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Generators for algebras dense in L^p-spaces

by

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Abstract. For various L^p-spaces ($1 \le p < \infty$) we investigate the minimum number of complex-valued functions needed to generate an algebra dense in the space. The results depend crucially on the regularity imposed on the generators. For μ a positive regular Borel measure on a compact metric space there always exists a single bounded measurable function that generates an algebra dense in $L^p(\mu)$. For M a Riemannian manifold-with-boundary of finite volume there always exists a single continuous function that generates an algebra dense in $L^p(M)$. These results are in sharp contrast to the situation when the generators are required to be smooth. For smooth generators we prove a result similar to a known fact about algebras uniformly dense in continuous functions: for M a smooth manifold-with-boundary of dimension n, at least n smooth functions are required in order to generate an algebra dense in $L^p(M)$. We also show that on every smooth manifold-with-boundary there exists a bounded continuous real-valued function that is one-to-one on the complement of a set of measure zero.

1. Introduction. A theorem of Andrew Browder [4] (or see [16, Theorem 10.6]) gives a lower bound on the number of elements needed to generate a dense subalgebra of a commutative Banach algebra. In this paper we consider the analogous problem for L^p-spaces: for various L^p-spaces ($1 \le p < \infty$) we investigate the minimum number of complex-valued functions needed to generate an algebra dense in the space. The results depend crucially on the regularity imposed on the generators. We consider separately the cases of measurable, continuous, and smooth generators. Throughout the paper, the word "smooth" shall always mean "of class C^1", and manifolds will always be assumed to be second countable. We will show a single generator always suffices both in the case of bounded measurable generators and "nice" measures and in the case of continuous generators on smooth manifolds. In contrast, for smooth generators on a smooth manifold one needs

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at least as many generators as the dimension of the manifold. The precise statements of these results are as follows.

THEOREM 1.1. *Let X be a compact metric space, and let μ be a positive regular Borel measure on X. Then there exists a real-valued function f in $L^\infty(\mu)$ such that the set of polynomials in f is dense in $L^p(\mu)$ for all $1 \leq p < \infty$.*

THEOREM 1.2. *Let M be a Riemannian manifold-with-boundary of finite volume. Then there exists a bounded continuous real-valued function f such that the set of polynomials in f is dense in $L^p(M)$ for all $1 \leq p < \infty$.*

THEOREM 1.3. *Let M be an n-dimensional Riemannian manifold-with-boundary. Then no set of fewer than n* smooth *complex-valued functions generates an algebra dense in $L^p(M)$ for $1 \leq p < \infty$.*

Note the similarity between Theorem 1.3 and the following rather well-known result about generators for $C(M)$ which is an easy consequence of the theorem of Browder mentioned above. Note also the contrast between these results and Theorem 1.2.

THEOREM 1.4 (Freeman [10, Corollary 2.2]). *Let M be a compact n-dimensional manifold-with-boundary. Then no set of fewer than n continuous complex-valued functions generates an algebra dense in $C(M)$.*

Of course, in Theorems 1.2 and 1.3, by $L^p(M)$ we mean L^p of the measure induced by the Riemannian structure. (See [17] for a discussion of integration on a Riemannian manifold.) In the case when the manifold is compact it is unnecessary to fix a Riemannian structure, for if μ_1 and μ_2 are two measures induced by two different Riemannian structures, then μ_1 and μ_2 are mutually absolutely continuous, and the Radon–Nikodym derivative of one with respect to the other is bounded away from zero and infinity. It follows that $L^p(\mu_1)$ and $L^p(\mu_2)$ are identical as sets, and the $L^p(\mu_1)$ and $L^p(\mu_2)$ norms are equivalent. Thus for M a compact manifold, $L^p(M)$ is well-defined as a topological vector space independent of the choice of Riemannian structure.

We also treat the case of homeomorphisms as generators. In that regard we prove the following results. Here and throughout the paper m will denote Lebesgue measure of whatever dimension is appropriate for the context, and S^k will denote the unit k-sphere in $\mathbb{R}^{k+1}$.

THEOREM 1.5. *There exists a homeomorphism $f : S^1 \to S^1$ such that the set of polynomials in f is dense in $L^p(S^1, m)$ for all $1 \leq p < \infty$.*

THEOREM 1.6. *There is no one-to-one* smooth *function f such that the set of polynomials in f is dense in $L^p(S^1, m)$ for some $1 \leq p < \infty$.*

THEOREM 1.7. *There exists a homeomorphism $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that the set of polynomials in f is dense in $L^p(\overline{\mathbb{D}}, m)$ for all $1 \leq p < \infty$.*

THEOREM 1.8. *Let X be a compact subset of $\mathbb{R}^n$. Then there exists an embedding $F : X \to \mathbb{C}^{\lceil n/2 \rceil}$ such that the set of the component functions $f_1, \ldots, f_{\lceil n/2 \rceil}$ of F generates an algebra dense in $L^p(X, m)$ for all $1 \leq p < \infty$.*

As pointed out by William Ross, the proof of Theorem 1.7 is reminiscent of the proof of Bram's theorem [3] (or see [7, Theorem VI.8.14]). One might therefore wonder whether it is possible to derive one of these theorems from the other. However, as far as the authors can tell, this is not the case.

Theorem 1.5 was the first result we proved. It was this result that led us to conjecture the assertion of Theorem 1.7, which in turn led us to Theorem 1.8. The question then naturally arose whether, in the context of Theorem 1.8, it is the case that $\lceil n/2 \rceil$ is the minimum number of complex-valued functions needed to generate an algebra dense in $L^p(X, m)$ for a suitable choice of X. We showed that this question is related to a question regarding a possible strengthening of the well-known theorem that there is no embedding of $\mathbb{R}^n$ into $\mathbb{R}^k$ for $k < n$. Before stating this question, it is convenient to make the following definition.

DEFINITION 1.9. We call a map F defined on a measure space X *one-to-one almost everywhere* if there is a subset E of X of measure zero such that the restriction of F to $X \setminus E$ is one-to-one.

We showed that if complex-valued functions $f_1, \ldots, f_k$ generate an algebra dense in $L^p(X, m)$, then the map $F = (f_1, \ldots, f_k) : X \to \mathbb{R}^k$ is one-to-one almost everywhere. This led us to ask whether, for $k < n$, there ever exists a continuous map $F : \mathbb{R}^n \to \mathbb{R}^k$ that is one-to-one almost everywhere. We initially conjectured that no such mappings exist and hence that $\lceil n/2 \rceil$ is the minimum number of complex-valued functions needed to generate an algebra dense in $L^p(X, m)$ for X a ball in $\mathbb{R}^n$. However, we subsequently proved the following result showing that this conjecture fails in the strongest possible way. The assertion of Theorem 1.2 that only one function is needed follows as a corollary.

THEOREM 1.10. *On every smooth manifold-with-boundary there exists a bounded continuous real-valued function that is one-to-one almost everywhere.*

Although this theorem is the exact opposite of what we at first expected, in hindsight the existence of such functions is not so surprising on account of the difference between measure-theoretic smallness and topological smallness in the sense of Baire category. The set on which our function will be shown to be one-to-one, while of full measure, will nevertheless be a set of first category. Thus from the point of view of category, the set on which our function is not required to be one-to-one is a large set. This leads us to make the following conjecture.

Conjecture 1.11. *If $k < n$, then there is no continuous map $F : \mathbb{R}^n \to \mathbb{R}^k$ that is one-to-one on the complement of a set of first category in $\mathbb{R}^n$.*

Our motivation for considering the question of how many functions are required to generate an algebra dense in an L^p-space arose from an erroneous paper by Guangfu Cao [5] related to the following theorem of Sheldon Axler, Željko Čučković, and N. V. Rao. Here $H^\infty(\Omega)$ denotes the algebra of all bounded holomorphic functions on Ω.

Theorem 1.12 (Axler–Čučković–Rao [1]). *Let Ω be a bounded connected open set in the complex plane. If $\phi \in H^\infty(\Omega)$ is non-constant and $\psi \in L^\infty(\Omega)$ is such that the Toeplitz operators T_ϕ and T_ψ on the Bergman space of Ω commute, then ψ is holomorphic.*

Axler, Čučković, and Rao [1] asked what happens in connection with their theorem in higher dimensions. In [5] Cao presents as a theorem the statement that the above theorem holds with the planar domain Ω replaced by an arbitrary bounded pseudoconvex domain with C^2-smooth boundary in $\mathbb{C}^n$. (It is somewhat unclear exactly what the hypotheses on the domain are, but they appear to be as stated in the preceding sentence.) We do not know whether this result of Cao is true although it seems unlikely since, as he notes, the result does not hold for the bidisk. Nevertheless, the argument he gives is definitely incorrect. Axler, Čučković, and Rao prove their result as a consequence of the following approximation theorem of Christopher Bishop (an alternative proof of which can be found in [13]).

Theorem 1.13 (Bishop [2, Theorem 1.2]). *Let Ω be an open set in the complex plane, and let $\phi \in H^\infty(\Omega)$ be non-constant on each component of Ω. Then $C(\overline{\Omega})$ is contained in the norm-closed subalgebra of $L^\infty(\Omega)$ generated by $H^\infty(\Omega)$ and $\overline{\phi}$.*

Cao presents as a lemma an assertion that is a weak form of Bishop's theorem for pseudoconvex domains in $\mathbb{C}^n$ and then correctly shows that from this assertion his main theorem follows by the same argument as that used by Axler, Čučković, and Rao. However, this weak form of Bishop's theorem for pseudoconvex domains, which we now state, is in fact false.

Erroneous Statement 1.14 (Cao [5, Theorem 5]). *Suppose Ω is a bounded pseudoconvex domain with C^2-smooth boundary in $\mathbb{C}^n$ and $\phi \in H^\infty(\Omega)$ is non-constant. Then $C(\overline{\Omega})$ is contained in the closure in $L^2(\Omega, m)$ of the algebra generated by $H^\infty(\Omega)$ and $\overline{\phi}$.*

It was this false assertion that first got us thinking about generators for algebras dense in L^p-spaces. A simple direct way to see that the above assertion is indeed false is to take Ω to be the open unit ball in $\mathbb{C}^2$ and note that then in $L^2(\Omega, m)$ the function $\overline{z}_2$ is orthogonal to the algebra generated by $H^\infty(\Omega)$ and $\overline{z}_1$. In fact, Erroneous Statement 1.14 is *never* true when $n > 1$. To see this note that if the conclusion of 1.14 held for some Ω and ϕ, then for a closed ball B contained in Ω, the algebra generated by the coordinate functions $z_1, \ldots, z_n$ and $\overline{\phi}$ would be dense in $L^2(B, m)$, whereas by Theorem 1.3 at least $2n$ smooth functions are needed to generate an algebra dense in $L^2(B, m)$.

2. Measurable generators. In this section we prove Theorem 1.1, and we show that the hypothesis that the space is metrizable cannot be omitted. The proof of Theorem 1.1 is essentially a repetition of an argument of Paul Halmos [12]. We thank Donald Sarason for directing our attention to Halmos' paper.

Proof of Theorem 1.1. Every compact metric space is a continuous image of the Cantor set. Let K denote the (standard middle thirds) Cantor set and let $\varphi : K \to X$ be a continuous surjective map. By a theorem of topology, there exists a Borel cross-section of φ, that is, a Borel function $f : X \to K$ such that the composition $\varphi \circ f$ is the identity on X. (See [12].)

Now let $f_*(\mu)$ be the push forward measure on K defined by $f_*(\mu)(E) = \mu(f^{-1}(E))$ for each measurable set E in K. Then for every Borel measurable function g on K,

$$\int_X (g \circ f)\, d\mu = \int_K g\, df_*(\mu).$$

It follows that the map $T : L^p(f_*(\mu)) \to L^p(\mu)$ given by $Tg = g \circ f$ is an isometry. Furthermore, T maps $L^p(f_*(\mu))$ onto $L^p(\mu)$; to see this define $S : L^p(\mu) \to L^p(f_*(\mu))$ by $Sg = g \circ \varphi$, and note that since $\varphi \circ f$ is the identity on X, the operator TS is the identity on $L^p(\mu)$. By the Weierstrass approximation theorem, the polynomials are uniformly dense in $C(K)$ and hence are dense in $L^p(f_*(\mu))$, so it follows that the polynomials in f are dense in $L^p(\mu)$. ∎

Remark 2.1. We note that the metrizability of the space X cannot be omitted from the hypotheses of Theorem 1.1. For example, consider the Cantor space $X = \{0, 1\}^J$ with the index set J uncountable. Then X is non-metrizable. Regard X as the infinite product group $(\mathbb{Z}/2)^J$, and let μ be the Haar measure on X. We now show that $L^p(\mu)$ is non-separable, which implies that the conclusion of Theorem 1.1 cannot hold. For each $\alpha \in J$, let χ_α be the characteristic function of the set $\{(x_\gamma)_{\gamma \in J} \in X : x_\alpha = 1\}$. Then the function $|\chi_\alpha - \chi_\beta|$ takes the value 1 on a set of measure $1/2$ and takes the value 0 on the rest of X. Hence,

$$\|\chi_\alpha - \chi_\beta\|_p = \Big[\int_X |\chi_\alpha - \chi_\beta|^p\, d\mu\Big]^{1/p} = 2^{-1/p}.$$

Thus $\{\chi_\alpha\}_{\alpha\in J}$ is an uncountable discrete set in $L^p(\mu)$, and hence $L^p(\mu)$ is non-separable.

One might be tempted to imagine that $L^p(\mu)$ is non-separable whenever the support of μ is non-metrizable, and hence that the conclusion of the theorem holds if and only if the support of μ is metrizable. However, this is not the case. Let X be a separable compact Hausdorff space that is not metrizable. For instance, X could be the product space I^I where $I = [0,1]$. Let $\{x_n\}$ be a countable dense subset of X. Choose a sequence (a_n) of positive numbers such that $\sum a_n < \infty$, and define a measure μ on X by

$$\mu(E) = \sum_{x_n\in E} a_n.$$

In other words, μ is counting measure with weight a_n at x_n. Then μ is a positive regular Borel measure on X whose support is X and hence is non-metrizable. But $L^p(\mu)$ is clearly isometrically isomorphic to the weighted l^p-space with weights (a_n), and hence $L^p(\mu)$ is separable. Furthermore, the isomorphism preserves the multiplication. By viewing the set of positive integers as a subset of its one-point compactification, we can apply Theorem 1.1 to conclude that there is a bounded sequence that generates an algebra dense in our weighted l^p-spaces. We conclude that there is a bounded measurable function on X that generates an algebra dense in $L^p(\mu)$ for all $1 \le p < \infty$.

3. Continuous generators and almost everywhere one-to-one functions. This section is devoted to proving Theorems 1.2 and 1.10. We first show that Theorem 1.2 follows from Theorem 1.10 and then turn to the proof of Theorem 1.10. We conclude the section by showing that dimension decreasing, almost everywhere one-to-one mappings can never be smooth.

Proof of Theorem 1.2. Let μ be the measure on M corresponding to the volume form given by the Riemannian metric on M. By Theorem 1.10 there is a bounded continuous real-valued function f on M that is one-to-one almost everywhere. Let $\widetilde{M}$ be the closure of $f(M)$ in $\mathbb{R}$, and let $f_*(\mu)$ be the push forward of μ under f. By the Weierstrass approximation theorem, the polynomials are uniformly dense in the continuous functions on $\widetilde{M}$ and hence dense in $L^p(\widetilde{M}, f_*(\mu))$. Now let $h \in L^p(M,\mu)$ be arbitrary and let E be a set of measure zero in M such that f is one-to-one on $M\setminus E$. Redefine h if necessary on E so that h is constant on each level set of f. Then h induces a function $\widetilde{h}$ in $L^p(\widetilde{M}, f_*(\mu))$ such that $\widetilde{h}\circ f = h$. For any $\epsilon > 0$, there is a polynomial q such that

$$\Big[\int_{\widetilde{M}} |\widetilde{h} - q|^p \, df_*(\mu)\Big]^{1/p} < \epsilon.$$

Then pulling back to M, we have

$$\Big[\int_M |h - q \circ f|^p \, d\mu\Big]^{1/p} < \epsilon.$$

Since $q \circ f$ is a polynomial in f, the assertion is proven. ■

We now proceed to the proof of Theorem 1.10. We begin with several easy lemmas. Throughout the paper, by "a Cantor set" we mean any space that is homeomorphic to the standard middle thirds Cantor set.

Lemma 3.1. *If C is a Cantor set and $\mathscr{U}$ is an open cover of C, then C can be written as a finite union of disjoint Cantor sets $C_1, \dots, C_N$ each of which lies in some member of $\mathscr{U}$.*

Proof. Without loss of generality we take $C = \{0,1\}^\omega$. By passing to a refinement of the open cover $\mathscr{U}$, we may assume that $\mathscr{U}$ consists of *basic* open sets, that is, sets of the form $\{(x_n) \in \{0,1\}^\omega : x_k = a_k \text{ for all } k \in F\}$ for some finite set F and some $a_k \in \{0,1\}$ for each $k \in F$. Furthermore, by compactness of C we may assume that $\mathscr{U}$ is a finite set $\{U_1, \dots, U_r\}$. For each $j = 1, \dots, r$, let F_j be the finite set such that U_j may be expressed in the form

$$U_j = \{(x_n) \in \{0,1\}^\omega : x_k = a_k \text{ for all } k \in F_j\}.$$

Let $F = \bigcup_{j=1}^r F_j$. Then F is finite and the equation

$$C = \{0,1\}^\omega = \bigcup_{(a_j)_{j\in F} \in \{0,1\}^F} \{(x_n) \in \{0,1\}^\omega : x_k = a_k \text{ for all } k \in F\}$$

expresses C as a finite union of disjoint Cantor sets each of which lies in some U_j. ■

Lemma 3.2. *Given an open set U in $\mathbb{R}^d$ with $m(U) < \infty$ and an $\varepsilon > 0$, there exists a Cantor set C in U such that $m(U \setminus C) < \varepsilon$.*

Proof. Every open set in $\mathbb{R}^d$ is a countable union of (d-dimensional) cubes with disjoint interiors [9, Lemma 2.43], so choose cubes $R_1, R_2, \dots$ with disjoint interiors such that $U = \bigcup_n R_n$. Then choose N such that $\sum_{n=N+1}^\infty m(R_n) < \varepsilon/2$. For each $j = 1, \dots, N$ choose a Cantor set C_j in R_j such that $m(R_j \setminus C_j) < \varepsilon/2^{j+1}$. Let $C = C_1 \cup \dots \cup C_n$. Then $m(U \setminus C) = \sum_{n=1}^N m(R_n \setminus C_n) + \sum_{n=N+1}^\infty m(R_n) < \varepsilon$, and being a finite union of disjoint Cantor sets, C is itself a Cantor set. ■

The next lemma follows immediately from Lemma 3.2 and a simple induction argument.

Lemma 3.3. *Given an open set U in $\mathbb{R}^d$ with $m(U) < \infty$, there exists a countable collection $\{C_n\}_{n=1}^\infty$ of disjoint Cantor sets in U such that $m(U \setminus \bigcup_{n=1}^\infty C_n) = 0$.*

LEMMA 3.4. *Let M be a smooth manifold-with-boundary. Then there exists a countable collection $\{C_n\}_{n=1}^\infty$ of disjoint Cantor sets in M such that $M \setminus \bigcup_{n=1}^\infty C_n$ has measure zero in M.*

Proof. Since the boundary of a smooth manifold always has measure zero, we may assume without loss of generality that M is boundaryless. Now choose an at most countable open cover $\{U_n\}$ of M such that for each n there is a coordinate system (ϕ_n, W_n) such that $\overline{U}_n \subset W_n$ and $\phi(U_n)$ is an open ball in Euclidean space. Then the topological boundary ∂U_n of each set U_n has measure zero in M. Now define a collection of disjoint open sets $\{V_n\}$ in M by setting $V_n = U_n \setminus (\overline{U}_1 \cup \cdots \cup \overline{U}_{n-1})$.

Suppose x is a point of $M \setminus \bigcup_n V_n$. Let k be the smallest value of n for which x is in U_n. Since x is not in V_k, it must be in $\overline{U}_j$ for some $j < k$. Then x is in ∂U_j. We conclude that $M \setminus \bigcup_n V_n$ is contained in $\bigcup_n \partial U_n$ and hence has measure zero in M.

By Lemma 3.3 we can choose, for each n, a countable collection $\{C_n^r\}_{r=1}^\infty$ of disjoint Cantor sets in V_n such that $V_n \setminus \bigcup_{r=1}^\infty C_n^r$ has measure zero. Then the collection $\{C_n^r\}$ of all the chosen Cantor sets is a countable collection of disjoint Cantor sets in M such that $M \setminus \bigcup_{n,r} C_n^r$ has measure zero in M. ∎

With these preliminaries we are ready for the proof of Theorem 1.10.

Proof of Theorem 1.10. Let M be a smooth manifold-with-boundary. By Lemma 3.4 there exists a countable collection $\{C_n\}_{n=1}^\infty$ of disjoint Cantor sets in M such that $M \setminus \bigcup_{n=1}^\infty C_n$ has measure zero in M. We will construct a sequence $(f_n)_{n=1}^\infty$ of continuous functions from M into $[0,1]$ such that, for each n,

(i) f_n is one-to-one on $C_1 \cup \cdots \cup C_n$,
(ii) f_{n+1} agrees with f_n on $C_1 \cup \cdots \cup C_n$, and
(iii) $\|f_{n+1} - f_n\|_\infty \le 1/2^n$.

Suppose for the moment that such a sequence of functions has been constructed. Then on account of condition (iii), the sequence (f_n) converges uniformly to a continuous limit function f. Due to condition (ii), f_m agrees with f_n on $C_1 \cup \cdots \cup C_n$ for all $m \ge n$, and hence the limit function f also agrees with f_n on $C_1 \cup \cdots \cup C_n$. Now given distinct points a and b in $\bigcup_{n=1}^\infty C_n$, choose N such that both a and b lie in $C_1 \cup \cdots \cup C_N$. Then $f(a) = f_N(a) \ne f_N(b) = f(b)$. Hence f is one-to-one on $\bigcup_{n=1}^\infty C_n$. Thus it suffices to construct a sequence of functions satisfying conditions (i)–(iii).

We will construct the sequence of functions f_n by induction. To carry out the induction we will also require the additional condition that, for each n,

(iv) $\{f_n(C_1), \ldots, f_n(C_n)\}$ is a collection of disjoint Cantor sets in $[0,1]$.

We begin by defining f_1. Choose a Cantor set $\widetilde{C}_1$ in $[0,1]$. Choose a homeomorphism g_1 of C_1 onto $\widetilde{C}_1$. By the Tietze extension theorem, there is an extension of g_1 to a continuous function of M into $[0,1]$. Let f_1 be the extension.

Now for the induction step, assume that functions $f_1, \dots, f_k$ have been defined so that conditions (i)–(iv) hold for those values of n for which they are meaningful. We wish to define f_{k+1}. By the continuity of f_k, there is an open cover $\mathscr{U}$ of C_{k+1} such that the image $f_k(U)$ of each member U of $\mathscr{U}$ is contained in an interval of length $1/2^k$. By Lemma 3.1, we can write C_{k+1} as a finite union $C_{k+1} = C_{k+1}^1 \cup \dots \cup C_{k+1}^N$ of disjoint Cantor sets $C_{k+1}^1, \dots, C_{k+1}^N$ each of which is contained in some member of $\mathscr{U}$. Then for each $j = 1, \dots, N$, the set $f_k(C_{k+1}^j)$ is contained in an interval I_{k+1}^j of length $1/2^k$. Since $f_k(C_1), \dots, f_k(C_k)$ are disjoint Cantor sets, their union is also a Cantor set and in particular has empty interior in $[0,1]$. Consequently, we can choose disjoint Cantor sets $\widetilde{C}_{k+1}^1, \dots, \widetilde{C}_{k+1}^N$ with $\widetilde{C}_{k+1}^j$ contained in $I_{k+1}^j \setminus (f_k(C_1) \cup \dots \cup f_k(C_k))$ for each j. Choose a homeomorphism g_{k+1}^j of C_{k+1}^j onto $\widetilde{C}_{k+1}^j$ for each j, and then define g_{k+1} on $C_1 \cup \dots \cup C_k \cup C_{k+1}$ by

$$g_{k+1}(x) = \begin{cases} f_k(x) & \text{for } x \in C_1 \cup \dots \cup C_k, \\ g_{k+1}^j(x) & \text{for } x \in C_{k+1}^j \ (j = 1, \dots, N). \end{cases}$$

Then g_{k+1} is a homeomorphism of $C_1 \cup \dots \cup C_{k+1}$ onto $f(C_1) \cup \dots \cup f(C_k) \cup \widetilde{C}_{k+1}^1 \cup \dots \cup \widetilde{C}_{k+1}^N$ taking C_{k+1} onto $\widetilde{C}_{k+1}^1 \cup \dots \cup \widetilde{C}_{k+1}^N$. Note that

$$\sup_{x \in C_1 \cup \dots \cup C_{k+1}} |f_k(x) - g_{k+1}(x)| \le 1/2^k$$

since for each j both $f_k(C_{k+1}^j)$ and $g_{k+1}(C_{k+1}^j)$ are contained in I_{k+1}^j. By the Tietze extension theorem, there is a continuous function h_{k+1} on M that agrees with $f_k - g_{k+1}$ on $C_1 \cup \dots \cup C_{k+1}$ and satisfies

$$\|h_{k+1}\|_\infty \le 1/2^k.$$

Set $f_{k+1} = f_k - h_{k+1}$. Then $f_{k+1} = g_{k+1}$ on $C_1 \cup \dots \cup C_{k+1}$, and $\|f_{k+1} - f_k\|_\infty \le 1/2^k$. It follows that $f_1, \dots, f_{k+1}$ satisfy the required conditions (i)–(iv) for those values of n for which the conditions are meaningful. Therefore, by induction we obtain the desired sequence (f_n). ∎

We conclude this section by showing that, not surprisingly, *smooth* maps one-to-one almost everywhere can never be dimension decreasing.

PROPOSITION 3.5. *If $k < n$, then there is no* smooth *map $F : \mathbb{R}^n \to \mathbb{R}^k$ that is one-to-one almost everywhere.*

Proof. Let $F = (f_1, \dots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map. Let r be the maximum rank achieved by the derivative dF of F, and let a be a point where this maximum is achieved. Then dF has rank r throughout some neighborhood U of a. Now without loss of generality we may assume that in U the differentials $df_1, \ldots, df_r$ are linearly independent, and $df_{r+1}, \ldots, df_k$ are in the span of $df_1, \ldots, df_r$. Then $(f_1, \ldots, f_r) : \mathbb{R}^n \to \mathbb{R}^r$ is a submersion at a, so by the local submersion theorem [11, p. 20], there are local coordinates about a and $\big(f_1(a), \ldots, f_r(a)\big)$ in which $(f_1, \ldots, f_r)$ becomes the canonical submersion $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_r)$. On each slice $\{(x_1, \ldots, x_r) = c\}$, for c a constant in $\mathbb{R}^r$, the functions $f_1, \ldots, f_r$ are constant. Hence $df_1, \ldots, df_r$ are identically zero when restricted to a slice. Since $df_{r+1}, \ldots, df_k$ are in the span of $df_1, \ldots, df_r$, we see that $df_{r+1}, \ldots, df_k$ are also identically zero when restricted to a slice. But then $f_{r+1}, \ldots, f_k$ must be constant on each slice. Thus $F = (f_1, \ldots, f_k)$ is constant on each slice. Consequently, any subset of U on which F is one-to-one contains at most one point of each slice. But any measurable set with this property has measure zero. Thus F cannot be one-to-one almost everywhere. ∎

4. Smooth generators. This section is devoted to proving Theorem 1.3. We noted in the introduction the similarity between Theorem 1.3 and the consequence of Browder's theorem stated as Theorem 1.4. In our context, the methods of Browder do not seem to be applicable. However, in the case of smooth generators, there is a more elementary proof of Theorem 1.4 due to Michael Freeman [10]. Our proof of Theorem 1.3 is an adaptation of Freeman's argument. Following Freeman, we make the following definition.

DEFINITION 4.1. Let M be an n-dimensional smooth manifold-with-boundary. Let F be a collection of complex-valued smooth functions on M. The *exceptional set* E of F is defined by the equation

$$E = \{p \in M : df_1 \wedge \cdots \wedge df_n(p) = 0 \quad \text{for each } n\text{-tuple } (f_1, \ldots, f_n) \text{ of functions in } F\}.$$

Note that in the set-up of the above definition, if the collection F has fewer than n elements, then necessarily the exceptional set E is all of M. Therefore, Theorem 1.3 is an immediate consequence of the following result.

THEOREM 4.2. *Let M be an n-dimensional Riemannian manifold-with-boundary, and let F be a collection of smooth functions on M that generates an algebra dense in $L^p(M)$ for some $1 \le p < \infty$. Then the exceptional set E of F has empty interior in M.*

Proof. Given an arbitrary open set U of M we are to show that there are functions $f_1, \ldots, f_n$ in F such that $df_1 \wedge \cdots \wedge df_n$ is not identically zero on U. Of course, by shrinking U we may assume that U is a set of finite measure that is diffeomorphic to a bounded open set in Euclidean space, and then we can pull everything back to Euclidean space. We may also assume that the

L^p-norm obtained from pulling back is comparable to the L^p-norm obtained by restricting Lebesgue measure to the image of U. The theorem therefore follows immediately from Lemma 4.3 below. ■

LEMMA 4.3. *If F is a collection of C^1 functions on a bounded open set U in $\mathbb{R}^n$ such that the polynomials in F are dense in $L^p(U,m)$ for some $1 \le p < \infty$, then the exceptional set for F is a* proper *subset of U.*

Proof. Since the measure of U is finite, the $L^1(U,m)$-norm is dominated by a multiple of the $L^p(U,m)$-norm, and $L^p(U,m)$ is a dense subset of $L^1(U,m)$. Thus density in $L^p(U,m)$ implies density in $L^1(U,m)$, so we only have to consider the case $p = 1$.

We need to show that there are functions $f_1, \ldots, f_n$ in F such that $df_1 \wedge \cdots \wedge df_n$ is not identically zero on U. We will choose these functions inductively so that for each $k, 1 \le k \le n$, the k-form $df_1 \wedge \cdots \wedge df_k$ is not identically zero on U.

When $k = 1$, the required function can clearly be chosen, since the functions in F cannot all be constant on U. Now assume that functions $f_1, \ldots, f_k \in F$ have been found so that $df_1 \wedge \cdots \wedge df_k$ does not vanish identically on U. Fix a point p_0 in U such that $df_1 \wedge \cdots \wedge df_k(p_0) \neq 0$. Then there must be some k coordinates $x_{j_1}, \ldots, x_{j_k}$ such that $(df_1 \wedge \cdots \wedge df_k(p_0))(\partial/\partial x_{j_1}, \ldots, \partial/\partial x_{j_k}) \neq 0$. By reordering the coordinates, we may assume that $(j_1, \ldots, j_k) = (1, \ldots, k)$. Now choose a neighborhood V of p_0 and an $r > 0$ such that $p + t \cdot (x_1, \ldots, x_k, 1 + x_{k+1}, 0, \ldots, 0)$ is in U for every $p \in V$, every $0 \le t \le r$, and every $(x_1, \ldots, x_{k+1}) \in S^k \subset \mathbb{R}^{k+1}$. Define $\alpha : S^k \times V \to U$ by

$$\alpha((x_1, \ldots, x_{k+1}), p) = p + r \cdot (x_1, \ldots, x_k, 1 + x_{k+1}, 0, \ldots, 0).$$

For each $p \in V$ the map α takes $S^k \times \{p\}$ diffeomorphically onto a k-sphere in U containing the point p and such that the tangent space to the sphere at p is spanned by $\partial/\partial x_1, \ldots, \partial/\partial x_k$. Thus letting N_p denote the closed ball whose boundary ∂N_p is the image sphere, we find that $(df_1 \wedge \cdots \wedge df_k)(p_0)$ is non-zero as a form on ∂N_{p_0}. Consequently, there is a continuous function g on U such that

$$\int_{\partial N_{p_0}} g \, df_1 \wedge \cdots \wedge df_k \neq 0. \tag{1}$$

We claim that inequality (1) together with the hypothesis that the polynomials in F are dense in $L^1(U,m)$ implies that for some function h that is a polynomial in functions in F and for some point p in V we have

$$\int_{\partial N_p} h \, df_1 \wedge \cdots \wedge df_k \neq 0. \tag{2}$$

To verify this, first note that (1) implies that there is a compact neighborhood $W \subset V$ of p_0 such that

$$\int_W \int_{\partial N_p} g\, df_1 \wedge \cdots \wedge df_k\, dm \neq 0. \tag{3}$$

Now let μ denote surface area measure on S^k, and note that the measure corresponding to $df_1 \wedge \cdots \wedge df_k$ on ∂N_p pulls back under α to $\sigma_p\, d\mu$ on S^k for some smooth function σ_p. Define σ on $S^k \times V$ by $\sigma(x,p) = \sigma_p(x)$. Then σ is smooth, and hence in particular the supremum $\|\sigma\|_\infty$ of σ over $S^k \times W$ is finite. Now pulling back and applying Fubini's theorem we have

$$\begin{aligned} \int_W \int_{\partial N_p} g\, df_1 \wedge \cdots \wedge df_k\, dm &= \int_W \int_{S^k} (g \circ \alpha)\sigma\, d\mu\, dm \\ &= \int_{S^k} \int_W (g \circ \alpha)\sigma\, dm\, d\mu. \end{aligned} \tag{4}$$

By (3) and (4) we have

$$\left| \int_{S^k} \int_W (g \circ \alpha)\sigma\, dm\, d\mu \right| > 0, \tag{5}$$

so by the hypothesis on the collection F, there is a function h that is a polynomial in functions in F such that

$$\|\sigma\|_\infty\, \|g - h\|_1\, \mu(S^k) < \left| \int_{S^k} \int_W (g \circ \alpha)\sigma\, dm\, d\mu \right|. \tag{6}$$

Furthermore,

$$\begin{aligned} &\left| \int_{S^k} \int_W (g \circ \alpha)\sigma\, dm\, d\mu - \int_{S^k} \int_W (h \circ \alpha)\sigma\, dm\, d\mu \right| \\ &\qquad \leq \int_{S^k} \int_W \big|(g \circ \alpha)\sigma - (h \circ \alpha)\sigma\big|\, dm\, d\mu \leq \|\sigma\|_\infty\, \|g - h\|_1\, \mu(S^k). \end{aligned} \tag{7}$$

From (6) and (7) we get

$$\int_{S^k} \int_W (h \circ \alpha)\sigma\, dm\, d\mu \neq 0. \tag{8}$$

Since equation (4) continues to hold with g replaced by h we get

$$\int_W \int_{\partial N_p} h\, df_1 \wedge \cdots \wedge df_k\, dm \neq 0. \tag{9}$$

Consequently, inequality (2) holds for some $p \in W \subset V$ as claimed.

Now Stokes' theorem gives

$$\int_{N_p} dh \wedge df_1 \wedge \cdots \wedge df_k = \int_{\partial N_p} h\, df_1 \wedge \cdots \wedge df_k \neq 0. \tag{10}$$

From the formula for the differential of a product, we see that dh is a linear combination (with coefficients that are smooth functions) of the differentials of functions in F. Thus (10) implies the existence of a function $f_{k+1} \in F$ such that $df_1 \wedge \cdots \wedge df_{k+1}$ is not identically zero on $N_p \subset U$. This completes the induction and the proof. ∎

5. Homeomorphism generators. In this section we prove Theorems 1.5–1.8. The proof of Theorem 1.7 is by far the longest and occupies the bulk of the section.

Proof of Theorem 1.5. Choose a self-homeomorphism α of S^1 that is continuously differentiable and such that

$$\int_{S^1} \log|\alpha'|\, dm = -\infty. \tag{11}$$

(For instance one could set $g(\theta) = \pi \int_0^\theta e^{-1/x^2}\, dx \,/ \int_0^\pi e^{-1/x^2}\, dx$, and take $\alpha(e^{i\theta}) = e^{ig(\theta)}$ for $\theta \in [-\pi, \pi]$.)

Let $d\mu = |\alpha'| dm$. Define an operator $T : L^p(S^1, m) \to L^p(S^1, \mu)$ by $Th = h \circ \alpha$. Then by the change of variables formula, T is an isometry:

$$\|Th\|_p^p = \int_{S^1} |h \circ \alpha|^p |\alpha'|\, dm = \int_{S^1} |h|^p\, dm = \|h\|_p^p.$$

Because of (11), Szegö's theorem [6, Theorem XX.6.6] shows that the set of polynomials in z is dense in $L^p(S^1, \mu)$. Now let f be the inverse of the homeomorphism α. Then since $Tf = z$, it follows that the set of polynomials in f is dense in $L^p(S^1, m)$. ∎

Proof of Theorem 1.6. We will denote the set of polynomials in a function f by $\mathcal{P}(f)$. Assume to get a contradiction that there is a one-to-one function $f \in C^1(S^1)$ such that $\mathcal{P}(f)$ is dense in $L^p(S^1, m)$. Let $\Gamma = f(S^1)$, which is a simple closed curve. By the Riemann mapping theorem there is a conformal mapping g that maps the interior of the unit circle onto the interior of Γ, and this map can be extended continuously to be a homeomorphism of S^1 onto Γ by Carathéodory's theorem [14, Theorem 5.1.1]. Since f is in $C^1(S^1)$, the curve Γ is rectifiable and consequently g is a function of bounded variation on S^1. Then by [8, Theorems 3.11 and 3.12], the function $g|_{S^1} : S^1 \to \Gamma$ is absolutely continuous and its derivative g' lies in the Hardy space H^1.

Let μ denote the arc length measure on Γ. Let $E = \{e^{i\psi} : f'(e^{i\psi}) = 0\}$. Then $\mu(f(E)) = \int_E |f'(e^{i\psi})|\, dm(\psi) = 0$. On $\Gamma \setminus f(E)$ we also have

$$(f^{-1})'(\tau) = \frac{1}{f'(f^{-1}(\tau))}.$$

Let $F = g^{-1}(f(E))$. Boundary sets of measure zero are preserved under the conformal mapping g [8, p. 45], so $m(F) = 0$. Let $\alpha(e^{i\theta}) = f^{-1} \circ g(e^{i\theta})$.

Then we know that α is a homeomorphism of S^1 onto S^1 and is differentiable almost everywhere on S^1 with

$$\alpha'(e^{i\theta}) = (f^{-1})'(g(e^{i\theta}))g'(e^{i\theta}) \quad \text{a.e.}$$

Let ν be the push forward measure $\alpha_*^{-1}(m)$ on S^1 induced by α^{-1}. Then for any $h \in L^1(S^1, m)$,

$$\int_{S^1} h(e^{i\psi})\, dm(\psi) = \int_{S^1} h \circ \alpha(e^{i\theta})\, d\nu(\theta).$$

Thus the operator $T : L^p(S^1, m) \to L^p(S^1, \nu)$ given by

$$(Th)(e^{i\theta}) = h \circ \alpha(e^{i\theta})$$

is an isometry, and T is invertible because α is a homeomorphism. Moreover,

$$Tf = g.$$

In particular, $T(\mathcal{P}(f)) = \mathcal{P}(g)$. By our assumption, $\mathcal{P}(f)$ is dense in $L^p(S^1, m)$, and hence $\mathcal{P}(g)$ is dense in $L^p(S^1, \nu)$. Note that g is in the disk algebra $A(S^1) = \overline{\mathcal{P}(z)}$. Consequently, $\mathcal{P}(g) \subset A(S^1) = \overline{\mathcal{P}(z)}$, which implies that $\mathcal{P}(z)$ is dense in $L^p(S^1, \nu)$. Hence by Szegö's theorem,

$$\int_{S^1} \log \frac{d\nu(\theta)}{dm(\theta)}\, dm(\theta) = -\infty. \tag{12}$$

Note that

$$\frac{d\nu(\theta)}{dm(\theta)} = |\alpha'(e^{i\theta})| = |(f^{-1})'(g(e^{i\theta}))g'(e^{i\theta})| \quad \text{a.e.} \tag{13}$$

on S^1. Since $g' \in H^1$, we know that $\log|g'| \in L^1(S^1, m)$ [15, Theorem 17.17]. Then by (12) and (13), we have

$$\int_{S^1} \log|(f^{-1})'(g(e^{i\theta}))|\, dm(\theta) = -\infty.$$

This implies that for some $e^{i\theta_0} \in S^1 \setminus F$,

$$(f^{-1})'(g(e^{i\theta_0})) = 0,$$

which contradicts the assumption that $f \in C^1(S^1)$. ∎

We now turn to the proof of Theorem 1.7. An important trick in the proof is to consider the intersection of the spaces $L^p(X, m)$ over all $1 \le p < \infty$. While the individual L^p-spaces are *not* algebras, their intersection *is* an algebra, and this enables us to apply the Stone–Weierstrass theorem to reduce the problem to proving the existence of a continuous function f such that $\overline{f}$ lies in the L^p-closure of the polynomials in f for every p. We isolate this part of the proof in the following two lemmas.

LEMMA 5.1. *Let μ be a positive measure and let A be an algebra of functions contained in the intersection over all $1 \le p < \infty$ of the spaces $L^p(\mu)$. Then the intersection over all $1 \le p < \infty$ of the $L^p(\mu)$-closures of A is also an algebra.*

Proof. Let B denote the intersection over all $1 \le p < \infty$ of the $L^p(\mu)$-closures of A. If g and h are in B, then so is gh, since applying Hölder's inequality $\|gh\|_r \le \|g\|_p \|h\|_q$ for $1/p + 1/q = 1/r$ (with $p = q = 2r$) gives, for any g_n and h_n in A,

$$\begin{aligned}\|gh - g_n h_n\|_r &\le \|gh - gh_n\|_r + \|gh_n - g_n h_n\|_r \\ &\le \|g\|_{2r}\|h - h_n\|_{2r} + \|g - g_n\|_{2r}\|h_n\|_{2r}.\end{aligned}$$

Since B is obviously a vector space, this shows that B is an algebra. ∎

LEMMA 5.2. *Let X be a compact space and μ be a positive regular Borel measure on X. If f is a one-to-one continuous function on X such that $\overline{f}$ is in the $L^p(\mu)$-closure of the polynomials in f for every $1 \le p < \infty$, then the polynomials in f are dense in $L^p(\mu)$ for every $1 \le p < \infty$.*

Proof. Let B denote the intersection over all $1 \le p < \infty$ of the L^p-closures of the polynomials in f. By Lemma 5.1, B is an algebra. Our hypothesis on f now implies that B contains the algebra $\mathcal{P}(f, \overline{f})$ generated by f and $\overline{f}$. The algebra $\mathcal{P}(f, \overline{f})$ is obviously self-adjoint, and since f is one-to-one, this algebra also separates points. Thus by the Stone–Weierstrass theorem, $\mathcal{P}(f, \overline{f})$ is uniformly dense in $C(X)$. It follows that B is dense in $L^p(\mu)$. ∎

As motivation for the proof of Theorem 1.7, we give a second proof of Theorem 1.5. The construction for Theorem 1.7 is somewhat similar but more elaborate.

Second proof of Theorem 1.5. We will construct a homeomorphism $\alpha : S^1 \to S^1$ such that the inverse of α is the desired homeomorphism f. We define α by prescribing its values at the points $0, \pi, 3\pi/2, 7\pi/4, \ldots$ $\ldots, (2^n - 1)\pi/2^{n-1}, \ldots$ for $n = 0, 1, 2, \ldots$, and then interpolating between the prescribed values.

Let $\langle a, b\rangle$ stand for the closed arc $\{e^{i\theta} : a \le \theta \le b\}$. It is well-known that on any proper closed subset of the unit circle, the polynomials in z are uniformly dense in the continuous functions. Thus for each $n = 1, 2, \ldots$, we can choose a polynomial q_n such that

$$|\overline{z} - q_n(z)| \le \frac{1}{\sqrt[n]{4\pi} \cdot n} \qquad \text{for all } z \in \langle 0, (2^n - 1)\pi/2^{n-1}\rangle.$$

Let

$$M_n = \sup_{z \in S^1} |\overline{z} - q_n(z)|.$$

Now we inductively define a sequence $(\theta_n)_{n=0}^\infty$ of angles by setting $\theta_0 = 0$ and in general choosing θ_n such that $\theta_{n-1} < \theta_n < 2\pi$ and

$$m(\langle \theta_n, 2\pi\rangle) < \frac{1}{2n^n M_n^n}.$$

Then we set $\alpha(e^{i(2^n-1)\pi/2^{n-1}}) = e^{i\theta_n}$. Finally, on each arc $\langle (2^n-2)\pi/2^{n-1}, (2^n-1)\pi/2^{n-1}\rangle$, with $n = 1, 2, \ldots$, we define α by interpolating between the values at the end points using any interpolation that maps the arc $\langle (2^n-2)\pi/2^{n-1}, (2^n-1)\pi/2^{n-1}\rangle$ continuously and bijectively onto the arc $\langle \theta_{n-1}, \theta_n\rangle$. Because $\theta_n \to 2\pi$ as $n \to \infty$, the map α is a continuous bijection of S^1 onto itself and hence is a homeomorphism. Set $f = \alpha^{-1}$.

Let μ denote the push forward measure $f_*(m)$. Then

$$\begin{aligned}
\int_{S^1} |\overline{z} - q_n(z)|^n \, d\mu &= \int_0^{(2^n-1)\pi/2^{n-1}} |\overline{z} - q_n(z)|^n \, d\mu \\
&\quad + \int_{(2^n-1)\pi/2^{n-1}}^{2\pi} |\overline{z} - q_n(z)|^n \, d\mu \\
&\le 2\pi \left(\frac{1}{\sqrt[n]{4\pi}\cdot n}\right)^n + M_n^n \int_{(2^n-1)\pi/2^{n-1}}^{2\pi} d\mu \\
&= \frac{1}{2n^n} + M_n^n m(\langle \theta_n, 2\pi\rangle) \\
&\le \frac{1}{2n^n} + M_n^n \frac{1}{2n^n M_n^n} = \frac{1}{n^n}.
\end{aligned}$$

Now the formula for integrating against a push forward measure gives

$$\|\overline{f} - q_n(f)\|_{L^n(m)} = \|\overline{z} - q_n(z)\|_{L^n(\mu)} \le \frac{1}{n}.$$

Since the $L^p(m)$-norms are increasing in p, it follows that $q_n(f) \to \overline{f}$ in $L^p(m)$ for every $1 \le p < \infty$. Hence Lemma 5.2 shows that the polynomials in f are dense in $L^p(m)$ for every $1 \le p < \infty$. ∎

Proof of Theorem 1.7. We will construct a sequence (α_n) of self-homeomorphisms of S^1 by induction and show that the sequence converges uniformly to a self-homeomorphism α. We will then define a self-homeomorphism H of $\overline{\mathbb{D}}$ by extending α linearly in radial directions (i.e., $H(re^{i\theta}) = r\alpha(e^{i\theta})$). Finally, the desired function f will be H^{-1}.

Every self-map of S^1 lifts to a self-map of $\mathbb{R}$ under the covering map $\theta \mapsto e^{i\theta}$. For convenience we will call a self-map of S^1 (strictly) increasing if the lift is (strictly) increasing.

We present the proof in steps.

STEP 1: We define the homeomorphism α_1.

Choose a Cantor set C_1 on S^1. Let $K_1 = C_1$, and let $\widetilde{K}_1$ be the cone of K_1 over the origin defined by

$$\widetilde{K}_1 = \{rz \in \overline{\mathbb{D}} : z \in K_1,\, 0 \le r \le 1\}.$$

Then $\widetilde{K}_1$ is a compact set in the plane whose complement is connected, so by Lavrentiev's theorem there is a polynomial $q_1(z)$ such that

$$|\overline{z} - q_1(z)| < \frac{1}{2\pi} \qquad \forall z \in \widetilde{K}_1.$$

Let $M_1 = \sup_{z\in\overline{\mathbb{D}}} |\overline{z} - q_1(z)|$. Choose a Cantor set C_1' on S^1 such that

$$m(S^1 \setminus C_1') < 1/M_1.$$

Let $K_1' = C_1'$, and choose an increasing self-homeomorphism α_1 of S^1 such that $\alpha_1(K_1) = K_1'$.

STEP 2: We define the homeomorphism α_2.

We will let $\langle a, b\rangle$ stand for the open arc $\{e^{i\theta} : a < \theta < b\}$. Write the complement of K_1 in S^1 as a countable union of disjoint open arcs

$$S^1 - K_1 = \bigcup_k \langle a_k^{(1)}, b_k^{(1)}\rangle.$$

In each arc $\langle a_k^{(1)}, b_k^{(1)}\rangle$ choose a Cantor set $C_{2,k}$ that contains the end points $a_k^{(1)}$ and $b_k^{(1)}$. Let $C_2 = \bigcup_k C_{2,k}$, let $K_2 = C_1 \cup C_2$, and let $\widetilde{K}_2$ be the cone of K_2 over the origin defined by

$$\widetilde{K}_2 = \{rz : z \in K_2,\, 0 \le r \le 1\}.$$

Then $\widetilde{K}_2$ is a compact set in the plane whose complement is connected, so by Lavrentiev's theorem there is a polynomial $q_2(z)$ such that

$$|\overline{z} - q_2(z)| < \frac{1}{\sqrt{2\pi}\cdot 2} \qquad \forall z \in \widetilde{K}_2.$$

Let $M_2 = \sup_{z\in\overline{\mathbb{D}}} |\overline{z} - q_2(z)|$. Write the complement of K_1' in S^1 as a countable union of disjoint open arcs

$$S^1 - K_1' = \bigcup_k \langle c_k^{(1)}, d_k^{(1)}\rangle.$$

In each arc $\langle c_k^{(1)}, d_k^{(1)}\rangle$ choose a Cantor set $C_{2,k}'$ that contains the end points $c_k^{(1)}$ and $d_k^{(1)}$ in such a way that

$$m\Big(\bigcup_k (\langle c_k^{(1)}, d_k^{(1)}\rangle \setminus C_{2,k}')\Big) < \frac{1}{2^2 M_2^2}.$$

Then there is a homeomorphism $\alpha_{2,k}$ of $\langle a_k^{(1)}, b_k^{(1)}\rangle$ onto $\langle c_k^{(1)}, d_k^{(1)}\rangle$ such that $\alpha_{2,k}$ is increasing and $\alpha_{2,k}(C_{2,k}) = C_{2,k}'$. We define α_2 on S^1 by taking α_2 to agree with α_1 on K_1 and, for each k, to agree with $\alpha_{2,k}$ on $\langle a_k^{(1)}, b_k^{(1)}\rangle$. Then α_2 is an increasing homeomorphism of S onto itself and letting $C_2' = \bigcup_k C_{2,k}'$ and $K_2' = C_1' \cup C_2'$ we have $\alpha_2(K_2) = K_2'$.

Step 3: Continuing as in Step 2, we inductively obtain sequences (C_n), (K_n), $(\widetilde{K}_n)$, (C_n'), (K_n') of sets, a sequence (q_n) of polynomials, a sequence (M_n) of positive numbers, and a sequence (α_n) of self-homeomorphisms of S^1 such that:

(i) $K_n = C_1 \cup \ldots \cup C_n$ is a Cantor set in S^1.

(ii) $\widetilde{K}_n = \{rz : z \in K_n,\ 0 \le r \le 1\}$ is the cone of K_n over the origin.

(iii) expressing $S^1 \setminus K_{n-1}$ as a countable union $\bigcup_k \langle a_k^{(n-1)}, b_k^{(n-1)}\rangle$ of disjoint arcs, we have $C_n = \bigcup_k C_{n,k}$ where each $C_{n,k}$ is a Cantor set in $\langle a_k^{(n-1)}, b_k^{(n-1)}\rangle$ containing the end points $a_k^{(n-1)}$ and $b_k^{(n-1)}$.

(iv) $|\overline{z} - q_n(z)| < \dfrac{1}{\sqrt[n]{2\pi}\cdot n}$ for all $z \in \widetilde{K}_n$.

(v) $M_n = \sup_{z\in\overline{\mathbb{D}}} |\overline{z} - q_n(z)|$.

(vi) $K_n' = C_1' \cup \cdots \cup C_n'$.

(vii) expressing $S^1 \setminus K_{n-1}'$ as a countable union $\bigcup_k \langle c_k^{(n-1)}, d_k^{(n-1)}\rangle$ of disjoint arcs, we have $C_n' = \bigcup_k C_{n,k}'$ where each $C_{n,k}'$ is a Cantor set in $\langle c_k^{(n-1)}, d_k^{(n-1)}\rangle$ containing the end points $c_k^{(n-1)}$ and $d_k^{(n-1)}$ and such that
$$m(S^1 \setminus K_n') = m\Big(\bigcup_k (\langle c_k^{(n-1)}, d_k^{(n-1)}\rangle \setminus C_{n,k}')\Big) < \frac{1}{n^n M_n^n}.$$

(viii) α_n is an increasing self-homeomorphism of S^1 such that α_n agrees with α_{n-1} on K_{n-1} and satisfies $\alpha_n(C_{n,k}) = C_{n,k}'$ and $\alpha_n(K_n) = K_n'$.

Step 4: We show that the sequence (α_n) converges uniformly to a self-homeomorphism α.

Note that whenever $m \ge n$, the homeomorphisms α_m and α_n agree on K_n, and consequently the monotonicity of α_m and α_n gives
$$|\alpha_m(z) - \alpha_n(z)| \le \sup_k |e^{id_k^{(n)}} - e^{ic_k^{(n)}}| < \frac{1}{n^n M_n^n}.$$
Thus the sequence (α_n) is uniformly Cauchy and hence converges uniformly to a continuous limit function which we denote by α. (Note that the sequence (M_n) is bounded away from zero since $\overline{z}$ is not a uniform limit of polynomials in z.)

Note that α must agree with α_n on K_n. Since each α_n is strictly increasing and the union $\bigcup_{n=1}^\infty K_n$ is dense in S^1, it follows that α is strictly increasing. Hence α is a self-homeomorphism of S^1.

STEP 5: We set $f = H^{-1}$ and show that the set of polynomials in f is dense in $L^p(\overline{\mathbb{D}}, m)$ for $1 \le p < \infty$.

Note that because α agrees with α_n on K_n, we have $\alpha(S^1 \setminus K_n) = S^1 \setminus K_n'$. Consequently, letting m_1 and m_2 denote the 1- and 2-dimensional Lebesgue measures, respectively, we obtain

$$m_2(H(\overline{\mathbb{D}} \setminus \widetilde{K}_n)) = \frac{1}{2} m_1(S^1 \setminus K_n') < \frac{1}{2n^n M_n^n}.$$

Now our construction and the formula for integrating against a push forward measure give

$$\begin{aligned}
&\int\limits_{\overline{\mathbb{D}}} |\bar{f} - q_n(f)|^n \, dm \\
&= \int\limits_{\overline{\mathbb{D}}} |\overline{z} - q_n(z)|^n \, df_*(m) = \int\limits_{\widetilde{K}_n} |\overline{z} - q_n(z)|^n \, df_*(m) + \int\limits_{\overline{\mathbb{D}} \setminus \widetilde{K}_n} |\overline{z} - q_n(z)|^n \, df_*(m) \\
&\le \sup_{z \in \widetilde{K}_n} |\overline{z} - q_n(z)|^n \int\limits_{\widetilde{K}_n} df_*(m) + \sup_{z \in \overline{\mathbb{D}}} |\overline{z} - q_n(z)|^n \int\limits_{\overline{\mathbb{D}} \setminus \widetilde{K}_n} df_*(m) \\
&\le \frac{1}{2\pi n^n} \int\limits_{H(\widetilde{K}_n)} dm + M_n^n \int\limits_{H(\overline{\mathbb{D}} \setminus \widetilde{K}_n)} dm \le \frac{1}{2n^n} + M_n^n \frac{1}{2n^n M_n^n} = \frac{1}{n^n}.
\end{aligned}$$

Thus $\|\bar{f} - q_n(f)\|_n \le 1/n$. Since the $L^p(m)$-norms are increasing in p, it follows that $q_n(f) \to \overline{f}$ in $L^p(m)$ for every $1 \le p < \infty$. Hence Lemma 5.2 implies that the polynomials in f are dense in $L^p(m)$ for every $1 \le p < \infty$. ∎

Proof of Theorem 1.8. Consider first the case when n is even. We begin by verifying the following:

CLAIM. *There is an embedding $F : \overline{\mathbb{D}}^k \to \mathbb{C}^k$ such that set of the component functions $f_1, \ldots, f_k$ of F generates an algebra dense in $L^p(\overline{\mathbb{D}}^k, m)$ for all $1 \le p < \infty$.*

By Theorem 1.7, there is a self-homeomorphism f of $\overline{\mathbb{D}}$ such that the set of polynomials in f is dense in $L^p(\overline{\mathbb{D}}, m)$ for $1 \le p < \infty$. Now define $f_1, \ldots, f_k$ on $\overline{\mathbb{D}}^k$ by $f_j(z) = f(z_j)$. Let B denote the intersection over all $1 \le p < \infty$ of the L^p-closures of the algebra generated by $f_1, \ldots, f_k$. By Lemma 5.1, B is an algebra. Because of the property of f, the algebra B contains every continuous function on $\overline{\mathbb{D}}^k$ that depends on only one variable. By an application of the Stone–Weierstrass theorem, it follows that B is dense in $L^p(\overline{\mathbb{D}}, m)$, and the Claim is proved.

Now identify $\mathbb{R}^n$ with $\mathbb{C}^{n/2}$ in the obvious way. Since X is compact, X is contained in some polydisk $\overline{\varDelta}^{n/2}$ in $\mathbb{C}^{n/2}$. By the Claim, there is an embedding of $\overline{\varDelta}^{n/2}$ into $\mathbb{C}^{n/2}$ whose component functions $f_1, \ldots, f_{n/2}$ generate an algebra dense in $L^p(\overline{\Delta}^{n/2}, m)$ for every $1 \leq p < \infty$. Restricting these functions to X gives the result.

When n is odd, we replace the above claim by the assertion that there is an embedding $F : \overline{\mathbb{D}}^k \times I \to \mathbb{C}^{k+1}$ such that set of the component functions $f_1, \dots, f_{k+1}$ of F generates an algebra dense in $L^p(\overline{\mathbb{D}}^k \times I, m)$ for all $1 \leq p < \infty$, where I denotes the closed unit interval. This is proved in the same way as the Claim but taking $f_1, \dots, f_{k+1}$ defined by $f_j(z, x) = f(z_j)$ for $j = 1, \dots, k$ and $f_{k+1}(z, x) = x$. We then identify $\mathbb{R}^n$ with $\mathbb{C}^{\lfloor n/2 \rfloor} \times \mathbb{R}$, note that X is contained in a set of the form $\overline{\Delta}^{\lfloor n/2 \rfloor} \times [a, b]$, and conclude the argument as before. ∎

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