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On L_p-L_q boundedness for convolutions with kernels having singularities on a sphere

by

ALEXEY N. KARAPETYANTS (México, D.F.)

Abstract. For the convolution operators A_a^α with symbols $a(|\xi|)|\xi|^{-\alpha}\exp i|\xi|$, $0 \le \operatorname{Re}\alpha < n$, $a(|\xi|) \in L_\infty$, we construct integral representations and give the exact description of the set of pairs $(1/p, 1/q)$ for which the operators are bounded from L_p to L_q.

1. Introduction. In this paper we consider the L_p-L_q estimates for the operators A_a^α with the symbols

$$m_{a,\alpha}(|\xi|) = a(|\xi|)|\xi|^{-\alpha}e^{i|\xi|}, \qquad 0 \le \operatorname{Re}\alpha < n.$$

They are realized for $(n+1)/2 < \operatorname{Re}\alpha < n$ as potential-type convolution operators with kernels

$$\Omega_{\alpha,a}(|x|) = (2\pi)^{-n/2}|x|^{\alpha-n}\int_0^\infty t^{n/2-\alpha}a\left(\frac{t}{|x|}\right)e^{it/|x|}\mathcal{J}_{n/2-1}(t)\,dt \tag{1.1}$$

($\mathcal{J}_\nu(z)$ is a Bessel function) and for $0 \le \operatorname{Re}\alpha < (n+1)/2$ we treat $A_a^\alpha f$ in the distributional sense, since the kernels $\Omega_{\alpha,a}(|x|)$ have, generally speaking, singularities on a sphere (nonsummable for $0 \le \operatorname{Re}\alpha \le (n-1)/2$). The singularities essentially depend on the behavior of $a(|\xi|)$ at infinity and, for example, in the case $a(|\xi|) = 1$ we have

$$\Omega_\alpha(|x|) \sim c(1-|x|)^{\operatorname{Re}\alpha-(n+1)/2}, \qquad \frac{n-1}{2} < \operatorname{Re}\alpha < \frac{n+1}{2},$$
$$\Omega_\alpha(|x|) \sim c\ln|1-|x||, \qquad \alpha = \frac{n+1}{2},$$

as $|x| \to 1$. We also note that $\Omega_\alpha(|x|) \sim c|x|^{\alpha-n}$ as $|x| \to \infty$.

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We mention the related work [6], [7], devoted to L_p-L_q estimates for the operators with symbols $b(|\xi|)|\xi|^\alpha e^{i|\xi|^\gamma}$, where $0 < \alpha < n$, $\gamma > 0$, and $b(t)$ is a fixed smooth function on $\mathbb{R}^1$ such that $0 \le b(t) \le 1$, $b(t) = 1$ if $|t| \ge 2$, and $b(t) = 0$ if $|t| \le 1$, and also the paper [17] on boundedness results for the convolution operator with kernel $k(x) = (1 - |x|^2)^{-\alpha/2}$ for $|x| \le 1$, and $k(x) = 0$ for $|x| > 1$, $0 < \alpha < (n+1)/2$. The above-mentioned operators appeared in the study of the Cauchy problem for the wave equation (see references). Contrary to our case their kernels are summable at infinity, which makes the problem rather different.

In [3] the L_p-L_q estimates for the operators $A^\alpha = A^\alpha_1$ were obtained and we use that result. But our case is quite different, since consideration of the "locally nonsummable" range of α and the function $a(|\xi|)$ in the symbol lead to considerable difficulties.

We give integral representations for the operators A^α_a on smooth functions. As the main result, we give sufficient and necessary conditions on the parameters p, q, α for the operator A^α_a to be bounded from L_p to L_q. The result is obtained under some additional conditions on the function $a(|\xi|)$ (Theorem 4.1). In fact, the conditions are used, mainly, for the necessity part, and without these (for $a(|\xi|) \in L_\infty$) we give sufficient conditions, which are essentially the same as those mentioned above (Theorem 4.3).

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2. Preliminaries. We begin with the definitions: $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$; $Ff(\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\,dx$ is the Fourier transform and $F^{-1}f(x) = (2\pi)^{-n}Ff(-x)$ is the inverse Fourier transform; throughout the paper we assume $b(|\xi|)$ to be a smooth function, described in the Introduction; $[a]$ denotes the integral part of a real number a; $L_p \equiv L_p(\mathbb{R}^n)$; $S \equiv S(\mathbb{R}^n)$ is the Schwartz class of rapidly decreasing smooth functions and S' is the space of tempered distributions; $C^\infty \equiv C^\infty(\mathbb{R}^n)$ is the class of infinitely differentiable functions; C^∞_0 is the subclass of C^∞ consisting of the functions with compact support; the space $\Phi \equiv \Phi(\mathbb{R}^n)$ is defined to consist of the Schwartz functions φ such that the Fourier transform $\widehat{\varphi}$ and all its derivatives vanish at the origin; the dual space $\Psi \equiv \Psi(\mathbb{R}^n) = F\Phi$ is equipped with the countable set of norms

$$\|\psi\|_N = \sup_{\substack{x\in\mathbb{R}^n\setminus 0\\ |k|\le N}} [\max\{\sqrt{1+|x|^2}, 1/|x|\}]^N |\mathcal{D}^k\psi(x)|$$

where

$$\mathcal{D}^k = \frac{\partial^{|k|}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}, \qquad k = (k_1, \ldots, k_n),\ |k| = k_1 + \ldots + k_n.$$

Thus the space Φ can be equipped with the dual topology. The spaces Φ, Ψ

were introduced by P. I. Lizorkin (see [4], [5]) and one may find information about them also in [10], [11]. The symbols Φ', Ψ' stand for the spaces of distributions on Φ, Ψ respectively. We recall that Φ is dense in L_p, $1 < p < \infty$ (see the above-mentioned references). As usual, L_p^q denotes the Banach space of S'-distributions T such that the closure in L_p of the convolution $T * U$, $U \in S$, is an L_p-L_q bounded translation invariant operator, and $M_p^q = FL_p^q$ stands for the space of L_p-L_q multipliers (see [2]).

By $\mathcal{J}_\nu(z)$ we denote the first kind Bessel function of order ν. From [13] (see also [17]) we have the following estimate:

$$|t^{-(a+ib)}\mathcal{J}_{a+ib}(t)| \le c_a e^{c|b|}(1+t)^{-a-1/2}, \qquad 0 < t < \infty. \tag{2.1}$$

Let

$$\Lambda = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}.$$

Suppose that to each $z \in \Lambda$ there is assigned a linear operator T_z on the space C_0^∞ into the measurable functions on $\mathbb{R}^n$ in such a way that $[T_z f(x)]g(x)$ is integrable on $\mathbb{R}^n$ whenever $f, g \in C_0^\infty$. The family $\{T_z\}$ is called *admissible* if the function

$$F(z) = \int_{\mathbb{R}^n} [T_z f(x)]g(x)\,dx$$

is analytic in the interior of Λ and continuous on Λ, and there exists a constant $\beta < \pi$ such that

$$\sup_{z\in\Lambda} e^{-\beta|\operatorname{Im} z|} \ln |F(z)| < \infty.$$

The above assumptions are just as the classical ones (see [14], [16]) with the only difference that we assume $f, g \in C_0^\infty$ instead of that f and g are both simple functions. This is required in (3.4) below. Nevertheless, it is easy to verify that the interpolation theorem for analytic families of operators holds under the above assumptions as well. Namely:

Theorem 2.1 ([14], [16]). *Suppose $\{T_z\}$, $z \in \Lambda$, is an admissible family of linear operators satisfying*

$$\|T_{m+iy} f\|_{q_m} \le M_m(y)\|f\|_{p_m}$$

for all $f \in C_0^\infty$, where $1 \le p_m, q_m \le \infty$ and $M_m(y)$, $m = 0, 1$, are independent of f and satisfy

$$\sup e^{-b|y|} \ln M_m(y) < \infty$$

for some $b < \pi$. Then for $0 \le t \le 1$ there exists a constant M_t such that

$$\|T_t f\|_{q_t} \le M_t\|f\|_{p_t}$$

for all $f \in C_0^\infty$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Our next goal is to give a boundedness result for some auxiliary operators, which we shall use below. Let B_1^α, B_2^α be the multiplier operators with symbols

$$b(|\xi|)|\xi|^{-\alpha}e^{i|\xi|}, \qquad (1-b(|\xi|))|\xi|^{-\alpha}e^{i|\xi|}$$

respectively. Here $b(|\xi|)$ is a smooth function, described above.

As was mentioned in the Introduction, the operator B_1^α (as well as more general ones) was deeply studied in [6] (see also [8]) for real α. A slight modification of a result from [6] (a particular case of Theorem 4.2 there) leads us to the following fact.

PROPOSITION 2.2. *Let* $0<\operatorname{Re}\alpha<n$. *The operator* B_1^α *is bounded*:

(1) *from* L_p *to* L_q, $1<p\le q<\infty$, *if and only if either*

$$\frac{1}{p}+\frac{1}{q}\le 1, \quad \frac{1}{p}-\frac{n}{q}\le \operatorname{Re}\alpha-\frac{n-1}{2}, \qquad \text{or}$$
$$\frac{1}{p}+\frac{1}{q}\ge 1, \quad \frac{n}{p}-\frac{1}{q}\le \operatorname{Re}\alpha+\frac{n-1}{2},$$

(2) *from* L_p *to* L_∞, $1<p<\infty$, *if and only if* $1/p<\operatorname{Re}\alpha-(n-1)/2$,
(3) *from* L_1 *to* L_q, $1<q<\infty$, *if and only if* $\operatorname{Re}\alpha-(n+1)/2>-1/q$.

Proof. The proof is straightforward: we only need to note that (see [6])

$$b(|\xi|)|\xi|^{-\alpha}e^{i|\xi|}\in M_p^p, \quad \frac{1}{2}-\frac{\operatorname{Re}\alpha}{n-1}\le\frac{1}{p}\le\frac{1}{2}+\frac{\operatorname{Re}\alpha}{n-1}, \qquad 1<p<\infty,$$

and next to take into account that M_p^p is a normed ring with the pointwise operations of multiplication and addition (see [2]). Thus, having established the equality

$$B_1^\alpha f = B_1^{\operatorname{Re}\alpha}M^{\operatorname{Im}\alpha}f, \qquad f\in L_p,\ 1<p<\infty,$$

with $M^{\operatorname{Im}\alpha}$ being the multiplier operator with symbol $|\xi|^{i\operatorname{Im}\alpha}$ (an isomorphism on L_p, $1<p<\infty$) we can now use the above-mentioned Theorem 4.2 from [6] to obtain (1) and (2). Next, (3) follows by duality arguments. ■

We now turn to the operator B_2^α.

PROPOSITION 2.3. *Let* $0<\operatorname{Re}\alpha<n$.

(1) *The operator* B_2^α *is bounded from* L_p *to* L_q *if*

$$1\le p<\frac{n}{\operatorname{Re}\alpha}, \qquad \frac{1}{q}\le\frac{1}{p}-\frac{\operatorname{Re}\alpha}{n}$$

except for the case $p=1$, $q=n/(n-\operatorname{Re}\alpha)$.

(2) *The operator* B_2^α *is not bounded from* L_p *to* L_q, $1 \le p \le q \le \infty$, *if* $1/q > 1/p - \operatorname{Re}\alpha/n$.

Proof. We first note that this proposition was proved in [3] for $(n-1)/2 < \alpha < n$, where the following representation was used (for part (2)):

$$B_2^\alpha f = AI^\alpha f, \quad f \in L_p,\ 1 < p < \frac{n}{\alpha},\ \frac{n-1}{2} < \alpha < n. \tag{2.2}$$

Here I^α is the Riesz potential and A is the multiplier operator with symbol $(1 - b(|\xi|))e^{i|\xi|}$.

The direct analysis of the result from [3] shows that this proposition is in fact valid for complex α in the corresponding strip, thus (2.2) holds for complex α at least in the strip $n/2 \le \operatorname{Re}\alpha < n$.

Let now $0 \le \operatorname{Re}\alpha < n/2$. It is sufficient to note that the representation (2.2) holds for $2 < p < n/\operatorname{Re}\alpha$. In fact, the symbol $|\xi|^{-\alpha}$ of the Riesz potential belongs to M_p^q if and only if $1/q = 1/p - \operatorname{Re}\alpha/n$, $1 < p < n/\operatorname{Re}\alpha$, and the symbol $(1 - b(|\xi|))e^{i|\xi|}$ of the operator A is in M_p^r for all $1 \le p \le r \le \infty$, since $(F^{-1}(1 - b(|\xi|))e^{i|\xi|})(|x|) \in L_1 \cap L_\infty$. It is well known (see [2]) that in this situation (with $r = p$) the translation invariant operator corresponding to the symbol $(1 - b(|\xi|))|\xi|^{-\alpha}e^{i|\xi|}$ is the composition of those corresponding to the symbols $(1 - b(|\xi|))e^{i|\xi|}$ and $|\xi|^{-\alpha}$.

Now, due to the above, from (2.2) we deduce that for $2 < p < n/\operatorname{Re}\alpha$ the operator B_2^α is bounded from L_p to L_q if and only if $1/q \le 1/p - \operatorname{Re}\alpha/n$.

To pass to the general situation, note that B_2^α is bounded from L_1 to L_∞ (since $(F^{-1}(1 - b(|\xi|))|\xi|^{-\alpha}e^{i|\xi|})(|x|) \in L_\infty$) and then apply convexity and duality arguments. ∎

3. Integral representation. For $f \in S$ and $(n+1)/2 < \operatorname{Re}\alpha < n$ we define

$$A_a^\alpha f(x) = \int_{\mathbb{R}^n} \Omega_{\alpha,a}(|y|) f(x-y)\, dy, \tag{3.3}$$

where $\Omega_{\alpha,a}(|x|)$ is the kernel (1.1).

For $0 \le \operatorname{Re}\alpha \le (n+1)/2$ the integral (3.3) no longer makes sense, as was mentioned above. But we can understand it for nonintegral α in the sense of distributions in the following way. Using the representation (3.3) for large $\operatorname{Re}\alpha$ ($(n+1)/2 < \operatorname{Re}\alpha < n$) we have

$$A_a^\alpha f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} |y|^{\alpha-n} f(x-y)\, dy \int_0^\infty t^{n/2-\alpha} a\left(\frac{t}{|y|}\right) e^{it/|y|} \mathcal{J}_{(n-2)/2}(t)\, dt.$$

As the above iterated integral is absolutely convergent for large $\operatorname{Re}\alpha$, interchanging the order of integration we obtain

$$A_a^\alpha f(x) = (2\pi)^{-n/2} \int_0^\infty t^{n/2-\alpha} \mathcal{J}_{(n-2)/2}(t)\, dt \int_{\mathbb{R}^n} |y|^{\alpha-n} a\left(\frac{t}{|y|}\right) e^{it/|y|} f(x-y)\, dy$$
$$= (2\pi)^{-n/2} \int_0^\infty t^{n/2} \mathcal{J}_{(n-2)/2}(t)\, dt \int_{\mathbb{R}^n} |y|^{\alpha-n} a\left(\frac{1}{|y|}\right) e^{i/|y|} f(x-ty)\, dy$$
$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} |y|^{\alpha-n} a\left(\frac{1}{|y|}\right) e^{i/|y|}\, dy \int_0^\infty t^{n/2} \mathcal{J}_{(n-2)/2}(t) f(x-ty)\, dt.$$

Applying the well known formula

$$\frac{d}{dt}[t^\nu \mathcal{J}_\nu(t)] = t^\nu \mathcal{J}_{\nu-1}(t),$$

integrating by parts and neglecting the boundary terms, we obtain

$$\int_0^\infty t^{n/2} \mathcal{J}_{(n-2)/2}(t) f(x-yt)\, dt$$
$$= (-1)^k \int_0^\infty t^{n/2+k} \mathcal{J}_{n/2+k-1}(t) \left(\frac{1}{t}\frac{d}{dt}\right)^k f(x-yt)\, dt.$$

Thus, for α in the range $\max\{0, (n+1)/2-k\} < \operatorname{Re}\alpha \le (n+1)/2-k+1$, we define, for $f \in S$,

$$A_a^\alpha f(x) = \int_{\mathbb{R}^n} \Omega_{\alpha,a}^k(|y|) \left(\frac{1}{s}\frac{d}{ds}\right)^k f(x-sy)\bigg|_{s=1} dy, \tag{3.4}$$

where $s \in (0,\infty)$ and

$$\Omega_{\alpha,a}^k(|x|) = (-1)^k (2\pi)^{-n/2} |x|^{\alpha-n} \int_0^\infty t^{n/2-\alpha-k} a\left(\frac{t}{|x|}\right) e^{it/|x|} \mathcal{J}_{n/2+k-1}(t)\, dt. \tag{3.5}$$

For $\operatorname{Re}\alpha = 0$ we set

$$A_a^\alpha f(x) = \int_{\mathbb{R}^n} \Omega_{\alpha,a}^{[(n+1)/2]+1}(|y|) \left(\frac{1}{s}\frac{d}{ds}\right)^{[(n+1)/2]+1} f(x-sy)\bigg|_{s=1} dy. \tag{3.6}$$

It is useful to note that for all $0 \le \operatorname{Re}\alpha < n$ we can use (3.6) as it coincides with the above definitions in the corresponding strips.

THEOREM 3.1. *Let* $0 \le \operatorname{Re}\alpha < n$ *and* $a(|x|) \in L_\infty$.

(1) *The "kernel"* $\Omega_{\alpha,a}^k(|x|)$ *is continuous on* $\mathbb{R}^n \setminus \{0\}$ *and satisfies the estimate*

$$|\Omega_{\alpha,a}^k(|x|)| \le c|x|^{\operatorname{Re}\alpha-n}. \tag{3.7}$$

(2) *The integral* $A_a^\alpha f(x)$, $f \in S$, $0 \le \operatorname{Re}\alpha < n$, *is absolutely convergent and a* C^∞ *function bounded on* $\mathbb{R}^n$.

(3) *The function*

$$\alpha \mapsto A_a^\alpha f(x), \qquad f \in S,$$

is analytic in the strip $0 < \operatorname{Re}\alpha < n$ *and continuous on* $0 \le \operatorname{Re}\alpha < n$.

Proof. The first assertion of the theorem follows directly from the estimate (2.1). The second one is obvious.

To prove the last assertion we note that the right side of (3.6) in fact represents a function analytic at least in the strip $-1/2 < \operatorname{Re}\alpha < n$. This follows from the fact that the formally differentiated (in α) integral $A_a^\alpha f(x)$ still converges absolutely uniformly in α on some neighborhood of each point of the strip, which can be seen by applying (2.1) as above. ∎

Now we are about to prove the Fourier transform formula for the operator A_a^α.

THEOREM 3.2. *Let* $0 \le \operatorname{Re}\alpha < n$, $a(|x|) \in L_\infty$, $f \in S$. *Then*

$$FA_a^\alpha f(\xi) \overset{(S')}{=} m_{\alpha,a}(|\xi|)Ff(\xi). \tag{3.8}$$

Moreover, *if* $a(|x|) \in C_{\mathrm{b}}^\infty(\mathbb{R}^n \setminus 0)$ *then for* $f \in \Phi$ *we can understand* (3.8) *in the regular sense*, *i.e.*

$$FA_a^\alpha f(\xi) = m_{\alpha,a}(|\xi|)Ff(\xi), \qquad \xi \in \mathbb{R}^n. \tag{3.9}$$

Proof. Let first $(n+1)/2 < \operatorname{Re}\alpha < n$. Then the inverse Fourier transform

$$\Omega_{\alpha,a}(|x|) = (F^{-1}m_{\alpha,a}(|\xi|))(|x|) \tag{3.10}$$

exists in the regular sense (as a conditionally convergent integral). In fact, applying the Bochner formula (see [12], p. 485) we have

$$\begin{aligned}
&(F^{-1}m_{\alpha,a}(|\xi|))(|x|)\\
&\quad = (2\pi)^n \lim_{N\to\infty} \int_{|\xi|<N} m_{\alpha,a}(|\xi|)e^{-ix\cdot\xi}\,d\xi\\
&\quad = (2\pi)^{-n/2}|x|^{\alpha-n} \lim_{N\to\infty} \int_0^{N|x|} t^{n/2-\alpha}e^{it/|x|}a\left(\frac{t}{|x|}\right)\mathcal{J}_{(n-2)/2}(t)\,dt\\
&\quad = (2\pi)^{-n/2}|x|^{\alpha-n} \int_0^\infty t^{n/2-\alpha}e^{it/|x|}a\left(\frac{t}{|x|}\right)\mathcal{J}_{(n-2)/2}(t)\,dt,
\end{aligned}$$

which agrees with (3.10). Thus, we can write

$$\int_{\mathbb{R}^n} \Omega_{\alpha,a}(|y|)f(y)\,dy = \langle \Omega_{\alpha,a}(|y|), \bar{f}(y)\rangle = (2\pi)^{-n}\int_{\mathbb{R}^n} m_{\alpha,a}(|\xi|)\widehat{f}(-\xi)\,d\xi, \qquad f\in S.$$

Since S is a space with continuous shift, according to the previous equality, we have

$$\int_{\mathbb{R}^n} \Omega_{\alpha,a}(|y|)f(x-y)\,dy = (2\pi)^{-n}\int_{\mathbb{R}^n} m_{\alpha,a}(|\xi|)\widehat{f}(\xi)e^{-ix\cdot\xi}\,d\xi. \tag{3.11}$$

Now, to obtain (3.8) for $(n+1)/2 < \operatorname{Re}\alpha < n$ it is sufficient to multiply both sides of (3.11) by $\overline{g}(x)$, integrate on $\mathbb{R}^n$ with respect to x and interchange the order of integration on the right side.

Therefore, for $f,g\in S$, in the strip $(n+1)/2 < \operatorname{Re}\alpha < n$ we have

$$\int_{\mathbb{R}^n} A_a^\alpha f(x)\overline{g}(x)\,dx = (2\pi)^{-n}\int_{\mathbb{R}^n} m_{\alpha,a}(|\xi|)\widehat{f}(\xi)\overline{\widehat{g}}(\xi)\,d\xi. \tag{3.12}$$

But in fact, (3.12) holds for all α with $0\le \operatorname{Re}\alpha < n$, since both the left- and right-hand sides of (3.12) are functions of α analytic for $0<\operatorname{Re}\alpha<n$ and continuous on $0\le\operatorname{Re}\alpha<n$. This can be seen for the right side (as in Theorem 3.1) from the fact that the formal derivative (with respect to α) is also an absolutely convergent integral (uniformly in α on some neighborhood of each α with $\operatorname{Re}\alpha<n$). For the left side this follows from Theorem 3.1. Hence, the equality (3.8) is proved.

To prove (3.9) we note that for a bounded together with all its derivatives, $a(|x|)\in C_{\mathrm{b}}^\infty(\mathbb{R}^n\setminus 0)$, the symbol $m_{\alpha,a}(|\xi|)$ is a Ψ-multiplier. Thus, for $(n+1)/2<\operatorname{Re}\alpha<n$ the equality (3.9) follows from (3.11) and then can be extended to $0\le\operatorname{Re}\alpha<n$ by analyticity as above. ■

Remark 3.3. Below, the particular case of the operators A_a^α when $a(|x|)=1$ will play an essential role. In this case we have the representation

$$A_1^\alpha f(x)\equiv A^\alpha f(x) = \int_{\mathbb{R}^n}\Omega_\alpha(|y|)f(x-y)\,dy, \qquad \frac{n-1}{2}<\operatorname{Re}\alpha<n, \tag{3.13}$$

where

$$\Omega_\alpha(|x|) = (-i)^{\alpha-n}\frac{\Gamma(n-\alpha)}{2^{n-1}\pi^{n/2}\Gamma(n/2)}F\left(\frac{n-\alpha}{2},\frac{n-\alpha+1}{2};\frac{n}{2};|x|^2\right) \tag{3.14}$$

for $|x|<1$, and

$$(3.15)\qquad \Omega_\alpha(|x|) = \frac{\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)}|x|^{\alpha-n}F\left(\frac{n-\alpha}{2}, \frac{2-\alpha}{2}; \frac{1}{2}; \frac{1}{|x|^2}\right) + i\frac{\Gamma((n-\alpha+1)/2)}{2^{\alpha-1}\pi^{n/2}\Gamma((\alpha-1)/2)}|x|^{\alpha-n-1}F\left(\frac{n-\alpha+1}{2}, \frac{3-\alpha}{2}; \frac{3}{2}; \frac{1}{|x|^2}\right)$$

for $|x| > 1$. Here $F(a, b; c; d)$ is the Gauss hypergeometric function.

The above representation as well as the Fourier transform formula were obtained in [9] (see also [3]). In fact, one can check at once that the corresponding Fourier integral (3.10) is still conditionally convergent for $(n-1)/2 < \operatorname{Re}\alpha < n$ and then apply the formulae 6.699.1, 6.699.2 from [1].

It is important that in this case we have the exact behavior of the kernel $\Omega_\alpha(|x|)$. Using (3.14), (3.15) one can easily show that $\Omega_\alpha(|x|)$ is continuous on $\mathbb{R}^n \setminus \{x \in \mathbb{R}^n : |x| = 1\}$ and has the following asymptotics:

$$(3.16)\qquad \Omega_\alpha(|x|) \sim c|x|^{\alpha-n}$$

as $|x| \to \infty$, and

$$(3.17)\qquad \Omega_\alpha(|x|) \sim c(1-|x|)^{\operatorname{Re}\alpha-(n+1)/2}, \qquad \frac{n-1}{2} < \operatorname{Re}\alpha < \frac{n+1}{2},$$

$$(3.18)\qquad \Omega_\alpha(|x|) \sim c\ln|1-|x||, \qquad \alpha = \frac{n+1}{2},$$

as $|x| \to 1$. For $(n+1)/2 < \operatorname{Re}\alpha < n$ or $\operatorname{Re}\alpha = (n+1)/2$, $\operatorname{Im}\alpha \neq 0$ the kernel $\Omega_\alpha(|x|)$ is bounded (see [9] for details).

4. The main result. Now we are in a position to prove the main result of the present paper.

THEOREM 4.1. *Let* $0 \le \operatorname{Re}\alpha < n$, *and* $a(|x|)$ *and* $1/a(|x|)$ *both be in* M_p^p, $1 < p < \infty$. *In the case* $\operatorname{Re}\alpha \ge (n+1)/2$, $p = 1$, $q = \infty$ *we additionally assume* $a(|x|), 1/a(|x|) \in M_1^1$. *The operator* A_a^α *is bounded*:

(1) *from* L_p *to* L_q, $1 < p \le q < \infty$, *if and only if*

$$\frac{1}{q} \le \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n}$$

and either

$$\frac{1}{p} + \frac{1}{q} \le 1, \quad \frac{1}{p} - \frac{n}{q} \le \operatorname{Re}\alpha - \frac{n-1}{2}, \qquad \textit{or}$$
$$\frac{1}{p} + \frac{1}{q} \ge 1, \quad \frac{n}{p} - \frac{1}{q} \le \operatorname{Re}\alpha + \frac{n-1}{2},$$

(2) *from* L_1 *to* L_q, $1 \le q < \infty$, *if and only if*

$$\frac{n+1}{2} - \operatorname{Re}\alpha < \frac{1}{q} < \frac{n-\operatorname{Re}\alpha}{n},$$

(3) *from* L_1 *to* L_∞ *if and only if* $\operatorname{Re}\alpha > (n+1)/2$ *or* $\operatorname{Re}\alpha = (n+1)/2$, $\operatorname{Im}\alpha \neq 0$,

(4) *from* L_p *to* L_∞, $1 < p < \infty$ *if and only if*

$$\frac{\operatorname{Re}\alpha}{n} < \frac{1}{p} < \operatorname{Re}\alpha - \frac{n-1}{2}.$$

Proof. I. We first prove a reduced form of the theorem, i.e. we consider the case of the operator A^α with the restrictions:

$$\frac{1}{2} \le \frac{1}{p} \le \frac{1}{2} + \frac{\operatorname{Re}\alpha}{n}, \qquad 0 < \operatorname{Re}\alpha < \frac{n}{2}, \tag{4.19}$$

$$\frac{\operatorname{Re}\alpha}{n} < \frac{1}{p} < 1, \qquad \frac{n}{2} \le \operatorname{Re}\alpha < n. \tag{4.20}$$

Since L_p^q is a Banach space, for p satisfying (4.19), (4.20) we have the representation

$$A^\alpha f = B_1^\alpha f + B_2^\alpha f, \qquad f \in L_p,\ 0 < \operatorname{Re}\alpha < n$$

(where B_1^α, B_2^α are the operators introduced above). Now the reduced form of the theorem immediately follows from Propositions 2.2 and 2.3.

II. We now turn to the general case. We first note that for p, α satisfying (4.19), (4.20) we have the representation

$$A_a^\alpha f = A^\alpha M_a f, \tag{4.21}$$

where M_a is the multiplier operator with symbol $a(|\xi|)$ (thus, it is an isomorphism on L_p, $1 < p < \infty$).

In fact, the representation (4.21) in the S'-distribution sense can be easily verified for $f \in S$, $0 \le \operatorname{Re}\alpha < n$ passing to the Fourier images according to Theorem 3.2. Next, this implies that (4.21) is valid for a.e. $x \in \mathbb{R}^n$, $f \in S$, since $A_a^\alpha f$, $A^\alpha M_a f$ are regular functionals from S' (see Theorem 3.1) which coincide in the distribution sense. Then it can be extended to L_p, since both the left- and right-hand sides of (4.21) are bounded operators from L_p to L_q, $1/q = 1/p - \operatorname{Re}\alpha/n$ with p, α from (4.19), (4.20) (and S is dense in L_p, $1 \le p < \infty$).

In fact, for the boundedness of the right side of (2.1), see I above. For the left side we refer to the Hardy–Littlewood theorem (see [2]), which states that a multiplier $m(\xi)$ is in M_p^q if

$$|m(\xi)| \le c|\xi|^b, \qquad 1 < p \le 2 \le q < \infty,\ \frac{1}{p} - \frac{1}{q} = -\frac{b}{n},\ -n < b < 0.$$

III. For p, α satisfying (4.19), (4.20), due to (4.21) the boundedness results for the operator A^α obviously imply those for A_a^α, as well as the "only if" part.

In the case $p = n/\operatorname{Re}\alpha$, $n/2 \le \operatorname{Re}\alpha < n$ we still obviously have

$$\|A_a^\alpha f\|_\infty = \|A^\alpha M_a f\|_\infty, \qquad f \in S. \tag{4.22}$$

The operator A^α is not bounded from $L_{n/\mathrm{Re}\,\alpha}$ to L_∞, since otherwise, from the converse of Hölder's inequality we would have $\Omega_\alpha \in L_{n/(n-\mathrm{Re}\,\alpha)}$, which is impossible in view of the asymptotics (3.16). Due to (4.22) and the conditions $a(|x|), 1/a(|x|) \in M_p^p$, $1 < p < \infty$, the same conclusion is true for the operator A_a^α as well.

IV. We now turn to the case

$$(4.23)\quad p=1,\quad \max\left\{0, \frac{n+1}{2} - \mathrm{Re}\,\alpha\right\} \le \frac{1}{q} \le 1 - \frac{\mathrm{Re}\,\alpha}{n},\quad \frac{n+1}{2} \le \mathrm{Re}\,\alpha < n.$$

In this situation we cannot obtain the information directly from (4.21), but for $q < \infty$ we can use the previous result, applying duality arguments. Obviously, A_a^α is bounded from L_1 to L_q if $0 < 1/q < 1 - \mathrm{Re}\,\alpha/n$ but cannot be bounded from L_1 to L_q with $q = n/(n - \mathrm{Re}\,\alpha)$.

Let us consider the excluded case $q = \infty$. For $\mathrm{Re}\,\alpha > (n+1)/2$ the operator A_a^α is bounded from L_1 to L_∞, since $\Omega_\alpha(|x|) \in L_\infty$ (see Remark 3.3).

What is left is the case $\mathrm{Re}\,\alpha = n + 1/2$. The operator A^α with $\mathrm{Re}\,\alpha = (n+1)/2$ is bounded from L_1 to L_∞ if and only if the kernel $\Omega_{(n+1)/2}(|x|)$ is in L_∞, which is the case if and only if $\mathrm{Im}\,\alpha \ne 0$ (see asymptotics (3.18)). Now, due to the additional conditions $a(|x|), 1/a(|x|) \in M_1^1$ from (4.22) we see that the operator A_a^α with $\mathrm{Re}\,\alpha = (n+1)/2$ is bounded from L_1 to L_∞ if and only if $\mathrm{Im}\,\alpha \ne 0$.

V. Summarizing the results obtained in II–IV we can draw the following conclusion. We have actually proved the "if" part of the theorem, except the case $\mathrm{Re}\,\alpha = 0$, but this case is obvious, since the symbol $m_{\alpha,a}(|x|) \in L_\infty$ for imaginary α. Thus, the sufficiency part of the theorem is proved.

We have also proved the "only if" part with the restrictions (4.19), (4.20) and also in the following particular cases:

(i) $p = n/\mathrm{Re}\,\alpha$, $q = \infty$,

(ii) $p = 1$, $q = n/(n - \mathrm{Re}\,\alpha)$,

provided $n/2 \le \mathrm{Re}\,\alpha < n$, and

(iii) $p = 1$, $q = \infty$, $\mathrm{Re}\,\alpha = (n+1)/2$.

Considering the case where p, q, α do not satisfy the conditions (4.19), (4.20), (i)–(iii), and $\mathrm{Re}\,\alpha \ne 0$ it is sufficient to make the following observations (obviously, we are now considering only those parameters which do not satisfy the sufficient conditions of the theorem). Each $(1/p, 1/q)$, $p \le q$, under consideration can be connected with a "point of boundedness" by an interval passing through a "point of unboundedness" . If A_a^α were bounded from L_p to L_q this would contradict the convexity of the set of "points of boundedness".

VI. The only point remaining concerns the “only if” part of the theorem in the case $\operatorname{Re}\alpha = 0$. We have to show that the operator A_a^α cannot be bounded from L_p to L_q, except in the case $p = q = 2$.

Suppose, contrary to our claim, that A_a^α is bounded from L_{p_0} to L_{q_0}, $p_0 \le q_0$ and $(1/p_0, 1/q_0)$ is not $(1/2, 1/2)$.

Let first $p_0 \ne q_0$. By symmetry we may actually assume that

$$\frac{1}{p_0} = \frac{1}{2} + \delta, \qquad \frac{1}{q_0} = \frac{1}{2} - \delta$$

with some fixed $\delta \in (0, (2n+1)/(4n))$. Next, consider the operators

$$B_a^t \equiv A_a^{\operatorname{Re}\alpha}, \qquad t = \frac{2n+1}{2n(n+1)\delta} \operatorname{Re}\alpha$$

(their symbols have the form $a(|\xi|)|\xi|^{-\operatorname{Re}\alpha} e^{i|\xi|}$). We claim that the family of operators $\{B_a^z\}$, $\operatorname{Re} z \in [0,1]$, is admissible (in the sense of the definition given before Theorem 2.1; here B^z is formally obtained from B^t by substituting z for t). Indeed, this follows from (3.12) and the arguments given below that formula. On account of the above remarks, we see that the operator $A_a^{\operatorname{Re}\alpha}$ $(= B_a^t)$ is bounded from L_{p_0} to L_{q_0} if $\operatorname{Re}\alpha = 0$ $(t = 0)$, and from L_{p_1} to L_{q_1} if $\operatorname{Re}\alpha = \frac{2n(n+1)}{2n+1}\delta$ $(t = 1)$, provided

$$\frac{1}{p_1} = \frac{1}{2} + \frac{2n\delta}{2n+1}, \qquad \frac{1}{q_1} = \frac{1}{2} - \frac{2n\delta}{2n+1}.$$

Now, interpolating between the points $(1/p_0, 1/q_0)$, $(1/p_1, 1/q_1)$ with respect to the parameter t, from Theorem 2.1 we deduce that $A_a^{\operatorname{Re}\alpha}$ is bounded from L_{p_t} to L_{q_t}, where

$$(4.24) \qquad \frac{1}{p_t} = \frac{1}{2} + \delta - \eta(\operatorname{Re}\alpha), \quad \frac{1}{q_t} = \frac{1}{2} - \delta + \eta(\operatorname{Re}\alpha), \quad \eta(\operatorname{Re}\alpha) = \frac{\operatorname{Re}\alpha}{2n(n+1)}.$$

Our next claim is that we have the representation

$$(4.25) \qquad A_a^\alpha f = A_a^{\operatorname{Re}\alpha} M^{\operatorname{Im}\alpha} f, \qquad f \in L_{p_t}, \ \frac{2n+1}{2n+3} < t \le 1,$$

with $M^{\operatorname{Im}\alpha}$ defined in the proof of Proposition 2.2. This can be proved similarly to (4.21), noticing that the operator A_a^α is bounded from L_{p_t} to L_q with $1/q = 1/p_t - \operatorname{Re}\alpha/n$, $(2n+1)/(2n+3) < t \le 1$. Finally, since $M^{\operatorname{Im}\alpha}$ is an isomorphism in L_p, $1 < p < \infty$, the relation (4.25) shows that A_a^α is bounded from L_{p_t} to L_{q_t} with $1/p_t$, $1/q_t$ from (4.24), which contradicts the results for $0 < \operatorname{Re}\alpha < n$, obtained above in II–V.

The case $p_0 = q_0$, $p_0 \ne 2$ can be considered in the same way. Here we may assume $p_0 < 2$, sufficiently close to 2, and for the second point we can take, for example, $(3/4, 1/2)$, which corresponds to $\operatorname{Re}\alpha = n/4$. The proof of the theorem is complete. ∎

REMARK 4.2. The additional conditions $a(|x|), 1/a(|x|) \in M_1^1$, assumed above in the case $\operatorname{Re}\alpha = (n+1)/2$, $p = 1$, $q = \infty$, are essential. For example, take

$$a(|x|) = |x|^{-i\operatorname{Im}\alpha}, \qquad \operatorname{Im}\alpha \neq 0.$$

Then, as is known, $a(|x|)$ belongs to M_p^p, $1 < p < \infty$, and is not in M_1^1. Evidently, $A_a^\alpha = A^{\operatorname{Re}\alpha}$ and by Remark 3.3 and the asymptotics (3.18) it is not bounded from L_1 to L_∞, but the operator A^α is.

In fact, under the assumption that $a(|x|) \in L_\infty$ we still essentially have the sufficiency part of Theorem 4.1. Namely:

THEOREM 4.3. *Let* $0 \le \operatorname{Re}\alpha < n$, $a(|x|) \in L_\infty$ *and for* $\operatorname{Re}\alpha > (n+1)/2$ *let* $\Omega_{\alpha,a}(|x|) \in L_\infty$. *The operator* A_a^α *is bounded from* L_p *to* L_q *if*:

(1) $(n+1)/2 < \operatorname{Re}\alpha < n$ *and* $1 \le p \le q \le \infty$, $1/q \le 1/p - \operatorname{Re}\alpha/n$,

(2) $0 \le \operatorname{Re}\alpha \le (n+1)/2$ *and* $1 < p \le q < \infty$, $1/q \le 1/p - \operatorname{Re}\alpha/n$ *and either* $[1/p + 1/q \le 1$ *and* $1/p - n/q < \operatorname{Re}\alpha - (n-1)/2]$, *or* $[1/p + 1/q \ge 1$ *and* $n/p - 1/q < \operatorname{Re}\alpha + (n-1)/2]$,

(3) $0 \le \operatorname{Re}\alpha < n/2$ *and either* $[p = 2$ *and* $q = 2n/(n - \operatorname{Re}\alpha)]$ *or* $[p = 2n/(n + \operatorname{Re}\alpha)$ *and* $q = 2]$.

Proof. Keep in mind that A_a^α, $0 \le \operatorname{Re}\alpha < n$, is bounded from L_p to L_q if

$$\frac{1}{q} = \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n}, \qquad 1 < p \le 2,\ 2 \le q < \infty. \tag{4.26}$$

In fact, this follows from the Hardy–Littlewood theorem mentioned above in II.

From (3.7) (with $k = 0$) we see that for $(n+1)/2 < \operatorname{Re}\alpha < n$ the operator A_a^α is bounded from L_1 to L_∞ and also (see [15], Comment on p. 121) is of weak type $(1, n/(n - \operatorname{Re}\alpha))$. Now, applying the Marcinkiewicz interpolation theorem ([16], [15]) we obtain (1).

We know that for $\operatorname{Re}\alpha = (n+1+\varepsilon)/2$ (with $\varepsilon > 0$ sufficiently small) the operator A_a^α is bounded from L_1 to L_∞, and for $\operatorname{Re}\alpha = 0$ it is bounded on L_2. Thus, applying Theorem 2.1 as in the proof of Theorem 4.1 (see VI) we conclude that A_a^α is bounded from L_{p_ε} to L_{q_ε}, where

$$\frac{1}{p_\varepsilon} = \frac{1}{2} + \frac{\operatorname{Re}\alpha}{n+1+\varepsilon}, \qquad \frac{1}{q_\varepsilon} = \frac{1}{2} - \frac{\operatorname{Re}\alpha}{n+1+\varepsilon}.$$

In the same way, noticing that for $\operatorname{Re}\alpha = (n-\delta)/2$ the operator A_a^α is bounded from L_p, $p = n/(n-\delta/2)$, to L_2 and, as already mentioned, for $\operatorname{Re}\alpha = (n+1+\delta)/2$, it is bounded from L_1 to L_∞, we obtain the boundedness of A_a^α, $(n-\delta)/2 \le \operatorname{Re}\alpha \le (n+1+\delta)/2$, from L_{p_δ} to L_{q_δ}, where as above $\delta > 0$ is sufficiently small and

$$\frac{1}{p_\delta} = 1 - \frac{\delta}{n}\left(\frac{1}{2} - \frac{1}{1+2\delta}\left(\operatorname{Re}\alpha - \frac{n-\delta}{2}\right)\right), \qquad \frac{1}{q_\delta} = \frac{1}{2} - \frac{1}{1+2\delta}\left(\operatorname{Re}\alpha - \frac{n-\delta}{2}\right).$$

We note that the parameters ε, δ may be chosen arbitrarily small. Now applying the Riesz–Thorin interpolation theorem ([16], [15]) we obtain (2). The statement (3) is a consequence of (4.26). ■

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Mathematics Department
Center for Research and Advanced Study
A.P. 14-740, 07000 México, D.F., México
E-mail: alexey@math.cinvestav.mx

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