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On $(C, 1)$ summability for Vilenkin-like systems

by

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Abstract. We give a common generalization of the Walsh system, Vilenkin system, the character system of the group of 2-adic (m-adic) integers, the product system of normalized coordinate functions for continuous irreducible unitary representations of the coordinate groups of noncommutative Vilenkin groups, the UDMD product systems (defined by F. Schipp) and some other systems. We prove that for integrable functions $\sigma_n f \to f$ $(n \to \infty)$ a.e., where $\sigma_n f$ is the nth $(C, 1)$ mean of f. (For the character system of the group of m-adic integers, this proves a more than 20 years old conjecture of M. H. Taibleson [24, p. 114].) Define the maximal operator $\sigma^* f := \sup_n |\sigma_n f|$. We prove that σ^* is of type (p, p) for all $1 < p \le \infty$ and of weak type $(1, 1)$. Moreover, $\|\sigma^* f\|_1 \le c\|f\|_H$, where H is the Hardy space.

Introduction and examples. Denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{P}$ the set of positive integers. Let $m := (m_k : k \in \mathbb{N})$ be a sequence of positive integers such that $m_k \ge 2$ for $k \in \mathbb{N}$, and let G_{m_k} be a set of cardinality m_k. Suppose that each (coordinate) set has the discrete topology and the measure μ_k which maps every singleton of G_{m_k} to $1/m_k$ ($\mu_k(G_{m_k}) = 1$) for $k \in \mathbb{N}$. Let G_m be the compact set formed by the complete direct product of G_{m_k} equipped with the product topology and product measure (μ). Thus each $x \in G_m$ is a sequence $x := (x_0, x_1, \ldots)$, where $x_k \in G_{m_k}$, $k \in \mathbb{N}$. Then G_m is called a *Vilenkin space*. It is a compact totally disconnected space, with normalized regular Borel measure μ. The Vilenkin space G_m is said to be *bounded* if the generating system m is bounded.

Throughout this paper we assume the boundedness of G_m; moreover, c, c_p denote absolute constants, the latter can depend (only) on p.

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A neighborhood base of G_m can be given as follows:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbb{P}$. Then

$$\mathcal{I} := \{I_n(x) : n \in \mathbb{N},\ x \in G_m\}$$

is the set of *intervals* on G_m.

Denote by $L^p(G_m)$ the usual Lebesgue spaces (with norms $\|\cdot\|_p$) ($1 \le p \le \infty$), by $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)\,(x \in G_m)$ and by E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$.

The maximal Hardy space $H^1(G_m)$ is defined by means of the maximal function $f^* := \sup_n |E_n f|$ $(f \in L^1(G_m))$: f is said to be in $H^1(G_m)$ if $f^* \in L^1(G_m)$. Then $H^1(G_m)$ is a Banach space with the norm

$$\|f\|_{H^1} := \|f^*\|_1.$$

This definition is suitable if the sequence m is bounded. In this case $H^1(G_m)$ has an atomic structure (for the dyadic case ($m_k = 2$, $k \in \mathbb{N}$) see [21, p. 104] and for the general case see [22, p. 92]). A function $g \in L^\infty(G_m)$ is an *atom* if either $g = 1$ or $\operatorname{supp} g \subset I_n(x)$, $\int_{I_n(x)} g\, d\mu = 0$, and $\|g\|_\infty \le 1/\mu(I_n(x))$ for some $x \in G_m$, $n \in \mathbb{N}$. By definition, $f \in H(G_m)$ iff $f = \sum_{i=0}^\infty \lambda_i g_i$, where $\sum_{i=0}^\infty |\lambda_i| < \infty$, $\lambda_i \in \mathbb{C}$ and g_i is an atom $(i \in \mathbb{N})$. Then $H(G_m)$ is a Banach space with the norm

$$\|f\|_H := \inf \sum_{i=0}^\infty |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^\infty \lambda_i g_i$ as above. If the sequence m is bounded (in this paper this is supposed), then $H(G_m) = H^1(G_m)$, moreover, the two norms are equivalent. (If the sequence m is not bounded, then the situation changes [22].)

We say that an operator $T : L^1 \to L^0$ (where $L^0(G_m)$ is the space of measurable functions on the Vilenkin space G_m) is of *type* (p,p) (for $1 \le p \le \infty$) if $\|Tf\|_p \le c_p \|f\|_p$ for all $f \in L^p(G_m)$ and the constant c_p depends only on p; T is of *type* (H, L) if $\|Tf\|_1 \le c\|f\|_H$ for all $f \in H(G_m)$; and T is of *weak type* $(1,1)$ if $\mu(|Tf| > \lambda) \le c\|f\|_1/\lambda$ for all $f \in L^1(G_m)$ and $\lambda > 0$.

Let $M_0 := 1$ and $M_{k+1} := m_k M_k$ for $k \in \mathbb{N}$ be the so-called *generalized powers*. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^\infty n_k M_k$, $0 \le n_k < m_k$, $n_k \in \mathbb{N}$. The sequence $(n_0, n_1, \ldots)$ is called the *expansion* of n with respect to m. We often use the following notations. Let $|n| := \max\{k \in \mathbb{N} : n_k \ne 0\}$ (that is, $M_{|n|} \le n < M_{|n|+1}$) and $n^{(k)} = \sum_{j=k}^\infty n_j M_j$. Next we introduce on G_m an orthonormal system which we call a Vilenkin-like system.

Complex-valued functions $r_k^n : G_m \to \mathbb{C}$ which we call *generalized Rademacher functions* have the following properties:

(i) r_k^n is $\mathcal{A}_{k+1}$-measurable (i.e. $r_k^n(x)$ depends only on $x_0, \dots, x_k$ $(x \in G_m)$), $r_k^0 = 1$ for all $k, n \in \mathbb{N}$.

(ii) If M_k is a divisor of n and l and if $n^{(k+1)} = l^{(k+1)}$ $(k, l, n \in \mathbb{N})$, then

$$E_k(r_k^n \overline{r}_k^l) = \begin{cases} 1 & \text{if } n_k = l_k, \\ 0 & \text{if } n_k \neq l_k \end{cases}$$

($\overline{z}$ is the complex conjugate of z).

(iii) If M_{k+1} is a divisor of n (that is, $n = n_{k+1}M_{k+1} + n_{k+2}M_{k+2} + \dots + n_{|n|}M_{|n|}$), then

$$\sum_{j=0}^{m_k-1} |r_k^{jM_k+n}(x)|^2 = m_k$$

for all $x \in G_m$.

(iv) There exists a $\delta > 1$ for which $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$.

Define a *Vilenkin-like* system $\psi = (\psi_n : n \in \mathbb{N})$ as follows:

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}}, \qquad n \in \mathbb{N}.$$

(Since $r_k^0 = 1$, we have $\psi_n = \prod_{k=0}^{|n|} r_k^{n^{(k)}}$.)

Example A (the Vilenkin and Walsh systems). Let $G_{m_k} := Z_{m_k}$ be the m_kth $(2 \leq m_k \in \mathbb{N})$ discrete cyclic group $(k \in \mathbb{N})$. That is, Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, where the group operation is mod m_k addition and every subset is open. The group operation $(+)$ on G_m is coordinatewise addition. G_m is called a *Vilenkin group*. The Vilenkin group for which $m_k = 2$ for all $k \in \mathbb{N}$ is the *Walsh–Paley group*. In this case let $r_k^n(x) := (\exp(2\pi \imath x_k/m_k))^{n_k}$, where $\imath := \sqrt{-1}, x \in G_m$. The system $\psi := (\psi_n : n \in \mathbb{N})$ is the *Vilenkin system*, where $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k M_k}$. For the Vilenkin group with $m_k = 2$ for all $k \in \mathbb{N}$, we get the *Walsh–Paley system*. Since $|r_k^n| = 1$, (iii) and (iv) are trivial and so are (i) and (ii). For more on the Vilenkin and Walsh systems and groups see e.g. [21, 1].

Example B (the group of 2-adic (m-adic) integers). Let $G_{m_k} := \{0, 1, \dots, m_k - 1\}$ for all $k \in \mathbb{N}$. Define on G_m the following (commutative) addition: Let $x, y \in G_m$. Then $x + y = z \in G_m$ is defined in a recursive way. First, $x_0 + y_0 = t_0 m_0 + z_0$, where (of course) $z_0 \in \{0, 1, \dots, m_0 - 1\}$ and $t_0 \in \mathbb{N}$. Suppose that $z_0, \dots, z_k$ and $t_0, \dots, t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1}m_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, \dots, m_{k+1} - 1\}$ and $t_{k+1} \in \mathbb{N}$. Then G_m is called the *group of m-adic integers* (if $m_k = 2$ for all

$k \in \mathbb{N}$, then these are the 2-*adic integers*). In this case let

$$r_k^n(x) := \left(\exp\left(2\pi\imath\left(\frac{x_k}{m_k} + \frac{x_{k-1}}{m_k m_{k-1}} + \ldots + \frac{x_0}{m_k m_{k-1} \ldots m_0}\right)\right)\right)^{n_k}.$$

Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k M_k}$. Then the system $\psi := (\psi_n : n \in \mathbb{N})$ is the character system of the group of m-adic (2-adic if $m_k = 2$ for each $k \in \mathbb{N}$) integers. Since $|r_k^n| = 1$, (i), (iii) and (iv) are trivial. (ii) is also easy to see and well known [22, p. 91]. For more on the group of m-adic integers see e.g. [9, 15, 24].

Example C (noncommutative Vilenkin groups). Let σ be an equivalence class of continuous irreducible unitary representations of a compact group G. Denote by Σ the set of all such σ. Then Σ is called the *dual object* of G. The dimension of a representation $U^{(\sigma)}$, $\sigma \in \Sigma$, is denoted by d_σ and we let

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \qquad i, j \in \{1, \ldots, d_\sigma\},$$

be the coordinate functions for $U^{(\sigma)}$, where $\xi_1, \ldots, \xi_{d_\sigma}$ is an orthonormal basis in the representation space of $U^{(\sigma)}$. (For the notations see [9, Vol. 2, p. 3].) According to the Weyl–Peter theorem (see e.g. [9, Vol. 2, p. 24]), the system of functions $\sqrt{d_\sigma} u_{i,j}^{(\sigma)}$, $\sigma \in \Sigma$, $i, j \in \{1, \ldots, d_\sigma\}$, is an orthonormal basis for $L^2(G)$. If G is a finite group, then Σ is also finite. If $\Sigma := \{\sigma_1, \ldots, \sigma_s\}$, then $|G| = d_{\sigma_1}^2 + \ldots + d_{\sigma_s}^2$.

Let G_{m_k} be a finite group of order m_k, $k \in \mathbb{N}$. Let $\{r_k^{sM_k} : 0 \le s < m_k\}$ be the set of all normalized coordinate functions of the group G_{m_k} and suppose that $r_k^0 \equiv 1$. Thus for every $0 \le s < m_k$ there exists a $\sigma \in \Sigma_k$ and $i, j \in \{1, \ldots, d_\sigma\}$ such that

$$r_k^{sM_k} = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_{m_k}).$$

Set $r_k^n := r_k^{n_k M_k}$. Let ψ be the product system of r_k^j, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n^{(k)}}(x_k) \qquad (x \in G_m),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \ldots)$. We remark that if G_{m_k} is the discrete cyclic group of order m_k, $k \in \mathbb{N}$, then G_m coincides with the Vilenkin group, and ψ is the Vilenkin system with respect to the corresponding order [8, 21, 26, 1]. In [8] it is proved that the system ψ has the properties (i)–(iii). Moreover, (iv) is satisfied because $m_k = |G_{m_k}| = d_{\sigma_{k,1}}^2 + \ldots + d_{\sigma_{k,s_k}}^2$, where $\{\sigma_{k,i} : i = 1, \ldots, k_s\} = \Sigma_k$ (the dual object of G_{m_k}) and $d_{\sigma_{k,i}}$ is the dimension of $\sigma_{k,i}$. We have $\|r_k^j\|_\infty \le \sqrt{d}$, where d is one of $d_{\sigma_{k,i}}$ and since d is a divisor of m_k [9, Vol. 2, p. 44], [8] and at least one of $d_{\sigma_{k,i}}$ is 1, it follows that $d < \sqrt{m_k}$. Since m is bounded, we conclude that there exists a $\delta > 1$ (possibly depending on the sequence m) such that (iv) holds for all $n, k \in \mathbb{N}$. For more on this system and noncommutative Vilenkin groups see [8, 6].

EXAMPLE D (a system in number theory). Let

$$r_k^n(x) := \exp\left(2\pi\imath \sum_{j=k}^{\infty} \frac{n_j}{M_{j+1}} \sum_{i=0}^{k} x_i M_i\right)$$

for $k, n \in \mathbb{N}$ and $x \in G_m$. Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}},\ n \in \mathbb{N}$.

Then $\psi := (\psi_n : n \in \mathbb{N})$ is a Vilenkin-like system (introduced in [7]) which is a useful tool in the approximation theory of limit periodic, almost even arithmetical functions [7]. (i) is trivial and since $|r_k^n| = 1$, so are (iii) and (iv). It is easy to prove (ii) (see [7]). This system (on Vilenkin groups) was a new tool for investigating limit periodic arithmetical functions. For their definition see also the book of Mauclaire [11, p. 25].

EXAMPLE E (UDMD product systems). The notion of UDMD product system was introduced by F. Schipp [20, p. 88] on the Walsh–Paley group. Let $\alpha_k : G_m \to \mathbb{C}$ satisfy $|\alpha_k| = 1$ and be $\mathcal{A}_k$-measurable. Let $r_k^n(x) := (-1)^{x_k n_k} \alpha_k(x)$. Then (i) is trivial and since $|r_k^n| = 1$, so are (iii) and (iv). The proof of (ii) is simple. Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k M_k}$ $(n \in \mathbb{N})$. The system $\psi := (\psi_n : n \in \mathbb{N})$ is called an *UDMD product system*. For more on such systems see [19, 20].

EXAMPLE F (universal contractive projections). The notion of universal contractive projection system (UCP) was introduced by F. Schipp [18] as follows. Let $\phi_n : G_m \to \mathbb{C}$ $(n \in \mathbb{N})$ be measurable functions with $|\phi_n| = 1$ $(n \in \mathbb{N})$ and $\phi_0 = 1$. Let $f \in L^1(G_m)$ and $P_{n^{(s)}} f := \phi_{n^{(s)}} E_s(f \overline{\phi}_{n^{(s)}})$ for $n, s \in \mathbb{N}$. Suppose that $P_{n^{(s)}} = P_{n^{(s)}} P_{n^{(s+j)}} = P_{n^{(s+j)}} P_{n^{(s)}}$ for all $j \in \mathbb{N}$. Also suppose that if $n^{(s)}$ and $k^{(t)}$ are incomparable, that is, if there are no $j \in \mathbb{N}$ such that $n^{(s+j)} = k^{(t)}$ or $k^{(t+j)} = n^{(s)}$, then $P_{n^{(s)}} P_{k^{(t)}} = P_{k^{(t)}} P_{n^{(s)}} = 0$.

We prove that the system $(\phi_n : n \in \mathbb{N})$ is also a Vilenkin-like system. Let $r_k^{n^{(k)}} := \phi_{n^{(k)}} \overline{\phi}_{n^{(k+1)}}$ for $k, n \in \mathbb{N}$. Since $|\phi_n| = 1$, (iii) and (iv) hold. Next, we prove that $r_k^{n^{(k)}}$ is $\mathcal{A}_{k+1}$-measurable. Since $P_{n^{(s)}}(\phi_{n^{(s)}}) = \phi_{n^{(s)}} E_s(\phi_{n^{(s)}} \overline{\phi}_{n^{(s)}})$ $= \phi_{n^{(s)}}$ we have

$$\begin{aligned}\phi_{n^{(s+1)}} E_{s+1}(\phi_{n^{(s)}} \overline{\phi}_{n^{(s+1)}}) &= P_{n^{(s+1)}}(\phi_{n^{(s)}}) = P_{n^{(s+1)}}(P_{n^{(s)}}(\phi_{n^{(s)}})) \\ &= P_{n^{(s)}}(\phi_{n^{(s)}}) = \phi_{n^{(s)}}.\end{aligned}$$

Consequently, $E_{s+1}(\phi_{n^{(s)}} \overline{\phi}_{n^{(s+1)}}) = \phi_{n^{(s)}} \overline{\phi}_{n^{(s+1)}}$, i.e. $r_s^{n^{(s)}}$ is $\mathcal{A}_{s+1}$-measurable. Since

$$\phi_n = \phi_{n^{(0)}} = \phi_{n^{(0)}} \overline{\phi}_{n^{(1)}} \phi_{n^{(1)}} = \prod_{k=0}^{\infty} \phi_{n^{(k)}} \overline{\phi}_{n^{(k+1)}} = \prod_{k=0}^{\infty} r_k^{n^{(k)}} = \psi_n,$$

to prove that $(\phi_n : n \in \mathbb{N}) = (\psi_n : n \in \mathbb{N})$ is a Vilenkin-like system we only need to verify (ii). We have $|r_k^{n^{(k)}}| = 1$, thus it remains to prove

$$E_k(r_k^{n^{(k+1)}+jM_k}\overline{r}_k^{n^{(k+1)}+lM_k}) = 0$$

for $j, l \in \{0, 1, \ldots, m_k - 1\}$ and $j \neq l$. Since for all $f \in L^1(G_m)$ we have $P_{n^{(k+1)}+jM_k}P_{n^{(k+1)}+lM_k}f = 0$, we can apply this for $f = \phi_{n^{(k+1)}+lM_k}$. Thus,

$$\phi_{n^{(k+1)}+jM_k}E_k(\overline{\phi}_{n^{(k+1)}+jM_k}\phi_{n^{(k+1)}+lM_k}E_k(\overline{\phi}_{n^{(k+1)}+lM_k}\phi_{n^{(k+1)}+lM_k})) = 0.$$

It follows that

$$0 = E_k(\overline{\phi}_{n^{(k+1)}+jM_k}\phi_{n^{(k+1)}+lM_k}) = E_k(\overline{r}_k^{n^{(k+1)+jM_k}}r_k^{n^{(k+1)+lM_k}}).$$

It is simple to prove that the partial sums of the Fourier series with respect to the system $\psi = \phi$ are $S_nf = \sum_{k=0}^{\infty}\sum_{j=0}^{n_k-1} P_{n^{(k+1)}+jM_k}f$ (cf. [18]).

Some more preliminaries. For $f \in L^1(G_m)$ we define the Fourier coefficients and partial sums by

$$\widehat{f}(k) := \int_{G_m} f\overline{\psi}_k \, d\mu \qquad (k \in \mathbb{N}),$$

$$S_nf := \sum_{k=0}^{n-1}\widehat{f}(k)\psi_k \qquad (n \in \mathbb{P},\ S_0f := 0).$$

The *Dirichlet kernels* are given by

$$D_n(y,x) := \sum_{k=0}^{n-1}\psi_k(y)\overline{\psi}_k(x) \qquad (n \in \mathbb{P},\ D_0 := 0).$$

It is clear that

$$S_nf(y) = \int_{G_m} f(x)D_n(y,x)\, d\mu(x).$$

Denote by

$$\sigma_nf = \frac{1}{n}\sum_{k=0}^{n-1}S_kf \qquad (n \in \mathbb{P},\ \sigma_0f := 0)$$

the *Fejér* (or $(C,1)$) *means* of the Fourier series, and by

$$K_n := \frac{1}{n}\sum_{k=0}^{n-1}D_k \qquad (n \in \mathbb{P},\ K_0 := 0)$$

the *Fejér kernels*. Then

$$\sigma_nf(y) = \int_{G_m} f(x)K_n(y,x)\, d\mu(x) \qquad (y \in G_m,\ n \in \mathbb{P}).$$

Results and proofs

PROPOSITION 1. *Every Vilenkin-like system ψ is orthonormal.*

Proof. Let $n \neq k$ $(n,k \in \mathbb{N})$. Set $s := \max(j \in \mathbb{N} : n_j \neq k_j)$. We can suppose that $n > k$. This implies that $|n| \geq s$. Then

$$\begin{aligned} E_0(\psi_n\overline{\psi}_k) &= E_0\Big(E_s\Big(\prod_{i=0}^{s-1}(r_i^{n^{(i)}}\bar{r}_i^{k^{(i)}})r_s^{n^{(s)}}\bar{r}_s^{k^{(s)}}\prod_{i=s+1}^{\infty}|r_i^{n^{(i)}}|^2\Big)\Big) \\ &= E_0\Big(\prod_{i=0}^{s-1}(r_i^{n^{(i)}}\bar{r}_i^{k^{(i)}})E_s\Big(r_s^{n^{(s)}}\bar{r}_s^{k^{(s)}}E_{s+1}\Big(\prod_{i=s+1}^{\infty}|r_i^{n^{(i)}}|^2\Big)\Big)\Big). \end{aligned}$$

Since

$$\begin{aligned} E_{s+1}\Big(\prod_{i=s+1}^{\infty}|r_i^{n^{(i)}}|^2\Big) &= E_{s+1}\Big(E_{|n|}\Big(\prod_{i=s+1}^{|n|}|r_i^{n^{(i)}}|^2\Big)\Big) \\ &= E_{s+1}\Big(\prod_{i=s+1}^{|n|-1}|r_i^{n^{(i)}}|^2E_{|n|}(|r_{|n|}^{n^{(|n|)}}|^2)\Big) \\ &= E_{s+1}\Big(\prod_{i=s+1}^{|n|-1}|r_i^{n^{(i)}}|^2\Big) = \ldots = E_{s+1}(|r_{s+1}^{n^{(s+1)}}|^2) = 1 \end{aligned}$$

and $E_s(r_s^{n^{(s)}}\bar{r}_s^{k^{(s)}}) = 0$ (property (ii)), we have $E_0(\psi_n\overline{\psi}_k) = 0$. On the other hand, for $n = k$,

$$\begin{aligned} E_0(|\psi_n|^2) &= E_0\Big(\prod_{i=0}^{|n|}|r_i^{n^{(i)}}|^2\Big) = E_0\Big(\prod_{i=0}^{|n|-1}|r_i^{n^{(i)}}|^2E_{|n|}(|r_{|n|}^{n^{(|n|)}}|^2)\Big) \\ &= E_0\Big(\prod_{i=0}^{|n|-1}|r_i^{n^{(i)}}|^2\Big) = \ldots = E_0(|r_0^{n^{(0)}}|^2) = 1. \end{aligned}$$

Note that this proof of Proposition 1 was based directly on properties (i) and (ii). ∎

The Dirichlet kernels play a prominent role in the convergence of Fourier series. The following two lemmas will be useful in this regard.

LEMMA 2. *Let $M_{n+1}\,|\,k,\ y \in I_n(x)$ $(n,k \in \mathbb{N},\ x,y \in G_m)$. Then*

$$\sum_{j=0}^{m_n-1} r_n^{k+jM_n}(y)\bar{r}_n^{k+jM_n}(x) = \begin{cases} 0 & \text{if } y \notin I_{n+1}(x), \\ m_n & \text{if } y \in I_{n+1}(x). \end{cases}$$

Proof. By properties (ii), (iii) we have

$$\frac{1}{m_n}\sum_{x_n=0}^{m_n-1}\Big|\sum_{j=0}^{m_n-1} r_n^{k+jM_n}(y)\bar{r}_n^{k+jM_n}(x)\Big|^2$$

$$=\sum_{j,l=0}^{m_n-1} r_n^{k+jM_n}(y)\bar{r}_n^{k+lM_n}(y)\frac{1}{m_n}\sum_{x_n=0}^{m_n-1}\bar{r}_n^{k+jM_n}(x)r_n^{k+lM_n}(x)$$

$$=\sum_{j=0}^{m_n-1}|r_n^{k+jM_n}(y)|^2=m_n.$$

Let $y\in I_{n+1}(x)$. Then by the $\mathcal{A}_{n+1}$-measurability of r_n^s $(n,s\in\mathbb{N})$, (i) and (iii) we get

$$\frac{1}{m_n}\Big|\sum_{j=0}^{m_n-1} r_n^{k+jM_n}(y)\bar{r}_n^{k+jM_n}(x)\Big|^2=\frac{1}{m_n}\Big|\sum_{j=0}^{m_n-1}|r_n^{k+jM_n}(y)|^2\Big|^2=\frac{1}{m_n}m_n^2=m_n.$$

Consequently, for $y\in I_n(x)$ we have

$$\frac{1}{m_n}\sum_{x_n=0,\,x_n\neq y_n}^{m_n-1}\Big|\sum_{j=0}^{m_n-1} r_n^{k+jM_n}(y)\bar{r}_n^{k+jM_n}(x)\Big|^2=0. \ \blacksquare$$

Lemma 3.

$$D_{M_n}(y,x)=\begin{cases} M_n & \text{if } y\in I_n(x),\\ 0 & \text{if } y\notin I_n(x).\end{cases}$$

Proof. Suppose that $y\notin I_n(x)$. Then $y\in I_a(x)\setminus I_{a+1}(x)$ for some $a\in\{0,1,\dots,n-1\}$. Lemma 2 gives

$$\sum_{k_i=0}^{m_i-1} r_i^{k^{(i)}}(y)\bar{r}_i^{k^{(i)}}(x)=\begin{cases} m_i & \text{for } i=0,1,\dots,a-1,\\ 0 & \text{for } i=a.\end{cases}$$

It follows that

$$D_{M_n}(y,x)=\sum_{k_{n-1}=0}^{m_{n-1}-1}\dots\sum_{k_0=0}^{m_0-1}\prod_{i=0}^{n-1} r_i^{k^{(i)}}(y)\bar{r}_i^{k^{(i)}}(x)$$

$$=\sum_{k_{n-1}=0}^{m_{n-1}-1} r_{n-1}^{k^{(n-1)}}(y)\bar{r}_{n-1}^{k^{(n-1)}}(x)\dots\sum_{k_1=0}^{m_1-1} r_1^{k^{(1)}}(y)\bar{r}_1^{k^{(1)}}(x)\sum_{k_0=0}^{m_0-1} r_0^{k^{(0)}}(y)\bar{r}_0^{k^{(0)}}(x)$$

$$=\sum_{k_{n-1}=0}^{m_{n-1}-1} r_{n-1}^{k^{(n-1)}}(y)\bar{r}_{n-1}^{k^{(n-1)}}(x)\dots\sum_{k_a=0}^{m_a-1} r_a^{k^{(a)}}(y)\bar{r}_a^{k^{(a)}}(x)M_a=0.$$

On the other hand, this expansion of D_{M_n} also shows that $D_{M_n}(x,y)=M_n$ for $y\in I_n(x)$. $\blacksquare$

LEMMA 4 (Calderón–Zygmund decomposition [23, p. 752]). *Let* $f \in L^1(G_m)$, $\lambda > \|f\|_1$. *Then there exists an absolute constant* $c > 0$ (*which may depend only on* $\sup m_n$), *a decomposition* $f = \sum_{j=0}^{\infty} f_j$, *and disjoint intervals* $I^j := I_{k_j}(u^j)$ *for which* $\operatorname{supp} f_j \subseteq I^j$, $\int_{I^j} f_j \, d\mu = 0$, $\int_{I^j} |f_j| \, d\mu \le c\lambda\mu(I^j)$ $(u^j \in G_m,\ k_j, j \in \mathbb{P})$, *and*

$$\|f_0\|_\infty \le c\lambda, \qquad \|f_0\|_1 \le c\|f\|_1, \qquad \mu(F) \le c\|f\|_1/\lambda,$$

where $F = \bigcup_{j\in\mathbb{P}} I^j$.

Define the maximal operator $S^* f := \sup_{n\in\mathbb{N}} |S_{M_n} f|$.

PROPOSITION 5. *The operator* S^* *is of type* (p,p) *for all* $1 < p \le \infty$ *and of weak type* $(1,1)$.

Proof. The proof is based on standard techniques of the theory of Vilenkin and Walsh systems. Lemma 3 gives

$$\begin{aligned}\|S^* f\|_\infty &= \Big\| \sup_{n\in\mathbb{N}} \Big| \int_{G_m} f(x) D_{M_n}(y,x)\, d\mu(x) \Big| \Big\|_\infty \\ &\le \|f\|_\infty \Big\| \sup_{n\in\mathbb{N}} \Big| \int_{G_m} |D_{M_n}(y,x)|\, d\mu(x) \Big| \Big\|_\infty = \|f\|_\infty.\end{aligned}$$

That is, S^* is of type (∞,∞). In order to prove the weak $(1,1)$ type we apply Lemma 4. We can suppose $\lambda > \|f\|_1$. Since S^* is sublinear, we have

$$\begin{aligned}\mu(S^* f > 2c\lambda) &\le \mu(S^* f_0 > c\lambda) + \mu\Big(S^*\Big(\sum_{i=1}^{\infty} f_i\Big) > c\lambda\Big) \\ &\le \mu(F) + \mu\Big(y \in G_m : y \notin F,\ S^*\Big(\sum_{i=1}^{\infty} f_i\Big)(y) > c\lambda\Big) \\ &\le c\|f\|_1/\lambda + \frac{c}{\lambda} \int_{G_m\setminus F} S^*\Big(\sum_{i=1}^{\infty} f_i\Big)\, d\mu \\ &\le c\|f\|_1/\lambda + \frac{c}{\lambda} \sum_{i=1}^{\infty} \int_{G_m\setminus F} S^* f_i \, d\mu \\ &\le c\|f\|_1/\lambda + \frac{c}{\lambda} \sum_{i=1}^{\infty} \int_{G_m\setminus I_{k_i}(u^i)} \sup_{n\in\mathbb{N}} \Big| \int_{I_{k_i}(u^i)} f_i(x) D_{M_n}(y,x)\, d\mu(x) \Big|\, d\mu(y) \\ &=: c\|f\|_1/\lambda + \frac{c}{\lambda} \sum_{i=1}^{\infty} B^i.\end{aligned}$$

We prove that $B^i = 0$ for $i \in \mathbb{P}$. If $n < k_i$, then $D_{M_n}(y,x)$ is $\mathcal{A}_{k_i}$-measurable (with respect to both x and y), thus

$$\int_{I_{k_i}(u^i)} f_i(x) D_{M_n}(y,x)\, d\mu(x) = D_{M_n}(y,u^i) \int_{I_{k_i}(u^i)} f_i(x) d\mu(x) = 0 \qquad (i \in \mathbb{P}).$$

If $n \geq k_i$, then $x \in I_{k_i}(u^i)$ and $y \in G_m \setminus I_{k_i}(u^i)$ give that $y \notin I_{k_i}(x)$. Consequently, $y \notin I_n(x)$ and by Lemma 3 we have $D_{M_n}(y,x) = 0$. That is, $B^i = 0$ for $i \in \mathbb{P}$. Hence, $\mu(S^* f > c\lambda) \leq c\|f\|_1/\lambda$. That is, the sublinear operator S^* is both of type (∞,∞) and of weak type $(1,1)$. The interpolation theorem of Marcinkiewicz [21, p. 479] shows that S^* is of type (p,p) for all $1 < p \leq \infty$. ∎

Set $\mathcal{P}_n := \{\sum_{k=0}^{n-1} b_k \psi_k : b_0, \dots, b_{n-1} \in \mathbb{C}\}$ $(n \in \mathbb{P})$ be the set of polynomials of degree less than n, and $\mathcal{P} := \bigcup_{n=1}^{\infty} \mathcal{P}_n$ the set of all polynomials (with respect to the system ψ). Then $\mathcal{P}$ is dense in $C(G_m)$ (the set of functions continuous on G_m). This can be proved in the following way. Let $f \in C(G_m)$, and $\varepsilon > 0$. Since G_m is compact, f is uniformly continuous on G_m. Thus, there exists $n \in \mathbb{N}$ such that $|f(x) - f(y)| < \varepsilon$ for all $y \in I_n(x)$. In view of Lemma 3 we can define the following polynomial:

$$P(y) := M_n \int_{I_n(y)} f(x)\, d\mu(x) = \int_{G_m} f(y) D_{M_n}(y,x)\, d\mu(x) = S_{M_n} f(y).$$

Then

$$|P(y) - f(y)| \leq M_n \int_{I_n(y)} |f(x) - f(y)|\, d\mu(x) < \varepsilon$$

for all $y \in G_m$. Moreover, we prove that $\mathcal{P}$ is dense in $L^p(G_m)$ $(1 \leq p < \infty)$. Let $G \subset G_m$ be an open set. For all $x \in G$ set $n = n(x) := \min(k \in \mathbb{N} : I_k(x) \subset G)$. Then $G = \bigcup_{x \in G} I_n(x)$. (It can be supposed that there are no $x^1, x^2 \in G$ for which $I_{n_1}(x^1) \supset I_{n_2}(x^2)$ (the latter can be omitted).) This union consists of disjoint intervals (since for any intervals $I, J \in \mathcal{I}$ we have $I \cap J = \emptyset$ or $I \subset J$ or $I \supset J$). Consequently, for each $\varepsilon > 0$ we have a finite number of disjoint intervals $I_{n_1}(x^1), \dots, I_{n_k}(x^k)$ the union of which is a subset of G and the difference of the measure of G and the sum of their measures is less than ε. Lemma 3 shows that the characteristic function of an interval is a polynomial. The set of step functions (finite (complex) linear combinations of characteristic functions of measurable sets) is dense in $L^p(G_m)$ $(1 \leq p < \infty)$. The Haar measure μ is regular. Thus, the set of finite (complex) linear combinations of characteristic functions of open sets is also dense in $L^p(G_m)$. By the above, this proves that the set of polynomials is dense in $L^p(G_m)$ $(1 \leq p < \infty)$.

Then by the usual density argument (see e.g. [21, p. 81]) we have

PROPOSITION 6. *$S_{M_n} f \to f$ a.e. for each $f \in L^1(G_m)$.* ∎

For the whole sequence of partial sums of Fourier series, the situation changes. For Vilenkin systems (also on unbounded G_m groups) it is known (see e.g. [16]) that $f \in L^p(G_m)$ $(1 < p < \infty)$ implies that $S_n f \to f$ in the L^p norm, and on bounded Vilenkin groups $S_n f \to f$ almost everywhere (see e.g. [17]). On the other hand, this is not the case on nonabelian Vilenkin groups. In [8, 6] it is proved that there exists a nonabelian Vilenkin group (even a bounded one) such that there exists $p > 1$ for which there exists an $f \in L^p(G_m)$ such that $S_n f$ converges to f neither in norm nor a.e. That is, the theorem of Carleson does not hold for all Vilenkin-like systems. From this point of view it seems to be more interesting that the $(C,1)$ means of an integrable function converge a.e. to the function.

Set

$$D_{a,b}(y,x) := \sum_{k=a}^{a+b-1} \psi_k(y)\overline{\psi}_k(x) \quad (x,y \in G_m,\ a \in \mathbb{N},\ b \in \mathbb{P},\ D_{a,0} := 0)$$

and

$$\psi_{k,n} := \prod_{s=n}^{\infty} r_s^{k^{(s)}}, \qquad \psi_{k,n,l} := \prod_{s=n}^{l} r_s^{k^{(s)}}.$$

Lemma 7. *Let $M_n \mid k$, that is, $k = k_n M_n + \ldots + k_{|k|} M_{|k|}$. Then*

$$D_{k,M_n}(y,x) = \begin{cases} 0 & \text{if } y \notin I_n(x), \\ \psi_{k,n}(y)\overline{\psi}_{k,n}(x) M_n & \text{if } y \in I_n(x). \end{cases}$$

Proof. We have

$$\begin{aligned} D_{k,M_n}(y,x) &= \sum_{j=0}^{M_n-1} \psi_{k+j}(y)\overline{\psi}_{k+j}(x) \\ &= \psi_{k,n}(y)\overline{\psi}_{k,n}(x) \sum_{j=0}^{M_n-1} \prod_{s=0}^{n-1} r_s^{k+j^{(s)}}(y)\overline{r}_s^{k+j^{(s)}}(x). \end{aligned}$$

To complete the proof, proceed as in the proof of Lemma 3, with the use of Lemma 2. ∎

Next, we give a formula for the Dirichlet kernels.

Proposition 8. *Let $x, y \in G_m,\ n \in \mathbb{N}$. Then*

$$D_n(y,x) = \sum_{s=0}^{\infty} \psi_{n,s+1}(y)\overline{\psi}_{n,s+1}(x) D_{M_s}(y,x) \sum_{j=0}^{n_s-1} r_s^{n^{(s+1)}+jM_s}(y)\overline{r}_s^{n^{(s+1)}+jM_s}(x).$$

Proof. Apply Lemma 7 and the equality

$$D_n(y,x)=\sum_{s=0}^{\infty}\sum_{j=0}^{n_s-1}D_{n^{(s+1)}+jM_s,M_s}(y,x).\ \blacksquare$$

Let $n,t\in\mathbb{N}$ and $y\in I_t(x)\setminus I_{t+1}(x)$. Then by Proposition 8,

$$\begin{aligned}(1)\quad D_n(y,x)=&\sum_{s=0}^{t-1}M_s\prod_{k=s+1}^{t-1}|r_k^{n^{(k)}}(x)|^2\psi_{n,t}(y)\overline{\psi}_{n,t}(x)\sum_{j=0}^{n_s-1}|r_s^{n^{(s+1)}+jM_s}(x)|^2\\ &+M_t\psi_{n,t+1}(y)\overline{\psi}_{n,t+1}(x)\sum_{j=0}^{n_t-1}r_t^{n^{(t+1)}+jM_t}(y)\overline{r}_t^{n^{(t+1)}+jM_t}(x)\end{aligned}$$

(recall that the empty sum is zero, e.g. $\sum_{j=0}^{n_s-1}|r_s^{n^{(s+1)}+jM_s}(x)|^2=0$ for $n_s=0$). By (1) and (iv) it follows ($y\in I_t(x)\setminus I_{t+1}(x)$) that

$$\begin{aligned}(2)\qquad |D_n(y,x)|\le&\sum_{s=0}^{t-1}M_s\prod_{k=s+1}^{|n|}(m_k/\delta)n_s(m_s/\delta)\\ &+M_t\prod_{k=t+1}^{|n|}(m_k/\delta)n_t(m_t/\delta)\\ \le& c\sum_{s=0}^{t}\frac{M_{|n|}}{\delta^{|n|-s}}\le cn\frac{1}{\delta^{|n|-t}}.\end{aligned}$$

Denote by $K_{a,b}:=\sum_{k=a}^{a+b-1}D_k$ the sum of Dirichlet kernels ($a\in\mathbb{N},\ b\in\mathbb{P}$). Using (2) we prove

LEMMA 9. *Let $s,n,t\in\mathbb{N},\ s\le t\le|n|$ and $y\in I_t(x)\setminus I_{t+1}(x)$. Then*

$$|K_{n^{(s+1)}+jM_s,M_s}(y,x)|\le c\delta^{t-|n|}M_sM_{|n|}\qquad (j\in\{0,1,\dots,m_s-2\}).$$

Proof. By (2) we have

$$\begin{aligned}|K_{n^{(s+1)}+jM_s,M_s}(y,x)|\le&\sum_{k=n^{(s+1)}+jM_s}^{n^{(s+1)}+(j+1)M_s-1}|D_k(y,x)|\\ \le& c\sum_{k=n^{(s+1)}+jM_s}^{n^{(s+1)}+(j+1)M_s-1}k\frac{1}{\delta^{|k|-t}}.\end{aligned}$$

If $s+1\le|n|$, or $s=|n|$ and $j>0$, then $|k|=|n|$ and

$$|K_{n^{(s+1)}+jM_s,M_s}(y,x)|\le cM_sM_{|n|}\delta^{t-|n|}.$$

It remains to discuss the case $s=|n|$ and $j=0$. Without loss of generality we can suppose $\delta < 2$. Hence

$$\sum_{k=0}^{M_{|n|}-1} k\delta^{t-|k|} \le \sum_{a=0}^{|n|-1}\sum_{k=M_a}^{M_{a+1}-1} cM_a\delta^{t-a} = c\delta^t \sum_{a=0}^{|n|-1} M_a^2\delta^{-a} \le c\delta^t M_{|n|}^2\delta^{-|n|}.$$

That is,

$$\begin{aligned}|K_{n^{(s+1)}+jM_s,M_s}(y,x)| &\le c \sum_{k=n^{(s+1)}+jM_s}^{n^{(s+1)}+(j+1)M_s-1} k\frac{1}{\delta^{|k|-t}}\\ &\le c\delta^t \sum_{k=0}^{M_{|n|}-1} k\delta^{t-|k|} \le c\delta^t M_{|n|}^2\delta^{-|n|}. \ \blacksquare\end{aligned}$$

What can be said for $t < s \le |n|$, $y \in I_t(x)\setminus I_{t+1}(x)$?

LEMMA 10. *Let $s,n,t\in\mathbb{N}$, $t<s\le|n|$ and $j\in\{0,1,\dots,m_s-2\}$. Then*

$$\int\limits_{I_t(y)\setminus I_{t+1}(y)} |K_{n^{(s+1)}+jM_s,M_s}(y,x)|^2\,d\mu(x) \le c\delta^{t-|n|}M_tM_sM_{|n|}.$$

Proof. By (1),

$$\begin{aligned}K_{n^{(s+1)}+jM_s,M_s}(y,x) = &\sum_{k=n^{(s+1)}+jM_s}^{n^{(s+1)}+(j+1)M_s-1}\ \sum_{i=0}^{t-1} M_i|\psi_{k,i+1,t-1}(x)|^2\\ &\times \psi_{k,t}(y)\overline{\psi}_{k,t}(x)\sum_{l=0}^{k_i-1}|r_i^{k^{(i+1)}+lM_i}(y)|^2\\ &+\sum_{k=n^{(s+1)}+jM_s}^{n^{(s+1)}+(j+1)M_s-1} M_t\psi_{k,t+1}(y)\overline{\psi}_{k,t+1}(x)\\ &\times\sum_{l=0}^{k_t-1} r_t^{k^{(t+1)}+lM_t}(y)\overline{r}_t^{k^{(t+1)}+lM_t}(x)\\ =:{}& A_1(y,x)+A_2(y,x).\end{aligned}$$

We start with a bound for $\int_{I_t(y)}|A_2(y,x)|^2\,d\mu(x)$. It follows from (iv) that

$$\Big|\sum_{l=0}^{k_t-1} r_t^{k^{(t+1)}+lM_t}(y)\overline{r}_t^{k^{(t+1)}+lM_t}(x)\Big| \le k_tm_t/\delta \le c. \tag{3}$$

If $k,l\in[n^{(s+1)}+jM_s, n^{(s+1)}+(j+1)M_s)$, then $s>t$ gives that ($v\in G_m$ is arbitrary)

$$(4)\quad \int_{I_{t+1}(v)} \overline{\psi}_{k,t+1}(x)\psi_{l,t+1}(x)\,d\mu(x) = \frac{1}{M_{t+1}}E_{t+1}(\overline{\psi}_{k,t+1}\psi_{l,t+1})(v) = \begin{cases} 1/M_{t+1} & \text{if } k^{(t+1)} = l^{(t+1)}, \\ 0 & \text{otherwise} \end{cases}$$

(as in Lemma 1). Let $u^{(z)} \in I_t(y)$, $z = 0, 1, \ldots, m_t - 1$, with $u_t^{(z)} = z$. Then

$$\int_{I_t(y)} |A_2(y,x)|^2\,d\mu(x) = \sum_{z=0}^{m_t-1} \int_{I_{t+1}(u^{(z)})} M_t^2 \sum_{k,l\in[n^{(s+1)}+jM_s,n^{(s+1)}+(j+1)M_s)} \psi_{k,t+1}(y)\overline{\psi}_{l,t+1}(y)\overline{\psi}_{k,t+1}(x)\psi_{l,t+1}(x)$$
$$\times \sum_{i=0}^{k_t-1} r_t^{k^{(t+1)}+iM_t}(y)\overline{r}_t^{k^{(t+1)}+iM_t}(x) \sum_{a=0}^{l_t-1} r_t^{l^{(t+1)}+aM_t}(y)\overline{r}_t^{l^{(t+1)}+aM_t}(x)\,d\mu(x).$$

This, (3), (4) and the $\mathcal{A}_{t+1}$-measurability of r_t^k $(t, k \in \mathbb{N})$ imply that

$$\int_{I_t(y)} |A_2(y,x)|^2\,d\mu(x) \le cM_t^2 \sum_{u_t=0}^{m_t-1} \frac{1}{M_{t+1}} \sup_{k\in\mathbb{N},\,|k|=|n|\ge t} \|\psi_{k,t+1}\|_\infty^2 \sum_{\substack{k,l\in[n^{(s+1)}+jM_s,n^{(s+1)}+(j+1)M_s) \\ k^{(t+1)}=l^{(t+1)}}} 1$$
$$\le cM_t^2 \frac{1}{M_{t+1}} \cdot \frac{M_{|n|}}{M_t}\delta^{t-|n|}M_sM_t = cM_tM_sM_{|n|}\delta^{t-|n|}.$$

On the other hand, since r_b^k is $\mathcal{A}_t$-measurable for $b < t$, we have

$$\int_{I_t(y)} |A_1(y,x)|^2\,d\mu(x) = \sum_{k,l\in[n^{(s+1)}+jM_s,n^{(s+1)}+(j+1)M_s)} \Big(\int_{I_t(y)} \overline{\psi}_{k,t}(x)\psi_{l,t}(x)\,d\mu(x)\Big)\psi_{k,t}(y)\overline{\psi}_{l,t}(y)$$
$$\times \sum_{i=0}^{t-1} M_i|\psi_{k,i+1,t-1}(y)|^2 \sum_{j=0}^{k_i-1} |r_i^{k^{(i+1)}+jM_i}(y)|^2 \sum_{a=0}^{t-1} M_a|\psi_{l,a+1,t-1}(y)|^2$$
$$\times \sum_{b=0}^{l_a-1} |r_a^{l^{(a+1)}+bM_a}(y)|^2.$$

As above, in (4),

$$\int_{I_t(y)} \overline{\psi}_{k,t}(x)\psi_{l,t}(x)\,d\mu(x) = \begin{cases} 1/M_t & \text{if } k^{(t)} = l^{(t)}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{i=0}^{t-1} M_i|\psi_{k,i+1,t-1}(y)|^2 \sum_{l=0}^{k_i-1} |r_i^{k^{(i+1)}+lM_i}(y)|^2$$
$$\leq c\sum_{i=0}^{t-1} M_i|\psi_{k,i+1,t-1}(y)|^2 \leq c\sum_{i=0}^{t-1} M_i \frac{M_t}{M_i}\delta^{i-t} \leq cM_t.$$

Thus,

$$\int\limits_{I_t(y)} |A_1(y,x)|^2\, d\mu(x)$$
$$\leq \sum_{\substack{k,l\in[n^{(s+1)}+jM_s,n^{(s+1)}+(j+1)M_s)\\ k^{(t)}=l^{(t)}}} \frac{1}{M_t} \sup_{k\in\mathbb{N},\, |k|=|n|\geq t} \|\psi_{k,t}\|_\infty^2 M_t^2$$
$$\leq cM_tM_s\frac{1}{M_t}M_t^2\frac{M_{|n|}}{M_t}\delta^{t-|n|} = cM_tM_sM_{|n|}\delta^{t-|n|}.$$

Since $I_t(y)\setminus I_{t+1}(y)\subset I_t(y)$, we have

$$\int\limits_{I_t(y)\setminus I_{t+1}(y)} |K_{n^{(s+1)}+jM_s,M_s}(y,x)|^2\, d\mu(x) \leq c\delta^{t-|n|}M_tM_sM_{|n|}. \blacksquare$$

Using Lemmas 9 and 10 we prove the following proposition on the Fejér kernels which is the base of the weak $(1,1)$ and (H,L^1) type of the maximal operator $\sigma^*f := \sup_{n\in\mathbb{P}}|\sigma_n f|$.

Propositon 11. *Let $y\in G_m, k\in\mathbb{N}$. Then*

$$\int\limits_{G_m\setminus I_k(y)} \sup_{n\geq M_k} |K_n(y,x)|\, d\mu(x) \leq c.$$

Proof. Since

$$nK_n = \sum_{s=0}^{|n|}\sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s,M_s},$$

we have

$$\int\limits_{G_m\setminus I_k(y)} \sup_{n\geq M_k} |K_n(y,x)|\, d\mu(x)$$
$$\leq \sum_{A=k}^{\infty} \int\limits_{G_m\setminus I_k(y)} \sup_{n:|n|=A} |K_n(y,x)|\, d\mu(x)$$
$$\leq c\sum_{A=k}^{\infty}\sum_{t=0}^{k-1} \int\limits_{I_t(y)\setminus I_{t+1}(y)} \frac{1}{M_A} \sup_{n:|n|=A} \Big|\sum_{s=0}^{A}\sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s,M_s}(y,x)\Big|\, d\mu(x)$$

$$\le c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} \sum_{s=0}^{t} \sum_{j=0}^{m_s-2} \int\limits_{I_t(y)\setminus I_{t+1}(y)} \frac{1}{M_A} \sup_{n:|n|=A} |K_{n^{(s+1)}+jM_s,M_s}(y,x)|\, d\mu(x)$$

$$+ c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} \sum_{s=t+1}^{A} \sum_{j=0}^{m_s-2} \int\limits_{I_t(y)\setminus I_{t+1}(y)} \frac{1}{M_A} \sup_{n:|n|=A} |K_{n^{(s+1)}+jM_s,M_s}(y,x)|\, d\mu(x)$$

$$=: B^1 + B^2.$$

From Lemma 9 it follows that

$$B^1 \le c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} \sum_{s=0}^{t} \sum_{j=0}^{m_s-2} \frac{1}{M_t} \frac{1}{M_A} M_s M_A \delta^{t-A} \le c.$$

By the Cauchy–Bunyakovskiĭ inequality and Lemma 10 we have (1_X denotes the characteristic function of the set X)

$$\begin{aligned}
B^2 &= c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} \sum_{s=t+1}^{A} \sum_{j=0}^{m_s-2} \frac{1}{M_A} \int\limits_{G_m} 1_{I_t(y)\setminus I_{t+1}(y)}(x) \\
&\qquad \times \sup_{n:|n|=A} |K_{n^{(s+1)}+jM_s,M_s}(y,x)|\, d\mu(x) \\
&\le c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} \sum_{s=t+1}^{A} \sum_{j=0}^{m_s-2} \frac{1}{M_A} (\mu(I_t(y) \setminus I_{t+1}(y)))^{1/2} \\
&\qquad \times \sqrt{\int\limits_{I_t(y)\setminus I_{t+1}(y)} \sup_{n:|n|=A} |K_{n^{(s+1)}+jM_s,M_s}(y,x)|^2\, d\mu(x)} \\
&\le c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} \sum_{s=t+1}^{A} \sum_{j=0}^{m_s-2} \frac{1}{M_A} \sqrt{\frac{1}{M_t}} \\
&\qquad \times \sqrt{\int\limits_{I_t(y)\setminus I_{t+1}(y)} \sum_{\substack{n_i=0,1,\dots,m_i-1 \\ i=s+1,s+2,\dots,A}} |K_{n^{(s+1)}+jM_s,M_s}(y,x)|^2\, d\mu(x)} \\
&\le c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} \sum_{s=t+1}^{A} \sum_{j=0}^{m_s-2} \frac{1}{M_A} \sqrt{\frac{1}{M_t}} \sqrt{\frac{M_A}{M_s} M_t M_s M_A \delta^{t-A}} \\
&= c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} \sum_{s=t+1}^{A} \delta^{(t-A)/2} \le c \sum_{A=k}^{\infty} \sum_{t=0}^{k-1} (A-t) \delta^{(t-A)/2} \le c. \quad \blacksquare
\end{aligned}$$

Proposition 11 gives the following.

THEOREM 12.

$$\int\limits_{G_m} |K_n(y,x)|\,d\mu(x) \le c \qquad (n\in\mathbb{P},\ y\in G_m).$$

Proof. Let $M_A \le n < M_{A+1}$. By Lemma 7 and Proposition 8 we have

$$|D_n(y,x)| \le c\sum_{s=0}^{|n|}\|\psi_{n,s+1}\|_\infty^2 M_s \le c\sum_{s=0}^{|n|}\frac{M_{|n|}}{M_s}\delta^{s-|n|}M_s \le cM_{|n|} \le cn.$$

Thus, $|K_n| \le cn$ for each $n\in\mathbb{P}$. Proposition 11 now gives

$$\begin{aligned}\int\limits_{G_m}|K_n(y,x)|\,d\mu(x) \le &\int\limits_{I_A(y)}|K_n(y,x)|\,d\mu(x)\\ &+\int\limits_{G_m\setminus I_A(y)}\sup_{n\ge M_A}|K_n(y,x)|\,d\mu(x)\\ &\le c\mu(I_A(y))n + c \le c. \ \blacksquare\end{aligned}$$

COROLLARY 3. *$\sigma_n f \to f$ in the $L^p(G_m)$-norm for each $f\in L^p(G_m)$ $(1\le p<\infty)$.*

Proof. It is sufficient to prove that the operators σ_n are uniformly of type (p,p) when $1\le p\le\infty$, since the convergence $\sigma_n f\to f$ is valid for each polynomial $f\in\mathcal{P}$ and we can apply the theorem of Banach–Steinhaus. From the Marcinkiewicz interpolation theorem [21, p. 479], it is sufficient to prove that the operators σ_n are uniformly of type $(1,1)$ and (∞,∞). These properties (by a standard argument) are straightforward consequences of Theorem 12. ■

THEOREM 14. *The operator σ^* is of weak type $(1,1)$ and of type (p,p) for all $1<p\le\infty$.*

Proof. Theorem 12 gives the (∞,∞) type. We prove that σ^* is of weak type $(1,1)$. We can suppose $\lambda > \|f\|_1$. Apply the notations of Lemma 4. By property (i) it follows that ψ_i is $\mathcal{A}_{k_j}$-measurable for $i<M_{k_j}$, thus $\widehat{f_j}(i)=0$ $(i\in\mathbb{N},\ j\in\mathbb{P})$. Consequently, $\sigma^* f_j = \sup_{n\ge M_{k_j}}|\sigma_n f_j|$. Applying the sublinearity of σ^*, Proposition 11 and the Fubini theorem, we get

$$\begin{aligned}\mu(\sigma^* f > 2c\lambda) &\le \mu(\sigma^* f_0 > c\lambda) + \mu(F)\\ &\quad+\mu\Big(\Big\{y\in G_m\setminus F : \sigma^*\Big(\sum_{j=1}^\infty f_j\Big)(y) > c\lambda\Big\}\Big)\\ &\le c\|f\|_1/\lambda + \frac{c}{\lambda}\int\limits_{G_m\setminus F}\sigma^*\Big(\sum_{j=1}^\infty f_j\Big)\,d\mu\end{aligned}$$

$$\le c\|f\|_1/\lambda + \frac{c}{\lambda}\sum_{j=1}^{\infty}\int\limits_{G_m\setminus I_{k_j}(u^j)}\sigma^* f_j\,d\mu$$

$$\le c\|f\|_1/\lambda + \frac{c}{\lambda}\sum_{j=1}^{\infty}\int\limits_{G_m\setminus I_{k_j}(u^j)}\sup_{n\ge M_{k_j}}|\sigma_n f_j|\,d\mu$$

$$\le c\|f\|_1/\lambda + \frac{c}{\lambda}\sum_{j=1}^{\infty}\int\limits_{I_{k_j}(u^j)}|f_j(x)|\int\limits_{G_m\setminus I_{k_j}(u^j)}\sup_{n\ge M_{k_j}}|K_n(y,x)|\,d\mu(y)\,d\mu(x)$$

$$\le c\|f\|_1/\lambda + \frac{c}{\lambda}\sum_{j=1}^{\infty}\|f_j\|_1 \le c\|f\|_1/\lambda + \frac{c}{\lambda}\sum_{j=1}^{\infty}\int\limits_{I_{k_j}(u^j)}\sum_{k=1}^{\infty}|f_k|\,d\mu$$

$$\le c\|f\|_1/\lambda + \frac{c}{\lambda}\int\limits_{G_m}\sum_{k=1}^{\infty}|f_k|\,d\mu \le c\|f\|_1/\lambda + \frac{c}{\lambda}\int\limits_{G_m}|f-f_0|\,d\mu$$

$$\le c\|f\|_1/\lambda.$$

(We have $G_m\setminus I_{k_j}(u) = G_m\setminus I_{k_j}(x)$ since $x\in I_{k_j}(u)$.) That is, the operator σ^* is of weak type $(1,1)$ and since it is of type (∞,∞), the Marcinkiewicz interpolation lemma [21, p. 479] completes the proof. ∎

Theorem 14 and the usual density argument (see e.g. [21, p. 81]) give the next result.

Theorem 15. *$\sigma_n f\to f$ a.e. for each $f\in L^1(G_m)$.* ∎

Theorem 16. *The operator σ^* is of type (H,L).*

Proof. Let g be an atom, $\operatorname{supp} g\subseteq I_k(u) =: I_g$, $\|g\|_\infty\le M_k$ for some $k\in\mathbb{N}$, $u\in G_m$. Then

$$\|\sigma^* g\|_1 \le \int\limits_{I_g}\sigma^* g\,d\mu + \int\limits_{G_m\setminus I_g}\sigma^* g d\mu =: I^1 + I^2.$$

As σ^* is of type $(2,2)$, we have

$$I^1 \le c\|\sigma^* g\|_2(\mu(I_g))^{1/2} \le c\|g\|_2(\mu(I_g))^{1/2}\le c.$$

Proposition 11 (as in the proof of Theorem 14) gives $I^2\le c\|g\|_1\le c$. By a standard argument (see e.g. [22, p. 95]) $\|\sigma^* g\|_1\le c$ implies the (H,L) type of σ^*.

Applications and Remarks. *Example A.* Fine [3] proved that every Walsh–Fourier series (in the Walsh case $m_j=2$ for all $j\in\mathbb{N}$) is a.e. (C,α) summable for $\alpha>0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [10]. Schipp [15] gave a simpler proof for the case $\alpha=1$, i.e. $\sigma_n f\to f$ a.e. ($f\in L^1(G_m)$). He proved that σ^* is of weak type $(1,1)$. That σ^* is of type (L^1, H^1) was discovered by Fujii [4]. The a.e. convergence of the $(C,1)$ means of integrable functions with respect to the p-series fields (which is a Vilenkin group with $m_k = p$ for all $k \in \mathbb{N}$) is due to Taibleson [25]. Later, this result was generalized to the so-called bounded Vilenkin groups with respect to the Vilenkin system by Pál and Simon [12].

Example B. The theorem of Schipp and Fujii for the character system of the group of 2-adic integers was proved by the author [5, p. 89]. Theorem 15 proves the more than 20 years old conjecture of M. H. Taibleson.

Example C. Theorems 14, 15 and 16 (in this case) were proved by the author [6]. Theorem 12 and Corollary 13 can be found in [8].

Example D. The results in this paper preceding Lemma 7 with respect to this system can be found in [7].

Example E. Theorem 12 and Corollary 13 can be found in [19]. Schipp also proved the a.e. convergence of the Fourier series of functions in L^2 [17].

Example F. Schipp [18] proved the a.e. convergence of the Fourier series of functions in L^2. The $(C,1)$ or Fejér means have not been investigated yet.

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