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CM-Selectors for pairs of oppositely semicontinuous multivalued maps with $\mathbb{L}_p$-decomposable values

by

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Abstract. We present a new continuous selection theorem, which unifies in some sense two well known selection theorems; namely we prove that if F is an H-upper semicontinuous multivalued map on a separable metric space X, G is a lower semicontinuous multivalued map on X, both F and G take nonconvex $L_p(T,E)$-decomposable closed values, the measure space T with a σ-finite measure μ is nonatomic, $1 \le p < \infty$, $L_p(T,E)$ is the Bochner–Lebesgue space of functions defined on T with values in a Banach space E, $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$, then there exists a CM-selector for the pair (F,G), i.e. a continuous selector for G (as in the theorem of H. Antosiewicz and A. Cellina (1975), A. Bressan (1980), S. Łojasiewicz, Jr. (1982), generalized by A. Fryszkowski (1983), A. Bressan and G. Colombo (1988)) which is simultaneously an ε-approximate continuous selector for F (as in the theorem of A. Cellina, G. Colombo and A. Fonda (1986), A. Bressan and G. Colombo (1988)).

Introduction. The purpose of this paper is to present a new joint continuous selection theorem, which synthesizes, in a sense, two well known selection theorems. This theorem (Theorem 2.2 of Section 2) states that if F is an H-upper semicontinuous multifunction on a separable metric space X, G is a lower semicontinuous multifunction on X, $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$, G has closed values, both F and G take (possibly nonconvex) $L_p(T,E)$-decomposable values, where the measure space T with a σ-finite measure μ is nonatomic, $1 \le p < \infty$, $L_p(T,E)$ is the Bochner–Lebesgue space of functions defined on T with values in a Banach space E, then there exists a CM-selector for the pair (F,G), i.e. there exists a continuous selector for G (as in the theorem due to H. A. Antosiewicz and A. Cellina

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[AntC, 1975], A. Bressan [Br, 1980] and S. Łojasiewicz, Jr. [Łoj, 1982], generalized by A. Fryszkowski [Fry, 1983], A. Bressan and G. Colombo [BrC, 1988]), which is simultaneously an ε-approximate continuous selector for F (as in the theorem of A. Cellina, G. Colombo and A. Fonda [CelCF, 1986], A. Bressan and G. Colombo [BrC, 1988]). Here, the notion of $L_p(T, E)$-decomposability of a set is understood in the well known sense of F. Hiai and H. Umegaki [HiU, 1977].

In our previous papers [NgJZ1-2], where we introduced the notion of CM-selectors, we proved a continuous selection theorem, unifying in an analogous sense two famous continuous selection theorems due to E. A. Michael [Mi] (1956) and to A. Cellina [Cel] (1969); we proved namely that if F is an H-upper semicontinuous multifunction on a metric space X, G is a lower semicontinuous multifunction on X, G takes closed values, both F and G take convex values in a Banach or complete metric locally convex space Y, $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$, then the pair (F, G) admits a CM-selector, i.e. G has a continuous selector (as in Michael's theorem) which is also an ε-approximate continuous selector for F (as in Cellina's theorem). In the case $G(x) \equiv Y$ the selection theorem of [NgJZ1-2] reduces to the Cellina approximate continuous selection theorem for F. In the case $F(x) \equiv Y$ it reduces to the Michael continuous selection theorem for G. In the general case our theorem can be interpreted as an "intermediate" theorem between the Michael and Cellina theorems.

It is well known that in theorems of Cellina's and Michael's type decomposability is a good substitute for convexity. Following this idea, we prove in the present paper a version of the selection theorem [NgJZ1-2], replacing the *convexity* condition for the values of F, G with the $L_p(T, E)$-*decomposability* condition, where the measure space T is nonatomic and σ-finite. All the proofs of the results of Section 2 are collected in Section 3. The core of the rather long proof of the main selection Theorem 2.2 is Theorem 2.1, in the proof of which we apply Lemma 2.1. Although the schemes and methods of our proofs are known (see the above cited papers), they are applied in a more complicated situation. As a by-product of the proof, we re-establish (see Corollaries 2.1 and 2.2) the fact that the above mentioned theorems: the theorem on continuous selectors for a lower semicontinuous multifunction G with $\mathbb{L}_1$-decomposable values due to Fryszkowski and Bressan–Colombo, and the theorem on ε-approximate continuous selectors for an H-upper semicontinuous multifunction F with $\mathbb{L}_1$-decomposable values due to Cellina–Colombo–Fonda and Bressan–Colombo, are also true in the case when the measure space is σ-finite, i.e. under weaker assumptions than in their original formulation (finite measure space).

The notion of CM-selectors and the problem of their existence find motivation in our recent research [Ng2-3], [NgJZ1-2] on the existence of solutions of "strongly nonlinear" Hammerstein multivalued equations (inclusions "with lack of compactness") and of elliptic boundary value problems with "strongly nonlinear" multivalued right-hand sides F which satisfy some *one-sided estimates* (e.g. the sign condition, generalized sign condition, Hammerstein one-sided estimate, etc.). We observed that each one-sided estimate in the multivalued case generates some pair (F, G), where the multifunction G is lower semicontinuous (cf. Theorem 2.2 and Theorem 3.1 of [NgJZ1]). The $\mathbb{L}_p$-decomposable-valued version of the above result appears here as Theorem 2.3, and is a direct consequence of the general selection Theorem 2.2.

Using Theorem 2.3, we have obtained in [NgJZ2] an existence theorem for both nonlinear (operator and integral) Hammerstein inclusions and elliptic differential inclusions with H-upper semicontinuous nonlinearities. There, we also apply the above theorem to get new existence results for periodic oscillations in a nonlinear control system with "undetermined noise", for Clarke's critical points of nonsmooth energy functionals as well as for solutions of some discontinuous elliptic problems (see references in [Cla], [Dei], [RW], [ADNZ], [Ng1]).

We have also obtained the existence of so-called "combinative CM-selectors" for several multivalued maps with convex closed values (see [NgJZ2]) or $\mathbb{L}_p$-decomposable values (see [NgJZ3]); the result unifies similarly two well known selection theorems: the above theorem on ε-approximate continuous selectors due to Cellina–Colombo–Fonda and Bressan–Colombo, and the existence theorem on continuous selectors proved by V. V. Goncharov and A. A. Tolstonogov in 1991, 1994 (see references in [CelCF], [BrC], [TolT], [Dei], [Ng1]). We have also proven the existence of so-called "extremal CMT-selectors", by means of which two well known selection theorems can be subsumed within one statement: the above theorem due to Cellina–Colombo–Fonda and Bressan–Colombo, and the existence theorem on extremal selectors obtained by A. A. Tolstonogov in 1995, 1996, and A. A. Tolstonogov and D. A. Tolstonogov in 1996 (see references in [CelCF], [BrC], [TolT], [Ng1]).

Other applications of the existence results for CM-selectors of the present paper and of our previous ones [NgJZ1-2] are presented in [Ng2], [NgJZ3]. We hope that all these results will have applications in other problems of set-valued nonlinear analysis.

1. Some terminology and notation. We give some terminology and notation following [AubC], [AubF], [Dei], [Kur], [CasV], [RW], [ADNZ], [Ng1].

Let (X, ϱ) be a metric space. For $x \in X$, $M \subset X$ and $\varepsilon > 0$ we denote by $d(x, M) = \inf\{\varrho(x, y) : y \in M\}$ the distance from x to M and by

$U_\varepsilon(M) = \{y \in X : d(y, M) < \varepsilon\}$ the ε-neighbourhood of M. $B_X(x, r)$ (or simply $B(x, r)$) is the open ball with centre x and radius r. The distance in the product $X \times Y$ of two metric spaces (X, ϱ_X) and (Y, ϱ_Y) is defined as follows: $d((x, y), (x_1, y_1)) = \max\{\varrho_X(x, x_1), \varrho_Y(y, y_1)\}$.

We assume that each multifunction considered has nonempty values, unless stated to the contrary. The graph of a multifunction $F : X \to 2^Y$ is the set $\operatorname{Gr} F = \{(x, y) \in X \times Y : y \in F(x)\}$. By $F(M)$ we denote the image of the set $M \subset X$ under the multifunction F; recall that $F(M) = \bigcup_{x \in M} F(x)$.

Let X, Y be metric spaces and let $F : X \to 2^Y$ be a multifunction. F is called *upper semicontinuous* (or *usc* or *Vietoris usc* or *V-usc*) *at* $x_0 \in X$ if for any open set $V \subset Y$ such that $F(x_0) \subset V$, one can find an open neighbourhood $U \subset X$ of x_0 such that $F(x) \subset V$ for all $x \in U$. F is called *lower semicontinuous* (or *lsc* or *Vietoris lsc* or *V-lsc*) *at* $x_0 \in X$ if for any open set $V \subset Y$ such that $F(x_0) \cap V \neq \emptyset$, there exists an open neighbourhood $U \subset X$ of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$. A multifunction F is upper semicontinuous (or usc) [resp. lower semicontinuous or lsc] if it is usc [resp. lsc] at every $x \in X$. We say that F is *Hausdorff upper semicontinuous* (or *H-usc* or *ε-δ-usc*) *at* $x_0 \in X$ if for any $\varepsilon > 0$ one can find a $\delta > 0$ such that $F(B(x_0, \delta)) \subset U_\varepsilon(F(x_0))$. F is said to be *Hausdorff lower semicontinuous* (or *H-lsc* or *ε-δ-lsc*) *at* $x_0 \in X$ if for any $\varepsilon > 0$ one can find a $\delta > 0$ such that $F(x_0) \subset U_\varepsilon(F(x))$ for all $x \in B(x_0, \delta)$. A multifunction F is Hausdorff upper semicontinuous (or H-usc) [resp. H-lower semicontinuous or H-lsc] if it is H-usc [resp. H-lsc] at every $x \in X$.

Recall that if F is usc, then F is H-usc; the converse is true if F takes compact values. If F is H-lsc, then F is lsc; the converse is true if F takes compact values. Recall that F is lsc at $x \in X$ iff for every closed subset C of Y, and for every sequence $(x_n)_{n \in \mathbb{N}}$ converging in X to x, the fact that $F(x_n) \subset C$ $(n \in \mathbb{N})$ implies $F(x) \subset C$. Recall also that if F is lsc, then so is $\overline{F}$, where $\overline{F}(x)$ is the closure of $F(x)$.

In order to present a new theorem, which unifies in some sense two famous continuous selection theorems: of E. A. Michael [Mi] and of A. Cellina [Cel], we introduced in [NgJZ1-2] the following new important notion.

Definition 1.1. Let $F, G : X \to 2^Y$ be two multifunctions, where X and Y are metric spaces. Let $\varepsilon > 0$. By a *CM-selector* or *Cellina–Michael selector* for the pair (F, G) we mean a continuous map $f : X \to Y$ which is a selector for G, i.e., $f(x) \in G(x)$ $(x \in X)$, and simultaneously an ε-*approximate selector* (ε-selector for short) for F, i.e., $\operatorname{Gr} f \subset U_\varepsilon(\operatorname{Gr} F)$.

Remark. If Y is a normed space, then $f : X \to Y$ is an ε-selector for F iff $f(x) \in F(B_X(x, \varepsilon)) + B_Y(0, \varepsilon)$ for all $x \in X$.

If Y is a metric locally convex space, we denote by $\operatorname{cl} D$, $\operatorname{conv} D$ and $\overline{\operatorname{conv}} D$ the closure, the convex hull and the closed convex hull of a subset D of Y, respectively.

2. CM**-Selectors in the case of** $\mathbb{L}_p$**-decomposable values.** Throughout Sections 2–3 of this paper $(T, \mathcal{T}, \mu)$ is always a measure space, where $\mathcal{T}$ is a σ-algebra of subsets of T and μ is a complete σ-finite nonatomic measure on $\mathcal{T}$.

Let E be a Banach space with norm $\|\cdot\|_E$. Given a function $u : T \to E$, we sometimes write $\|u\|_E$ to denote the function $\|u(\cdot)\|_E : T \to \mathbb{R}_+$ if this causes no misunderstanding.

By $\mathbb{L}_p = L_p(T, E) = L_p(T, \mu, E)$ $(1 \le p < \infty)$ we denote [DunS] the Bochner–Lebesgue space (Bochner space) of (equivalence classes of) strongly measurable functions $u : T \to E$ with the norm $\|u\|_p = (\int_T \|u(t)\|_E^p \, d\mu(t))^{1/p}$ $= (\int_T \|u\|_E^p \, d\mu)^{1/p} < \infty$.

Note that all results of this section are also true for the weighted Bochner –Lebesgue space $L_p(T, E; \varrho)$ with a weight ϱ, since we have $L_p(T, E; \varrho) = L_p(T, \widetilde{\mu}, E)$ with $d\widetilde{\mu}(t) = \varrho(s)^p d\mu(t)$.

A set $K \subset L_p(T, E)$ is said to be *decomposable* (or $\mathbb{L}_p$-*decomposable*) in the sense of Hiai–Umegaki [HiU] if $u\chi_S + v\chi_{T\setminus S} \in K$ whenever $u, v \in K$ and $S \in \mathcal{T}$, where χ_D is the characteristic function of a set D. Recall (see [HiU]) that the closure $\operatorname{cl} K$ of a decomposable set is also a decomposable set. If $1 \le p < \infty$ and K is nonempty and closed then K is decomposable if and only if there exists a measurable multifunction $F : T \to 2^E$ with closed nonempty values such that $K = \{f \in L_p(T, E) : f(t) \in F(t)\ \mu\text{-a.e. in } T\}$.

The collection of all nonempty decomposable (resp. nonempty decomposable and closed) subsets of $L_p(T, E)$ is denoted by $\operatorname{Dc}(L_p(T, E))$ (resp. by $\operatorname{Cl}\operatorname{Dc}(L_p(T, E))$. The smallest decomposable set which contains a set $H \subset L_p(T, E)$ is denoted by $\operatorname{dec} H$ and called the *decomposable hull* of H.

The following lemma is the main tool in proving Theorem 2.1.

LEMMA 2.1. *Assume that X is a separable metric space. Let $(c_{n,k})_{n,k\ge1}$ be a sequence of nonnegative elements of $L_1(T, \mathbb{R})$ and $(\phi_n)_{n\ge1}$ be a sequence of continuous mappings $\phi_n : X \to L_1(T, \mathbb{R})$ with nonnegative values (i.e. $\phi_n(x)(t) \ge 0$ for every $x \in X$ and a.a. $t \in T$). Let $(h_n)_{n\ge1}$ be a sequence of continuous functions $h_n : X \to [0, 1]$ such that $\{\operatorname{supp} h_n : n \ge 1\}$ is a locally finite covering of X. Then for every continuous function $\lambda : X \to \mathbb{R}_+ \setminus \{0\}$ there exist a continuous function $\tau : X \to \mathbb{R}_+$ and a map $\Phi : \mathbb{R}_+ \times [0, 1] \to \mathcal{T}$ with the following properties*:

(a) $\Phi(r, \alpha_1) \subset \Phi(r, \alpha_2)$, $\mu(\Phi(r, 0)) = \mu(T \setminus \Phi(r, 1)) = 0$ *for* $r \ge 0$, $0 \le \alpha_1 \le \alpha_2 \le 1$;

(b) *for all* $x \in X$, $\alpha \in [0,1]$ *and* $n,k \geq 1$, *if* $h_n(x) = 1$ *then*

$$\Big| \int\limits_{\Phi(\tau(x),\alpha)} \phi_n(x)\,d\mu - \alpha \int\limits_T \phi_n(x)\,d\mu \Big| < \lambda(x), \tag{2.1a}$$

and if $h_n(x) = h_k(x) = 1$ *then*

$$\int\limits_{\Phi(\tau(x),\alpha)} c_{n,k}\,d\mu = \alpha \int\limits_T c_{n,k}\,d\mu; \tag{2.1b}$$

(c) *the map* $\psi : X \to L_p(T,E)$ *defined by*

$$\psi(x) = \phi(x)\chi_{\Phi(\beta(x),\alpha(x))} \tag{2.2}$$

is continuous whenever the functions $\alpha : X \to [0,1]$, $\beta : X \to \mathbb{R}_+$ *and the map* $\phi : X \to L_p(T,E)$ *are continuous.*

The following theorem, from which the main Theorem 2.2 of this section will be derived, is of independent interest.

THEOREM 2.1. *Assume that* X *is a separable metric space and* $1 \leq p < \infty$. *Let* $F : X \to \mathrm{Dc}(L_p(T,E))$ *be an* H*-usc multifunction and* $G : X \to \mathrm{Cl}\,\mathrm{Dc}(L_p(T,E))$ *be a lsc multifunction. If* $F(x) \cap G(x) \neq \emptyset$ *for all* $x \in X$, *then for every* $\varepsilon > 0$ *there exist continuous maps* $f_\varepsilon : X \to L_p(T,E)$ *and* $\phi_\varepsilon : X \to L_p(T,\mathbb{R})$ *such that* $\mathrm{Gr}\, f_\varepsilon \subset U_\varepsilon(\mathrm{Gr}\, F)$, $f_\varepsilon(X) \subset \mathrm{dec}\, F(X)$, $\|\phi_\varepsilon(x)\|_p < \varepsilon$ *for each* $x \in X$, *all sets*

$$G_\varepsilon(x) := \{u \in G(x) : \|u(t) - f_\varepsilon(x)(t)\|_E < \phi_\varepsilon(x)(t) \quad \mu\text{-a.e. in } T\} \quad (x \in X) \tag{2.3}$$

are nonempty, and the multifunction $G_\varepsilon : X \to \mathrm{Dc}(L_p(T,E))$ *is lsc.*

The following, main theorem in this section states that under appropriate assumptions for a pair of multifunctions (F,G), of which the former is H-usc and the latter is lsc, there exists a continuous function which is an ε-approximate selector for F and a selector for G. For this kind of selector common for two multifunctions we coined in [NgJZ1] the term CM-selector (see Definition 1.1).

THEOREM 2.2. *Assume that* X *is a separable metric space and* $1 \leq p < \infty$. *Let* $F : X \to \mathrm{Dc}(L_p(T,E))$ *be an* H*-usc multifunction and* $G : X \to \mathrm{Cl}\,\mathrm{Dc}(L_p(T,E))$ *be a lsc multifunction. If* $F(x) \cap G(x) \neq \emptyset$ *for all* $x \in X$, *then for every* $\varepsilon > 0$ *there exists a* CM*-selector* f *for the pair* (F,G). *Moreover, if* $A \subset X$ *is a fixed closed set, and* $f_0 : A \to L_p(T,E)$ *is a fixed continuous map such that* $f_0(a) \in F(a) \cap G(a)$ $(a \in A)$, *then there exists a* CM*-selector* f *for the pair* (F,G) *such that* $f(a) = f_0(a)$ $(a \in A)$.

The following Theorem 2.3 is an example of using Theorem 2.2 to prove the existence of an approximate selector satisfying some additional one-sided conditions.

THEOREM 2.3. *Let $F : C \to \mathrm{Dc}(L_p(T, \mathbb{R}^m))$ be an H-usc multifunction, where C is a separable subset of $L_q(T, \mathbb{R}^m)$, $1 \le p < \infty$, $1 \le q \le \infty$. Assume that for every $x \in C$ there exists a function $w \in F(x)$ such that*

$$(w(t), x(t)) \le \alpha(x(t), x(t)) + h(t) \quad \text{for a.a. } t \in T, \tag{2.4}$$

where $\alpha \in \mathbb{R}_+ \setminus \{0\}$ and $h : T \to \mathbb{R}_+ \setminus \{0\}$ is a measurable function. Then for every $\varepsilon > 0$ there exists a continuous map $f : C \to L_p(T, \mathbb{R}^m)$ which is an ε-approximate selector for F such that

$$(f(x)(t), x(t)) \le \alpha(x(t), x(t)) + h(t) \quad \text{for all } x \in C \text{ and for a.a. } t \in T. \tag{2.5}$$

Moreover, if $A \subset C$ is a fixed closed set, $f_0 : A \to L_p(T, \mathbb{R}^m)$ is a fixed continuous map such that $f_0(a) \in F(a)$ $(a \in A)$ and $(f_0(a)(t), a(t)) \le \alpha(a(t), a(t)) + h(t)$ for all $a \in A$ and for a.a. $t \in T$, then there exists a continuous ε-approximate selector f for F satisfying (2.5) *with $f(a) = f_0(a)$ $(a \in A)$.*

From Theorem 2.2 we obtain the following known selection theorems as easy corollaries. These are versions of the Michael and Cellina theorems for multifunctions with $\mathbb{L}_p$-decomposable values. It is sufficient to take $F(x) \equiv L_p(T, E)$ and $G(x) \equiv L_p(T, E)$, respectively.

COROLLARY 2.1 (Bressan–Colombo–Fryszkowski theorem). *Assume that X is a separable metric space and $1 \le p < \infty$. Let $G : X \to \mathrm{Cl}\,\mathrm{Dc}(L_p(T, E))$ be a lsc multifunction. Then there exists a continuous map $g : X \to L_p(T, E)$ which is a selector for G. Moreover, if $A \subset X$ is a fixed closed set and $g_0 : A \to L_p(T, E)$ is a fixed continuous map such that $g_0(a) \in G(a)$ $(a \in A)$, then there exists a continuous selector g of G such that $g(a) = g_0(a)$ $(a \in A)$.*

COROLLARY 2.2. *Assume that X is a separable metric space and $1 \le p < \infty$. Let $F : X \to \mathrm{Dc}(L_p(T, E))$ be an H-usc multifunction. Then for every $\varepsilon > 0$ there exists a continuous ε-selector f of F with $f(X) \subset \mathrm{dec}\, F(X)$. Moreover, if $A \subset X$ is a fixed closed set and $f_0 : A \to L_p(T, E)$ is a fixed continuous map such that $f_0(a) \in F(a)$ $(a \in A)$, then there exists a continuous ε-selector f of F such that $f(a) = f_0(a)$ $(a \in A)$.*

REMARK 2.1. The additional condition $f(X) \subset \mathrm{dec}\, F(X)$ in Corollary 2.2 is a consequence of Theorem 2.1. The first part of Corollary 2.2 is the selection theorem due to Bressan–Colombo [BrC] and Cellina–Colombo–Fonda [CelCF]. Further, *the existence of an ε-selector f with $f(a) = f_0(a)$ $(a \in A)$ as in the second part of Corollary* 2.2 *is a new fact* in comparison with the theorem of [CelCF], [BrC], and it cannot be deduced directly from those papers.

3. Proofs of the results of Section 2. In the calculations below, c *denotes the constant* 2^{p-1} *with* $1 \le p < \infty$.

We first formulate (without proofs) standard Lemmas 3.1–3.3 (which are easy adaptations of Propositions 2.1–2.3 of [Fry1]). Then we give the proofs for Lemma 2.1, Theorem 2.1 (both of which are essential for the proof of the main Theorem 2.2), Theorem 2.2 and Theorem 2.3.

Recall that by the Kantorovich order-completeness theorem for the space $S(T,\mathbb{R})$ of all measurable scalar functions [KaA, Corollary 1.6.2] any set $\mathcal{K}$ of nonnegative measurable scalar functions has an essential infimum (which is equal to the infimum of $\mathcal{K}$ with respect to the natural ordering in $S(T,\mathbb{R})$), which will be denoted by $\psi = \operatorname{ess\,inf}\{a : a \in \mathcal{K}\}$.

LEMMA 3.1. *Let* $K \in \mathrm{Cl}\,\mathrm{Dc}(L_p(T,E))$. *Then for every real-valued* $\mathcal{T}$-*measurable function* $a_0 : T \to \mathbb{R}$ *such that* $a_0(t) > \psi(t)$ μ-*a.e. in* T, *where* $\psi = \operatorname{ess\,inf}\{\|u(\cdot)\|_E : u \in K\}$, *there exists* $u_0 \in K$ *such that* $\|u_0(t)\|_E < a_0(t)$ μ-*a.e. in* T.

LEMMA 3.2. *Let* X *be a metric space and let* $G : X \to \mathrm{Cl}\,\mathrm{Dc}(L_p(T,E))$ *be a lsc multifunction. For all* $x \in X$ *set* $\psi_x = \operatorname{ess\,inf}\{\|u(\cdot)\|_E : u \in G(x)\}$. *Then the multifunction* $P : X \to 2^{L_p(T,\mathbb{R})} \setminus \{\emptyset\}$ *defined as*

$$P(x) = \{a \in L_p(T,\mathbb{R}) : a(t) > \psi_x(t)\ \mu\text{-a.e. in } T\} \tag{3.1}$$

is lsc.

LEMMA 3.3. *Let* X *be a metric space and* $G : X \to \mathrm{Cl}\,\mathrm{Dc}(L_p(T,E))$ *be a lsc multifunction. Assume that* $g : X \to L_p(T,E)$ *and* $\phi : X \to L_p(T,\mathbb{R})$ *are continuous maps such that for every* $x \in X$ *the set*

$$\widehat{G}(x) = \{u \in G(x) : \|u(t) - g(x)(t)\|_E < \phi(x)(t)\ \mu\text{-a.e. in } T\} \tag{3.2}$$

is nonempty. Then the multifunction $\widehat{G} : X \to \mathrm{Dc}(L_p(T,E))$ *is lsc.*

Proof of Lemma 2.1. Let $\lambda : X \to \mathbb{R}_+ \setminus \{0\}$ be given. For every $x \in X$ choose an open neighbourhood U_x of x which intersects the supports of finitely many functions h_n and set $I(x) = \{n \in \mathbb{N} : U_x \cap \operatorname{supp} h_n \ne \emptyset\}$. By our choice the set $I(x)$ is finite. For $x \in X$ put

$$V_x = \bigcap_{n \in I(x)} \{x' \in U_x : \|\phi_n(x') - \phi_n(x)\|_1 < \lambda(x')/2\}. \tag{3.3}$$

The family $(V_x)_{x\in X}$ is an open covering of the separable metric space X, which is paracompact by the Stone Theorem [Kur]. Hence there exists a sequence $(q_m)_{m\ge1}$ of continuous functions $q_m : X \to [0,1]$ such that $\{\operatorname{supp} q_m : m \ge 1\}$ is a locally finite refinement of $(V_x)_{x\in X}$ and the sets $W_m = \{x \in X : q_m(x) = 1\}$, $m \in \mathbb{N}$, cover X. For every $m \ge 1$ choose x_m such that $W_m \subset V_{x_m}$. Define the sequence $(a_j)_{j\ge0}$ of nonnegative functions in $L_1(T,\mathbb{R})$ by

$$a_j = \begin{cases} \phi_n(x_m) & \text{if } j = 2^n 3^m \text{ for some } n,m \in \mathbb{N}, \\ c_{n,k} & \text{if } j = 3^n 5^k \text{ for some } n,k \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases} \tag{3.4}$$

Moreover put

$$\tau_1(x) = \sum_{n,m\ge 1} h_n(x) q_m(x) 2^n 3^m, \quad \tau_2(x) = \sum_{n,k\ge 1} h_n(x) h_k(x) 3^n 5^k, \tag{3.5}$$

$$\tau(x) = \tau_1(x) + \tau_2(x). \tag{3.6}$$

The function τ is continuous, because the summations in $\tau_1(x)$ and $\tau_2(x)$ are locally finite.

Since μ is σ-finite, there is a sequence $(T_i)_{i\in\mathbb{N}} \subset \mathcal{T}$ of disjoint measurable sets of finite measure such that $\bigcup_{i\in\mathbb{N}} T_i = T$. For $i \in \mathbb{N}$ define $\mathcal{T}_i = \{A \cap T_i : A \in \mathcal{T}\}$ and observe that $(a_{j|T_i})_{j\ge 0} \subset L_1(T_i,\mathbb{R})$. Thus by [BrC, Lemma 1, p. 73–75] for the sequence $(a_j)_{j\ge 0}$ (it is important that $a_0 = 1$) there exist maps $\Phi_i : \mathbb{R}_+ \times [0,1] \to \mathcal{T}_i$, $i \in \mathbb{N}$, with the following properties:

(a′) $\Phi_i(r,\alpha_1) \subset \Phi_i(r,\alpha_2)$, $\mu(\Phi_i(r,0)) = \mu(T \setminus \Phi_i(r,1)) = 0$ for $r \ge 0$, $0 \le \alpha_1 \le \alpha_2 \le 1$;

(b′) $\mu(\Phi_i(r_1,\alpha_1)) \triangle \Phi_i(r_2,\alpha_2)) \le 2|r_1 - r_2| + |\alpha_1 - \alpha_2|$ for $r_1, r_2 \ge 0$, $\alpha_1, \alpha_2 \in [0,1]$;

(c′)

$$\int_{\Phi_i(r,\alpha)} a_j \, d\mu = \alpha \int_{T_i} a_j \, d\mu \qquad (0 \le j \le r,\ \alpha \in [0,1]). \tag{3.7}$$

Define the map $\Phi : \mathbb{R}_+ \times [0,1] \to \mathcal{T}$ to be the union of disjoint measurable sets:

$$\Phi(r,\alpha) = \bigcup_{i\ge 1} \Phi_i(r,\alpha). \tag{3.8}$$

It is easy to see that Φ satisfies the condition (a) of Lemma 2.1.

To prove (b) it suffices to prove (2.1a), (2.1b). Fix $x \in X$, $\alpha \in [0,1]$ and $n,k \ge 1$. We can find m such that $x \in W_m$ (and so $q_m(x) = 1$). If $h_n(x) = 1$ then, by (3.4),

$$\begin{aligned}\Big| \int_{\Phi(\tau(x),\alpha)} & \phi_n(x) \, d\mu - \alpha \int_T \phi_n(x) \, d\mu \Big| \\ &\le \int_{\Phi(\tau(x),\alpha)} |\phi_n(x) - \phi_n(x_m)| \, d\mu \\ &\quad + \Big| \int_{\Phi(\tau(x),\alpha)} \phi_n(x_m) \, d\mu - \alpha \int_T \phi_n(x_m) \, d\mu \Big| + \alpha \int_T |\phi_n(x_m) - \phi_n(x)| \, d\mu\end{aligned}$$

$$\le 2\|\phi_n(x) - \phi_n(x_m)\|_1 + \Big| \int\limits_{\Phi(\tau(x),\alpha)} a_{2^n 3^m}\, d\mu - \alpha \int\limits_T a_{2^n 3^m}\, d\mu \Big|.$$

But $n \in I(x_m)$, because $x \in W_m \subset V_{x_m} \subset U_{x_m}$ and $h_n(x) = 1$. Hence, by (3.3), $x \in V_{x_m} \subset \{x' \in U_{x_m} : \|\phi_n(x') - \phi_n(x_m)\|_1 < \lambda(x')/2\}$. Moreover, $\tau(x) \ge \tau_1(x) \ge 2^n 3^m$ (since $q_m(x) = 1$, $h_n(x) = 1$), and consequently, by (3.7), (3.8) and the Lebesgue Dominated Convergence Theorem [DunS], we get

$$\Big| \int\limits_{\Phi(\tau(x),\alpha)} a_{2^n 3^m}\, d\mu - \alpha \int\limits_T a_{2^n 3^m}\, d\mu \Big| = 0.$$

Hence (2.1a) follows.

If $h_n(x) = h_k(x) = 1$, then $\tau(x) \ge \tau_2(x) \ge 3^n 5^k$. Consequently, by (3.4), (3.7), (3.8) and the Lebesgue Dominated Convergence Theorem, we get

$$\int\limits_{\Phi(\tau(x),\alpha)} c_{n,k}\, d\mu = \int\limits_{\Phi(\tau(x),\alpha)} a_{3^n 5^k}\, d\mu = \alpha \int\limits_T a_{3^n 5^k}\, d\mu = \alpha \int\limits_T c_{n,k}\, d\mu$$

and so we have (2.1b).

For the proof of (c) take arbitrary $x_0 \in X$, $\varepsilon_0 > 0$, continuous functions $\alpha : X \to [0,1]$, $\beta : X \to \mathbb{R}_+$ and a continuous map $\phi : X \to L_p(T, E)$. Note that for $x \in X$ the following inequalities hold for the map ψ defined in (2.2) [for simplicity we write $\Phi(x)$ for $\Phi(\beta(x), \alpha(x))$ and $\chi(x)$ for $\chi_{\Phi(x)}$]:

$$\begin{aligned}
&\|\psi(x) - \psi(x_0)\|_p^p \\
&\qquad = \|\phi(x)\chi(x) - \phi(x_0)\chi(x_0)\|_p^p \\
&\qquad \le c\|\phi(x)\chi(x) - \phi(x_0)\chi(x)\|_p^p + c\|\phi(x_0)\chi(x) - \phi(x_0)\chi(x_0)\|_p^p \\
&\qquad = c\|(\phi(x) - \phi(x_0))\chi(x)\|_p^p + c\|\phi(x_0)|\chi(x) - \chi(x_0)|\|_p^p \\
&\qquad \le c\|\phi(x) - \phi(x_0)\|_p^p + c\|\phi(x_0)\chi_{\Phi(x)\triangle\Phi(x_0)}\|_p^p \\
&\qquad = c\|\phi(x) - \phi(x_0)\|_p^p + c\sum_{i\ge 1} \int\limits_{\Phi_i(x)\triangle\Phi_i(x_0)} \|\phi(x_0)\|_E^p\, d\mu
\end{aligned}$$

(recall that $c = 2^{p-1}$). By the absolute continuity of the indefinite Lebesgue integral there exist $\delta_0 > 0$ and $i_0 \in \mathbb{N}$ such that

$$\int\limits_A \|\phi(x_0)\|_E^p\, d\mu < \frac{\varepsilon_0}{3ci_0} \quad (\mu(A) < \delta_0), \qquad \sum_{i\ge i_0+1} \int\limits_{T_i} \|\phi(x_0)\|_E^p\, d\mu < \frac{\varepsilon_0}{3c}.$$

By the continuity of α, β and ϕ and by (b$'$), we can find $\delta > 0$ such that

$$\|\phi(x)-\phi(x_0)\|_p^p < \frac{\varepsilon_0}{3c}, \quad \mu(\Phi_i(x)\triangle\Phi_i(x_0)) < \delta_0 \quad (x \in B(x_0,\delta),\ 1 \le i \le i_0).$$

Consequently, we get

$$\begin{aligned}\|\psi(x)-\psi(x_0)\|_p^p &\le c\|\phi(x)-\phi(x_0)\|_p^p + c\sum_{1\le i\le i_0}\ \int\limits_{\Phi_i(x)\triangle\Phi_i(x_0)} \|\phi(x_0)\|_E^p\,d\mu\\ &\quad + c\sum_{i\ge i_0+1}\int\limits_{T_i}\|\phi(x_0)\|_E^p\,d\mu\\ &< \frac{\varepsilon_0}{3}+i_0\frac{\varepsilon_0}{3i_0}+\frac{\varepsilon_0}{3}=\varepsilon_0 \qquad (x\in B(x_0,\delta)).\end{aligned}$$

Hence ψ is continuous. ■

Proof of Theorem 2.1. Fix $\varepsilon > 0$. Let u_x be an arbitrary element of $F(x)\cap G(x)$, $x\in X$. Since F is H-usc, for every $x \in X$ there is $\delta_1(x) > 0$ such that $\delta_1(x) < \varepsilon$ and $F(B(x,\delta_1(x))) \subset U_{\varepsilon/2}(F(x))$. On the other hand, G is lsc and has closed decomposable values. Thus, by Lemma 3.2, for every $x \in X$ the multifunction $\overline{P}_x : X \to 2^{L_p(T,\mathbb{R})}$ defined by

$$\overline{P}_x(z)=\{a\in L_p(T,\mathbb{R}) : a(t)\ge (\operatorname*{ess\,inf}_{u\in G(z)}\|u(\cdot)-u_x(\cdot)\|_E)(t)\ \mu\text{-a.e. in } T\} \tag{3.9}$$

is lsc with nonempty closed convex values, and clearly $0\in\overline{P}_x(x)$. Applying now the famous Michael selection theorem [Mi] to the multifunction $\overline{P}_x$ we can find a continuous mapping $\phi_x : X\to L_p(T,\mathbb{R})$ such that

$$\begin{aligned}&\phi_x(x)=0,\\ &\phi_x(z)(t)\ge (\operatorname*{ess\,inf}_{u\in G(z)}\|u(\cdot)-u_x(\cdot)\|_E)(t)\qquad \mu\text{-a.e. in } T.\end{aligned} \tag{3.10a}$$

The continuity of ϕ_x implies that for every $x\in X$ there is $\delta_2(x) > 0$ such that

$$\|\phi_x(z)-\phi_x(x)\|_p^p=\int\limits_T(\phi_x(z))^p\,d\mu<\frac{\varepsilon^p}{3c}\qquad (z\in B(x,\delta_2(x))). \tag{3.10b}$$

Define $\delta(x)=\min(\delta_1(x),\delta_2(x))$, $U_x=B(x,\delta(x)/3)$ $(x\in X)$.

Since $(U_x)_{x\in X}$ is an open covering of the separable metric space X, and X is paracompact by the Stone theorem [Kur], we can find a countable locally finite refinement $(W_n)_{n\in\mathbb{N}}$ of $(U_x)_{x\in X}$ and a continuous partition of unity $(p_n)_{n\in\mathbb{N}}$ subordinate to $(W_n)_{n\in\mathbb{N}}$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $h_n : X \to [0,1]$ such that $h_n\equiv 1$ on $\operatorname{supp} p_n$ and $\operatorname{supp} h_n \subset W_n$. For every $n\ge 1$ choose $x_n\in X$ such that $W_n\subset U_{x_n}$. Set

$$\delta(x_n)=\delta_n,\quad U_{x_n}=U_n,\quad u_{x_n}=u_n,\quad \phi_{x_n}=\phi_n\quad \text{for } n\in\mathbb{N}.$$

The mappings ϕ_n satisfy the following inequalities:

(3.11a) $$\phi_n(x)(t) \geq (\operatorname*{ess\,inf}_{u\in G(x)} \|u(\cdot) - u_n(\cdot)\|_E)(t) \quad \mu\text{-a.e. in } T,$$

(3.11b) $$p_n(x)\|\phi_n(x)\|_p^p \leq \frac{\varepsilon^p}{3c} p_n(x) \quad (x \in X).$$

For all $n, k \in \mathbb{N}$ choose $v_{n,k} \in F(x_k)$ such that

(3.12) $$d_p(u_n, v_{n,k}) = \Big(\int_T \|u_n - v_{n,k}\|_E^p \, d\mu\Big)^{1/p} < d_p(u_n, F(x_k)) + \frac{\varepsilon}{2}$$

and define $w_{n,k} = u_n - v_{n,k}$ for $n, k \geq 1$. The continuity of ϕ_x implies the continuity of $(\phi_x)^p$ in $L_1(T, \mathbb{R})$.

By Lemma 2.1 applied to the sequences $(\|w_{n,k}(\cdot)\|_E^p)_{n,k\geq 1}$, $(\phi_n^p)_{n\geq 1}$, $(h_n)_{n\geq 1}$ and to the function

(3.13) $$x \mapsto \lambda(x) = \frac{\varepsilon^p}{6c\sum_{n\geq 1} h_n(x)},$$

there exist a continuous function $\tau : X \to \mathbb{R}_+$ and a family $(\Phi(\tau, \alpha))_{\tau\geq 0,\, \alpha\in[0,1]}$ of measurable subsets of T satisfying (a), (b), (c). Put $\alpha_0 \equiv 0$ and $\alpha_n(x) = \sum_{m\leq n} p_m(x)$ for $n \geq 1$.

Define

(3.14a) $$\chi_n(x) = \chi_{\Phi(\tau(x),\alpha_n(x))\setminus\Phi(\tau(x),\alpha_{n-1}(x))}.$$

Then, by the condition (a), we have

(3.14b) $$\sum_{n\geq 1} \chi_n(x)(t) = \chi_{\Phi(\tau(x),1)}(t) - \chi_{\Phi(\tau(x),0)}(t) = 1$$

for all $x \in X$ and a.a. $t \in T$.

Define maps $f_\varepsilon : X \to L_p(T, E)$ and $\phi_\varepsilon : X \to L_p(T, \mathbb{R})$ by

(3.15) $$f_\varepsilon(x) = \sum_{n\geq 1} u_n \chi_n(x), \qquad \phi_\varepsilon(x) = \eta + \sum_{n\geq 1} \phi_n(x)\chi_n(x),$$

where $\eta \in L_p(T, \mathbb{R})$, $\eta(t) > 0$ a.e., $\|\eta\|_p^p < \varepsilon^p/(3c)$ (such a function exists as μ is σ-finite). Clearly, by Lemma 2.1(c), the maps f_ε and ϕ_ε are continuous, because the above summations are locally finite and the functions τ, α_n, ϕ_n are continuous. Moreover, by (3.14a)–(3.14b) clearly we get $f_\varepsilon(X) \subset \operatorname{dec} F(X)$.

We claim that $\operatorname{Gr} f_\varepsilon \subset U_\varepsilon(\operatorname{Gr} F)$. Indeed, fix $x \in X$ and define $I(x) = \{n \in \mathbb{N} : p_n(x) \neq 0\}$. Since $I(x)$ is finite, there exists $\overline{n} \in I(x)$ such that $\delta_{\overline{n}} = \max\{\delta_n : n \in I(x)\}$. But $p_n(x) \neq 0$ means that $x \in U_n$. Thus $x \in \bigcap_{n\in I(x)} U_n$ and hence $\bigcup_{n\in I(x)} U_n \subset B(x_{\overline{n}}, \delta_{\overline{n}})$. It follows that $\varrho_X(x, x_{\overline{n}}) < \delta_{\overline{n}} < \varepsilon$. Moreover, we have $\{x_n : n \in I(x)\} \subset B(x_{\overline{n}}, \delta_{\overline{n}})$. Therefore $u_n \in F(x_n) \subset F(B(x_{\overline{n}}, \delta_{\overline{n}})) \subset U_{\varepsilon/2}(F(x_{\overline{n}}))$ and hence, by (3.12),

$$\text{(3.16a)}\qquad \begin{aligned}\|w_{n,\overline{n}}\|_p = \|u_n - v_{n,\overline{n}}\|_p &< d_p(u_n, F(x_{\overline{n}})) + \varepsilon/2\\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad (n \in I(x)).\end{aligned}$$

Let

$$\text{(3.16b)}\qquad v_{\overline{n}} = \sum_{n\in I(x)} v_{n,\overline{n}}\chi_n(x).$$

Since $\{v_{n,\overline{n}} : n \in I(x)\} \subset F(x_{\overline{n}})$ and $F(x_{\overline{n}})$ is decomposable, by (3.14a)–(3.14b) we get $v_{\overline{n}} \in F(x_{\overline{n}})$. By our choice of h_n and $\overline{n}$ we obtain

$$\text{(3.16c)}\qquad h_n(x) = h_{\overline{n}}(x) = 1 \qquad (n \in I(x)).$$

Finally, by Lemma 2.1(b) and (3.16a)–(3.16c),

$$\begin{aligned}\|f_\varepsilon(x) - v_{\overline{n}}\|_p^p &= \Big\|\sum_{n\in I(x)} (u_n - v_{n,\overline{n}})\chi_n(x)\Big\|_p^p = \sum_{n\in I(x)} \|(u_n - v_{n,\overline{n}})\chi_n(x)\|_p^p\\ &= \sum_{n\in I(x)} \int_T \|w_{n,\overline{n}}\|_E^p \chi_n(x)\, d\mu\\ &= \sum_{n\in I(x)} \Big(\int_{\Phi(\tau(x),\alpha_n(x))} - \int_{\Phi(\tau(x),\alpha_{n-1}(x))}\Big)\|w_{n,\overline{n}}\|_E^p\, d\mu\\ &= \sum_{n\in I(x)} (\alpha_n(x) - \alpha_{n-1}(x)) \int_T \|w_{n,\overline{n}}\|_E^p\, d\mu\\ &= \sum_{n\in I(x)} p_n(x)\|w_{n,\overline{n}}\|_p^p < \varepsilon^p.\end{aligned}$$

Hence $\|f_\varepsilon(x) - v_{\overline{n}}\|_p < \varepsilon$, and therefore $d_p(f_\varepsilon(x), F(x_{\overline{n}})) < \varepsilon$.

Now we claim that $\|\phi_\varepsilon(x)\|_p < \varepsilon$ for $x \in X$. Fix $x \in X$. From (3.16c), (3.11b), (3.13) and Lemma 2.1(b) we have

$$\begin{aligned}\|\phi_\varepsilon(x)\|_p^p &= \Big\|\eta + \sum_{n\in I(x)} \phi_n(x)\chi_n(x)\Big\|_p^p \le c\|\eta\|_p^p + c\Big\|\sum_{n\in I(x)} \phi_n(x)\chi_n(x)\Big\|_p^p\\ &< c\varepsilon^p/(3c) + c\sum_{n\in I(x)} \|\phi_n(x)\chi_n(x)\|_p^p\\ &= \varepsilon^p/3 + c\sum_{n\in I(x)} \int_T (\phi_n(x)\chi_n(x))^p\, d\mu\\ &= \varepsilon^p/3 + c\sum_{n\in I(x)} \Big(\int_{\Phi(\tau(x),\alpha_n(x))} - \int_{\Phi(\tau(x),\alpha_{n-1}(x))}\Big)(\phi_n(x))^p\, d\mu\\ &\le \varepsilon^p/3 + c\sum_{n\in I(x)} \Big[(\alpha_n(x) - \alpha_{n-1}(x)) \int_T (\phi_n(x))^p\, d\mu + 2\lambda(x)\Big]\end{aligned}$$

$$= \varepsilon^p/3 + c \sum_{n \in I(x)} (p_n(x) \|\phi_n(x)\|_p^p + 2\lambda(x))$$

$$\leq \varepsilon^p/3 + c \sum_{n \in I(x)} \left(\frac{\varepsilon^p}{3c} p_n(x) + 2\lambda(x) \right)$$

$$\leq \varepsilon^p/3 + \varepsilon^p/3 + 2c \sum_{n \geq 1} h_n(x)\lambda(x)$$

$$= \varepsilon^p/3 + \varepsilon^p/3 + \varepsilon^p/3 = \varepsilon^p.$$

Finally, we claim that G_ε (defined in (2.3) in the statement of Theorem 2.1) is lsc with nonempty decomposable values. By Lemma 3.3, it suffices to show that all the sets $G_\varepsilon(x)$, $x \in X$, are nonempty. Indeed, let $x \in X$. By Lemma 3.1 for every $n \geq 1$ there is an element $v_x^n \in G(x)$ such that

$$\|v_x^n(t) - u_n(t)\|_E < \eta(t) + (\operatorname*{ess\,inf}_{u \in G(x)} \|u(\cdot) - u_n(\cdot)\|_E)(t) \quad \mu\text{-a.e. in } T.$$

Set

$$v_x = \sum_{n \in I(x)} v_x^n \chi_n(x).$$

By (3.14a)–(3.14b) we get $v_x \in G(x)$, because $G(x)$ is decomposable. Moreover, by (3.11a),

$$\begin{aligned} \|v_x(t) - f_\varepsilon(x)(t)\|_E &= \sum_{n \in I(x)} \|v_x^n(t) - u_n(t)\|_E \chi_n(x)(t) \\ &< \sum_{n \in I(x)} [\eta(t) + (\operatorname*{ess\,inf}_{u \in G(x)} \|u(\cdot) - u_n(\cdot)\|_E)(t)] \chi_n(x)(t) \\ &\leq \eta(t) + \sum_{n \in I(x)} \phi_n(x)(t)\chi_n(x)(t) = \phi_\varepsilon(x)(t) \quad \mu\text{-a.e. in } T, \end{aligned}$$

i.e. $v_x \in G_\varepsilon(x)$. ■

Proof of Theorem 2.2. Fix $\varepsilon > 0$. By Theorem 2.1 there exist sequences $(f_n)_{n \geq 1}$ and $(\varphi_n)_{n \geq 1}$ of continuous maps $f_n : X \to L_p(T, E)$ and $\varphi_n : X \to L_p(T, \mathbb{R})$ such that for all $x \in X$:

(1) $\operatorname{Gr} f_1 \subset U_{\varepsilon/2}(\operatorname{Gr} F)$, $\|\varphi_1(x)\|_p < \varepsilon/2$,

$G_1(x) = \{u \in G(x) : \|u(t) - f_1(x)(t)\|_E < \varphi_1(x)(t) \ \mu\text{-a.e. in } T\} \neq \emptyset$,

(2) $G_n(x) = \{u \in \overline{G_1(x)} : \|u(t) - f_n(x)(t)\|_E < \varphi_n(x)(t) \ \mu\text{-a.e. in } T\} \neq \emptyset$, $n \geq 2$,

(3) $\|f_n(x)(t) - f_{n-1}(x)(t)\|_E \leq \varphi_n(x)(t) + \varphi_{n-1}(x)(t)$ μ-a.e. in T, $n \geq 2$,

(4) $\|\varphi_n(x)\|_p < 2^{-n}$, $n \geq 2$.

To do this define f_1 and φ_1 by applying Theorem 2.1 with $\varepsilon/2$. Let now f_m, φ_m and G_m be defined so that (1)–(4) hold for $m = 1, \ldots, n-1$. To construct f_n and φ_n apply again Theorem 2.1 (jointly with Lemma 3.3) with ε as 2^{-n} for the pair $(\widetilde{F}, \widetilde{G})$, where $\widetilde{F}(x) \equiv L_p(T, E)$ and $\widetilde{G}(x) = \overline{G_{n-1}(x)}$ for $x \in X$.

By (3) and (4) the sequence $(f_n)_{n\geq 1}$ is uniformly Cauchy. Therefore it uniformly converges to some continuous map $f : X \to L_p(T, E)$. By (2) and (4) we have $d_p(f_n(x), G(x)) \to 0$ and hence $f(x) \in \overline{G_1(x)}$ for $x \in X$. Note that f is a selector of G, because $\overline{G_1(x)} \subset \overline{G(x)} = G(x)$ for $x \in X$.

It remains to show that $\operatorname{Gr} f \subset U_\varepsilon(\operatorname{Gr} F)$. Indeed, let $x \in X$. Since $(x, f_1(x)) \in U_{\varepsilon/2}(\operatorname{Gr} F)$, we have $\varrho_X(x, x') < \varepsilon/2 < \varepsilon$ and $\|f_1(x) - u'\|_p < \varepsilon/2$ for some $x' \in X$ and $u' \in F(x')$. Hence

$$\begin{aligned}\|f(x) - u'\|_p &\leq \|f(x) - f_1(x)\|_p + \|f_1(x) - u'\|_p \\ &< \Big(\int_T \|f(x) - f_1(x)\|_E^p \, d\mu\Big)^{1/p} + \varepsilon/2 \\ &\leq \|\varphi_1(x)\|_p + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon,\end{aligned}$$

because $f(x) \in \overline{G_1(x)}$.

Finally, let A and f_0 be as in the formulation of Theorem 2.2. Put $G_0(x) = \{f_0(x)\}$ for $x \in A$ and $G_0(x) = G(x)$ for $x \notin A$. By [Mi] (see also [AubC]), G_0 is lsc just as G. Applying the statement proved above for the pair (F, G_0), we get its CM-selector f, which clearly is a CM-selector for the pair (F, G) with the additional property $f(x) = f_0(x)$ $(x \in A)$. ∎

Proof of Theorem 2.3. Define

$$g(u,b) = \{v \in \mathbb{R}^m : (u,v) \leq \alpha(u,u) + b\} \ (u \in \mathbb{R}^m,\ b \in \mathbb{R}_+ \setminus \{0\}), \tag{3.17}$$

$$G(x) = \{y \in L_p(T, \mathbb{R}^m) : y(t) \in g(x(t), h(t)) \ \mu\text{-a.e. in } T\} \quad (x \in L_q(T, \mathbb{R}^m)). \tag{3.18}$$

We remark that

$$\operatorname{int} g(u,b) = \{v \in \mathbb{R}^m : (u,v) < \alpha(u,u) + b\}, \tag{3.19a}$$

$$\operatorname{bdry} g(u,b) = \{v \in \mathbb{R}^m : (u,v) = \alpha(u,u) + b\}. \tag{3.19b}$$

The multifunction G obviously has closed decomposable values, and so, by Theorem 2.2, to complete the proof it suffices to show that G is lsc. We need to prove that for any fixed $x_0 \in L_q(T, \mathbb{R}^m)$, we have $y_0 \in D$ if $y_0 \in G(x_0)$, whenever $(x_n)_{n\geq 1} \subset X$ is any sequence converging to x_0 and D is any closed subset of $L_p(T, \mathbb{R}^m)$ such that $G(x_n) \subset D$ for $n \geq 1$. Taking subsequences, without loss of generality we may assume that $x_n(t) \to x_0(t)$ μ-a.e. in T. Define the measurable set

$$I = \{t \in T : y_0(t) \in \operatorname{int} g(x_0(t), h(t))\}.$$

For $t \in I$ denote by $i(t)$ the least n such that $y_0(t) \in \operatorname{int} g(x_k(t), h(t))$ $(k \geq n)$.

Let $P(v, u, b)$ denote the orthogonal projection of $v \in \mathbb{R}^m$ on the hyperplane $\operatorname{bdry} g(u, b)$, where $u \neq 0$. For $t \in T \setminus I$ denote by $j(t)$ the least n such that $x_k(t) \neq 0$ and $|y_0(t) - P(y_0(t), x_k(t), h(t))| \leq \eta(t)$ $(k \geq n)$, where $\eta \in L_p(T, \mathbb{R})$ and $\eta(t) > 0$ for every $t \in T$. The above definition makes sense in view of the following facts (which are direct consequences of (3.19a), (3.19b)):

(1) if $v \in \operatorname{int} g(u, b)$ and $u_n \to u$ then $v \in \operatorname{int} g(u_n, b)$ for sufficiently large n;

(2) if $v \in \operatorname{bdry} g(u, \beta(u))$ and $u_n \to u$, $u_n \neq 0$, then $P(v, u_n, \beta(u_n)) \to v$, where $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Define

$$y_n(t) = \begin{cases} y_0(t) & \text{if } t \in I \text{ and } n \geq i(t), \\ 0 & \text{if } t \in I \text{ and } n < i(t), \\ P(y_0(t), x_n(t), h(t)) & \text{if } t \in T \setminus I \text{ and } n \geq j(t), \\ 0 & \text{if } t \in T \setminus I \text{ and } n < j(t). \end{cases}$$

It is easy to see that the functions $i(\cdot)$ and $j(\cdot)$ are measurable (we omit the standard but cumbrous proof of this fact), so y_n $(n \in \mathbb{N})$ is also a measurable function. We obviously have $y_n \in G(x_n)$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n(t) = y_0(t)$ for a.a. $t \in T$. As $|y_n(t)| \leq |y_0(t)| + \eta(t)$ for $n \in \mathbb{N}$ and for μ-a.a. $t \in T$, it follows from the Lebesgue Dominated Convergence Theorem that $y_n \to y_0$ in $L_p(T, \mathbb{R}^m)$. Hence, by closedness of D, $y_0 \in D$. ∎

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The present paper is an extended exposition with complete proofs of some results announced in [Ng3] (in September 1997).

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