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Isolated points of some sets of bounded cosine families, bounded semigroups, and bounded groups on a Banach space

by

ADAM BOBROWSKI (Warszawa) and
WOJCIECH CHOJNACKI (Adelaide and Warszawa)

Dedicated to Professor Jan Kisyński on the occasion of his 80th birthday

Abstract. We show that if the set of all bounded strongly continuous cosine families on a Banach space X is treated as a metric space under the metric of the uniform convergence associated with the operator norm on the space $\mathscr{L}(X)$ of all bounded linear operators on X, then the isolated points of this set are precisely the scalar cosine families. By definition, a scalar cosine family is a cosine family whose members are all scalar multiples of the identity operator. We also show that if the sets of all bounded cosine families and of all bounded strongly continuous cosine families on an infinite-dimensional separable Banach space X are viewed as topological spaces under the topology of the uniform convergence associated with the strong operator topology on $\mathscr{L}(X)$, then these sets have no isolated points. We present counterparts of all the above results for semigroups and groups of operators, relating to both the norm and strong operator topologies.

1. Introduction. This paper is a continuation of our earlier work [2] in which, among other things, we discussed the convergence of sequences of cosine families. One question that arose previously was whether, given a cosine family $C = \{C(t)\}_{t\in\mathbb{R}}$ on a Banach space X, there is a sequence $C_n = \{C_n(t)\}_{t\in\mathbb{R}}$, $n = 1, 2, \ldots$, of cosine families on X such that $C_n \neq C$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} C_n(t) = C(t)$ strongly and uniformly in $t \in \mathbb{R}$. Here our main interest lies in a similar question but with the strong operator topology replaced by the norm operator topology: Given a cosine family $C = \{C(t)\}_{t\in\mathbb{R}}$ on a Banach space X, is there a sequence $C_n = \{C_n(t)\}_{t\in\mathbb{R}}$, $n = 1, 2, \ldots$, of cosine families on X such that $C_n \neq C$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} C_n(t) = C(t)$ in operator norm uniformly in $t \in \mathbb{R}$?

We show that the answer to the latter question is in the affirmative if C is bounded and not scalar in the sense that, for some $t \in \mathbb{R}$, $C(t)$ is not

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a scalar multiple of the identity operator on X. As a partial converse, we prove that if C is bounded, strongly continuous, and scalar (meaning, of course, that each $C(t)$ is a scalar multiple of the identity operator), then the answer is in the negative. In addition, we show that in the case where X is a Hilbert space, the negative answer holds without C necessarily being strongly continuous.

An immediate consequence of the first two of the above results is the identification of the isolated points of the set of all bounded strongly continuous cosine families on a Banach space X when this set is endowed with the metric of the uniform convergence corresponding to the operator norm on $\mathscr{L}(X)$—these are precisely the scalar cosine families. Here, of course, $\mathscr{L}(X)$ denotes the Banach space of all bounded linear operators on X. Contrasting this result is an analogous result that we derive for semigroups of operators: it turns out that only rather special scalar semigroups constitute the isolated points of the set of all bounded strongly continuous semigroups on a Banach space, namely those that are extendable to scalar groups. Our companion result reveals that the isolated points of the set of all bounded strongly continuous groups on a Banach space are precisely the scalar groups.

As a supplementary contribution and one that provides a link with our earlier work, we show that if the sets of all bounded cosine families and of all bounded strongly continuous cosine families on an infinite-dimensional separable Banach space X are viewed as topological spaces under the topology of the uniform convergence associated with the strong operator topology on $\mathscr{L}(X)$, then these sets have no isolated points. We derive similar results for semigroups and groups of operators.

2. Preliminaries. We first review some of the concepts and terminology used in the paper.

Let $\mathbf{A}$ be a Banach algebra with a unity e and let G be an Abelian group, written additively, with a neutral element 0. A family $\{C(g)\}_{g\in G}$ in $\mathbf{A}$ is said to be a *cosine family* if

(i) $2C(g)C(h) = C(g+h) + C(g-h)$ for all $g, h \in G$ (d'Alembert's functional equation, also called the cosine functional equation),
(ii) $C(0) = e$.

As is customary, a cosine family for which the indexing group is the additive group of integers $\mathbb{Z}$ will be referred to as a *cosine sequence*.

Given a Banach space X, we shall henceforth consider $\mathscr{L}(X)$ as a Banach algebra with unity, the unity element being the identity operator I_X on X.

We shall be mainly concerned with cosine families indexed by $\mathbb{R}$ in Banach algebras of the form $\mathscr{L}(X)$, where X is a Banach space. An $\mathscr{L}(X)$-valued cosine family, where X is a Banach space, will be termed a cosine family on X. Related objects of interest will be *semigroups* and *groups* of operators on a Banach space. We recall that a family $\{S(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is a semigroup of operators on X if

(i) $S(s)S(t) = S(s+t)$ for all $s, t \geq 0$ (Cauchy's equation, also called the exponential equation),
(ii) $S(0) = I_X$.

Likewise, a family $\{G(t)\}_{t\in\mathbb{R}}$ of bounded linear operators on X is a group of operators if

(i) $G(s)G(t) = G(s+t)$ for all $s, t \in \mathbb{R}$,
(ii) $G(0) = I_X$.

Our main focus will be on bounded strongly continuous cosine families and semigroups on a Banach space. We recall that a family $F = \{F(t)\}_{t\in T}$ of bounded linear operators on a Banach space X indexed by $T \subset \mathbb{R}$ is *bounded* if $\sup_{t\in T} \|F(t)\| < \infty$, and is *strongly continuous* if, for each $x \in X$, the function $T \ni t \mapsto F(t)x \in X$ is continuous in norm. We also recall that strongly continuous semigroups and cosine families on a Banach space are uniquely characterised by their respective generators. The *generator* A of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space X is defined by

$$(2.1)\qquad Ax = \frac{d}{dt}\bigg|_0 S(t)x = \lim_{s\to 0} \frac{S(s)x - x}{s} \quad (x \in D(A)),$$

where $D(A)$, the domain of A, is the set of all $x \in X$ for which the derivative (2.1) exists. In turn, the generator A of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ on X is defined by

$$(2.2)\qquad Ax = \frac{d^2}{dt^2}\bigg|_0 C(t)x = \lim_{s\to 0} \frac{2}{s^2}(C(s)x - x) \quad (x \in D(A)),$$

where $D(A)$ is the set of all $x \in X$ for which the second derivative (2.2) exists. For standard results concerning strongly continuous semigroups and cosine families and their corresponding generators, the reader is referred to, e.g., [1], [13], or [16].

Finally, we recall the rudiments of the theory of almost periodic functions [5]. A continuous function f on $\mathbb{R}$ with values in a Banach space X is said to be (*uniformly*) *almost periodic* if the set of its translates $\{T_t f\}_{t\in\mathbb{R}}$ is relatively compact in the metric $\rho(f, g) = \sup_{t\in\mathbb{R}} \|f(t) - g(t)\|$; the translate $T_t f$ of f by $t \in \mathbb{R}$ is, by definition, given by $T_t f(s) = f(t + s), s \in \mathbb{R}$. Let $AP(\mathbb{R}, X)$ be the space of all X-valued almost periodic functions on $\mathbb{R}$. For any $f \in AP(\mathbb{R}, X)$ and any $\alpha \in \mathbb{R}$, the mean value

$$M_t\{e^{-i\alpha t} f(t)\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\alpha t} f(t)\, dt$$

exists and defines the *Fourier–Bohr coefficient* of f for the *Fourier exponent* α, $\widehat{f}(\alpha)$. The Fourier–Bohr coefficients of f vanish for all but at most countably many Fourier exponents. The set

$$\Sigma(f) = \{\alpha \in \mathbb{R} \mid \widehat{f}(\alpha) \neq 0\}$$

constitutes the *Bohr spectrum* of f and is non-empty if f is non-zero. The function f is uniquely determined by its Fourier–Bohr coefficients, this property being implicitly meant when one refers to f via its expansion into a formal *Fourier–Bohr series*

$$f(t) \sim \sum_{\alpha \in \Sigma(f)} e^{i\alpha t} \widehat{f}(\alpha).$$

3. Main results. As already indicated, this paper is primarily concerned with the question of when a given cosine family $C = \{C(t)\}_{t\in\mathbb{R}}$ on a Banach space X satisfies the following condition:

(AN) there exists a sequence $C_n = \{C_n(t)\}_{t\in\mathbb{R}}$, $n = 1, 2, \ldots$, of cosine families on X such that $C_n \neq C$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} C_n(t) = C(t)$ in operator norm uniformly in $t \in \mathbb{R}$.

If the given cosine family C is strongly continuous, it is also of interest to consider the following variant of the above condition:

(ACN) there exists a sequence $C_n = \{C_n(t)\}_{t\in\mathbb{R}}$, $n = 1, 2, \ldots$, of strongly continuous cosine families on X such that $C_n \neq C$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} C_n(t) = C(t)$ in operator norm uniformly in $t \in \mathbb{R}$.

Below we give a full characterisation of bounded strongly continuous cosine families satisfying **(ACN)**. Some statements which we shall make on the way will, in fact, be of a more general nature and will concern bounded cosine families which are not necessarily strongly continuous.

A cosine family $C = \{C(t)\}_{t\in\mathbb{R}}$ on a Banach space X will be called *scalar* if, for every $t \in \mathbb{R}$, $C(t)$ is a scalar multiple of I_X. A cosine family which is not scalar will be termed *non-scalar*. It is immediate that a cosine family $C = \{C(t)\}_{t\in\mathbb{R}}$ is scalar if and only if there exists a scalar-valued cosine family $c = \{c(t)\}_{t\in\mathbb{R}}$ such that $C(t) = c(t)I_X$ for every $t \in \mathbb{R}$. If C is a scalar cosine family on a non-zero Banach space X, then the scalar-valued cosine family $c = \{c(t)\}_{t\in\mathbb{R}}$ satisfying $C(t) = c(t)I_X$ for every $t \in \mathbb{R}$ is uniquely determined.

We first show that every bounded (strongly continuous or not) non-scalar cosine family on a Banach space satisfies **(AN)** and that every bounded strongly continuous non-scalar cosine family on a Banach space satisfies **(ACN)**.

THEOREM 1. *Let $C = \{C(t)\}_{t\in\mathbb{R}}$ be a bounded non-scalar cosine family on a Banach space X. Then there exists a sequence $C_n = \{C_n(t)\}_{t\in\mathbb{R}}$, $n = 1, 2, \ldots$, of cosine families on X such that $C_n \neq C$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} C_n(t) = C(t)$ in operator norm uniformly in $t \in \mathbb{R}$. Moreover, if C is strongly continuous, then each C_n may be assumed to be strongly continuous.*

To establish this theorem, we need an auxiliary result which seems to be part of the folklore of operator theory; for the sake of completeness, we state it with a proof.

For a Banach space X, let X' be the dual space of X. Given $x \in X$ and $x' \in X'$, we denote by $\langle x, x'\rangle$ the value of the functional x' at x. Let $Z(\mathscr{L}(X))$ be the centre of the algebra of $\mathscr{L}(X)$, that is,

$$Z(\mathscr{L}(X)) = \{A \in \mathscr{L}(X) \mid AB = BA \text{ for each } B \in \mathscr{L}(X)\}.$$

LEMMA 1. *If X is a Banach space, then $Z(\mathscr{L}(X))$ consists precisely of all scalar multiples of I_X.*

Proof. Without loss of generality, we may assume that X is non-zero. It is clear that any scalar multiple of I_X is in $Z(\mathscr{L}(X))$. To prove the converse statement, suppose that $A \in Z(\mathscr{L}(X))$. For any $x' \in X'$ and any $y \in X$, let $T_{x',y}$ be the operator in $\mathscr{L}(X)$ given by

$$T_{x',y}x = \langle x, x'\rangle y \quad (x \in X).$$

Then $AT_{x',y} = T_{x',y}A$ for all $x' \in X'$ and all $y \in X$, or equivalently,

$$\langle x, x'\rangle Ay = \langle Ax, x'\rangle y \tag{3.1}$$

for all $x' \in X'$ and all $x, y \in X$. Fix $x_0 \in X \setminus \{0\}$ arbitrarily and next, employing the Hahn–Banach theorem, select $x_0' \in X'$ so that $\langle x_0, x_0'\rangle = 1$. If we now let $\lambda = \langle Ax_0, x_0'\rangle$, then (3.1) yields $Ay = \lambda y$ for all $y \in X$, or equivalently, $A = \lambda I_X$. The lemma follows. ∎

We are now ready to establish Theorem 1.

Proof of Theorem 1. Since C is non-scalar, there exists $s \in \mathbb{R}$ such that $C(s)$ is not a scalar multiple of I_X. By Lemma 1, there exists a bounded linear operator B on X such that $C(s)B \neq BC(s)$. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence in $(0, \|B\|^{-1})$ converging to 0, and let $I_n = I_X + \epsilon_n B$ for every $n \in \mathbb{N}$. Then, clearly, each operator I_n has a bounded inverse and

$$C_n(t) = I_n^{-1}C(t)I_n \quad (t \in \mathbb{R},\, n \in \mathbb{N})$$

defines a sequence of cosine families on X. Moreover, each C_n is strongly continuous whenever C is strongly continuous. Since C is bounded and $\lim_{n\to\infty} I_n = \lim_{n\to\infty} I_n^{-1} = I_X$ in operator norm, we see that $\lim_{n\to\infty} C_n(t) = C(t)$ in operator norm uniformly in $t \in \mathbb{R}$. Taking into account that, for each $n \in \mathbb{N}$, $C_n(s) = C(s)$ holds if and only if $C(s)B = BC(s)$, we finally deduce that $C_n(s) \neq C(s)$ for all $n \in \mathbb{N}$. ∎

We remark that it is immediate that the cosine families C_n appearing in Theorem 1 are equibounded, that is, $\sup_{n\in\mathbb{N},\, t\in\mathbb{R}} \|C_n(t)\| < \infty$.

We now investigate to what extent the converse of Theorem 1 holds. To tackle this question, we begin with a preliminary technical result.

Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.

Lemma 2. *Let $a, b \in \mathbb{T}$ be such that*

$$\limsup_{n\to\infty} |a^n + a^{-n} - (b^n + b^{-n})| < 1.$$

Then either $a = b$ or $a = b^{-1}$.

Proof. Arguing contrapositively, assume that neither $a = b$ nor $a = b^{-1}$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a^n b^n = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a^{-n} b^n = 0.$$

In addition,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} b^{2n} = \begin{cases} 0 & \text{if } b \notin \{-1, 1\}, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore

$$\Bigl| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \bigl(a^n b^n + a^{-n} b^n - (b^{2n} + 1)\bigr) \Bigr|$$

equals either 1 or 2. On the other hand,

$$\begin{aligned} &\Bigl| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \bigl(a^n b^n + a^{-n} b^n - (b^{2n} + 1)\bigr) \Bigr| \\ &\qquad \le \limsup_{n\to\infty} |a^n b^n + a^{-n} b^n - (b^{2n} + 1)| \\ &\qquad = \limsup_{n\to\infty} |a^n + a^{-n} - (b^n + b^{-n})| < 1, \end{aligned}$$

where the last inequality holds by assumption. The resulting contradiction establishes the lemma. ■

If X is a Banach space and $f\colon \mathbb{R} \to \mathscr{L}(X)$ is a bounded function, we let

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

Given a bounded linear operator A on a Hilbert space, we denote by A^* the adjoint of A.

Theorem 2. *Let $c = \{c(t)\}_{t\in\mathbb{R}}$ be a bounded scalar-valued cosine family. If $C = \{C(t)\}_{t\in\mathbb{R}}$ is a bounded cosine family on a Hilbert space H such that*

$$4\|C\|_\infty^2 \limsup_{t\to\infty} \|C(t) - c(t)I_H\| < 1,$$

then $C(t) = c(t)I_H$ for every $t \in \mathbb{R}$.

Proof. By a result of Fattorini [12], there exists an invertible operator S in $\mathscr{L}(H)$ such that the cosine family defined by

$$\tilde{C}(t) = SC(t)S^{-1} \quad (t \in \mathbb{R})$$

satisfies $\tilde{C}(t) = \tilde{C}^*(t)$ for each $t \in \mathbb{R}$. In view of [4, Theorem 1], one may safely assume that $\|S\|\,\|S^{-1}\| \le 2\|C\|_\infty^2$. It is clear that

$$\limsup_{t\to\infty} \|\tilde{C}(t) - c(t)I_H\| \le \|S\|\,\|S^{-1}\| \limsup_{t\to\infty} \|C(t) - c(t)I_H\|,$$

and hence

$$\limsup_{t\to\infty} \|\tilde{C}(t) - c(t)I_H\| \le 2\|C\|_\infty^2 \limsup_{t\to\infty} \|C(t) - c(t)I_H\|. \tag{3.2}$$

Let $\mathbf{A}$ be the smallest complex Banach subalgebra of $\mathscr{L}(H)$ containing all the operators $\tilde{C}(t)$. The algebra $\mathbf{A}$ is a commutative C^*-algebra with unity. Let $\Delta(\mathbf{A})$ be the set of all complex-valued homomorphisms on $\mathbf{A}$. Fix $\phi \in \Delta(\mathbf{A})$ and $s \in \mathbb{R}$ arbitrarily. Consider two scalar-valued cosine sequences $c_1 = \{c_1(n)\}_{n\in\mathbb{N}}$ and $c_2 = \{c_2(n)\}_{n\in\mathbb{N}}$ defined by

$$c_1(n) = \phi(\tilde{C}(ns)) \quad \text{and} \quad c_2(n) = c(ns)$$

for all $n \in \mathbb{N}$. The sequences c_1 and c_2 are both bounded. This is clear for c_2, and for c_1 it follows from the fact that $\|\phi\| = 1$ and from the estimate

$$|\phi(\tilde{C}(ns))| \le \|\phi\|\,\|\tilde{C}(ns)\| \le \|\tilde{C}(ns)\| \le \|\tilde{C}\|_\infty \quad (n \in \mathbb{N}).$$

By a result of Kannappan [19], there exist $a, b \in \mathbb{C} \setminus \{0\}$ such that

$$c_1(n) = \tfrac{1}{2}(a^n + a^{-n}) \quad \text{and} \quad c_2(n) = \tfrac{1}{2}(b^n + b^{-n})$$

for all $n \in \mathbb{N}$. In fact, both a and b have unit modulus; for if $|a| \ne 1$, say, then $a^n + a^{-n}$ diverges in modulus to infinity as $n \to \infty$, contradicting the boundedness of c_1. Since, for each $n \in \mathbb{N}$,

$$|c_1(n) - c_2(n)| = |\phi(\tilde{C}(ns) - c(ns)I_H)| \le \|\tilde{C}(ns) - c(ns)I_H\|,$$

we see that

$$\limsup_{n\to\infty} |c_1(n) - c_2(n)| \le \limsup_{t\to\infty} \|\tilde{C}(t) - c(t)I_H\|.$$

Combining this with (3.2), we get

$$\limsup_{n\to\infty} |c_1(n) - c_2(n)| \le 2\|C\|_\infty^2 \limsup_{t\to\infty} \|C(t) - c(t)I_H\|.$$

Consequently,

$$\limsup_{n\to\infty} |a^n + a^{-n} - (b^n + b^{-n})| \le 4\|C\|_\infty^2 \limsup_{t\to\infty} \|C(t) - c(t)I_H\| < 1,$$

where the rightmost inequality holds by assumption. An application of Lemma 2 now shows that either $a = b$ or $a = b^{-1}$. This in turn implies that $c_1 = c_2$. In particular, $\phi(\tilde{C}(s)) = \phi(c(s)I_H)$. Since ϕ was arbitrarily chosen and since $\mathbf{A}$, as any other commutative C^*-algebra, is semisimple (which means that, for any $A \in \mathbf{A}$, if $\psi(A) = 0$ holds for every $\psi \in \Delta(\mathbf{A})$, then $A = 0$), it follows that $\tilde{C}(s) = c(s)I_H$. Hence also $C(s) = c(s)I_H$. As s was arbitrarily chosen, the theorem follows. ∎

As an immediate consequence of Theorem 2, we obtain the following partial converse to Theorem 1:

THEOREM 3. *No bounded scalar cosine family on a Hilbert space satisfies* **(AN)**.

We next establish an analogue of Theorem 3 for bounded scalar cosine families on arbitrary Banach spaces. It will concern exclusively cosine families that are strongly continuous. We begin with a handful of technical results.

LEMMA 3. *Let $c = \{c(t)\}_{t\in\mathbb{R}}$ be a bounded continuous scalar-valued cosine family. If $C = \{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous bounded cosine family on a Banach space X such that*

$$\sup_{t\in\mathbb{R}} \|C(t) - c(t)I_X\| < 1,$$

then the generator of C is bounded.

Proof. Let $a \geq 0$ be such that $c(t) = \cos at$ for each $t \in \mathbb{R}$. Select $0 < \delta \leq 1/2$ so that

$$\sup_{t\in\mathbb{R}} \|C(t) - \cos(at)I_X\| \leq 1 - 2\delta.$$

Next choose $\lambda > 0$ so that $1 - \lambda^2/(\lambda^2 + a^2) \leq \delta$; any λ satisfying $\lambda \geq a\sqrt{\delta^{-1}(1-\delta)}$ will do. Let A be the generator of C. Then

$$\left\|\lambda^2(\lambda^2 - A)^{-1} - \frac{\lambda^2}{\lambda^2 + a^2} I_X\right\| = \left\|\int_0^\infty \lambda e^{-\lambda t}(C(t) - \cos(at)I_X)\,dt\right\| \leq 1 - 2\delta,$$

since $\int_0^\infty \lambda e^{-\lambda t}\,dt = 1$. Consequently,

$$\begin{aligned}\|\lambda^2(\lambda^2 - A)^{-1} - I_X\| &\leq \left\|\lambda^2(\lambda^2 - A)^{-1} - \frac{\lambda^2}{\lambda^2 + a^2} I_X\right\| + \left\|\frac{\lambda^2}{\lambda^2 + a^2} I_X - I_X\right\| \\ &\leq 1 - \delta.\end{aligned}$$

It follows that $\lambda^2(\lambda^2 - A)^{-1}$ is invertible in $\mathscr{L}(X)$, and hence so is $(\lambda^2 - A)^{-1}$. Consequently, A is bounded. ∎

Let δ_0 denote the Dirac function on $\mathbb{R}$ concentrated at the origin, i.e.,

$$\delta_0(\alpha) = \begin{cases} 1 & \text{if } \alpha = 0, \\ 0 & \text{otherwise.}\end{cases}$$

LEMMA 4. *Let $\mathbf{A}$ be a Banach algebra with unity and let $\{C(t)\}_{t\in\mathbb{R}}$ be an $\mathbf{A}$-valued cosine family such that the function $C\colon t \mapsto C(t)$ is almost* *periodic. Then*

$$\tfrac{1}{2}(1+\delta_0(\alpha))\widehat{C}(\alpha) = \widehat{C}(\alpha)^2$$

for each $\alpha \in \mathbb{R}$.

Proof. By the cosine functional equation, for each $\alpha \in \mathbb{R}$ and each $s \in \mathbb{R}$,

$$\begin{aligned} M_t\{e^{-i\alpha t}C(t+s)\} + M_t\{e^{-i\alpha t}C(t-s)\} &= 2M_t\{e^{-i\alpha t}C(t)C(s)\} \\ &= 2M_t\{e^{-i\alpha t}C(t)\}C(s) \\ &= 2\widehat{C}(\alpha)C(s). \end{aligned} \tag{3.3}$$

Clearly,

$$\begin{aligned} M_t\{e^{-i\alpha t}C(t+s)\} &= e^{i\alpha s}M_t\{e^{-i\alpha(t+s)}C(t+s)\} \\ &= e^{i\alpha s}M_t\{e^{-i\alpha t}C(t)\} = e^{i\alpha s}\widehat{C}(\alpha) \end{aligned}$$

and, likewise,

$$M_t\{e^{-i\alpha t}C(t-s)\} = e^{-i\alpha s}\widehat{C}(\alpha).$$

Therefore (3.3) can be rewritten as

$$e^{i\alpha s}\widehat{C}(\alpha) + e^{-i\alpha s}\widehat{C}(\alpha) = 2\widehat{C}(\alpha)C(s).$$

Consequently,

$$(1+e^{-2i\alpha s})\widehat{C}(\alpha) = 2e^{-i\alpha s}\widehat{C}(\alpha)C(s)$$

and further

$$(1+M_s\{e^{-2i\alpha s}\})\widehat{C}(\alpha) = 2\widehat{C}(\alpha)M_s\{e^{-i\alpha s}C(s)\} = 2\widehat{C}(\alpha)^2.$$

To complete the proof, it suffices to note that $M_s\{e^{-2i\alpha s}\} = \delta_0(\alpha)$. ∎

LEMMA 5. *Let* $\mathbf{A}$ *be a Banach algebra with unity and let* $\{C(t)\}_{t\in\mathbb{R}}$ *be an* $\mathbf{A}$*-valued cosine family such that the function* $C\colon t \mapsto C(t)$ *is almost periodic. Then, for each* $\alpha \in \mathbb{R}$, *if* $\|\widehat{C}(\alpha)\| < 1/2$, *then* $\widehat{C}(\alpha) = 0$.

Proof. By Lemma 4, for each $\alpha \in \mathbb{R}$,

$$\tfrac{1}{2}\|\widehat{C}(\alpha)\| \le \tfrac{1}{2}(1+\delta_0(\alpha))\|\widehat{C}(\alpha)\| = \|\widehat{C}(\alpha)^2\| \le \|\widehat{C}(\alpha)\|^2.$$

Hence

$$0 \le \|\widehat{C}(\alpha)\|(\|\widehat{C}(\alpha)\| - 1/2).$$

We see that if $\|\widehat{C}(\alpha)\| < 1/2$, then, necessarily, $\|\widehat{C}(\alpha)\| \le 0$, implying that $\widehat{C}(\alpha) = 0$. ∎

LEMMA 6. *Let* $(\epsilon_k)_{k\in\mathbb{N}}$ *be a sequence of mutually independent Rademacher random variables satisfying* $\mathsf{P}(\epsilon_k = -1) = \mathsf{P}(\epsilon_k = 1) = 1/2$ *for each* $k \in \mathbb{N}$. *Let* $\mathbf{A}$ *be an algebra with unity and let* $\{C(t)\}_{t\in\mathbb{R}}$ *be an* $\mathbf{A}$*-valued cosine family. Then*

$$C(t)^n = \mathsf{E}[C(\epsilon_1 t + \cdots + \epsilon_n t)]$$

for each $t \in \mathbb{R}$ *and each* $n \in \mathbb{N}$.

Proof. We proceed by induction on n, with $t \in \mathbb{R}$ being fixed arbitrarily. The statement is true for $n = 1$ because

$$\mathsf{E}[C(\epsilon_1 t)] = \tfrac{1}{2}(C(t) + C(-t)) = C(t).$$

Assume that the statement holds for n. Then

$$\begin{aligned} C(t)^{n+1} &= C(t)^n C(t) = \mathsf{E}[C(\epsilon_1 t + \cdots + \epsilon_n t)C(t)] \\ &= \mathsf{E}\big[\tfrac{1}{2}\big(C(\epsilon_1 t + \cdots + \epsilon_n t + t) + C(\epsilon_1 t + \cdots + \epsilon_n t - t)\big)\big]. \end{aligned}$$

Since the ϵ_k's are mutually independent, we have

$$\begin{aligned} &\tfrac{1}{2}\big(C(\epsilon_1 t + \cdots + \epsilon_n t + t) + C(\epsilon_1 t + \cdots + \epsilon_n t - t)\big) \\ &\qquad = \mathsf{E}[C(\epsilon_1 t + \cdots + \epsilon_{n+1} t) \mid \epsilon_1, \ldots, \epsilon_n], \end{aligned}$$

where $\mathsf{E}[C(\epsilon_1 t + \cdots + \epsilon_{n+1} t) \mid \epsilon_1, \ldots, \epsilon_n]$ denotes the conditional expectation of $C(\epsilon_1 t + \cdots + \epsilon_{n+1} t)$ given $\epsilon_1, \ldots, \epsilon_n$. Now the statement for $n + 1$ follows from the law of total expectation (which is a particular case of the tower property of conditional expectation):

$$\mathsf{E}[C(\epsilon_1 t + \cdots + \epsilon_{n+1} t)] = \mathsf{E}\big[\mathsf{E}[C(\epsilon_1 t + \cdots + \epsilon_{n+1} t) \mid \epsilon_1, \ldots, \epsilon_n]\big].$$

The induction is complete and so is the proof. ■

LEMMA 7. *Let* $\mathbf{A}$ *be a Banach algebra with unity and let* $C_1 = \{C_1(t)\}_{t\in\mathbb{R}}$ *and* $C_2 = \{C_2(t)\}_{t\in\mathbb{R}}$ *be two* $\mathbf{A}$*-valued bounded cosine families. Then, for each* $n \in \mathbb{N}$,

$$\|C_1^n - C_2^n\|_\infty \le \|C_1 - C_2\|_\infty,$$

where C_i^n, $i = 1, 2$, *denotes the mapping* $\mathbb{R} \ni t \mapsto C_i(t)^n \in \mathbf{A}$.

Proof. Let $(\epsilon_k)_{k\in\mathbb{N}}$ be a sequence of mutually independent Rademacher random variables. By Lemma 6, for each $t \in \mathbb{R}$ and each $n \in \mathbb{N}$,

$$\begin{aligned} \|C_1^n(t) - C_2^n(t)\| &= \|\mathsf{E}[C_1(\epsilon_1 t + \cdots + \epsilon_n t) - C_2(\epsilon_1 t + \cdots + \epsilon_n t)]\| \\ &\le \mathsf{E}[\|C_1(\epsilon_1 t + \cdots + \epsilon_n t) - C_2(\epsilon_1 t + \cdots + \epsilon_n t)\|] \\ &\le \mathsf{E}[\|C_1 - C_2\|_\infty] = \|C_1 - C_2\|_\infty. \end{aligned}$$

Hence $\|C_1^n - C_2^n\|_\infty \le \|C_1 - C_2\|_\infty$, as desired. ■

LEMMA 8. *Let* X *be a Banach space and let* $f\colon \mathbb{R} \to X$ *be a periodic function with a period* t_0. *Suppose that there exists a bounded function* $g\colon \mathbb{R} \to X$ *such that*

$$f(t) = g(t + t_0) - g(t)$$

for each $t \in \mathbb{R}$. *Then* $f(t) = 0$ *for each* $t \in \mathbb{R}$.

Proof. For each $t \in \mathbb{R}$, we have

$$g(t + t_0) = g(t) + f(t)$$

and also

$$g(t) = g(t - t_0) + f(t - t_0) = g(t - t_0) + f(t).$$

Hence

$$g(t+t_0)-g(t)=g(t)-g(t-t_0),$$

showing that the function $t\mapsto g(t+t_0)-g(t)$ is periodic with period t_0. Now, for each $n\in\mathbb{N}$ and each $t\in\mathbb{R}$,

$$\begin{aligned}g(t+nt_0)-g(t)&=\sum_{k=1}^{n}\big(g(t+kt_0)-g(t+(k-1)t_0)\big)\\&=n(g(t+t_0)-g(t))=nf(t).\end{aligned}$$

Since $g(t+nt_0)-g(t)$ stays bounded as $n\to\infty$, it follows that $f(t)=0$. ∎

LEMMA 9. *Let* $\mathbf{A}$ *be a Banach algebra with a unity* e *and let* $\{C(t)\}_{t\in\mathbb{R}}$ *be a bounded continuous* $\mathbf{A}$*-valued cosine family. Suppose that* $C(t_0)=e$ *for some* $t_0\in\mathbb{R}$. *Then the function* $t\mapsto C(t)$ *is periodic with period* t_0.

Proof. For each $t\in\mathbb{R}$, set

$$F(t)=C(t+t_0)-C(t).$$

We have, for each $t\in\mathbb{R}$,

$$C(t+2t_0)+C(t)=2C(t+t_0)C(t_0)=2C(t+t_0),$$

and so

$$C(t+2t_0)-C(t+t_0)=C(t+t_0)-C(t),$$

which means that $F(t+t_0)=F(t)$ for each $t\in\mathbb{R}$. Applying Lemma 8 with F as f and C as g, we deduce that $F(t)=0$ for every $t\in\mathbb{R}$. Hence $C(t+t_0)=C(t)$ for every $t\in\mathbb{R}$. ∎

Before formulating the main result of the remainder of this section, we recall some definitions and facts that will be needed in its proof.

Let X be a Banach space. Recall that a subset $\Gamma\subset X'$ is *total* if, for any $x\in X$, $\langle x,x'\rangle=0$ for all $x'\in\Gamma$ implies $x=0$.

Let $\mathscr{M}$ be a σ-algebra of subsets of a set Ω and let Γ be a total subset of X'. A *spectral measure of class* Γ is a map $E\colon\mathscr{M}\to\mathscr{L}(X)$ such that

(i) $E(\emptyset)=0$ and $E(\Omega)=I_X$;
(ii) $E(\omega\cap\omega')=E(\omega)E(\omega')$ for any $\omega,\omega'\in\mathscr{M}$;
(iii) $\omega\mapsto\langle E(\omega)x,x'\rangle$ is σ-additive for any $x\in X$ and $x'\in\Gamma$;
(iv) $\sup_{\omega\in\Omega}\|E(\omega)\|<\infty$.

It follows from the Orlicz–Pettis theorem that any spectral measure of class X' is strongly σ-additive—that is, the function $\mathscr{M}\ni\omega\mapsto E(\omega)x\in E$ is σ-additive for each $x\in X$.

The spectrum of an operator $T\in\mathscr{L}(X)$ is denoted by $\sigma(T)$. For $T\in\mathscr{L}(X)$ and a linear subspace Y of X such that $T(Y)\subset Y$, $T|_Y$ denotes the restriction of T to Y.

The Borel σ-algebra of a topological space Y is designated by $\mathscr{B}(Y)$.

Following Dunford [9] (cf. also [8, 10, 11]), an operator $T \in \mathscr{L}(X)$ is called *prespectral of class Γ* if there is a spectral measure $E\colon \mathscr{B}(\mathbb{C}) \to \mathscr{L}(X)$ of class Γ such that

(i) $TE(\omega) = E(\omega)T$ for each $\omega \in \mathscr{B}(\mathbb{C})$,
(ii) $\sigma(T|_{E(\omega)X}) \subset \overline{\omega}$ for each $\omega \in \mathscr{B}(\mathbb{C})$, with the bar denoting set closure.

The spectral measure $E\colon \mathscr{M} \to \mathscr{L}(X)$ of class Γ satisfying (i) and (ii) is uniquely determined by T and is called the *resolution of the identity of class Γ for T* [8, Theorem 5.13]. Any resolution of the identity E for a prespectral operator $T \in \mathscr{L}(X)$, of some class, is supported on $\sigma(T)$ in the sense that $E(\sigma(T)) = I_X$. In general, a prespectral operator of some class can also be a prespectral operator of another class, with a possibly different resolution of the identity [14] (see also [8, Example 5.35]).

If $T \in \mathscr{L}(X)$ has the form

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda),$$

where $E\colon \mathscr{B}(\mathbb{C}) \to \mathscr{L}(X)$ is a spectral measure of class Γ, then T is a prespectral operator of class Γ and E is its resolution of the identity of class Γ. In this case, T is termed a *scalar-type operator of class Γ*.

An operator $Q \in \mathscr{L}(X)$ is called *quasinilpotent* if $\lim_{n\to\infty} \|Q^n\|^{1/n} = 0$, which is equivalent to $\sigma(Q) = \{0\}$.

If $T \in \mathscr{L}(X)$ is a prespectral operator with resolution of the identity E of class Γ and if

$$S = \int_{\sigma(T)} \lambda \, dE(\lambda), \quad Q = T - S, \tag{3.4}$$

then S is a scalar-type operator with resolution of the identity E of class Γ and Q is a quasinilpotent operator commuting with $\{E(\omega) \mid \omega \in \mathscr{B}(\mathbb{C})\}$; moreover $\sigma(T) = \sigma(S)$. This characterisation of prespectral operators has a partial converse: If $S \in \mathscr{L}(X)$ is a scalar-type operator with resolution of the identity E of class Γ and Q is a quasinilpotent operator commuting with $\{E(\omega) \mid \omega \in \mathscr{B}(\mathbb{C})\}$, then $S+Q$ is prespectral with resolution of the identity E of class Γ; moreover, $\sigma(S+Q) = \sigma(S)$ [8, Theorem 5.15].

The decomposition $T = S+Q$ in (3.4) is called the *Jordan decomposition* of T. It does not depend on the spectral measure E used to define S (and, effectively, also Q)—all spectral measures for which T is prespectral yield the same S and Q. This follows from the fact that if an operator $T \in \mathscr{L}(X)$, prespectral or not, can be represented as $T = S+Q = S_0+Q_0$, where $S, S_0 \in \mathscr{L}(X)$ are scalar-type prespectral, and $Q, Q_0 \in \mathscr{L}(X)$ are quasinilpotent and satisfy $SQ = QS$ and $S_0Q_0 = Q_0S_0$, then $S = S_0$ and $Q = Q_0$ [8, Theorem 5.23]. If $T \in \mathscr{L}(X)$ can be written as $T = S + Q$ with $S \in \mathscr{L}(X)$ of scalar type and $Q \in \mathscr{L}(X)$ quasinilpotent with $SQ = QS$, then S is said to be the *scalar part* of T and Q is its *radical part.*

An operator $T \in \mathscr{L}(X)$ is a *spectral operator* if it is a prespectral operator of class X'. In this case, T has a unique resolution of the identity [8, Theorem 6.7]. An operator $T \in \mathscr{L}(X)$ is spectral if and only if it has the form $T = S + Q$, where $S \in \mathscr{L}(X)$ is a scalar-type spectral operator (i.e., a scalar-type operator of class X') and $Q \in \mathscr{L}(X)$ is a quasinilpotent operator which commutes with S. The operators T and S have the same spectrum and the same resolution of the identity [8, Theorem 6.8].

THEOREM 4. *Let $c = \{c(t)\}_{t\in\mathbb{R}}$ be a bounded continuous scalar-valued cosine family. If $C = \{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous bounded cosine family on a Banach space X such that*

$$\sup_{t\in\mathbb{R}} \|C(t) - c(t)I_X\| < 1/2,$$

then $C(t) = c(t)I_X$ for every $t \in \mathbb{R}$.

Proof. Let $\mathbf{A}$ be the smallest complex Banach subalgebra of $\mathscr{L}(X)$ containing all the operators $C(t)$. The algebra $\mathbf{A}$ is a commutative Banach algebra with I_X as its identity. Let $\Delta(\mathbf{A})$ be the set of all complex-valued homomorphisms on $\mathbf{A}$. Fix $\phi \in \Delta(\mathbf{A})$ and $s \in \mathbb{R}$ arbitrarily. Consider two scalar-valued cosine sequences $c_1 = \{c_1(n)\}_{n\in\mathbb{N}}$ and $c_2 = \{c_2(n)\}_{n\in\mathbb{N}}$ defined by

$$c_1(n) = \phi(C(ns)) \quad \text{and} \quad c_2(n) = c(ns)$$

for all $n \in \mathbb{N}$. Using the same argument as in the proof of Theorem 2, we see that there exist $a, b \in \mathbb{T}$ such that

$$c_1(n) = \tfrac{1}{2}(a^n + a^{-n}) \quad \text{and} \quad c_2(n) = \tfrac{1}{2}(b^n + b^{-n})$$

and also

$$|c_1(n) - c_2(n)| \le \|C(ns) - c(ns)I_X\|$$

for every $n \in \mathbb{N}$. Consequently,

$$\limsup_{n\to\infty} |a^n + a^{-n} - (b^n + b^{-n})| \le 2\sup_{t\in\mathbb{R}} \|C(t) - c(t)I_X\| < 1,$$

where the rightmost inequality holds by assumption. Reprising the familiar argument, we have either $a = b$ or $a = b^{-1}$ by Lemma 2, and this in turn implies that $c_1 = c_2$. In particular, $\phi(C(s)) = \phi(c(s)I_X)$. Since ϕ was arbitrarily chosen, it follows that the operator

$$Q(s) = C(s) - c(s)I_X$$

is a quasinilpotent element of $\mathbf{A}$: $\phi(Q(s)) = 0$ holds for every $\phi \in \Delta(\mathbf{A})$, or equivalently, $\lim_{n\to\infty} \|Q(s)^n\|^{1/n} = 0$. By virtue of the representation $C(s) = c(s)I_X + Q(s)$ and the fact that $c(s)I_X$ commutes with $Q(s)$, $C(s)$ is a spectral operator, with $c(s)I_X$ being the scalar part of $C(s)$ and $Q(s)$

being the radical part of $C(s)$. We remark here that the scalarity of any operator of the form aI_X, where $a \in \mathbb{C}$, follows from the fact that

$$aI_X = \int_{\{a\}} \lambda \, dE(\lambda),$$

where E is the spectral measure given by

$$E(\omega) = \delta_a(\omega) I_X \qquad (\omega \in \mathscr{B}(\mathbb{C}))$$

with δ_a being the Dirac measure concentrated on a, i.e.,

$$\delta_a(\omega) = \begin{cases} 1 & \text{if } a \in \omega, \\ 0 & \text{otherwise} \end{cases}$$

for each $\omega \in \mathscr{B}(\mathbb{C})$.

Let t_0 be a period of c. Then, of course, $c(t_0) = 1$. By Lemma 7 and the assumption, for each $n \in \mathbb{N}$,

$$\|C(t_0)^n - I_X\| = \|C(t_0)^n - c(t_0)^n I_X\| \le \sup_{t \in \mathbb{R}} \|C(t) - c(t) I_X\| < 1/2.$$

Hence, firstly,

$$\|C(t_0)^n\| \le 1 + 1/2$$

and, secondly, $C(t_0)^n$ is invertible in $\mathscr{L}(X)$ and

$$\|C(t_0)^{-n}\| \le (1 - 1/2)^{-1} = 2.$$

It follows that $C(t_0)$ is doubly power bounded, i.e.,

$$\sup_{n \in \mathbb{Z}} \|C(t_0)^n\| < \infty.$$

According to a theorem proved independently by Fixman [14] and Foguel [15] and further extended by Dowson [7] (see also [8, Theorem 10.17]), every doubly power bounded spectral operator is of scalar type. Hence $C(t_0)$ is scalar-type spectral. By the uniqueness of the Jordan decomposition,

$$C(t_0) = c(t_0) I_X = I_X.$$

With this result in hand, Lemma 9 now ensures that the function $C \colon t \mapsto C(t)$ is periodic with period t_0. By Lemma 3, C is continuous under the operator norm topology on $\mathscr{L}(X)$. Thus C is an $\mathscr{L}(X)$-valued almost periodic function. Let $a \in \mathbb{R}$ be such that $c(t) = \cos at$ for each $t \in \mathbb{R}$. Then, clearly, $\widehat{c}(\alpha) = 0$ unless $\alpha = a$ or $\alpha = -a$. Since

$$\begin{aligned} \|\widehat{C}(\alpha) - \widehat{c}(\alpha) I_X\| &= \|M_t\{e^{-i\alpha t} C(t) - e^{-i\alpha t} c(t) I_X\}\| \\ &\le M_t\{\|e^{-i\alpha t} C(t) - e^{-i\alpha t} c(t) I_X\|\} \\ &\le \sup_{t \in \mathbb{R}} \|C(t) - c(t) I_X\| < 1/2, \end{aligned}$$

it follows that $\|\widehat{C}(\alpha)\| < 1/2$ whenever $\alpha \in \mathbb{R} \setminus \{-a, a\}$. In view of Lemma 5, $\widehat{C}(\alpha) = 0$ for each $\alpha \in \mathbb{R} \setminus \{-a, a\}$. By the uniqueness of the Fourier–Bohr expansion,

$$C(t) = e^{iat}\widehat{C}(a) + e^{-iat}\widehat{C}(-a)$$

for each $t \in \mathbb{R}$. Since C is even, we have $\widehat{C}(a) = \widehat{C}(-a)$, and so

$$C(t) = (e^{iat} + e^{-iat})\widehat{C}(a)$$

for each $t \in \mathbb{R}$. Letting $t = 0$ in the above formula yields

$$\widehat{C}(a) = \tfrac{1}{2} I_X.$$

Therefore, finally,

$$C(t) = \tfrac{1}{2}(e^{iat} + e^{-iat}) I_X = c(t) I_X$$

for each $t \in \mathbb{R}$, as was to be proved. ∎

As an immediate consequence of Theorem 5, we obtain the following partial converse to Theorem 1:

Theorem 5. *No bounded strongly continuous scalar cosine family on a Banach space satisfies* **(ACN)**.

4. Isolated points within cosine families, semigroups, and groups. We are now going to look at the results obtained in the previous section from a slightly different perspective.

Let X be a Banach space and let $Cos_{b,sc}^{norm}(X)$ denote the set of all bounded strongly continuous cosine families on X turned into a metric space by defining the distance $d(C, \tilde{C})$ between two cosine functions C and $\tilde{C}$ in $Cos_{b,sc}^{norm}(X)$ as

$$d(C, \tilde{C}) = \sup_{t \in \mathbb{R}} \|C(t) - \tilde{C}(t)\|.$$

It follows from Theorem 4 that scalar cosine families form isolated points of $Cos_{b,sc}^{norm}(X)$, whereas Theorem 1 guarantees that there are no other isolated points in this space. In other words, we have the following result:

Theorem 6. *If X is a Banach space, then the isolated points of $Cos_{b,sc}^{norm}(X)$ are precisely the scalar cosine families in $Cos_{b,sc}^{norm}(X)$.*

Remarkably, the picture displayed in Theorem 6 changes when we turn our attention to semigroups of operators. Let $Semi_{b,sc}^{norm}(X)$ denote the set of all bounded strongly continuous semigroups on X made into a metric space by means of the metric

$$d(S, \tilde{S}) = \sup_{t \geq 0} \|S(t) - \tilde{S}(t)\| \quad (S, \tilde{S} \in Semi_{b,sc}^{norm}(X)).$$

It transpires that the isolated points of $Semi_{b,sc}^{norm}(X)$ constitute only a small fraction of the set of all scalar semigroups in $Semi_{b,sc}^{norm}(X)$, where the term "scalar semigroup" has a meaning analogous to the corresponding term for cosine families.

THEOREM 7. *If X is a complex Banach space, then the isolated points of $Semi_{b,sc}^{norm}(X)$ are precisely the scalar semigroups of the form*

$$S(t) = e^{iat} I_X \quad (t \geq 0,\, a \in \mathbb{R}).$$

If X is a real Banach space, then the only isolated point of $Semi_{b,sc}^{norm}(X)$ is the identity semigroup defined by $S(t) = I_X$ for each $t \geq 0$.

The first step in the proof of Theorem 7 is based on our Theorem 8 below. We omit the proof of the latter result as it is essentially the same as the proof of Theorem 1.

THEOREM 8. *Let $S = \{S(t)\}_{t\geq 0}$ be a bounded non-scalar semigroup on a Banach space X. Then there exists a sequence $S_n = \{S_n(t)\}_{t\geq 0}$, $n = 1, 2, \ldots$, of semigroups on X such that $S_n \neq S$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} S_n(t) = S(t)$ in operator norm uniformly in $t \geq 0$. Moreover, if S is strongly continuous, then each S_n may be assumed to be strongly continuous.*

Proof of Theorem 7. It follows from Theorem 8 that any isolated point of $Semi_{b,sc}^{norm}(X)$ is necessarily a scalar semigroup.

Assume now that the space X is complex. For any $z \in \mathbb{C}$, let $S_{[z]} = \{S_{[z]}(t)\}_{t\geq 0}$ be the semigroup on X given by

$$S_{[z]}(t) = e^{zt} I_X \quad (t \geq 0).$$

Every scalar semigroup in $Semi_{b,sc}^{norm}(X)$ is of the form $S_{[z]}$ for some z with $\Re z \leq 0$. We next show that if a semigroup $S_{[z]}$ is an isolated point of $Semi_{b,sc}^{norm}(X)$, then necessarily $\Re z = 0$.

Fix $\lambda > 0$ arbitrarily. If $0 < \mu < \lambda$, then, as is easily checked, the function $t \mapsto e^{-\mu t} - e^{-\lambda t}$ on $[0, \infty)$ is non-negative and attains its maximum at $t_{\lambda,\mu} = (\ln \mu - \ln \lambda)/(\mu - \lambda)$. Since $\lim_{\mu\to\lambda} t_{\lambda,\mu} = \lambda^{-1}$, we see that, for any $a \in \mathbb{R}$, the expression

$$\sup_{t\geq 0} \|S_{[-\mu+ia]}(t) - S_{[-\lambda+ia]}(t)\| = e^{-\mu t_{\lambda,\mu}} - e^{-\lambda t_{\lambda,\mu}}$$

converges to 0 as $\mu \to \lambda$. Hence $S_{[-\lambda+ia]}$ is not an isolated point of $Semi_{b,sc}^{norm}(X)$.

We now show that, for each $a \in \mathbb{R}$, $S_{[ia]}$ is an isolated point of $Semi_{b,sc}^{norm}(X)$. The proof will rely on the following result.

LEMMA 10. *For any semigroup $T = \{T(t)\}_{t\geq 0}$ in $Semi_{b,sc}^{norm}(X)$, if*

$$\sup_{t\geq 0} \|T(t) - I_X\| < 1,$$

then $T(t) = I_X$ for every $t \geq 0$.

Assuming this for now, note that if S is a semigroup in $Semi_{b,sc}^{norm}(X)$ such that $\sup_{t\geq 0} \|S(t) - e^{iat} I_X\| < 1$, then $\sup_{t\geq 0} \|e^{-iat} S(t) - I_X\| < 1$ and this then implies that $e^{-iat}S(t) = I_X$ for each $t \geq 0$ so that $S = S_{[ia]}$, with the immediate consequence that $S_{[ia]}$ is an isolated point of $Semi_{b,sc}^{norm}(X)$.

Proof of Lemma 10. Let $T = \{T(t)\}_{t\geq 0}$ be a semigroup in $Semi_{b,sc}^{norm}(X)$ satisfying $\sup_{t\geq 0} \|T(t) - I_X\| = 1 - \delta$ for some $0 < \delta < 1$. Let A be the generator of T. Arguing as in the proof of Lemma 3, we first conclude that A is bounded, so that $T(t) = e^{tA}$ for all $t \geq 0$. Next, for each $t \geq 0$, we let

$$U(t) = -\sum_{n=1}^{\infty} \frac{1}{n}(I_X - e^{tA})^n \in \mathscr{L}(X).$$

Then the mapping $t \mapsto U(t)$ is differentiable on $(0, \infty)$ and

$$\frac{dU(t)}{dt} = \sum_{n=1}^{\infty}(I_X - e^{tA})^{n-1}e^{tA}A = e^{-tA}e^{tA}A = A.$$

Since $U(0) = 0$, it follows that $U(t) = At$ for all $t \geq 0$. On the other hand,

$$\|U(t)\| \leq \sum_{n=1}^{\infty} \frac{1}{n}(1 - \delta)^n = -\ln\delta < \infty$$

for every $t \geq 0$, implying that U is bounded. This, however, is impossible unless $A = 0$. (We remark that the condition $\sup_{t\geq 0} \|T(t) - I_X\| < 1$ is optimal: for any $\lambda > 0$, we have $\sup_{t\geq 0} \|I_X - e^{-\lambda t}I_X\| = 1$, while $e^{-\lambda t} \neq 1$ whenever $t > 0$.) ∎

Finally, we note that if the space X is real, then every scalar semigroup in $Semi_{b,sc}^{norm}(X)$ is of the form $S_{[-\lambda]}$ for some $\lambda \geq 0$. This observation along with straightforward modifications of the proof given thus far establishes the result in the real case. ∎

We remark that Lemma 10 can be readily deduced from the following generalisation of a result of Cox [6]: If $\mathbf{A}$ is a normed algebra with an identity e and a is an element of $\mathbf{A}$ such that $\sup_{n\in\mathbb{N}} \|a^n - e\| < 1$, then $a = e$. Cox's original result concerned the case of square matrices of a given size. This was later extended to bounded operators on Hilbert space by Nakamura and Yoshida [22] and to an arbitrary normed algebra by Hirschfeld [17] and Wallen [26]. The latter author gave an elementary argument using a significantly weaker hypothesis, namely that $\|a^n - 1\| = o(n)$ and $\liminf_{n\to\infty} n^{-1}(\|a-e\|+\|a^2-e\|+\cdots+\|a^n-e\|) < 1$. Wils [27], Chernoff [3], Nagisa and Wada [21], and Kalton et al.[18] provided further generalisations.

Lemma 10 allows us, serendipitously, to improve on the constant describing the neighbourhood of a scalar cosine family in Theorem 4 in the case $c(t) \equiv 1$. We claim that for any strongly continuous cosine family $C = \{C(t)\}_{t\in\mathbb{R}}$ on a Banach space X, the condition

$$\sup_{t\in\mathbb{R}} \|C(t) - I_X\| < 1 \tag{4.1}$$

implies $C(t) = I_X$ for every $t \in \mathbb{R}$. Let A be the generator of C. If $S = \{S(t)\}_{t\geq 0}$ is the strongly continuous semigroup on X related to C by the abstract Weierstrass formula

$$S(t)x = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\tau^2/4t} C(\tau)x\, d\tau \quad (t > 0,\, x \in X),$$

then the generator of S coincides with A (see, e.g., [1, proof of Theorem 3.14.17] or [16, Theorem 8.7]). Moreover, on account of (4.1), we have $\sup_{t\geq 0} \|S(t) - I_X\| < 1$. This last condition guarantees, by Lemma 10, that $S(t) = I_X$ for each $t \geq 0$, and this in turn implies that $A = 0$ and further that $C(t) = I_X$ for each $t \geq 0$, establishing the claim.

We finally note that Theorem 7 has a natural counterpart for groups of operators. Let X be a Banach space and let $Grp_{b,sc}^{norm}(X)$ denote the set of all bounded strongly continuous groups on X converted into a metric space by means of the metric

$$d(G, \tilde{G}) = \sup_{t\in\mathbb{R}} \|G(t) - \tilde{G}(t)\| \quad (G, \tilde{G} \in Grp_{b,sc}^{norm}(X)).$$

A minor modification of the proof of Theorem 7 (requiring, among other things, the use of a natural analogue of Theorem 8 for groups of operators) yields the following result.

THEOREM 9. *If X is a Banach space, then the isolated points of $Grp_{b,sc}^{norm}(X)$ are precisely the scalar groups in $Grp_{b,sc}^{norm}(X)$; these are of the form*

$$G(t) = e^{iat} I_X \quad (t \in \mathbb{R},\, a \in \mathbb{R})$$

when X is complex, and reduce to the single identity group, defined by $G(t) = I_X$ for every $t \in \mathbb{R}$, when X is real.

5. Uniform convergence in the strong operator topology. Let us introduce the following two conditions, of which the first is applicable to any given bounded cosine family $C = \{C(t)\}_{t\in\mathbb{R}}$ on a Banach space X, and the second is of relevance when the given cosine family C is bounded and strongly continuous:

- **(AS)** there exists a sequence $C_n = \{C_n(t)\}_{t\in\mathbb{R}}$, $n = 1, 2, \ldots$, of equibounded cosine families on X such that $C_n \neq C$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} C_n(t) = C(t)$ strongly and uniformly in $t \in \mathbb{R}$.
- **(ACS)** there exists a sequence $C_n = \{C_n(t)\}_{t\in\mathbb{R}}$, $n = 1, 2, \ldots$, of strongly continuous equibounded cosine families on X such that $C_n \neq C$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} C_n(t) = C(t)$ strongly and uniformly in $t \in \mathbb{R}$.

In this final section of the paper, we show that a bounded scalar cosine family may satisfy **(AS)** without satisfying **(AN)**, and a bounded strongly continuous scalar cosine family may satisfy **(ACS)** without satisfying **(ACN)**. We also present similar results for operator semigroups and groups.

We start our discussion with a few preliminaries. Let X be a Banach space and let Γ be a non-empty set. A family $\{(x_\gamma, x'_\gamma)\}_{\gamma\in\Gamma}$ of pairs in $X \times X'$ is called a *biorthogonal system* in $X \times X'$ if $\langle x_\alpha, x'_\beta\rangle = \delta_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$, where $\delta_{\alpha\beta}$ denotes the Kronecker delta. For simplicity, we shall write a biorthogonal system $\{(x_\gamma, x'_\gamma)\}_{\gamma\in\Gamma}$ in $X \times X'$ as $\{x_\gamma; x'_\gamma\}_{\gamma\in\Gamma}$. A biorthogonal system $\{x_\gamma; x'_\gamma\}_{\gamma\in\Gamma}$ is called *fundamental* if the closed linear span $\overline{\operatorname{span}}\{x_\gamma \mid \gamma \in \Gamma\}$ of $\{x_\gamma\}_{\gamma\in\Gamma}$ is equal to X, and *total* if $\{x'_\gamma\}_{\gamma\in\Gamma}$ separates the points of X, i.e., if for each $x \in X \setminus \{0\}$, there exists $\gamma \in \Gamma$ such that $\langle x, x_\gamma\rangle \neq 0$. A fundamental and total biorthogonal system in $X \times X'$ is called a *Markushevich basis* for X. If the set of functional coefficients is understood, a Markushevich basis $\{x_\gamma; x'_\gamma\}_{\gamma\in\Gamma}$ is abbreviated to $\{x_\gamma\}_{\gamma\in\Gamma}$. Clearly, every Schauder basis of a Banach space X is a Markushevich basis of X. An example of a Markushevich basis that is not a Schauder basis is the family of trigonometric polynomials $t \mapsto e^{2\pi int}$, $n \in \mathbb{Z}$, in the space $C[0, 1]$ of all complex continuous functions on $[0, 1]$ whose values at 0 and 1 are equal, endowed with the supremum norm.

A biorthogonal system $\{x_\gamma; x'_\gamma\}_{\gamma\in\Gamma}$ is called λ-*bounded* for some $\lambda \geq 1$ if $\sup\{\|x_\gamma\|\,\|x'_\gamma\| \mid \gamma \in \Gamma\} \leq \lambda$. A biorthogonal system is called *bounded* if it is λ-bounded for some $\lambda \geq 1$. Markushevich [20] proved that every separable Banach space has a countable Markushevich basis. Ovsepian and Pełczyński [23] showed the existence of a bounded countable Markushevich basis for any separable Banach space. Pełczyński [24] and Plichko [25] established independently that every separable Banach space has a $(1 + \epsilon)$-bounded countable Markushevich basis for every $\epsilon > 0$.

THEOREM 10. *Every bounded scalar cosine family on an infinite-dimensional separable Banach space satisfies* **(AS)**. *Likewise, every bounded strongly continuous scalar cosine family on an infinite-dimensional separable Banach space satisfies* **(ACS)**.

Proof. Let X be an infinite-dimensional separable Banach space. Let $c = \{c(t)\}_{t\in\mathbb{R}}$ be a bounded scalar-valued cosine family and let $C = \{C(t)\}_{t\in\mathbb{R}}$ be the corresponding scalar cosine family on X, i.e.,

$$C(t) = c(t)I_X$$

for each $t \in \mathbb{R}$. Let $\{x_n; x'_n\}_{n\in\mathbb{N}}$ be a λ-bounded Markushevich basis for X for some $\lambda > 1$. For each $n \in \mathbb{N}$, let P_n be the one-dimensional operator in

$\mathscr{L}(X)$ given by

$$P_n(x) = \langle x, x_n' \rangle x_n.$$

Clearly, each P_n is a projection, i.e., $P_n^2 = P_n$, and, moreover, we have $\|P_n\| \le \lambda$. Let $\tilde{c} = \{\tilde{c}(t)\}_{t\in\mathbb{R}}$ be an arbitrary bounded continuous scalar-valued cosine family different from c. For each $n \in \mathbb{N}$ and each $t \in \mathbb{R}$, set

$$C_n(t) = c(t)(I_X - P_n) + \tilde{c}(t)P_n.$$

It is evident that, for each $n \in \mathbb{N}$, $C_n = \{C_n(t)\}_{t\in\mathbb{R}}$ is a cosine family on X different from C, and, moreover, if C is continuous, then C_n is continuous, too. In addition, the C_n's are equibounded:

$$\|C_n(t)\| \le 1 + 2\|P_n\| \le 1 + 2\lambda$$

for each $n \in \mathbb{N}$ and each $t \in \mathbb{R}$. Here the first inequality makes use of the fact that any bounded (continuous or not) scalar cosine family $c = \{c(t)\}_{t\in\mathbb{R}}$ automatically satisfies

$$|c(t)| \le 1 \tag{5.1}$$

for all $t \in \mathbb{R}$. One way to see this is as follows. By the result of Kannappan quoted earlier, if $c = \{c(t)\}_{t\in\mathbb{R}}$ is a scalar cosine family, then there exists a function $g\colon \mathbb{R} \to \mathbb{C} \setminus \{0\}$ such that $g(s+t) = g(s)g(t)$ for all $s, t \in \mathbb{R}$ and

$$c(t) = \tfrac{1}{2}(g(t) + g(t)^{-1}) \tag{5.2}$$

for all $t \in \mathbb{R}$. Now, if c is also bounded, then necessarily

$$|g(t)| = 1 \tag{5.3}$$

for every $t \in \mathbb{R}$, for otherwise, should $|g(t_0)| \ne 1$ hold for some t_0, the representation

$$c(nt_0) = \tfrac{1}{2}(g(nt_0) + g(nt_0)^{-1}) = \tfrac{1}{2}(g(t_0)^n + g(t_0)^{-n}) \quad (n \in \mathbb{N})$$

would imply that $c(nt_0)$ diverges in modulus to infinity as $n \to \infty$. At this point, (5.1) readily follows from (5.2) and (5.3).

Returning to the main line of the proof, note that if $x = \sum_{k=1}^{K} \alpha_k x_k$ is a finite linear combination of the x_n's, then $P_n x = 0$ whenever $n > K$, and so $C_n(t)x = C(t)x$ for all $n > K$ and all $t \in \mathbb{R}$. Since the C_n's are equibounded and $\overline{\operatorname{span}}\{x_n \mid n \in \mathbb{N}\} = X$, it follows that, for each $x \in X$, $\lim_{n\to\infty} C_n(t)x = C(t)x$ uniformly in t. ∎

An immediate consequence of Theorems 3 and 10 is the following.

Theorem 11. *Any bounded scalar cosine family on an infinite-dimensional separable Hilbert space satisfies* **(AS)** *without satisfying* **(AN)**.

In turn, Theorems 5 and 10 immediately imply the following.

THEOREM 12. *Any bounded strongly continuous scalar cosine family on an infinite-dimensional separable Banach space satisfies* **(ACS)** *without satisfying* **(ACN)**.

Let X be a Banach space and let $Cos_b^{strong}(X)$ and $Cos_{b,sc}^{strong}(X)$ denote the sets of all bounded cosine families on X and of all bounded strongly continuous cosine families on X, respectively, each converted into a topological space by means of the collection of pseudometrics

$$d_x(C,\tilde{C}) = \sup_{t\in\mathbb{R}} \|C(t)x - \tilde{C}(t)x\| \quad (x \in X).$$

Combining Theorems 1 and 10, we obtain the following.

THEOREM 13. *If X is an infinite-dimensional separable Banach space, then neither $Cos_b^{strong}(X)$ nor $Cos_{b,sc}^{strong}(X)$ has isolated points.*

We finally discuss the extent to which the results obtained thus far in this section carry over to semigroups and groups of operators.

By consistently replacing each occurrence of "cosine family" or "cosine families" by "semigroup" or "semigroups" in one case, and "group" or "groups" in another, conditions **(AN)**, **(ACN)**, **(AS)**, and **(ACS)** can be extended to apply to semigroups and groups of operators. With this extension in effect, it takes a minor modification of the proof to obtain the following companion result to Theorem 10.

THEOREM 14. *Every bounded scalar semigroup or group on an infinite-dimensional separable Banach space satisfies* **(AS)**. *Likewise, every bounded strongly continuous scalar semigroup or group on an infinite-dimensional separable Banach space satisfies* **(ACS)**.

Not every bounded strongly continuous scalar semigroup on an infinite-dimensional separable Banach space satisfies **(ACS)** without satisfying **(ACN)**. Indeed, only semigroups of the form $S_{[ia]}$ with $a \in \mathbb{R}$ are in that category, whereas any semigroup of the form $S_{[-\lambda+ia]}$ with $\lambda > 0$ and $a \in \mathbb{R}$ satisfies both **(ACN)** and **(ACS)** (as is clear from the proof of Theorem 7 and from Theorem 14). In contrast, for groups of operators the following consequence of Theorems 9 and 14 holds.

THEOREM 15. *Any bounded strongly continuous scalar group on an infinite-dimensional separable Banach space satisfies* **(ACS)** *without satisfying* **(ACN)**.

Given a Banach space X, let $Semi_b^{strong}(X)$ and $Semi_{b,sc}^{strong}(X)$ be the sets of all bounded semigroups on X and of all bounded strongly continuous semigroups on X, and let $Grp_b^{strong}(X)$ be the sets of all bounded groups on X and of all bounded strongly continuous groups on X, respectively, each equipped with the topology determined by a collection of seminorms analogous to the collection defining the topology of $Cos_b^{strong}(X)$ or $Cos_{b,sc}^{strong}(X)$. Combining Theorem 8, the natural analogue of Theorem 8 for groups of operators, and Theorem 14, we obtain the following as our final result.

THEOREM 16. *If X is an infinite-dimensional separable Banach space, then none of the spaces $Semi_b^{strong}(X)$, $Semi_{b,sc}^{strong}(X)$, $Grp_b^{strong}(X)$, and $Grp_{b,sc}^{strong}(X)$ has isolated points.*

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Adam Bobrowski
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: a.bobrowski@pollub.pl

Wojciech Chojnacki
School of Computer Science
The University of Adelaide
Adelaide, SA 5005, Australia
and
Wydział Matematyczno-Przyrodniczy
Szkoła Nauk Ścisłych
Uniwersytet Kardynała Stefana Wyszyńskiego
Dewajtis 5
01-815 Warszawa, Poland
E-mail: wojciech.chojnacki@adelaide.edu.au

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