finite k-graph and $\widetilde{\Lambda}$ is the graph with no sources obtained by applying the construction of Section 4 to Λ, then the embedding of Λ in $\widetilde{\Lambda}$ induces an isomorphism π of $C^*(\Lambda)$ onto a full corner of $C^*(\widetilde{\Lambda})$.

Section 7 contains results about the diagonal C^*-subalgebra of a k-graph C^*-algebra: the C^*-algebra generated by range projections associated to paths in the k-graph. We identify the boundary-path space of a finitely aligned higher-rank graph with the spectrum of its diagonal C^*-algebra. We then show that the isomorphism π of Section 6 restricts to an isomorphism of diagonals which implements the homeomorphism of Section 5.

2. Preliminaries

DEFINITION 2.1. Given $k \in \mathbb{N}$, a *k-graph* is a pair (Λ, d) consisting of a countable category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Mor}(\Lambda), r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, which satisfies the *factorisation property*: for every $\lambda \in \mathrm{Mor}(\Lambda)$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \mathrm{Mor}(\Lambda)$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$. Elements $\lambda \in \mathrm{Mor}(\Lambda)$ are called *paths*. We follow the usual abuse of notation and write $\lambda \in \Lambda$ to mean $\lambda \in \mathrm{Mor}(\Lambda)$. For $m \in \mathbb{N}^k$ we define $\Lambda^m := \{\lambda \in \Lambda :$

$d(\lambda) = m\}$. For subsets $F \subset \Lambda$ and $V \subset \mathrm{Obj}(\Lambda)$, we write $VF := \{\lambda \in F : r(\lambda) \in V\}$ and $FV := \{\lambda \in F : s(\lambda) \in V\}$. If $V = \{v\}$, we drop the braces and write vF and Fv. A morphism between two k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \to \Lambda_2$ which respects the degree maps. The factorisation property allows us to identify $\mathrm{Obj}(\Lambda)$ with Λ^0. We refer to elements of Λ^0 as *vertices*.

Remark 2.2. To visualise a k-graph we draw its 1-*skeleton*: a directed graph with vertices Λ^0 and edges $\bigcup_{i=1}^{k} \Lambda^{e_i}$. To each edge we assign a colour determined by the edge's degree. We tend to use 2-graphs for examples, and we draw edges of degree $(1,0)$ as solid lines, and edges of degree $(0,1)$ as dashed lines.

Example 2.3. For $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, we define k-graphs $\Omega_{k,m}$ as follows. Set $\mathrm{Obj}(\Omega_{k,m}) = \{p \in \mathbb{N}^k : p_i \le m_i \text{ for all } i \le k\}$,

$$\mathrm{Mor}(\Omega_{k,m}) = \{(p,q) : p, q \in \mathrm{Obj}(\Omega_{k,m}) \text{ and } p_i \le q_i \text{ for all } i \le k\},$$

$r(p,q) = p$, $s(p,q) = q$ and $d(p,q) = q - p$, with composition given by $(p,q)(q,t) = (p,t)$. If $m = (\infty)^k$, we drop m from the subscript and write Ω_k. The 1-skeleton of $\Omega_{2,2}$ is depicted in Figure 1.

$$\begin{tikzcd}
(0,2) \arrow[d, dashed] & (1,2) \arrow[l] \arrow[d, dashed] & (2,2) \arrow[l] \arrow[d, dashed] \\
(0,1) \arrow[d, dashed] & (1,1) \arrow[l] \arrow[d, dashed] & (2,1) \arrow[l] \arrow[d, dashed] \\
(0,0) & (1,0) \arrow[l] & (2,0) \arrow[l]
\end{tikzcd}$$

Fig. 1. The 2-graph $\Omega_{2,2}$

Remark 2.4. The graphs $\Omega_{k,m}$ provide an intuitive model for paths: every path λ of degree m in a k-graph Λ determines a k-graph morphism $x_\lambda : \Omega_{k,m} \to \Lambda$. To see this, let $p, q \in \mathbb{N}^k$ be such that $p \le q \le m$. Define $x_\lambda(p,q) = \lambda''$, where $\lambda = \lambda'\lambda''\lambda'''$, and $d(\lambda') = p$, $d(\lambda'') = q - p$ and $d(\lambda''') = m - q$. In this way, paths in Λ are often identified with the graph morphisms $x_\lambda : \Omega_{k,m} \to \Lambda$. We refer to the segment λ'' of λ (as factorised above) as $\lambda(p,q)$, and for $n \le m$, we refer to the vertex $r(\lambda(n,m)) = s(\lambda(0,n))$ as $\lambda(n)$. By analogy, for $m \in (\mathbb{N} \cup \{\infty\})^k$ we define $\Lambda^m := \{x : \Omega_{k,m} \to \Lambda : x \text{ is a graph morphism}\}$. For clarity of notation, if $m = (\infty)^k$ we write Λ^∞.

Define

$$W_\Lambda := \bigcup_{n \in (\mathbb{N} \cup \{\infty\})^k} \Lambda^n.$$

We call W_Λ the *path space* of Λ. We drop the subscript when confusion is unlikely.

For $m, n \in \mathbb{N}^k$, we denote by $m \wedge n$ the coordinate-wise minimum, and by $m \vee n$ the coordinate-wise maximum. With no parentheses, $\vee$ and $\wedge$ take priority over the group operation: $a - b \wedge c$ means $a - (b \wedge c)$.

Since finite and infinite paths are fundamentally different, that one can compose them is not immediately obvious.

Lemma 2.5 ([19, Proposition 3.0.1.1]). *Let Λ be a k-graph. Suppose $\lambda \in \Lambda$ and suppose that $x \in W$ satisfies $r(x) = s(\lambda)$. Then there exists a unique k-graph morphism $\lambda x : \Omega_{k,d(\lambda)+d(x)} \to \Lambda$ such that $(\lambda x)(0, d(\lambda)) = \lambda$ and $(\lambda x)(d(\lambda), n + d(\lambda)) = x(0, n)$ for all $n \leq d(x)$.*

Definition 2.6. For $\lambda, \mu \in \Lambda$, write

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \lambda\alpha = \mu\beta,\ d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}$$

for the collection of pairs which give *minimal common extensions* of λ and μ, and denote the set of minimal common extensions by

$$\mathrm{MCE}(\lambda, \mu) := \{\lambda\alpha : (\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)\} = \{\mu\beta : (\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)\}.$$

Definition 2.7. A k-graph Λ is *row-finite* if for each $v \in \Lambda^0$ and $m \in \mathbb{N}^k$, the set $v\Lambda^m$ is finite; Λ has *no sources* if $v\Lambda^m \neq \emptyset$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

We say that Λ is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$.

As in [12, Definition 3.1], a k-graph Λ is *locally convex* if for all $v \in \Lambda^0$, all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, all $\lambda \in v\Lambda^{e_i}$ and all $\mu \in v\Lambda^{e_j}$, the sets $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are non-empty. Roughly speaking, local convexity stipulates that Λ contains no subgraph resembling

$$\begin{tikzcd} u \arrow[d, dashed, "\mu"'] & \\ v & w \arrow[l, "\lambda"] \end{tikzcd}$$

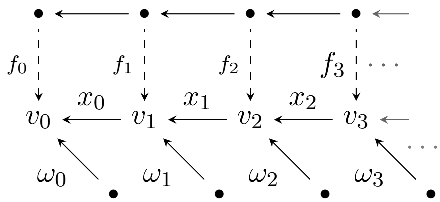
Definition 2.8. For $v \in \Lambda^0$, a subset $E \subset v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$ there exists a $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. We denote the set of all *finite exhaustive subsets* of Λ by $\mathcal{FE}(\Lambda)$.

Definition 2.9. An element $x \in W$ is a *boundary path* if for all $n \in \mathbb{N}^k$ with $n \leq d(x)$ and for all $E \in x(n)\mathcal{FE}(\Lambda)$ there exists $m \in \mathbb{N}^k$ such that $x(n, m) \in E$. We write $\partial\Lambda$ for the set of all boundary paths.

Define the set $\Lambda^{\leq\infty}$ as follows. A k-graph morphism $x : \Omega_{k,m} \to \Lambda$ is an element of $\Lambda^{\leq\infty}$ if there exists $n_x \leq d(x)$ such that for $n \in \mathbb{N}^k$ satisfying $n_x \leq n \leq d(x)$ and $n_i = d(x)_i$, we have $x(n)\Lambda^{e_i} = \emptyset$.

Remark 2.10. Raeburn, Sims and Yeend introduced $\Lambda^{\leq\infty}$ to construct a non-zero Cuntz–Krieger Λ-family [13, Proposition 2.12]. Farthing, Muhly and Yeend introduced $\partial\Lambda$ in [6]; in order to construct a groupoid to which Renault's theory of groupoid C^*-algebras [15] applied, they required a path space which was locally compact and Hausdorff in an appropriate topology, and $\Lambda^{\leq\infty}$ did not suffice. The differences between $\partial\Lambda$ and $\Lambda^{\leq\infty}$ can be easily seen if Λ contains any infinite receivers (e.g. any path in a 1-graph Λ with source an infinite receiver is an element of $\partial\Lambda \setminus \Lambda^{\leq\infty}$), but can even show themselves in the row-finite case if Λ is not locally convex.

Example 2.11. Suppose Λ is the 2-graph with the skeleton pictured below:



Consider the paths $x = x_0x_1\ldots$, and $\omega^n = x_0x_1\ldots x_{n-1}\omega_n$ for $n = 0, 1, 2, \ldots$. Observe that $x \notin \Lambda^{\leq\infty}$: for each $n \in \mathbb{N}$, we have $d(x)_2 = 0 = (n, 0)_2$, and $x((n, 0))\Lambda^{e_2} = v_n\Lambda^{e_2} \neq \emptyset$.

We claim that $x \in \partial\Lambda$. Fix $m \in \mathbb{N}$ and $E \in v_m\mathcal{FE}(\Lambda)$. Since E is exhaustive, for each $n \geq m$, there exists $\lambda^n \in E$ such that $\mathrm{MCE}(\lambda^n, x_m \ldots x_{n-1}\omega_n)$ is non-empty. Since E is finite, it cannot contain $x_m \ldots x_{n-1}\omega_n$ for every $n \geq m$, so it must contain $x_m \ldots x_p$ for some $p \in \mathbb{N}$. So $x((m, 0), (m + p)) = x_m \ldots x_p$ belongs to E.

The 2-graph of Example 2.11 first appeared in Robertson's honours thesis [16] to illustrate a subtlety arising in Farthing's procedure [5] for removing sources in k-graphs when the k-graphs in question are not locally convex. It was for this reason that only locally convex k-graphs were considered in the main results of [16, 17].

Proposition 2.12. *Suppose Λ is a finitely aligned k-graph. Then $\Lambda^{\leq\infty} \subset \partial\Lambda$. If Λ is row-finite and locally convex, then $\Lambda^{\leq\infty} = \partial\Lambda$.*

To prove this we use the following lemma.

Lemma 2.13. *Let Λ be a row-finite, locally convex k-graph, and suppose that $v \in \Lambda^0$ satisfies $v\Lambda^{e_i} \neq \emptyset$ for some $i \leq k$. Then $v\Lambda^{e_i} \in v\mathcal{FE}(\Lambda)$.*

Proof. Since Λ is row-finite, $v\Lambda^{e_i}$ is finite. To see that it is exhaustive, let $\mu \in v\Lambda$. If $d(\mu)_i > 0$, then $g = \mu(0, e_i) \in v\Lambda^{e_i}$ implies that $\Lambda^{\min}(\mu, g) \neq \emptyset$. Suppose that $d(\mu)_i = 0$. Let $\mu = \mu_1 \ldots \mu_n$ be a factorisation of μ such that $|d(\mu_j)| = 1$ for each $j \leq n$. Since Λ is locally convex, $s(\mu)\Lambda^{e_i} = s(\mu_n)\Lambda^{e_i} \neq \emptyset$.

Fix $g \in s(\mu)\Lambda^{e_i}$. Let $f := (\mu g)(0, e_i)$. Then $f \in v\Lambda^{e_i}$. Since $d(\mu_i) = 0$, we have $d(\mu g) = d(\mu) \vee d(f)$. Hence $(g, (\mu g)(e_i, d(\mu g))) \in \Lambda^{\min}(\mu, f)$ as required. ∎

Proof of Proposition 2.12. Fix $x \in \Lambda^{\leq\infty}$, $m \leq d(x)$ and $E \in x(m)\mathcal{FE}(\Lambda)$. Define $t \in \mathbb{N}^k$ by

$$t_i := \begin{cases} d(x)_i & \text{if } d(x)_i < \infty, \\ \max_{\lambda \in E}(n_x \vee (m + d(\lambda)))_i & \text{if } d(x)_i = \infty. \end{cases}$$

Then $x(m, t) \in x(m)\Lambda$, so there exists $\lambda \in E$ such that $\Lambda^{\min}(x(m, t), \lambda)$ is non-empty. Let $(\alpha, \beta) \in \Lambda^{\min}(x(m, t), \lambda)$. We first show that $d(\alpha) = 0$. Since $x \in \Lambda^{\leq\infty}$ and $n_x \leq t \leq d(x)$, if $d(x)_i < \infty$ then $x(t)\Lambda^{e_i} = \emptyset$. So for each i such that $d(x)_i < \infty$, we have $d(\alpha)_i = 0$. Now suppose that $d(x)_i = \infty$. Then $d(x(m, t))_i = t_i - m_i \geq d(\lambda)_i$. So $d(x(m, t)\alpha)_i = \max\{d(x(m, t))_i, d(\lambda)_i\} = d(x(m, t))_i$, giving $d(\alpha)_i = 0$. Hence $x(m, t) = \lambda\beta$, so $x(m, m + d(\lambda)) = \lambda$.

Now suppose that Λ is row-finite and locally convex. We want to show $\partial\Lambda \subset \Lambda^{\leq\infty}$. Fix $x \in \partial\Lambda$, and $n \in \mathbb{N}^k$ such that $n \leq d(x)$ and $n_i = d(x)_i$. It suffices to show that $x(n)\Lambda^{e_i} = \emptyset$. Since $n_i = d(x)_i$, we have $x(n)\Lambda^{e_i} \notin x(n)\mathcal{FE}(\Lambda)$. Lemma 2.13 then implies that $x(n)\Lambda^{e_i} = \emptyset$. ∎

3. Path space topology. Following the approach of Paterson and Welch in [9], we construct a locally compact Hausdorff topology on the path space W of a finitely aligned k-graph Λ. The *cylinder set* of $\mu \in \Lambda$ is $\mathcal{Z}(\mu) := \{\nu \in W : \nu(0, d(\mu)) = \mu\}$. Define $\alpha : W \to \{0, 1\}^\Lambda$ by $\alpha(w)(y) = 1$ if $w \in \mathcal{Z}(y)$ and 0 otherwise. For a finite subset $G \subset s(\mu)\Lambda$ we define

$$\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\mu\nu). \tag{3.1}$$

Our goals for this section are the following two theorems. The basis we end up with is slightly different to that in [9, Corollary 2.4], revealing a minor oversight of the authors.

THEOREM 3.1. *Let Λ be a finitely aligned k-graph. Then the collection*

$$\Big\{\mathcal{Z}(\mu \setminus G) : \mu \in \Lambda \text{ and } G \subset \bigcup_{i=1}^{k}(s(\mu)\Lambda^{e_i}) \text{ is finite}\Big\}$$

is a base for the initial topology on W induced by $\{\alpha\}$.

THEOREM 3.2. *Let Λ be a finitely aligned higher-rank graph. With the topology described in Theorem* 3.1, *W is a locally compact Hausdorff space.*

Let F be a set of paths in a k-graph Λ. A path $\beta \in W$ is a *common extension of the paths in F* if for each $\mu \in F$, we can write $\beta = \mu\beta_\mu$ for some $\beta_\mu \in W$. If in addition $d(\beta) = \bigvee_{\mu \in F} d(\mu)$, then β is a *minimal common extension of the paths in F*. We denote the set of all minimal common extensions of the paths in F by $\mathrm{MCE}(F)$. Since $\mathrm{MCE}(\{\mu,\nu\}) = \mathrm{MCE}(\mu,\nu)$, this definition is consistent with Definition 2.6.

REMARK 3.3. If $F \subset \Lambda$ is finite, then $\bigcap_{\mu\in F} \mathcal{Z}(\mu) = \bigcup_{\beta\in\mathrm{MCE}(F)} \mathcal{Z}(\beta)$.

Proof of Theorem 3.1. We first describe the topology on $\{0,1\}^\Lambda$. Given disjoint finite subsets $F, G \subset \Lambda$ and $\mu \in \Lambda$, define sets $U_\mu^{F,G}$ to be $\{1\}$ if $\mu \in F$, $\{0\}$ if $\mu \in G$ and $\{0,1\}$ otherwise. Then the sets $N(F,G) := \prod_{\mu\in\Lambda} U_\mu^{F,G}$, where F, G range over all finite disjoint pairs of subsets of Λ, form a base for the topology on $\{0,1\}^\Lambda$.

Clearly, α is a homeomorphism onto its range, so the sets $\alpha^{-1}(N(F,G))$ are a base for a topology on W. Routine calculation shows that

$$\alpha^{-1}(N(F,G)) = \Big(\bigcup_{\mu\in\mathrm{MCE}(F)} \mathcal{Z}(\mu)\Big) \setminus \Big(\bigcup_{\nu\in G} \mathcal{Z}(\nu)\Big),$$

so the sets $\mathcal{Z}(\mu) \setminus \bigcup_{\nu\in G} \mathcal{Z}(\mu\nu) = \mathcal{Z}(\mu \setminus G)$ are a base for our topology.

To finish the proof, it suffices to show that for $\mu \in \Lambda$, a finite subset $G \subset s(\mu)\Lambda$ and $\lambda \in \mathcal{Z}(\mu \setminus G)$, there exist $\alpha \in \Lambda$ and a finite $F \subset \bigcup_{i=1}^k (s(\alpha)\Lambda^{e_i})$ such that $\lambda \in \mathcal{Z}(\alpha \setminus F) \subset \mathcal{Z}(\mu \setminus G)$. Let $N := (\bigvee_{\nu\in G} d(\mu\nu)) \wedge d(\lambda)$ and $\alpha = \lambda(0,N)$. To define F, we first define a set F_ν associated to each $\nu \in G$, then take $F = \bigcup_{\nu\in G} F_\nu$. Fix $\nu \in G$. We consider the following cases:

(1) If $N \geq d(\mu\nu)$ or $\mathrm{MCE}(\alpha,\mu\nu) = \emptyset$, let $F_\nu = \emptyset$.
(2) If $N \not\geq d(\mu\nu)$ and $\mathrm{MCE}(\alpha,\mu\nu) \neq \emptyset$, define F_ν as follows: Since $N \not\geq d(\mu\nu)$, there exists $j_\nu \leq k$ such that $N_{j_\nu} < d(\mu\nu)_{j_\nu}$. Hence each $\gamma \in \mathrm{MCE}(\alpha,\mu\nu)$ satisfies $d(\gamma)_{j_\nu} = (N \vee d(\mu\nu))_{j_\nu} > N_{j_\nu}$. Define $F_\nu = \{\gamma(N, N+e_{j_\nu}) : \gamma \in \mathrm{MCE}(\alpha,\mu\nu)\}$. Since Λ is finitely aligned, F_ν is finite.

We now show that $\lambda \in \mathcal{Z}(\alpha \setminus F)$. We have $\lambda \in \mathcal{Z}(\alpha)$ by choice of α. If $F = \emptyset$ we are done. If not, then fix $\nu \in G$ such that $F_\nu \neq \emptyset$, and fix $e \in F_\nu$. Then $e = \gamma(N, N+e_{j_\nu})$ for some $\gamma \in \mathrm{MCE}(\alpha,\mu\nu)$. Therefore $d(\lambda)_{j_\nu} = N_{j_\nu} < (N+e_{j_\nu})_{j_\nu} = d(\alpha e)_{j_\nu}$. So $\lambda \notin \mathcal{Z}(\alpha e)$, hence $\lambda \in \mathcal{Z}(\alpha \setminus F)$.

We now show that $\mathcal{Z}(\alpha \setminus F) \subset \mathcal{Z}(\mu \setminus G)$. Fix $\beta \in \mathcal{Z}(\alpha \setminus F)$. Since $\alpha \in \mathcal{Z}(\mu)$, we have $\beta \in \mathcal{Z}(\mu)$. Fix $\nu \in G$. We show that $\beta \notin \mathcal{Z}(\mu\nu)$ in cases:

(1) Suppose that $N \geq d(\mu\nu)$. Since $\beta \in \mathcal{Z}(\alpha) = \mathcal{Z}(\lambda(0,N))$ and $\lambda \notin \mathcal{Z}(\mu\nu)$, it follows that $\beta \notin \mathcal{Z}(\mu\nu)$.
(2) If $N \not\geq d(\mu\nu)$, then either
    (a) $\mathrm{MCE}(\alpha,\mu\nu) = \emptyset$, in which case $\beta \in \mathcal{Z}(\alpha)$ implies that $\beta \notin \mathcal{Z}(\mu\nu)$; or
    (b) $\mathrm{MCE}(\alpha,\mu\nu) \neq \emptyset$. Then for each $\gamma \in \mathrm{MCE}(\alpha,\mu\nu)$, we know $\beta(N, N+e_{j_\nu}) \neq \gamma(N, N+e_{j_\nu})$. Hence $\beta \notin \mathcal{Z}(\mu\nu)$. ■

LEMMA 3.4. *Let $\{\nu^{(n)}\}$ be a sequence of paths in Λ such that*

(i) $d(\nu^{(n+1)}) \geq d(\nu^{(n)})$ *for all* $n \in \mathbb{N}$,
(ii) $\nu^{(n+1)}(0, d(\nu^{(n)})) = \nu^{(n)}$ *for all* $n \in \mathbb{N}$.

Then there exists a unique $\omega \in W$ *such that* $d(\omega) = \bigvee_{n\in\mathbb{N}} d(\nu^{(n)})$ *and* $\omega(0, d(\nu^{(n)})) = \nu^{(n)}$ *for all* $n \in \mathbb{N}$.

Proof. Let $m = \bigvee_{n\in\mathbb{N}} d(\nu^{(n)}) \in (\mathbb{N} \cup \{\infty\})^k$. Then

(3.2) For $a \in \mathbb{N}^k$ with $a \leq m$, there exists $N_a \in \mathbb{N}$ such that $d(\nu^{(N_a)}) \geq a$.

For each $(p, q) \in \Omega_{k,m}$ apply (3.2) with $a = q$ and define $\omega(p, q) = \nu^{(N_q)}(p, q)$. Routine calculations using (3.2) show that $\omega : \Omega_{k,m} \to \Lambda$ is a well-defined graph morphism with the required properties. ∎

Proof of Theorem 3.2. Fix $v \in \Lambda^0$. We follow the strategy of [9, Theorem 2.2] to show $\mathcal{Z}(v)$ is compact: since α is a homeomorphism onto its range, and since $\{0, 1\}^\Lambda$ is compact, it suffices to prove that $\alpha(\mathcal{Z}(v))$ is closed in $\{0, 1\}^\Lambda$. Suppose that $(\omega^{(n)})_{n\in\mathbb{N}}$ is a sequence in $\mathcal{Z}(v)$ converging to $f \in \{0, 1\}^\Lambda$. We seek $\omega \in \mathcal{Z}(v)$ such that $f = \alpha(\omega)$. Define $A = \{\nu \in \Lambda : \alpha(\omega^{(n)})(\nu) \to 1 \text{ as } n \to \infty\}$. Then $A \neq \emptyset$ since $v \in A$. Let $d(A) := \bigvee_{\nu\in A} d(\nu)$.

CLAIM 3.2.1. *There exists* $\omega \in v\Lambda^{d(A)}$ *such that:*

- $d(\omega) \geq d(\mu)$ *for all* $\mu \in A$,
- $\omega(0, n) \in A$ *for all* $n \in \mathbb{N}^k$ *with* $n \leq d(A)$.

Proof. To define ω we construct a sequence of paths and apply Lemma 3.4. We first show that for each pair $\mu, \nu \in A$, $\mathrm{MCE}(\mu, \nu) \cap A$ contains exactly one element. Fix $\mu, \nu \in A$. Then for large enough n, there exist $\beta^n \in \mathrm{MCE}(\mu, \nu)$ such that $\omega^n = \beta^n(\omega^n)'$. Since $\mathrm{MCE}(\mu, \nu)$ is finite, there exists M such that $\omega^n = \beta^M(\omega^n)'$ for infinitely many n. Define $\beta_{\mu,\nu} := \beta^M$. Then $\beta_{\mu,\nu} \in A$. For uniqueness, suppose that $\phi \in \mathrm{MCE}(\mu, \nu) \cap A$. Then for large n we have $\beta_{\mu,\nu} = \omega^n(0, d(\mu) \vee d(\nu)) = \phi$.

Since A is countable, we can list $A = \{\nu^1, \nu^2, \ldots\}$. Let $y^1 := \nu^1$, and iteratively define $y^n = \beta_{y^{n-1},\nu^n}$. Then $d(y^n) = d(y^{n-1}) \vee d(\nu^n) \geq d(y^{n-1})$, and $y^n(0, y^{n-1}) = y^{n-1}$. By Lemma 3.4, there exists a unique $\omega \in W$ satisfying $d(\omega) = d(A)$ and $\omega(0, d(y^n)) = y^n$ for all n. It then follows from (3.2) that $\omega(0, n) \in A$ for all $n \leq d(A)$. ∎Claim

To see $\alpha(\mathcal{Z}(v))$ is closed, fix $\lambda \in \Lambda$. We show that $\alpha(\omega^{(n)})(\lambda) \to \alpha(\omega)(\lambda)$. If $\alpha(\omega)(\lambda) = 1$, then $\lambda = \omega(0, d(\lambda)) \in A$ by Claim 3.2.1, and thus $\alpha(\omega^{(n)})(\lambda)$ approaches 1 as n approaches ∞. Now suppose that $\alpha(\omega)(\lambda) = 0$. If $d(\lambda) \not\leq d(\omega)$, then $\lambda \notin A$ by Claim 3.2.1, forcing $\alpha(\omega^{(n)})(\lambda) \to 0$. Suppose that $d(\lambda) \leq d(\omega)$. Since $\omega(0, d(\lambda)) \in A$, we have $\omega^{(n)}(0, d(\lambda)) = \omega(0, d(\lambda))$ for large n. Then $\alpha(\omega)(\lambda) = 0$ implies that $\omega(0, d(\lambda)) \neq \lambda$. So for large enough n we have $\omega^{(n)}(0, d(\lambda)) \neq \lambda$, forcing $\alpha(\omega^{(n)})(\lambda) \to 0$. ∎

Remark 3.5. It has been shown that $\partial\Lambda$ is a closed subset of W [6, Lemma 5.12]. Hence $\partial\Lambda$, with the relative topology, is a locally compact Hausdorff space. Consider the 2-graph of Example 2.11. For each $n \in \mathbb{N}$, we have $\omega^n \in \Lambda^{\leq\infty}$. Notice that $\omega^n \to x \notin \Lambda^{\leq\infty}$. So $\Lambda^{\leq\infty}$ is not closed in general, and hence is not locally compact.

4. Removing sources

Theorem 4.1. *Let Λ be a finitely aligned k-graph. Then there exists a finitely aligned k-graph $\widetilde{\Lambda}$ with no sources, and an embedding ι of Λ in $\widetilde{\Lambda}$. If Λ is row-finite, then so is $\widetilde{\Lambda}$.*

Definition 4.2. Define a relation $\approx$ on $V_\Lambda := \{(x; m) : x \in \partial\Lambda,\ m \in \mathbb{N}^k\}$ by: $(x; m) \approx (y; p)$ if and only if

(V1) $x(m \wedge d(x)) = y(p \wedge d(y))$,
(V2) $m - m \wedge d(x) = p - p \wedge d(y)$.

Definition 4.3. Define a relation $\sim$ on $P_\Lambda := \{(x; (m, n)) : x \in \partial\Lambda,\ m \leq n \in \mathbb{N}^k\}$ by: $(x; (m, n)) \sim (y; (p, q))$ if and only if

(P1) $x(m \wedge d(x), n \wedge d(Px)) = y(p \wedge d(y), q \wedge d(y))$,
(P2) $m - m \wedge d(x) = p - p \wedge d(y)$,
(P3) $n - m = q - p$.

It is clear from the definitions that both $\approx$ and $\sim$ are equivalence relations.

Lemma 4.4. *Suppose that $(x; (m, n)) \sim (y; (p, q))$. Then $n - n \wedge d(x) = q - q \wedge d(y)$.*

Proof. It follows from (P1) and (P3) that

$$n - n \wedge d(x) - (m - m \wedge d(x)) = q - q \wedge d(y) - (p - p \wedge d(y)).$$

The result is then a consequence of (P2). ∎

Let $\widetilde{P}_\Lambda := P_\Lambda/\sim$ and $\widetilde{V}_\Lambda := V_\Lambda/\approx$. The class in $\widetilde{P}_\Lambda$ of $(x; (m, n)) \in P_\Lambda$ is denoted $[x; (m, n)]$, and similarly the class in $\widetilde{V}_\Lambda$ of $(x; m) \in V_\Lambda$ is denoted $[x; m]$.

To define the range and source maps, observe that if $(x; (m, n)) \sim (y; (p, q))$, then $(x; m) \approx (y; p)$ by definition of $\sim$, and $(x; n) \approx (y; q)$ by Lemma 4.4. We define range and source maps as follows.

Definition 4.5. Define $\widetilde{r}, \widetilde{s} : \widetilde{P}_\Lambda \to \widetilde{V}_\Lambda$ by

$$\widetilde{r}([x; (m, n)]) = [x; m] \quad \text{and} \quad \widetilde{s}([x; (m, n)]) = [x, n].$$

We now define composition. For each $m \in \mathbb{N}^k$, we define the *shift map* $\sigma^m : \bigcup_{n \geq m} \Lambda^n \to \Lambda$ by $\sigma^m(\lambda)(p, q) = \lambda(p + m, q + m)$.

PROPOSITION 4.6. *Suppose that Λ is a k-graph, and that $[x; (m, n)]$ and $[y; (p, q)]$ are elements of $\widetilde{P}_\Lambda$ satisfying $[x; n] = [y; p]$. Let $z := x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y$. Then*

(1) $z \in \partial\Lambda$,
(2) $m \wedge d(x) = m \wedge d(z)$ *and* $n \wedge d(x) = n \wedge d(z)$,
(3) $x(m \wedge d(x), n \wedge d(x)) = z(m \wedge d(z), n \wedge d(z))$ *and* $y(p \wedge d(y), q \wedge d(y)) = z(n \wedge d(z), (n + q - p) \wedge d(z))$.

Proof. Part (1) follows from [6, Lemma 5.13], and (2) and (3) can be proved as in [5, Proposition 2.11]. ■

Fix $[x; (m, n)], [y; (p, q)] \in \widetilde{P}_\Lambda$ such that $[x; n] = [y; p]$, and let $z = x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y$. That the formula

$$[x; (m, n)] \circ [y; (p, q)] = [z; (m, n + q - p)] \tag{4.1}$$

determines a well-defined composition follows from Proposition 4.6.

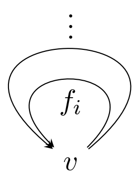
Define $\mathrm{id} : \widetilde{V}_\Lambda \to \widetilde{P}_\Lambda$ by $\mathrm{id}_{[x;m]} = [x; (m, m)]$.

PROPOSITION 4.7 ([5, Lemma 2.19]). $\widetilde{\Lambda} := (\widetilde{V}_\Lambda, \widetilde{P}_\Lambda, \widetilde{r}, \widetilde{s}, \circ, \mathrm{id})$ *is a category.*

DEFINITION 4.8. Define $\widetilde{d} : \widetilde{\Lambda} \to \mathbb{N}^k$ by $\widetilde{d}(v) = \star$ for all $v \in \widetilde{V}_\Lambda$, and $\widetilde{d}([x; (m, n)]) = n - m$ for all $[x; (m, n)] \in \widetilde{P}_\Lambda$.

PROPOSITION 4.9 ([5, Theorem 2.22]). *The map $\widetilde{d}$ defined above has the factorisation property. Hence with $\widetilde{\Lambda}$ as in Proposition 4.7, $(\widetilde{\Lambda}, \widetilde{d})$ is a k-graph with no sources.*

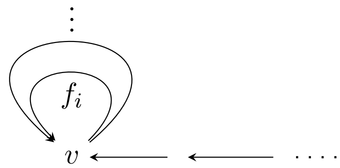
EXAMPLE 4.10. If we allow infinite receivers, our construction yields a different k-graph to Farthing's construction in [5, §2]: consider the 1-graph E with an infinite number of loops f_i on a single vertex v:



Here we have $E^{\leq \infty} = \emptyset$, so Farthing's construction yields a 1-graph $\overline{E} \cong E$. Since v belongs to every finite exhaustive set in E, we have $\partial E = E$. Furthermore $[f_j; p] = [f_i; p] = [v; p]$ for all $i, j, p \in \mathbb{N}$, and

$$[f_j; (p, q)] = [f_i; (p, q)] = [v; (p - 1, q - 1)]$$

for all i, j, p, q such that $1 < p \leq q$. Thus there is exactly one path between any two of the added vertices, resulting in a head at v, yielding the graph illustrated below:



It is intriguing that following Drinen and Tomforde's desingularisation, a head is also added at infinite receivers like this, and then the ranges of the edges f_i are distributed along this head—we cannot help but wonder whether this might suggest an approach to a Drinen–Tomforde desingularisation for k-graphs.

4.1. Row-finite 1-graphs. While one expects this style of desourcification to agree with adding heads to a row-finite 1-graph as in [1], this appears not to have been checked anywhere.

PROPOSITION 4.11. *Let E be a row-finite directed graph and F be the graph obtained by adding heads to sources, as in* [1, p. 4]. *Let Λ be the* 1*-graph associated to E. Then $\widetilde{\Lambda} \cong F^*$, where F^* is a the path-category of F.*

Proof. Define $\eta' : P_\Lambda \to F^*$ as follows. Fix $x \in \partial E$ and $m, n \in \mathbb{N}$. Then either $x \in E^\infty$, or $x \in E^*$ and $s(x)$ is a source in E. If $x \in E^\infty$, define $\eta'((x; (m, n))) = x(m, n)$. For $x \in E^*$, let μ_x be the head added to $s(x)$, and define $\eta'((x; (m, n))) = (x\mu_x)(m, n)$. It is straightforward to check that η' respects the equivalence relation $\sim$ on P_Λ. Define $\eta : \widetilde{\Lambda} \to F^*$ by $\eta([x; (m, n)]) = \eta'((x; (m, n)))$. Easy but tedious calculations show that η is a graph morphism.

We now construct a graph morphism $\xi : F^* \to \widetilde{\Lambda}$. Let $\nu \in F^*$. To define ξ we first need some preliminary notation. We will define ξ casewise, broken up as follows:

(i) $\nu \in E^*$,
(ii) $r(\nu) \in E^*$ and $s(\nu) \in F^* \setminus E^*$, or
(iii) $r(\nu), s(\nu) \in F^* \setminus E^*$.

If $\nu \in E^*$, fix $\alpha_\nu \in s(\nu)\partial E$. If ν has $r(\nu) \in E^*$ and $s(\nu) \in F^* \setminus E^*$, let $p_\nu = \max\{p \in \mathbb{N} : \nu(0, p) \in E^*\}$. Then $\nu(p_\nu)$ is a source in E^*, and $\nu(0, p_\nu) \in \partial E$. If $\nu \in F^* \setminus E^*$, then ν is a segment of a head μ_ν added to a source in E^*, and we let q_ν be such that $\nu = \mu_\nu(q_\nu, q_\nu + d(\mu))$.

We then define ξ by

$$\xi(\nu) = \begin{cases} [\nu\alpha_\nu; (0, d(\nu))] & \text{if } \nu \in E^*, \\ [\nu(0, p_\nu); (0, d(\nu))] & \text{if } r(\nu) \in E^* \text{ and } s(\nu) \notin E^*, \\ [r(\mu_\nu); (q_\nu, q_\nu + d(\nu))] & \text{if } r(\nu), s(\nu) \in F^* \setminus E^*. \end{cases}$$

Again, tedious but straightforward calculations show that ξ is a well-defined graph morphism, and that $\xi \circ \eta = 1_{\widetilde{\Lambda}}$ and $\eta \circ \xi = 1_{F^*}$. ∎

When Λ is row-finite and locally convex, Proposition 2.12 implies that $\Lambda^{\leq\infty} = \partial\Lambda$. In this case our construction is essentially the same as that of Farthing [5, §2], with notation as in [17]. If Λ is row-finite but not locally convex, then $\Lambda^{\leq\infty} \subset \partial\Lambda$ (Example 2.11 shows that this may be a strict containment). Thus it is reasonable to suspect that our construction could result in a larger path space than Farthing's. Interestingly, this is not the case.

Proposition 4.12. *Let Λ be a row-finite k-graph. Suppose that $x \in \partial\Lambda \setminus \Lambda^{\leq\infty}$ and $m \leq n \in \mathbb{N}^k$. Then there exists $y \in \Lambda^{\leq\infty}$ such that $(x; (m, n)) \sim (y; (m, n))$.*

Proof. Since $x \notin \Lambda^{\leq\infty}$, there exists $q \geq n \wedge d(x)$ and $i \leq k$ such that $q \leq d(x)$, $q_i = d(x)_i$, and $x(q)\Lambda^{e_i} \neq \emptyset$. Let

$$J := \{i \leq k : q_i = d(x)_i \text{ and } x(q)\Lambda^{e_i} \neq \emptyset\}.$$

Since $x \in \partial\Lambda$, for each $E \in x(q)\mathcal{FE}(\Lambda)$ there exists $t \in \mathbb{N}^k$ such that $x(q, q + t) \in E$. Since $q_i = d(x)_i$ for all $i \in J$, the set $\bigcup_{i\in J} x(q)\Lambda^{e_i}$ contains no such segments of x, and thus cannot be finite exhaustive. Since Λ is row-finite, $\bigcup_{i\in J} x(q)\Lambda^{e_i}$ is finite, so $\bigcup_{i\in J} x(q)\Lambda^{e_i}$ is not exhaustive. Thus there exists $\mu \in x(q)\Lambda$ such that $\mathrm{MCE}(\mu, \nu) = \emptyset$ for all $\nu \in \bigcup_{i\in J} x(q)\Lambda^{e_i}$. By [13, Lemma 2.11], $s(\mu)\Lambda^{\leq\infty} \neq \emptyset$. Let $z \in s(\mu)\Lambda^{\leq\infty}$, and define $y := x(0, q)\mu z$. Then $y \in \Lambda^{\leq\infty}$ by [13, Lemma 2.10].

Now we show that $(x; (m, n)) \sim (y; (m, n))$. Condition (P3) is trivially satisfied. To see that (P1) and (P2) hold, it suffices to show that $n \wedge d(x) = n \wedge d(y)$. Firstly, let $i \in J$. If $d(\mu z)_i \neq 0$, then $(\mu z)(0, d(\mu) + e_i) \in \mathrm{MCE}(\mu, \nu)$ for $\nu = (\mu z)(0, e_i) \in r(\mu)\Lambda^{e_i} = x(q)\Lambda^{e_i}$, a contradiction. So for each $i \in J$, we have $d(\mu z)_i = 0$, and hence $d(y)_i = d(x)_i$. Now suppose that $i \notin J$. Then either $x(q)\Lambda^{e_i} = \emptyset$ or $q_i < d(x)_i$. If $x(q)\Lambda^{e_i} = \emptyset$ then $d(y)_i = d(x)_i$. So suppose that $q_i < d(x)_i$. Since $n \wedge d(x) \leq q$, it follows that $n_i < d(x)_i$ and $n_i \leq q_i \leq d(y)_i$, hence $(n \wedge d(x))_i = n_i = (n \wedge d(y))_i$. So $n \wedge d(x) = n \wedge d(y)$. ∎

The following result allows us to identify Λ with a subgraph of $\widetilde{\Lambda}$.

Proposition 4.13. *Suppose that Λ is a k-graph, and that $\lambda \in \Lambda$. Then $s(\lambda)\partial\Lambda \neq \emptyset$. If $x, y \in s(\lambda)\partial\Lambda$, then $\lambda x, \lambda y \in \partial\Lambda$ and $(\lambda x; (0, d(\lambda))) \sim (\lambda y; (0, d(\lambda)))$. Moreover, there is an injective k-graph morphism $\iota : \Lambda \to \widetilde{\Lambda}$*

such that for $\lambda \in \Lambda$,

$$\iota(\lambda) = [\lambda x; (0, d(\lambda))] \quad \text{for any } x \in s(\lambda)\partial\Lambda.$$

Proof. By [6, Lemma 5.15], we have $v\partial\Lambda \neq \emptyset$ for all $v \in \Lambda^0$. In particular, $s(\lambda)\partial\Lambda \neq \emptyset$. Let $x, y \in s(\lambda)\partial\Lambda$. Then [6, Lemma 5.13(ii)] says that $\lambda x, \lambda y \in \partial\Lambda$. It follows from the definition of $\sim$ that $(\lambda x; (0, d(\lambda))) \sim (\lambda y; (0, d(\lambda)))$. Then straightforward calculations show that ι is an injective k-graph morphism. ∎

We want to extend ι to an injection of W_Λ into $W_{\widetilde{\Lambda}}$. The next proposition shows that any injective k-graph morphism defined on Λ can be extended to W_Λ.

PROPOSITION 4.14. *Let Λ, Γ be k-graphs and $\phi : \Lambda \to \Gamma$ be a k-graph morphism. Let $x \in W_\Lambda \backslash \Lambda$. Then $\phi(x) : \Omega_{k,d(x)} \to W_\Gamma$ defined by $\phi(x)(p, q) = \phi(x(p, q))$ belongs to W_Γ.*

Proof. This follows from ϕ being a k-graph morphism. ∎

In particular, we can extend ι to paths with non-finite degree. We need to know that composition works as expected for non-finite paths.

PROPOSITION 4.15. *Let Λ, Γ be k-graphs and $\phi : \Lambda \to \Gamma$ be a k-graph morphism. Let $\lambda \in \Lambda$, $x \in s(\lambda)W_\Lambda$, and suppose that $n \in \mathbb{N}^k$ satisfies $n \leq d(x)$. Then*

(1) $\phi(\lambda)\phi(x) = \phi(\lambda x)$,
(2) $\sigma^n(\phi(x)) = \phi(\sigma^n(x))$.

Proof. Again this follows from ϕ being a k-graph morphism. ∎

REMARK 4.16. We deduce that the extension of an injective k-graph morphism to W_Λ is also injective. In particular, the map $\iota : \Lambda \to \widetilde{\Lambda}$ has an injective extension $\iota : W_\Lambda \to W_{\widetilde{\Lambda}}$.

We need to be able to 'project' paths from $\widetilde{\Lambda}$ onto the embedding $\iota(\Lambda)$ of Λ. For $y \in \partial\Lambda$ define

$$\pi([y; (m, n)]) = [y; (m \wedge d(y), n \wedge d(y))]. \tag{4.2}$$

Straightforward calculations show that π is a surjective functor, and is a projection in the sense that $\pi(\pi([y; (m, n)])) = \pi([y; (m, n)])$ for all $[y; (m, n)] \in \widetilde{\Lambda}$. In particular, $\pi|_{\iota(\Lambda)} = \mathrm{id}_{\iota(\Lambda)}$.

LEMMA 4.17. *Let Λ be a k-graph. Suppose that $\lambda, \mu \in \widetilde{\Lambda}$, and that $\lambda \in \mathcal{Z}(\mu)$. Then $\pi(\lambda) \in \mathcal{Z}(\pi(\mu))$. If $d(\pi(\lambda))_i > d(\pi(\mu))_i$ for some $i \leq k$, then $d(\mu)_i = d(\pi(\mu))_i$.*

Proof. Write $\lambda = [x; (m, m + d(\lambda))]$. Then $\mu = [x; (m, m + d(\mu))]$, so

$$\pi(\lambda) = [x; (m \wedge d(x), (m + d(\lambda)) \wedge d(x))],$$
$$\pi(\mu) = [x; (m \wedge d(x), (m + d(\mu)) \wedge d(x))].$$

Since $d(\lambda) \geq d(\mu)$, it follows that $\pi(\lambda) \in \mathcal{Z}(\pi(\mu))$.

If $d(\pi(\lambda))_i > d(\pi(\mu))_i$, then $d(x)_i > m_i + d(\mu)_i$, so

$$d(\pi(\mu))_i = m_i + d(\mu)_i - m_i = d(\mu)_i. \blacksquare$$

LEMMA 4.18. *Let Λ be a k-graph and $\mu, \nu \in \widetilde{\Lambda}$. Then*

$$\pi(\mathrm{MCE}(\mu, \nu)) \subset \mathrm{MCE}(\pi(\mu), \pi(\nu)).$$

Proof. Suppose that $\lambda \in \mathrm{MCE}(\mu, \nu)$. By Lemma 4.17 we have $\pi(\lambda) \in \mathcal{Z}(\pi(\mu)) \cap \mathcal{Z}(\pi(\nu))$, hence $d(\pi(\lambda)) \geq d(\pi(\mu)) \vee d(\pi(\nu))$.

It remains to prove that $d(\pi(\lambda)) = d(\pi(\mu)) \vee d(\pi(\nu))$. Suppose, for a contradiction, that $i \leq k$ is such that $d(\pi(\lambda))_i > \max\{d(\pi(\mu))_i, d(\pi(\nu))_i\}$. By Lemma 4.17 we then have $d(\pi(\mu))_i = d(\mu)_i$ and $d(\pi(\nu))_i = d(\nu)_i$. So $d(\lambda)_i \geq d(\pi(\lambda))_i > \max\{d(\mu)_i, d(\nu)_i\}$, contradicting $\lambda \in \mathrm{MCE}(\mu, \nu)$. $\blacksquare$

LEMMA 4.19. *Let Λ be a k-graph, and let $\mu, \lambda \in \iota(\Lambda^0)\widetilde{\Lambda}$ be such that $d(\lambda) = d(\mu)$ and $\pi(\lambda) = \pi(\mu)$. Then $\lambda = \mu$.*

Proof. Since $\mu, \lambda \in \iota(\Lambda^0)\widetilde{\Lambda}$ and $d(\lambda) = d(\mu)$, we can write $\lambda = [x; (0, n)]$ and $\mu = [y; (0, n)]$ for some $x, y \in \partial\Lambda$ and $n \in \mathbb{N}^k$. We will show that $(x; (0, n)) \sim (y; (0, n))$. Conditions (P2) and (P3) are trivially satisfied. Since

$$[x; (0, n \wedge d(x))] = \pi(\lambda) = \pi(\mu) = [y; (0, n \wedge d(y))],$$

we have $(x; (0, n \wedge d(x))) \sim (y; (0, n \wedge d(y)))$. Hence $x(0, n \wedge d(x)) = y(0, n \wedge d(y))$, and (P1) is satisfied. $\blacksquare$

Proof of Theorem 4.1. The existence of $\widetilde{\Lambda}$ follows from Proposition 4.9, and the embedding from Proposition 4.13.

To check that $\widetilde{\Lambda}$ is finitely aligned, fix $\mu, \nu \in \widetilde{\Lambda}$, and $\alpha \in \iota(\Lambda^0)\widetilde{\Lambda}r(\mu)$. Then $|\mathrm{MCE}(\mu, \nu)| = |\mathrm{MCE}(\alpha\mu, \alpha\nu)|$. We know that $|\mathrm{MCE}(\pi(\alpha\mu), \pi(\alpha\nu))|$ is finite since Λ is finitely aligned. We will show that $|\mathrm{MCE}(\alpha\mu, \alpha\nu)| = |\mathrm{MCE}(\pi(\alpha\mu), \pi(\alpha\nu))|$.

It follows from Lemma 4.18 that

$$|\mathrm{MCE}(\alpha\mu, \alpha\nu)| \geq |\mathrm{MCE}(\pi(\alpha\mu), \pi(\alpha\nu))|.$$

For the opposite inequality, suppose that λ and β are distinct elements of $\mathrm{MCE}(\alpha\mu, \alpha\nu)$. Then $d(\lambda) = d(\beta)$. Since $r(\alpha\mu), r(\alpha\nu) \in \iota(\Lambda^0)$, Lemma 4.19 implies that $\pi(\lambda) \neq \pi(\beta)$. So $|\mathrm{MCE}(\alpha\mu, \alpha\nu)| = |\mathrm{MCE}(\pi(\alpha\mu), \pi(\alpha\nu))|$.

For the last part of the statement, we prove the contrapositive. Suppose that $\widetilde{\Lambda}$ is not row-finite. Let $[x; m] \in \widetilde{\Lambda}^0$ and $i \leq k$ be such that $|[x; m]\widetilde{\Lambda}^{e_i}| = \infty$. Then for each $[y; (n, n + e_i)] \in [x; m]\widetilde{\Lambda}^{e_i}$ we have $[y; n] =$

$[x;m]$, so $[x;(m,m+e_i)] \neq [y;(n,n+e_i)]$ only if (P1) fails. That is,

$$x(m \wedge d(x),(m+e_i) \wedge d(x)) \neq y(n \wedge d(y),(n+e_i) \wedge d(y)). \tag{4.3}$$

Since $|[x;m]\widetilde{\Lambda}^{e_i}| = \infty$, there are infinitely many $[y;(n,n+e_i)] \in [x;m]\widetilde{\Lambda}^{e_i}$ satisfying (4.3). Hence $|x(m \wedge d(x))\Lambda^{e_i}| = \infty$. ∎

REMARK 4.20. Suppose that Λ is a finitely aligned k-graph, that $x \in \partial\Lambda$ and that $E \subset x(0)\Lambda$. Since $\iota : \Lambda \to \iota(\Lambda)$ is a bijective k-graph morphism, we have $E \in x(0)\mathcal{FE}(\Lambda)$ if and only if $\iota(E) \in [x;0]\mathcal{FE}(\iota(\Lambda))$.

The following results show how sets of minimal common extensions and finite exhaustive sets in a k-graph Λ relate to those in $\widetilde{\Lambda}$.

PROPOSITION 4.21 ([5, Lemma 2.25]). *Suppose that Λ is a finitely aligned k-graph, and that $v \in \iota(\Lambda^0)$. Then $E \in v\mathcal{FE}(\iota(\Lambda))$ implies that $E \in v\mathcal{FE}(\widetilde{\Lambda})$.*

LEMMA 4.22. *Let Λ be a finitely aligned k-graph and let $\mu,\nu \in \iota(\Lambda)$. Then* $\mathrm{MCE}_{\iota(\Lambda)}(\mu,\nu) = \mathrm{MCE}_{\widetilde{\Lambda}}(\mu,\nu)$.

Proof. Since $\iota(\Lambda) \subset \widetilde{\Lambda}$, we have $\mathrm{MCE}_{\iota(\Lambda)}(\mu,\nu) \subset \mathrm{MCE}_{\widetilde{\Lambda}}(\mu,\nu)$. Suppose that $\lambda \in \mathrm{MCE}_{\widetilde{\Lambda}}(\mu,\nu)$. It suffices to show that $\lambda \in \iota(\Lambda)$. Write $\mu = [x;(0,n)]$, $\nu = [y;(0,q)]$ and $\lambda = [z;(0,n \vee q)]$. Then $\lambda \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)$ implies that $d(z) \geq n \vee q$, hence $\lambda \in \iota(\Lambda)$. ∎

REMARK 4.23. Since there is a bijection from $\Lambda^{\min}(\mu,\nu)$ onto $\mathrm{MCE}(\mu,\nu)$, it follows from Lemma 4.22 that $\widetilde{\Lambda}^{\min}(\mu,\nu) = \iota(\Lambda)^{\min}(\mu,\nu)$ for all $\mu,\nu \in \iota(\Lambda)$.

5. Topology of path spaces under desourcification. We extend the projection π defined in (4.2) to the set of infinite paths in $\widetilde{\Lambda}$, and prove that its restriction to $\iota(\Lambda^0)\widetilde{\Lambda}^\infty$ is a homeomorphism onto $\iota(\partial\Lambda)$. For $x \in \iota(\Lambda^0)\widetilde{\Lambda}^\infty$, let $p_x = \bigvee\{p \in \mathbb{N}^k : x(0,p) \in \iota(\Lambda)\}$, and define $\pi(x)$ to be the composition of x with the inclusion of Ω_{k,p_x} in $\Omega_{k,d(x)}$. Then $\pi(x)$ is a k-graph morphism. Our goal for this section is the following theorem.

THEOREM 5.1. *Let Λ be a row-finite k-graph. Then $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ is a homeomorphism.*

We first show that the range of π is a subset of $\iota(\partial\Lambda)$.

PROPOSITION 5.2. *Let Λ be a finitely aligned k-graph. Let $x \in \iota(\Lambda^0)\widetilde{\Lambda}^\infty$. Suppose that $\{y_n : n \in \mathbb{N}^k\} \subset \partial\Lambda$ satisfy $[y_n;(0,n)] = x(0,n)$. Then*

1. $\lim_{n\in\mathbb{N}^k} \iota(y_n) = \pi(x)$ *in* $W_{\widetilde{\Lambda}}$,
2. *there exists $y \in \partial\Lambda$ such that $\pi(x) = \iota(y)$, and for $m,n \in \mathbb{N}^k$ with $m \leq n \leq p_x$ we have $\pi(x)(m,n) = \iota(y(m,n))$.*

Proof. For part (i), fix a basic open set $\mathcal{Z}(\mu \setminus G) \subset W_{\widetilde{\Lambda}}$ containing $\pi(x)$. Fix $n \geq N := \bigvee_{\nu\in G} d(\mu\nu)$. We first show that $\iota(y_n) \in \mathcal{Z}(\mu)$. Since $\pi(x) \in \mathcal{Z}(\mu)$, we have $\mu \in \iota(\Lambda)$. Since $n \geq d(\mu)$, we have $[y_n;(0,d(\mu))] = \mu$.

Let $\alpha = \iota^{-1}(\mu)$ and $z \in s(\alpha)\partial\Lambda$. Then $[y_n; (0, d(\mu))] = \mu = [\alpha z; (0, d(\mu))]$, and (P1) gives $\iota(y_n(0, d(\mu) \wedge d(y_n))) = \iota((\alpha z)(0, d(\mu))) = \iota(\alpha) = \mu$. So $\iota(y_n) \in \mathcal{Z}(\mu)$.

We now show that $\iota(y_n) \notin \bigcup_{\nu\in G} \mathcal{Z}(\mu\nu)$. Fix $\nu \in G$. If $d(y_n) \not\geq d(\mu\nu)$, then trivially we have $\iota(y_n) \notin \mathcal{Z}(\mu\nu)$. Suppose that $d(y_n) \geq d(\mu\nu)$. Since $n \geq d(\mu\nu)$, we have

$$x(0, d(\mu\nu)) = [y_n; (0, n)](0, d(\mu\nu)) = \iota(y_n)(0, d(\mu\nu)) \in \iota(\Lambda).$$

So $\iota(y_n)(0, d(\mu\nu)) = x(0, d(\mu\nu)) = \pi(x)(0, d(\mu\nu)) \neq \mu\nu$.

For part (ii), recall that ι is injective, hence we can define $y : \Omega_{k,p_x} \to \Lambda$ by $\iota(y(m, n)) = \pi(x)(m, n)$. So $\iota(y) = \pi(x)$. To see that $y \in \partial\Lambda$, fix $m \in \mathbb{N}^k$ such that $m \leq d(y)$ and fix $E \in y(m)\mathcal{FE}(\Lambda)$. We seek $t \in \mathbb{N}^k$ such that $y(m, m+t) \in E$. Let $p := m + \bigvee_{\mu\in E} d(\mu)$. Then since $m \leq d(y) = p_x$, we get

$$[y_p; (0, m)] = x(0, m) = \pi(x)(0, m) = \iota(y(0, m)) = [y(0, m)y'; (0, m)]$$

for some $y' \in y(m)\partial\Lambda$. So $(y_p; (0, m)) \sim (y(0, m)y'; (0, m))$, hence

$$y_p(0, m \wedge d(y_p)) = (y(0, m)y')(0, m \wedge d(y(0, m)y')) = y(0, m)$$

by (P1). In particular, this implies that $y_p(m) = y(m)$. Since $y_p \in \partial\Lambda$, there exists $t \in \mathbb{N}^k$ such that $y_p(m, m+t) \in E$. So $m + t \leq p$, and we have

$$\iota(y_p(m, m+t)) = [y_p; (0, p)](m, m+t) = x(0, p)(m, m+t) = x(m, m+t).$$

So $x(m, m+t) \in \iota(\Lambda)$, giving

$$\iota(y_p(m, m+t)) = x(m, m+t) = \pi(x)(m, m+t) = \iota(y(m, m+t)).$$

Finally, injectivity of ι gives $y(m, m+t) = y_p(m, m+t) \in E$. ∎

The next few results ensure that our definition of π on $\widetilde{\Lambda}^\infty$ is compatible with (4.2) when we regard finite paths as k-graph morphisms. The following lemma is also crucial in showing that π is injective on $\iota(\Lambda^0)\widetilde{\Lambda}^\infty$.

LEMMA 5.3. *Let Λ be a finitely aligned k-graph. Let $x \in \iota(\Lambda^0)\widetilde{\Lambda}^\infty$. Suppose that $w \in \partial\Lambda$ satisfies $\pi(x) = \iota(w)$. Then $x(0, n) = [w; (0, n)]$ for all $n \in \mathbb{N}^k$.*

Proof. Fix $n \in \mathbb{N}^k$. Let $z \in \partial\Lambda$ be such that $x(0, n) = [z; (0, n)]$. We aim to show that $(z; (0, n)) \sim (w; (0, n))$. That (P2) and (P3) hold follows immediately from their definitions. It remains to verify condition (P1):

$$z(0, n \wedge d(z)) = w(0, n \wedge d(\omega)). \tag{5.1}$$

Since $\pi(x) = \iota(w)$ we have $d(w) = p_x$. Thus

$$[w; (0, n \wedge p_x)] = \iota(w(0, n \wedge p_x)) = x(0, n \wedge p_x) = [z; (0, n \wedge p_x)].$$

So $(w; (0, n \wedge p_x)) \sim (z; (0, n \wedge p_x))$. It then follows from (P1) that

$$w(0, n \wedge p_x) = z(0, n \wedge p_x). \tag{5.2}$$

Hence $n \wedge d(z) \geq n \wedge p_x$. Furthermore,

$$x(0, n \wedge d(z)) = [z; (0, n \wedge d(z))] = \iota(z(0, n \wedge d(z))) \in \iota(\Lambda)$$

implies that $n \wedge p_x \geq n \wedge d(z)$. So $n \wedge d(z) = n \wedge p_x$, and (5.2) becomes (5.1), as required. ∎

Remark 5.4. Suppose that Λ is a finitely aligned k-graph, and that $y \in \partial\Lambda$ and $m, n \in \mathbb{N}^k$ satisfy $m \leq n \leq d(y)$. Then

$$[y; (m, n)] = [\sigma^m(y); (0, n - m)] = \iota(\sigma^m(y)(0, n - m)) = \iota(y(m, n)),$$

so $[y; (m, n)] = \iota(y(m, n))$.

The next proposition shows that our definitions of π for finite and infinite paths are compatible:

Proposition 5.5. *Let Λ be a finitely aligned k-graph. Suppose that $x \in \widetilde{\Lambda}^\infty$, and $m \leq n \in \mathbb{N}^k$. Then $\pi(x(m, n)) = \pi(x)(m \wedge p_x, n \wedge p_x)$.*

Proof. Fix $y \in \partial\Lambda$ such that $\pi(x) = \iota(y)$. Then

$$\begin{aligned}
(x(m, n)) &= \pi([y; (m, n)]) && \text{by Lemma 5.3} \\
&= [y; (m \wedge p_x, n \wedge p_x)] && \text{since } d(y) = p_x \\
&= \iota(y(m \wedge p_x, n \wedge p_x)) && \text{by Remark 5.4} \\
&= \pi(x)(m \wedge p_x, n \wedge p_x) && \text{by Proposition 5.2(ii).} \ ∎
\end{aligned}$$

We can now show that π restricts to a homeomorphism of $\iota(\Lambda^0)\widetilde{\Lambda}^\infty$ onto $\iota(\partial\Lambda)$. We first show that it is a bijection, then show it is continuous. Openness is the trickiest part, and the proof of it completes this section.

Proposition 5.6. *Let Λ be a finitely aligned k-graph. Then the map $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ is a bijection.*

Proof. That π is injective follows from Lemma 5.3. To see that π is onto $\iota(\partial\Lambda)$, let $w \in \partial\Lambda$ and define $x : \Omega_k \to \widetilde{\Lambda}$ by $x(p, q) = [w; (p, q)]$. Then $p_x = d(w)$, and $r(x) \in \iota(\Lambda)$. To see that $\pi(x) = \iota(w)$, fix $m, n \in \mathbb{N}^k$ with $m \leq n \leq d(w)$. Then

$$\begin{aligned}
(x)(m, n) &= x(m, n) && \text{by Proposition 5.5} \\
&= [w; (m, n)] && \text{by Lemma 5.3} \\
&= \iota(w(m, n)) && \text{by Remark 5.4} \\
&= \iota(w)(m, n) && \text{by Proposition 4.14.} \ ∎
\end{aligned}$$

Proposition 5.7. *Let Λ be a finitely aligned k-graph. Then $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ is continuous.*

Proof. Fix a basic open set $\mathcal{Z}(\mu \setminus G) \subset W_{\widetilde{\Lambda}}$. If $\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda) = \emptyset$, then $\pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda)) = \emptyset$ is open. Suppose that $\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda) \neq \emptyset$, and fix $y \in \mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda)$. Let $F = G \cap \iota(\Lambda)$. We will show that

$$\pi^{-1}(y) \in \mathcal{Z}(\mu \setminus F) \cap (\widetilde{\Lambda}^{\infty} \cap r^{-1}(\iota(\Lambda))) \subset \pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda)). \tag{5.3}$$

Since $y \in \mathcal{Z}(\mu)$, we have $\pi^{-1}(y) \in \mathcal{Z}(\mu)$. To see that $\pi^{-1}(y) \notin \bigcup_{\beta\in F} \mathcal{Z}(\mu\beta)$, fix $\beta \in F$. First suppose that $d(\mu\beta) \not\leq d(y)$. Then $\pi^{-1}(y)(0, d(\mu\beta)) \notin \iota(\Lambda)$. Since $\mu\beta \in \iota(\Lambda)$, we have $\pi^{-1}(y)(0, d(\mu\beta)) \neq \mu\beta$. Now suppose that $d(\mu\beta) \leq d(y)$; then

$$\pi^{-1}(y)(0, d(\mu\beta)) = y(0, d(\mu\beta)) \neq \mu\beta.$$

We now show that $\mathcal{Z}(\mu \setminus F) \cap \iota(\Lambda^0)\widetilde{\Lambda}^{\infty} \subset \pi^{-1}(\mathcal{Z}(\mu \setminus G) \cap \iota(\partial\Lambda))$. Let $z \in \mathcal{Z}(\mu \setminus F) \cap \iota(\Lambda^0)\widetilde{\Lambda}^{\infty}$. It suffices to show that $\pi(z) \in \mathcal{Z}(\mu \setminus G)$. Firstly, $\pi(z)(0, d(\mu)) = z(0, d(\mu)) = \mu \in \iota(\Lambda)$. To see that $\pi(z) \notin \bigcup_{\nu\in G} \mathcal{Z}(\mu\nu)$, fix $\nu \in G$. If $d(\mu\nu) \not\leq d(\pi(z))$, then trivially $\pi(z) \notin \mathcal{Z}(\mu\nu)$. Suppose that $d(\mu\nu) \leq d(\pi(z))$. If $\nu \notin \iota(\Lambda)$, then $\pi(z)(0, d(\mu\nu)) \neq \mu\nu$. Otherwise, $\nu \in \iota(\Lambda)$, so $\nu \in F$ and we have $\pi(z)(0, d(\mu\nu)) = z(0, d(\mu\nu)) \neq \mu\nu$. ∎

Proposition 5.8. *Let Λ be a row-finite k-graph. Then $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^{\infty} \to \iota(\partial\Lambda)$ is open.*

Proof. Fix $\pi(y) \in \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^{\infty})$. Let $\omega \in \partial\Lambda$ be such that $\pi(y) = \iota(\omega)$. Define $\lambda := y(0, \bigvee_{\nu\in G} d(\mu\nu))$, and

$$F := \bigcup\{s(\pi(\lambda))\iota(\Lambda^{e_i}) : d(\lambda)_i > d(\pi(y))_i\}.$$

We claim that

$$\pi(y) \in \mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial\Lambda) \subset \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^{\infty}).$$

First we show that $\pi(y) \in \mathcal{Z}(\pi(\lambda))$. It follows from Lemma 5.3 that $\pi(\lambda) = [\omega; (0, d(\lambda) \wedge d(\omega))]$. Since $d(\omega) = d(\pi(y))$, Proposition 5.5 implies that

$$\pi(y)(0, d(\pi(\lambda))) = \pi(y)(0, d(\lambda) \wedge d(\omega)) = \pi(y(0, d(\lambda))) = \pi(\lambda).$$

Now we show that $\pi(y) \notin \bigcup_{f\in F} \mathcal{Z}(\pi(\lambda)f)$. Fix $f \in F$; say $d(f) = e_i$. Then by definition of F, $d(\lambda)_i > d(\pi(y))_i = d(\omega)_i$, and thus

$$d(\pi(\lambda))_i = \min\{d(\lambda)_i, d(\omega)_i\} = d(\omega)_i = d(\pi(y))_i.$$

So $d(\pi(y)) \not\geq d(\pi(\lambda)f)$, and hence $\pi(y) \notin \mathcal{Z}(\pi(\lambda)f)$ as required.

Now we show that $\mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial\Lambda) \subset \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^{\infty})$. Let $\pi(\beta) \in \mathcal{Z}(\pi(\lambda) \setminus F) \cap \iota(\partial\Lambda)$. We aim to show that $\beta \in \mathcal{Z}(\mu \setminus G)$. Since $\mathcal{Z}(\lambda) \subset \mathcal{Z}(\mu \setminus G)$, it suffices to show that $\beta \in \mathcal{Z}(\lambda)$. Clearly $\beta \in \mathcal{Z}(\pi(\lambda) \setminus F)$. If $d(\lambda) = d(\pi(\lambda))$ then $\pi(\lambda) = \lambda$ and we are done. Suppose that $d(\lambda) > d(\pi(\lambda))$, and let $\tau = \beta(d(\pi(\lambda)), d(\lambda))$. We know that $\beta \in \mathcal{Z}(\pi(\lambda))$. We aim to use Lemma 4.19 to show that $\tau = \lambda(d(\pi(\lambda)), d(\lambda))$. Fix $i \leq k$ such that $d(\lambda)_i > d(\pi(\lambda))_i$. Then since $d(\pi(\lambda)) = d(\lambda) \wedge d(\omega)$, we have $d(\lambda)_i > d(\omega)_i = d(\pi(y))_i$. Now $\beta \in \mathcal{Z}(\pi(\lambda) \setminus F)$ implies that $\tau(0, e_i) \notin F$. In particular,

$\tau(0,e_i) \notin \iota(\Lambda)$. We claim that $d(\pi(\tau)) = 0$. Suppose, for a contradiction, that $d(\pi(\tau))_j > 0$ for some $j \le k$. Then $\pi(\tau)(0,e_j) = \tau(0,e_j) \notin \iota(\Lambda)$. But $\pi(\tau) \in \iota(\Lambda)$ by definition of π. So we must have $d(\pi(\tau)) = 0$, which implies that

$$\pi(\tau) = r(\tau) = s(\pi(\lambda)) = \pi(\lambda(d(\pi(\lambda)), d(\lambda))).$$

Now Lemma 4.19 implies that $\tau = \lambda(d(\pi(\lambda)), d(\lambda))$. Then

$$\beta(0,\lambda) = \beta(0, d(\pi(\lambda)))\tau = \pi(\lambda)\lambda(d(\pi(\lambda)), d(\lambda)) = \lambda. \blacksquare$$

EXAMPLE 5.9. We can see that π is not open for non-row-finite graphs by considering the 1-graph E from Example 4.10 with 'desourcification' $\widetilde{E}$. Observe that $\mathcal{Z}(\mu_1) \cap \iota(E^0)\widetilde{\Lambda}^\infty = \{\mu_1\mu_2\cdots\}$ is open in $\widetilde{E}$, and $\pi(\mathcal{Z}(\mu_1) \cap \iota(E^0)\widetilde{E}^\infty) = \{v\}$. Since $\partial E = E$, any basic open set in ∂E containing v is of the form $\mathcal{Z}(v \setminus G)$ for some finite $G \subset E^1$. Since E^1 is infinite, there is no finite $G \subset E^1$ such that $\mathcal{Z}(v \setminus G) \subset \{v\}$. Hence $\{v\}$ is not open in E, and π is not an open map.

Proof of Theorem 5.1. Propositions 5.6, 5.7 and 5.8 say precisely that π is a bijection, is continuous, and is open. $\blacksquare$

REMARK 5.10. Although $\pi|_{\iota(\Lambda^0)\widetilde{\Lambda}^\infty}$ is open for all row-finite k-graphs, it behaves particularly well with respect to cylinder sets for locally convex k-graphs. The following discussion and example arose in preliminary work on a proof that π is open when Λ is row-finite and locally convex. We have retained this example since it helps illustrate some of the issues surrounding the map π.

Denote our standard topology for a finitely k-graph by τ_1. The collection $\{\mathcal{Z}(\mu) : \mu \in \Lambda\}$ of cylinder sets also forms a base for a topology: they cover W_Λ, and if $x \in \mathcal{Z}(\lambda) \cap \mathcal{Z}(\nu)$, then $x \in \mathcal{Z}(x(0, d(\lambda) \vee d(\nu))) \subset \mathcal{Z}(\lambda) \cap \mathcal{Z}(\nu)$. This topology, denoted τ_2, is not necessarily Hausdorff: we cannot separate any edge from its range: if $r(f) \in \mathcal{Z}(\mu)$ then $\mu = r(f)$, and thus $f \in \mathcal{Z}(\mu)$.

It may seem reasonable to expect that $\{\mathcal{Z}(\mu) \cap \partial\Lambda : \mu \in \Lambda\}$ is a base for the restriction of τ_1 to $\partial\Lambda$. However, this is not so. To see why, consider the 2-graph of Example 2.11. Let y be the boundary path beginning with f_0. So $x, y \in \partial\Lambda$. Let μ be such that $x \in \mathcal{Z}(\mu)$. Then $\mu = x_0 \dots x_n$ for some $n \in \mathbb{N}$, so $y \in \mathcal{Z}(\mu)$ as well. So the topology τ_1 is not Hausdorff even when restricted to $\partial\Lambda$. In the topology τ_2, it is easy to see how to separate these two points: $y \in \mathcal{Z}(f_0) \cap \partial\Lambda$ and $x \in \mathcal{Z}(r(x) \setminus \{f_0\}) \cap \partial\Lambda$, and these two sets are disjoint.

If we restrict ourselves to locally convex k-graphs, τ_1 and τ_2 do restrict to the same topology on $\partial\Lambda$: certainly, for each $\mu \in \Lambda$, we can realise a cylinder set $\mathcal{Z}(\mu)$ as a set of the form $\mathcal{Z}(\mu \setminus G)$ by taking $G = \emptyset$. Now suppose that

$x \in \mathcal{Z}(\mu \setminus G) \cap \partial\Lambda$. We claim that with

$$\nu_x := x\Big(0, \Big(\bigvee_{\alpha\in G} d(\mu\alpha)\Big) \wedge d(x)\Big),$$

we have $x \in \mathcal{Z}(\nu_x)\cap\partial\Lambda \subset \mathcal{Z}(\mu\setminus G)\cap\partial\Lambda$. Clearly $x \in \mathcal{Z}(\nu_x)\cap\partial\Lambda$. The containment requires a little more work. Clearly $y \in \mathcal{Z}(\mu)$. Fix $\alpha \in G$. We will show that $y \notin \mathcal{Z}(\mu\alpha)$. If $d(y) \not\geq d(\mu\alpha)$, then trivially $y \notin \mathcal{Z}(\mu\alpha)$. Suppose that $d(y) \geq d(\mu\alpha)$. We claim that $d(x) \geq d(\mu\alpha)$. Suppose, for a contradiction, that $d(x) \not\geq d(\mu\alpha)$. Then there exists $i \leq k$ such that $d(x)_i < d(\mu\alpha)_i$. Then $d(x)_i = d(\nu_x)_i$. Since $x \in \partial\Lambda$, we must have $x(d(\nu_x))\Lambda^{e_i} \notin x(d(\nu_x))\mathcal{FE}(\Lambda)$. Since Λ is locally convex, Lemma 2.13 implies that $y(d(\nu_x))\Lambda^{e_i} = x(d(\nu_x))\Lambda^{e_i} = \emptyset$. So $d(y)_i = d(\nu_x)_i = d(x)_i < d(\mu\alpha)_i$, a contradiction. Hence $d(x) \geq d(\mu\alpha)$. This implies that $d(\nu_x) \geq d(\mu\alpha)$. So

$$y(0, d(\mu\alpha)) = v_x(0, d(\mu\alpha)) = x(0, d(\mu\alpha)) \neq \mu\alpha.$$

PROPOSITION 5.11. *Suppose that Λ is a row-finite, locally convex k-graph, and let $\mu \in \iota(\Lambda^0)\widetilde{\Lambda}$. Then $\pi(\mathcal{Z}(\mu) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty) = \mathcal{Z}(\pi(\mu)) \cap \iota(\partial\Lambda)$. In particular, π is open.*

Proof. We first show that $\pi(\mathcal{Z}(\mu)\cap\iota(\Lambda^0)\widetilde{\Lambda}^\infty) \subset \mathcal{Z}(\pi(\mu))\cap\iota(\partial\Lambda)$. Suppose that $\pi(y) \in \pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty)$. Trivially $\pi(y) \in \iota(\partial\Lambda)$. We will show that $\pi(y) \in \mathcal{Z}(\pi(\mu) \setminus \pi(G))$. Since $y(0, d(\mu)) = \mu$, we have

$$\pi(\mu) = \pi(y(0, d(\mu))) = \pi(y)(0, d(\mu) \wedge d(\pi(y))).$$

So $\pi(y) \in \mathcal{Z}(\pi(\mu))$. Furthermore, $d(\pi(\mu)) = d(\mu) \wedge d(\pi(y))$.

Fix $\nu \in G$. We will show that $\pi(y) \notin \mathcal{Z}(\pi(\mu\nu))$. Since $y \in \mathcal{Z}(\mu \setminus G)$, we have $y(0, d(\mu\nu)) \neq \mu\nu$. Since $d(y(0, d(\mu\nu))) = d(\mu\nu)$ and $r(y) = r(\mu\nu) \in \iota(\Lambda^0)$, Lemma 4.19 implies that

$$\pi(\mu\nu) \neq \pi(y(0, d(\mu\nu))) = \pi(y)(0, d(\mu\nu) \wedge d(\pi(y))).$$

So $\pi(\mathcal{Z}(\mu \setminus G) \cap \iota(\Lambda^0)\widetilde{\Lambda}^\infty) \subset \mathcal{Z}(\pi(\mu) \setminus \pi(G)) \cap \iota(\partial\Lambda)$.

Now suppose that $\iota(\omega) \in \mathcal{Z}(\pi(\mu)) \cap \iota(\partial\Lambda)$, and let $y = \pi^{-1}(\iota(\omega))$. We show that $y \in \mathcal{Z}(\mu)$. Write $\mu = [z; (0, d(\mu))]$. Then $\pi(\mu) = [z; (0, d(\mu) \wedge d(z))]$ and $y(0, d(\mu)) = [\omega; (0, d(\mu))]$. We claim that $(z; (0, d(\mu))) \sim (\omega; (0, d(\mu)))$. That (P2) and (P3) hold follows immediately from their definition. To show that (P1) is satisfied, we must show that $z(0, d(\mu)\wedge d(z)) = w(0, d(\mu)\wedge d(w))$. Since $\pi(y) \in \mathcal{Z}(\pi(\mu))$, we have $y \in \mathcal{Z}(\pi(\mu))$. Then

$$[\omega; (0, d(\pi(\mu)))] = y(0, d(\pi(\mu))) = \pi(\mu) = [z; (0, d(\mu) \wedge d(z))].$$

So $(\omega; (0, d(\pi(\mu)))) \sim (z; (0, d(\mu) \wedge d(z)))$. Therefore (P1) implies that

$$\omega(0, d(\pi(\mu))) = \omega(0, d(\pi(\mu)) \wedge d(\omega)) = z(0, d(\mu) \wedge d(z)),$$

and $d(\pi(\mu)) = d(\mu) \wedge d(z)$. We will show $d(\mu) \wedge d(w) = d(\pi(\mu))$. Fix $i \leq k$. We argue the following cases separately:

(1) If $d(\pi(\mu))_i = d(\mu)_i$, we have $d(w) \geq d(\pi(\mu)) = d(\mu)_i$. It follows that $(d(\mu) \wedge d(w))_i = d(\mu)_i = d(\pi(\mu))_i$.

(2) If $d(\pi(\mu))_i < d(\mu)_i$, it requires a little more work: Since $d(\mu)_i > d(\pi(\mu))_i = \min\{d(\mu)_i, d(z)_i\}$, we have $d(\pi(\mu))_i = d(z)_i$. So $z \in \partial\Lambda$ implies that $z(d(\pi(\mu)))\Lambda^{e_i} \notin z(d(\pi(\mu)))\mathcal{FE}(\Lambda)$. By Lemma 2.13, we have $z(d(\pi(\mu)))\Lambda^{e_i} = \emptyset$, and hence $\omega(d(\pi(\mu)))\Lambda^{e_i} = \emptyset$. So $d(\omega)_i = d(\pi(\mu))_i < d(\mu)_i$, giving $(d(\mu) \wedge d(\omega))_i = d(\omega)_i = d(\pi(\mu))_i$. ∎

6. High-rank graph C^*-algebras

Definition 6.1. Let Λ be a finitely aligned k-graph. A *Cuntz–Krieger Λ-family* in a C^*-algebra B is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying

(CK1) $\{s_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections,
(CK2) $s_\mu s_\nu = s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$,
(CK3) $s_\mu^* s_\nu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} s_\alpha s_\beta^*$ for all $\mu, \nu \in \Lambda$,
(CK4) $\prod_{\mu\in E}(s_v - s_\mu s_\mu^*) = 0$ for all $v \in \Lambda^0$ and $E \in v\mathcal{FE}(\Lambda)$.

The C^*-algebra $C^*(\Lambda)$ of a k-graph Λ is the universal C^*-algebra generated by a Cuntz–Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$.

Remark 6.2. The following theorem appears as [5, Theorem 2.28]. Farthing alerted us to an issue in the proof of the theorem. It contains a claim which is proved in cases, and in the proof of Case 1 of the claim (on page 189), there is an error when i_0 is such that $m_{i_0} = d(x)_{i_0} + 1$. Then $a_{i_0} = d(x)_{i_0}$, and [5, equation (2.13)] gives $t_{i_0} \leq d(z)_{i_0}$; not $t_{i_0} \geq d(z)_{i_0}$ as stated.

Theorem 6.3. *Let Λ be a row-finite k-graph. Let $C^*(\Lambda)$ and $C^*(\widetilde{\Lambda})$ be generated by the Cuntz–Krieger families $\{s_\lambda : \lambda \in \Lambda\}$ and $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$. Then the sum $\sum_{v\in\iota(\Lambda^0)} t_v$ converges strictly to a full projection $p \in M(C^*(\widetilde{\Lambda}))$ such that $pC^*(\widetilde{\Lambda})p = C^*(\{t_{\iota(\lambda)} : \lambda \in \Lambda\})$, and $s_\lambda \mapsto t_{\iota(\lambda)}$ determines an isomorphism $\varsigma : C^*(\Lambda) \cong pC^*(\widetilde{\Lambda})p$.*

Before proving Theorem 6.3, we need the following results.

Proposition 6.4 ([5, Theorem 2.26]). *Let Λ be a finitely aligned k-graph. If $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$ is a Cuntz–Krieger $\widetilde{\Lambda}$-family, then $\{t_\lambda : \lambda \in \iota(\Lambda)\}$ is a Cuntz–Krieger $\iota(\Lambda)$-family.*

Remark 6.5. Let Λ be a finitely aligned k-graph. It follows from the universal properties of $C^*(\Lambda)$ and $C^*(\iota(\Lambda))$ that $C^*(\Lambda) \cong C^*(\iota(\Lambda))$.

Proposition 6.6 ([5, Theorem 2.27]). *Let Λ be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$ be the universal Cuntz–Krieger $\widetilde{\Lambda}$-family which* *generates $C^*(\widetilde{\Lambda})$. Then $C^*(\Lambda)$ is isomorphic to the subalgebra of $C^*(\widetilde{\Lambda})$ generated by $\{t_\lambda : \lambda \in \iota(\Lambda)\}$.*

LEMMA 6.7. *Suppose that Λ is a finitely aligned k-graph. Let $\lambda \in \widetilde{\Lambda}$, and let $\lambda' = \lambda(d(\pi(\lambda)), d(\lambda))$. Suppose that $x \in \partial\Lambda$ satisfies $\iota(r(x)) = r(\lambda')$ and $d(x) \wedge d(\lambda') = 0$. Then $\lambda' = [x; (0, d(\lambda'))]$.*

Proof. Write $\lambda = [y; (0, d(\lambda))]$ then $\lambda' = [y; (d(\lambda) \wedge d(y), d(\lambda))]$. We must show that $(y; (d(\lambda) \wedge d(y), d(\lambda))) \sim (x; (0, d(\lambda')))$. That conditions (P2) and (P3) hold follows immediately from their definitions. It remains to show that (P1) is satisfied. Since $d(x) \wedge d(\lambda') = 0$, it suffices to show that $y(d(\lambda) \wedge d(y)) = x(0)$. We have

$$\iota(x(0)) = \iota(r(x)) = r(\lambda') = [y; d(\lambda) \wedge d(y)] = \iota(y(d(\lambda) \wedge d(y))).$$

Injectivity of ι then gives $y(d(\lambda) \wedge d(y)) = x(0)$. ∎

LEMMA 6.8. *Let $\lambda \in \widetilde{\Lambda}$. Let $\lambda' = \lambda(d(\pi(\lambda)), d(\lambda))$ and define*

$$G_\lambda := \bigcup_{i=1}^{k} \{\alpha \in s(\pi(\lambda))\iota(\Lambda)^{e_i} : \mathrm{MCE}(\alpha, \lambda') = \emptyset\}.$$

Then $G_\lambda \cup \{\lambda'\} \in s(\pi(\lambda))\mathcal{FE}(\widetilde{\Lambda})$.

Proof. Fix $\mu \in s(\pi(\lambda))\widetilde{\Lambda}$, and suppose that $\mathrm{MCE}(\mu, \alpha) = \emptyset$ for all $\alpha \in G_\lambda$. We will show that $\mathrm{MCE}(\mu, \lambda') \neq \emptyset$. Fix $\nu \in s(\mu)\widetilde{\Lambda}^{d(\mu) \vee d(\lambda') - d(\mu)}$. Then $d(\mu\nu) \geq d(\lambda')$. It suffices to show that $\mathrm{MCE}(\mu\nu, \lambda') \neq \emptyset$. Write $\mu\nu = [z; (0, d(\mu\nu))]$.

We first show that $d(\lambda') \wedge d(\pi(\mu\nu)) = 0$. Suppose for a contradiction that $d(\lambda') \wedge d(\pi(\mu\nu)) > 0$. So we have $d(\lambda') \wedge d(\mu\nu) \wedge d(z) > 0$, hence there exists $i \leq k$ such that $d(\lambda')_i$, $d(\mu\nu)_i$, and $d(z)_i$ are all greater than zero. Then $(\mu\nu)(0, e_i) = [z; (0, e_i)] = \iota(z)(0, e_i) \in \iota(\Lambda)$. Since $\pi|_{\iota(\Lambda)} = \mathrm{id}_{\iota(\Lambda)}$ and $\pi(\lambda') = s(\pi(\lambda)) \neq \lambda'$, we have $\lambda' \notin \iota(\Lambda)$. This implies that $(\mu\nu)(0, e_i) \neq \lambda'(0, e_i)$. So $\mathrm{MCE}((\mu\nu)(0, e_i), \lambda') = \emptyset$, and thus $(\mu\nu)(0, e_i) \in G_\lambda$. But $\mathrm{MCE}(\mu\nu(0, e_i), \mu\nu) \neq \emptyset$, which implies that $\mathrm{MCE}(\mu, \mu\nu(0, e_i)) \neq \emptyset$. This contradicts our supposition that $\mathrm{MCE}(\mu, \alpha) = \emptyset$ for all $\alpha \in G_\lambda$. So $d(\lambda') \wedge d(\pi(\mu\nu)) = 0$.

Since $d(\mu\nu) \geq d(\lambda')$, we have

$$d(z) \wedge d(\lambda') = d(z) \wedge d(\mu\nu) \wedge d(\lambda') = d(\pi(\mu\nu)) \wedge d(\lambda') = 0.$$

Since $r(\lambda') = r(\mu\nu) = \iota(r(z))$, it follows from Lemma 6.7 that $\lambda' = [z; (0, \lambda')]$. Thus $\mu\nu = [z; (0, \mu\nu)] \in \mathrm{MCE}(\mu\nu, \lambda')$. ∎

Proof of Theorem 6.3. Let $A := C^*(\{t_\lambda : \lambda \in \iota(\Lambda)\})$. Then $A \cong C^*(\Lambda)$ by Proposition 6.6. We will show that A is a full corner of $C^*(\widetilde{\Lambda})$.

Following the argument of [10, Lemma 2.10], the sum $\sum_{v\in\iota(\Lambda^0)} t_v$ converges strictly in $M(C^*(\widetilde{\Lambda}))$ to a projection p satisfying

$$pt_\lambda t_\mu^* p = \begin{cases} t_\lambda t_\mu^* & \text{if } \widetilde{r}(\lambda), \widetilde{r}(\mu) \in \iota(\Lambda^0), \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

The standard argument shows that p is a full projection in $M(C^*(\widetilde{\Lambda}))$. It follows from (6.1) that $A \subset pC^*(\widetilde{\Lambda})p$. Now suppose that $\lambda, \mu \in \iota(\Lambda^0)\widetilde{\Lambda}$. We will show that $pt_\lambda t_\mu^* p \in A$. If $\widetilde{s}(\lambda) \neq \widetilde{s}(\mu)$, then (CK1) implies that $pt_\lambda t_\mu^* p = 0 \in A$. Suppose that $\widetilde{s}(\lambda) = \widetilde{s}(\mu)$. We first show

$$\lambda(d(\pi(\lambda)), d(\lambda)) = \mu(d(\pi(\mu)), d(\mu)). \tag{6.2}$$

Let $x, y \in \partial\Lambda$ be such that $\lambda = [x; (0, d(\lambda))]$ and $\mu = [y; (0, d(\mu))]$. Let

$$\begin{aligned} \lambda' &= \lambda(d(\pi(\lambda)), d(\lambda)) = [x; (d(\lambda) \wedge d(x), d(\lambda)], \\ \mu' &= \mu(d(\pi(\mu)), d(\mu)) = [y; (d(\mu) \wedge d(y), d(\mu))]. \end{aligned}$$

We claim that $\lambda' = \mu'$. Condition (P2) is trivially satisfied, and (P1) and (P3) follow from the vertex equivalence $[x; d(\lambda)] = \widetilde{s}(\lambda) = \widetilde{s}(\mu) = [y; d(\mu)]$. Hence $\lambda' = \mu'$.

Claim 6.3.1. *Let $G_\lambda := \bigcup_{i=1}^k \{\alpha \in s(\pi(\lambda))\iota(\Lambda)^{e_i} : \mathrm{MCE}(\alpha, \lambda') = \emptyset\}$. Then*

$$t_{\lambda'} t_{\lambda'}^* = \prod_{\alpha\in G_\lambda} (t_{s(\pi(\lambda))} - t_\alpha t_\alpha^*).$$

Proof. Lemma 6.8 implies that $G_\lambda \cup \{\lambda'\}$ is finite exhaustive, so (CK4) implies

$$\prod_{\beta\in G_\lambda\cup\{\lambda'\}} (t_{s(\pi(\lambda))} - t_\beta t_\beta^*) = 0.$$

Furthermore,

$$\begin{aligned} \prod_{\beta\in G_\lambda\cup\{\lambda'\}} (t_{s(\pi(\lambda))} - t_\beta t_\beta^*) &= \Big(\prod_{\alpha\in G_\lambda} (t_{s(\pi(\lambda))} - t_\alpha t_\alpha^*)\Big)(t_{s(\pi(\lambda))} - t_{\lambda'} t_{\lambda'}^*) \\ &= \Big(\prod_{\alpha\in G_\lambda} (t_{s(\pi(\lambda))} - t_\alpha t_\alpha^*)\Big) - \Big(t_{\lambda'} t_{\lambda'}^* \prod_{\alpha\in G_\lambda} (t_{s(\pi(\lambda))} - t_\alpha t_\alpha^*)\Big). \end{aligned}$$

Fix $\alpha \in G_\lambda$. By [13, Lemma 2.7(i)],

$$t_{\lambda'} t_{\lambda'}^* (t_{s(\pi(\lambda))} - t_\alpha t_\alpha^*) = t_{\lambda'} t_{\lambda'}^* - \sum_{\gamma\in\mathrm{MCE}(\lambda',\alpha)} t_\gamma t_\gamma^* = t_{\lambda'} t_{\lambda'}^*.$$

So

$$0 = \prod_{\beta\in G_\lambda\cup\{\lambda'\}} (t_{s(\pi(\lambda))} - t_\beta t_\beta^*) = \prod_{\alpha\in G_\lambda} (t_{s(\pi(\lambda))} - t_\alpha t_\alpha^*) - t_{\lambda'} t_{\lambda'}^*. \quad \blacksquare_{\text{Claim}}$$

Now we put the pieces together:

$$\begin{aligned} pt_\lambda t_\mu^* p &= t_\lambda t_\mu^* \\ &= t_{\pi(\lambda)} t_{\lambda'} t_{\lambda'}^* t_{\pi(\mu)}^* && \text{by (6.2)} \\ &= t_{\pi(\lambda)} \prod_{\alpha \in G_\lambda} (t_{s(\pi(\lambda))} - t_\alpha t_\alpha^*) t_{\pi(\mu)}^* && \text{by Claim 6.3.1.} \end{aligned}$$

This is an element of A since $\pi(\lambda), \pi(\mu), \alpha \in \iota(\Lambda)$ for all $\alpha \in G_\lambda$. So $A = pC^*(\widetilde{\Lambda})p$. ∎

7. The diagonal and the spectrum. For a k-graph Λ, we call $C^*\{s_\mu s_\mu^* : \mu \in \Lambda\} \subset C^*(\Lambda)$ the *diagonal* C^*-algebra of Λ and denote it by D_Λ, dropping the subscript when confusion is unlikely. For a commutative C^*-algebra A, denote by $\Delta(A)$ the spectrum of A. Given a homomorphism $\pi : A \to B$ of commutative C^*-algebras, denote by π^* the induced map from $\Delta(B)$ to $\Delta(A)$ such that $\pi^*(f)(y) = f(\pi(y))$ for all $f \in \Delta(B)$ and $y \in A$.

THEOREM 7.1. *Let Λ be a row-finite higher-rank graph. Let $p \in M(C^*(\widetilde{\Lambda}))$ and $\varsigma : C^*(\Lambda) \cong pC^*(\widetilde{\Lambda})p$ be from Theorem* 6.3. *Then the restriction $\varsigma|_{D_\Lambda} =: \rho$ is an isomorphism of D_Λ onto $pD_{\widetilde{\Lambda}}p$. Let $\pi : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \iota(\partial\Lambda)$ be the homeomorphism from Theorem* 5.1. *Then there exist homeomorphisms $h_\Lambda : \partial\Lambda \to \Delta(D_\Lambda)$ and $\eta : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \Delta(pD_{\widetilde{\Lambda}}p)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \iota(\Lambda^0)\widetilde{\Lambda}^\infty & \xrightarrow{\pi} & \iota(\partial\Lambda) \\ {\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle h_\Lambda \circ \iota^{-1}} \\ \Delta(pD_{\widetilde{\Lambda}}p) & \xrightarrow{\rho^*} & \Delta(D_\Lambda) \end{array}$$

As in [11], for a finite subset $F \subset \Lambda$, define

$$\vee F := \bigcup_{G \subset F} \mathrm{MCE}(G) = \bigcup_{G \subset F} \Big\{\lambda \in \bigcap_{\mu \in G} \mu\Lambda : d(\lambda) = \bigvee_{\mu \in G} d(\mu)\Big\}.$$

LEMMA 7.2. *Let Λ be a finitely aligned k-graph and let F be a finite subset of Λ. Suppose that $r(\lambda) \in F$ for each $\lambda \in F$. For $\mu \in F$, define*

$$q_\mu^{\vee F} := s_\mu s_\mu^* \prod_{\mu\mu' \in \vee F \setminus \{\mu\}} (s_\mu s_\mu^* - s_{\mu\mu'} s_{\mu\mu'}^*).$$

Then the $q_\mu^{\vee F}$ are mutually orthogonal projections in $\mathrm{span}\{s_\mu s_\mu^* : \mu \in \vee F\}$, *and for each $\nu \in \vee F$,*

$$s_\nu s_\nu^* = \sum_{\nu\nu' \in \vee F} q_{\nu\nu'}^{\vee F}. \tag{7.1}$$

Proof. Since

$$s_\mu s_\mu^* \prod_{\mu\mu'\in\vee F\setminus\{\mu\}} (s_\mu s_\mu^* - s_{\mu\mu'}s_{\mu\mu'}^*) = s_\mu s_\mu^* \prod_{\mu\mu'\in\vee F,\, d(\mu')\neq 0} (s_{r(\mu)} - s_{\mu\mu'}s_{\mu\mu'}^*),$$

[11, Proposition 8.6] says precisely that the $q_\mu^{\vee F}$ are mutually orthogonal projections. That

$$s_\nu s_\nu^* = \sum_{\nu\nu'\in\vee F} q_{\nu\nu'}^{\vee F}$$

is established in the proof of [11, Proposition 8.6] on page 421. ∎

REMARK 7.3. We have

$$q_\mu^{\vee F} = s_\mu\Big(\prod_{\substack{\mu'\in s(\mu)\Lambda\setminus\{s(\mu)\}\\ \mu\mu'\in\vee F}} (s_{s(\mu)} - s_{\mu'}s_{\mu'}^*)\Big)s_\mu^*.$$

This follows from a straightforward induction on $|\vee F|$.

The following lemma can be verified through routine calculation. The reader is referred to the author's PhD thesis for details.

LEMMA 7.4 ([19, Lemma A.0.7]). *Let A be a C^*-algebra, let p be a projection in A, let Q be a finite set of commuting subprojections of p and let q_0 be a non-zero subprojection of p. Then $\prod_{q\in Q}(p-q)$ is a projection. If q_0 is orthogonal to each $q\in Q$, then $q_0\prod_{q\in Q}(p-q) = q_0$, so in particular, $\prod_{q\in Q}(p-q)\neq 0$.*

PROPOSITION 7.5. *Let Λ be a finitely aligned k-graph. Then $D = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu\in\Lambda\}$, and for each $x\in\partial\Lambda$ there exists a unique $h(x)\in\Delta(D)$ such that*

$$h(x)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } x = \mu\mu', \\ 0 & \text{otherwise.}\end{cases}$$

Moreover, $x\mapsto h(x)$ is a homeomorphism $h:\partial\Lambda\to\Delta(D)$.

Proof. Let $\mu,\nu\in\Lambda$. It follows from (CK3) that

$$(s_\mu s_\mu^*)(s_\nu s_\nu^*) = \sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} s_\lambda s_\lambda^*,$$

hence $D = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu\in\Lambda\}$.

Fix $x\in\partial\Lambda$ and $\sum_{\mu\in F} b_\mu s_\mu s_\mu^* \in \operatorname{span}\{s_\mu s_\mu^* : \mu\in\Lambda\}$. By setting extra coefficients to zero we can assume that each path in F has its range in F, and write

$$\sum_{\mu\in F} b_\mu s_\mu s_\mu^* = \sum_{\mu\in\vee F} b_\mu s_\mu s_\mu^*.$$

Let $n = \bigvee\{p\in\mathbb{N}^k : x(0,p)\in\vee F\}$. Since $\vee F$ is a finite set of finite paths, n is finite. Since $\vee F$ is closed under minimal common extensions, $x(0,n)\in\vee F$.

Furthermore, since $x \in \partial\Lambda$, we have

$$F_x := \{\mu' \in x(n)\Lambda \setminus \{x(n)\} : x(0,n)\mu' \in \vee F\} \notin x(n)\mathcal{FE}(\Lambda).$$

So there exists $\nu \in x(n)\Lambda$ such that for each $\mu' \in F_x$, $\mathrm{MCE}(\nu,\mu') = \emptyset$. Then $s_\nu s_\nu^* s_{\mu'} s_{\mu'}^* = 0$ for all $\mu' \in F_x$. Applying Lemma 7.4 with $p = s_{x(n)}$, $q_0 = s_\nu s_\nu^*$ and $Q = \{s_{\mu'} s_{\mu'}^* : \mu' \in F_x\}$, we have $\prod_{\mu' \in F_x}(s_{x(n)} - s_{\mu'} s_{\mu'}^*) \neq 0$. So

$$q_{x(0,n)}^F = s_{x(0,n)} \prod_{\mu' \in F_x} (s_{x(n)} - s_{\mu'} s_{\mu'}^*) s_{x(0,n)}^* \neq 0.$$

We have

$$\begin{aligned}
\Big\| \sum_{\nu \in \vee F} b_\mu s_\mu s_\mu^* \Big\| &= \Big\| \sum_{\nu \in \vee F} \Big(\sum_{\substack{\mu \in \vee F \\ \nu \in \mathcal{Z}(\mu)}} b_\mu \Big) q_\nu^{\vee F} \Big\| && \text{by (7.1)} \\
&= \max_{\{\nu \in \vee F : q_\nu^{\vee F} \neq 0\}} \Big| \sum_{\substack{\mu \in \vee F \\ \nu \in \mathcal{Z}(\mu)}} b_\mu \Big| \\
&\geq \Big| \sum_{\substack{\mu \in \vee F \\ x(0,n) \in \mathcal{Z}(\mu)}} b_\mu \Big| && \text{since } q_{x(0,n)}^{\vee F} \neq 0 \\
&= \Big| \sum_{\substack{\mu \in F \\ x(0,n) \in \mathcal{Z}(\mu)}} b_\mu \Big| && \text{since } b_\mu = 0 \text{ for } \mu \in \vee F \setminus F.
\end{aligned}$$

Hence the formula

$$h(x)\Big(\sum_{\mu \in F} b_\mu s_\mu s_\mu^* \Big) = \sum_{\substack{\mu \in F \\ x \in \mathcal{Z}(\mu)}} b_\mu \tag{7.2}$$

determines a norm-decreasing linear map on $\mathrm{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}$.

To see that $h(x)$ is a homomorphism, it suffices to show that

$$h(x)(s_\mu s_\mu^* s_\alpha s_\alpha^*) = h(x)(s_\mu s_\mu^*)\ h(x)(s_\alpha s_\alpha^*). \tag{7.3}$$

Calculating the right-hand side of (7.3) yields

$$h(x)(s_\mu s_\mu^*) h(x)(s_\alpha s_\alpha^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the left-hand side of (7.3) gives

$$h(x)(s_\mu s_\mu^* s_\alpha s_\alpha^*) = h(x)\Big(\sum_{\lambda \in \mathrm{MCE}(\mu,\alpha)} s_\lambda s_\lambda^* \Big).$$

There exists at most one $\lambda \in \mathrm{MCE}(\mu,\alpha)$ such that $x \in \mathcal{Z}(\lambda)$. Such a λ exists if and only if $x \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\alpha)$, so

$$h(x)(s_\mu s_\mu^* s_\alpha s_\alpha^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\alpha) \cap \mathcal{Z}(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have established (7.3), hence $h(x)$ is a homomorphism, and thus extends uniquely to a non-zero homomorphism $h(x) : D \to \mathbb{C}$.

We claim the map $h : \partial\Lambda \to \Delta(D)$ is a homeomorphism. The trickiest part is to show h is onto:

Claim 7.5.1. *The map h is surjective.*

Proof. Fix $\phi \in \Delta(D)$. We seek $x \in \partial\Lambda$ such that $h(x) = \phi$. For each $n \in \mathbb{N}^k$, the $\{s_\mu s_\mu^* : d(\mu) = n\}$ are mutually orthogonal projections. It follows that for each $n \in \mathbb{N}^k$ there exists at most one $\nu^n \in \Lambda^n$ such that $\phi(s_{\nu^n} s_{\nu^n}^*) = 1$. Let S denote the set of n for which such ν^n exist. If $\nu = \mu\nu'$ and $\phi(s_\nu s_\nu^*) = 1$, then

$$1 = \phi(s_\nu s_\nu^*) = \phi(s_\nu s_\nu^* s_\mu s_\mu^*) = \phi(s_\nu s_\nu^*)\phi(s_\mu s_\mu^*) = \phi(s_\mu s_\mu^*).$$

This implies that if $n \in S$ and $m \leq n$, then $m \in S$ and $\nu^m = \nu^n(0, m)$. Set $N := \vee S$, and define $x : \Omega_{k,N} \to \Lambda$ by $x(p, q) = \nu^q(p, q)$. Then since each ν^q is a k-graph morphism, so is x.

We now show that $x \in \partial\Lambda$. Fix $n \in \mathbb{N}^k$ such that $n \leq d(x)$, and $E \in x(n)\mathcal{FE}(\Lambda)$. We seek $m \in \mathbb{N}^k$ such that $x(n, n+m) \in E$. Since E is finite exhaustive, (CK4) implies that $\prod_{\lambda\in E}(s_{x(n)} - s_\lambda s_\lambda^*) = 0$. Multiplying on the left by $s_{x(0,n)}$ and on the right by $s_{x(0,n)}^*$ yields

$$\prod_{\lambda\in E}(s_{x(0,n)} s_{x(0,n)}^* - s_{x(0,n)\lambda} s_{x(0,n)\lambda}^*) = 0.$$

Thus, since ϕ is a homomorphism, there exists $\lambda \in E$ such that

$$\begin{aligned} 0 &= \phi(s_{x(0,n)} s_{x(0,n)}^*) - \phi(s_{x(0,n)\lambda} s_{x(0,n)\lambda}^*) = \phi(s_{\nu^n} s_{\nu^n}^*) - \phi(s_{x(0,n)\lambda} s_{x(0,n)\lambda}^*) \\ &= 1 - \phi(s_{x(0,n)\lambda} s_{x(0,n)\lambda}^*). \end{aligned}$$

So $\phi(s_{x(0,n)\lambda} s_{x(0,n)\lambda}^*) = 1$. Thus $x(0, n)\lambda = \nu^{n+d(\lambda)} = x(0, n + d(\lambda))$, and hence $x \in \partial\Lambda$.

Now we must show that $h(x) = \phi$. For each $\mu \in \Lambda$ we have

$$\begin{aligned} \phi(s_\mu s_\mu^*) = 1 &\Leftrightarrow\ d(\mu) \in S \text{ and } \nu^{d(\mu)} = \mu \\ &\Leftrightarrow x(0, d(\mu)) = \mu \\ &\Leftrightarrow\ h(x)(s_\mu s_\mu^*) = 1. \end{aligned}$$

As $\phi(s_\mu s_\mu^*)$ and $h(x)(s_\mu s_\mu^*)$ take values in $\{0, 1\}$, we have $h(x) = \phi$. $\blacksquare_{\text{Claim}}$

To see that h is injective, suppose that $h(x) = h(y)$. Then for each $n \in \mathbb{N}^k$, we have

$$h(y)(s_{x(0,n\wedge d(x))} s_{x(0,n\wedge d(x))}^*) = h(x)(s_{x(0,n\wedge d(x))} s_{x(0,n\wedge d(x))}^*) = 1.$$

Hence $y(0, n \wedge d(x)) = x(0, n \wedge d(x))$. By symmetry, we also have $y(0, n \wedge d(y)) = x(0, n \wedge d(y))$ for all n. In particular, $d(x) = d(y)$ and $y(0, n) = x(0, n)$ for all $n \leq d(x)$. Thus $x = y$.

Recall that $\Delta(D)$ carries the topology of pointwise convergence. For openness, it suffices to check that h^{-1} is continuous. Suppose that $h(x^n) \to h(x)$. Fix a basic open set $\mathcal{Z}(\mu)$ containing x, so $h(x)(s_\mu s_\mu^*) = 1$. Since $h(x^n)(s_\mu s_\mu^*) \in \{0,1\}$ for all n, for large enough n we have $h(x^n)(s_\mu s_\mu^*) = 1$. So $x^n \in \mathcal{Z}(\mu)$. For continuity, a similarly straightforward argument shows that if $x^n \to x$, then $h(x^n)(s_\mu s_\mu^*) \to h(x)(s_\mu s_\mu^*)$. This convergence extends to $\operatorname{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}$ by linearity, and to D by an $\varepsilon/3$ argument. ∎

We can now prove our main result.

Proof of Theorem 7.1. Let Λ be a row-finite k-graph, and $\widetilde{\Lambda}$ be the desourcification described in Proposition 4.9. Let $\{s_\lambda : \lambda \in \Lambda\}$ and $\{t_\lambda : \lambda \in \widetilde{\Lambda}\}$ be universal Cuntz–Krieger families in $C^*(\Lambda)$ and $C^*(\widetilde{\Lambda})$. Let A be the C^*-subalgebra of $C^*(\widetilde{\Lambda})$ generated by $\{t_\lambda : \lambda \in \iota(\Lambda)\}$, and define the diagonal subalgebra of A by $D_A := \overline{\operatorname{span}}\{t_\lambda t_\lambda^* : \lambda \in \iota(\Lambda)\}$. Replacing $t_\lambda t_\mu^*$ with $t_\lambda t_\lambda^*$ in the proof Theorem 6.3 yields $D_A \cong pD_{\widetilde{\Lambda}}p$. Since $A \cong C^*(\Lambda)$, it follows that $D_A \cong D_\Lambda$. Thus $D_\Lambda \cong pD_{\widetilde{\Lambda}}p$ as required.

We now construct η and show that it is a homeomorphism. That p commutes with $D_{\widetilde{\Lambda}}$ implies that $pD_{\widetilde{\Lambda}}p$ is an ideal in $D_{\widetilde{\Lambda}}$. Then [14, Propositions A26(a) and A27(b)] imply that the map $k : \phi \mapsto \phi|_{pD_{\widetilde{\Lambda}}p}$ is a homeomorphism of $\{\phi \in \Delta(D_{\widetilde{\Lambda}}) : \phi|_{pD_{\widetilde{\Lambda}}p} \neq 0\}$ onto $\Delta(pD_{\widetilde{\Lambda}}p)$. Since $\widetilde{\Lambda}$ is row-finite with no sources, $\partial\widetilde{\Lambda} = \widetilde{\Lambda}^\infty$. Let $h_{\widetilde{\Lambda}} : \widetilde{\Lambda}^\infty \to \Delta(D_{\widetilde{\Lambda}})$ be the homeomorphism obtained from Proposition 7.5. Then $h_{\widetilde{\Lambda}}(x) \in \operatorname{dom}(k)$ for all $x \in \iota(\Lambda^0)\widetilde{\Lambda}^\infty$. Define $\eta := k \circ h_{\widetilde{\Lambda}}|_{\iota(\Lambda^0)\widetilde{\Lambda}^\infty} : \iota(\Lambda^0)\widetilde{\Lambda}^\infty \to \Delta(pD_{\widetilde{\Lambda}}p)$.

We now show that $h_\Lambda \circ \iota^{-1} \circ \pi = \rho^* \circ \eta$. Since ρ is an isomorphism, it suffices to fix $x \in \iota(\Lambda^0)\widetilde{\Lambda}^\infty$ and $\mu \in \Lambda$ and show that

$$(h_\Lambda \circ \iota^{-1} \circ \pi)(x)(s_\mu s_\mu^*) = (\rho^* \circ \eta)(x)(s_\mu s_\mu^*). \tag{7.4}$$

Let $\omega \in \partial\Lambda$ be such that $\pi(x) = \iota(\omega)$. Then the left-hand side of (7.4) becomes

$$(h_\Lambda \circ \iota^{-1} \circ \pi)(x)(s_\mu s_\mu^*) = h_\Lambda(w)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } \omega \in \mathcal{Z}(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Since $r(x) \in \iota(\Lambda^0)$, the right-hand side of (7.4) simplifies to

$$(\rho^* \circ \eta)(x)(s_\mu s_\mu^*) = \eta(x)(\rho(s_\mu s_\mu^*)) = h_{\widetilde{\Lambda}}(x)(t_{\iota(\mu)} t_{\iota(\mu)}^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\iota(\mu)), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $x \in \mathcal{Z}(\iota(\mu))$ if and only if $\omega \in \mathcal{Z}(\mu)$. Suppose that $x \in \mathcal{Z}(\iota(\mu))$. Since $\mu \in \Lambda$ and $\pi(x) = \iota(\omega)$, we have $\pi(x(0, d(\mu))) = \pi(\iota(\mu)) = \iota(\mu)$. So $d(\pi(x(0, d(\mu)))) = d(\mu)$, and thus $d(x) \wedge d(w) \geq d(\mu)$. So $d(\omega)$

$\geq d(\mu)$. Then we have

$$\begin{aligned} x \in \mathcal{Z}(\iota(\mu)) &\Leftrightarrow x(0, d(\mu)) = \iota(\mu) && \text{since } \iota \text{ preserves degree} \\ &\Leftrightarrow [\omega; (0, d(\mu))] = \iota(\mu) && \text{by Lemma 5.3} \\ &\Leftrightarrow \iota(\omega(0, d(\mu))) = \iota(\mu) && \text{by Remark 5.4} \\ &\Leftrightarrow \omega(0, d(\mu)) = \mu && \text{since } \iota \text{ is a injective} \\ &\Leftrightarrow \omega \in \mathcal{Z}(\mu).1 \end{aligned}$$

So equation (7.4) holds, and we are done. ∎

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