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Quasi-constricted linear operators on Banach spaces

by

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Abstract. Let X be a Banach space over $\mathbb{C}$. The bounded linear operator T on X is called quasi-constricted if the subspace $X_0 := \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}$ is closed and has finite codimension. We show that a power bounded linear operator $T \in L(X)$ is quasi-constricted iff it has an attractor A with Hausdorff measure of noncompactness $\chi_{\|\cdot\|_1}(A) < 1$ for some equivalent norm $\|\cdot\|_1$ on X. Moreover, we characterize the essential spectral radius of an arbitrary bounded operator T by quasi-constrictedness of scalar multiples of T. Finally, we prove that every quasi-constricted operator T such that $\overline{\lambda}T$ is mean ergodic for all λ in the peripheral spectrum $\sigma_\pi(T)$ of T is constricted and power bounded, and hence has a compact attractor.

1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space over the field $\mathbb{C}$ of complex numbers. We denote the space of all bounded linear operators on X by $L(X)$. Let $T \in L(X)$. A subset $A \subseteq X$ is called an *attractor* for T if $\lim_{n\to\infty} \operatorname{dist}(T^n x, A) = 0$ for each $x \in B_X$, where B_X is the closed unit ball of X and $\operatorname{dist}(y, A)$ denotes the distance $\inf\{\|y-z\| : z \in A\}$ from y to A. We will denote the set of all attractors for T by $\mathrm{Attr}_{\|\cdot\|}(T)$. It was established by many authors (see, for example, [LLY], [Ba], [Si]) that a power bounded operator $T \in L(X)$ has a compact attractor iff there exists a decomposition $X := X_0 \oplus X_r$ of the Banach space X into T-invariant subspaces X_0 and X_r such that $X_0 = \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}$ and $\dim(X_r) < \infty$. Operators of this type are called *constricted.*

Here we will study bounded linear operators T on X which satisfy the weaker condition that the subspace $X_0 := \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}$ of X is closed and has finite codimension. We call these operators *quasi-constricted.* Our first main result (Theorem 1) characterizes these operators in the following way: a power bounded $T \in L(X)$ is quasi-constricted iff there exists an attractor for T which has Hausdorff measure of noncompactness $\chi_{\|\cdot\|_1}(A) < 1$ for some equivalent norm $\|\cdot\|_1$ on X. Moreover, we show that for each $\varepsilon > 0$ there exists an ε-T-invariant subspace complementary

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to X_0. Here a subspace Y is called *ε-T-invariant* if for every $y \in Y$ we have $\operatorname{dist}(Ty, Y) \leq \varepsilon \|Ty\|$.

We study the spectrum of a quasi-constricted operator and conclude that for such an operator T, $\lim_{n\to\infty} \operatorname{dist}(T^n x, X_0) = 0$ always implies $x \in X_0$. Moreover, we characterize the essential spectral radius $r_{\mathrm{ess}}(T)$ of an arbitrary bounded operator T by the quasi-constrictedness of scalar multiples of T.

Let T be quasi-constricted. Then T need not be constricted as easy examples show. However, in Theorem 7 we show that if the operators $\overline{\lambda} T$ are mean ergodic for all λ in the peripheral spectrum $\sigma_\pi(T)$ of T then T must be power bounded and constricted.

2. Quasi-constricted operators and their attractors

THEOREM 1. *Let X be a Banach space and let $T \in L(X)$ be a power bounded operator. Then the following conditions are equivalent*:

(i) *T is quasi-constricted.*

(ii) *For every $\varepsilon > 0$ there exists a finite-dimensional ε-T-invariant subspace Y with $X = X_0 \oplus Y$.*

(iii) *For every finite-dimensional subspace Y with $X = X_0 \oplus Y$ and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $T^n(Y)$ is ε-T-invariant and $X = X_0 \oplus T^n(Y)$.*

(iv) *For every $\varepsilon > 0$ there exists an equivalent norm $\|\cdot\|_\varepsilon$ on X and $A_\varepsilon \in \operatorname{Attr}_{\|\cdot\|_\varepsilon}(T)$ such that its Hausdorff measure $\chi_{\|\cdot\|_\varepsilon}(A_\varepsilon) \leq \varepsilon$.*

(v) *There exists an equivalent norm $\|\cdot\|_1$ on X and $A \in \operatorname{Attr}_{\|\cdot\|_1}(T)$ such that $\chi_{\|\cdot\|_1}(A) < 1$.*

Proof. (i)$\Rightarrow$(iv). Take some $\varepsilon > 0$ and let P_0 be a projection of X onto X_0. Let $P := I - P_0$. Define $M := \|P_0\| \sup_{n\in\mathbb{N}} \|T^n\|$ and consider the equivalent norm $\|\cdot\|_\varepsilon$ on X:

$$\|y\|_\varepsilon := \|Py\| + \varepsilon M^{-1} \|P_0 y\| \qquad (y \in X).$$

Take $y \in X$, $\|y\|_\varepsilon \leq 1$. Then $PT^nPy \in PT^n(B_X) \subseteq MP(B_X)$ for all $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}
\varlimsup_{n\to\infty} \operatorname{dist}_{\|\cdot\|_\varepsilon}(T^n y, MP(B_X)) &\leq \varlimsup_{n\to\infty} \|T^n y - PT^nPy\|_\varepsilon \\
&= \varlimsup_{n\to\infty} \|T^n Py - PT^nPy\|_\varepsilon \\
&= \varlimsup_{n\to\infty} \|P_0 T^n Py\|_\varepsilon = \varepsilon M^{-1} \varlimsup_{n\to\infty} \|P_0 T^n Py\| \\
&\leq \varepsilon \|Py\| \leq \varepsilon \|y\|_\varepsilon \leq \varepsilon.
\end{aligned}$$

The set $MP(B_X)$ is compact, since $\dim(P(X)) < \infty$. So we deduce that $A_\varepsilon := MP(B_X) + \{y \in X : \|y\|_\varepsilon \leq \varepsilon\}$ is an attractor for T that satisfies $\chi_{\|\cdot\|_\varepsilon}(A_\varepsilon) \leq \varepsilon$.

(iv)⇒(v). Obvious.

(v)⇒(i). Without loss of generality we assume that $\|\cdot\|_1$ is the initial norm $\|\cdot\|$ on X. Take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and consider the bounded ultrapower $X_{\mathcal{U}} := \ell^\infty(X)/c_{\mathcal{U}}(X)$, where $\ell^\infty(X) := \{(x_n) \in X^{\mathbb{N}} : \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty\}$ and $c_{\mathcal{U}}(X) := \{(x_n) \in X^{\mathbb{N}} : \lim_{\mathcal{U}} \|x_n\| = 0\}$. Then $X_{\mathcal{U}}$ is a Banach space with respect to the norm $\|\widehat{(x_n)}\| := \lim_{\mathcal{U}} \|x_n\|$. We will identify X with the subspace in $X_{\mathcal{U}}$ of all equivalence classes of constant sequences. Consider the linear operator $\mathcal{T} : X \to X_{\mathcal{U}}$ defined by $\mathcal{T}x := \widehat{(T^n x)}$ for all $x \in X$ and let $Z := \mathcal{T}(X)$. Take a real $\delta > 0$ such that $(1+\delta)\alpha < 1$ where $\chi(A) < \alpha < 1$ and let the set $\{a_i\}_{i=1}^p \subseteq X$ satisfy $A \subseteq \bigcup_{i=1}^p B(a_i, \alpha)$.

Take an arbitrary $x = \mathcal{T}y \in B_Z$. Then $\{n : \|T^n y\| < 1 + \delta/2\} \in \mathcal{U}$, and consequently, there exists some $m \in \mathbb{N}$ that satisfies $\|T^m y\| < 1+\delta/2$. Since

$$\lim_{n\to\infty} \operatorname{dist}\Big(T^n T^m y, \bigcup_{i=1}^p B\Big(b_i, (1+\delta/2)\alpha\Big) = 0$$

where $b_i := (1+\delta/2)a_i$, there exists $i_x \in \overline{1,p}$ such that

$$\{k \in \mathbb{N} : \|T^k y - b_{i_x}\| < (1+\delta)\alpha\} \in \mathcal{U}.$$

So, we have $\|x - b_{i_x}\| \le (1+\delta)\alpha$ in $X_{\mathcal{U}}$. Consequently,

$$B_Z \subseteq \bigcup_{i=1}^p B(b_i, (1+\delta)\alpha)).$$

Now [LT, Lemma 2.c.8] implies that $\dim Z < \infty$, since $(1+\delta)\alpha < 1$.

Since the operator T is power bounded, we may assume without loss of generality that $\|T\| \le 1$. Therefore for each $x \in X$ the (nonnegative) sequence $(\|T^n x\|)_{n=1}^\infty$ is decreasing, and hence converges. Assume that $\lim_{\mathcal{U}} \|T^n x\| = 0$. Since the limit of a sequence according to an ultrafilter is a limit point of the sequence and since the sequence $(\|T^n x\|)_{n=1}^\infty$ converges, it follows that $\lim_{n\to\infty} \|T^n x\| = 0$, which proves that $\ker \mathcal{T} \subseteq X_0 := \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}$. The inverse inclusion is trivial. The equality $X_0 = \ker \mathcal{T}$ implies that X_0 is closed and that $\operatorname{codim}(X_0) = \dim Z < \infty$. Consequently, T is quasi-constricted.

(iii)⇒(ii)⇒(i). Obvious.

(i)⇒(iii). Let $X_{\mathcal{U}}$ and $\mathcal{T} : X \to X_{\mathcal{U}}$ be defined as in part (v)⇒(i) of the proof. Let Y be an arbitrary algebraic complement of X_0 in X. Obviously, $T^n(Y) \oplus X_0 = X$ for all $n \in \mathbb{N}$.

Now let $\varepsilon > 0$. If the assertion does not hold then for every $n \in \mathbb{N}$ there exists a normalized $y_n \in Y$ such that $\operatorname{dist}(T(T^n y_n), T^n(Y)) \ge \varepsilon \|T^{n+1} y_n\|$.

The unit sphere of Y is compact, so $\lim_{\mathcal{U}} y_n = y$ exists. But then

$$\begin{aligned}\lim_{\mathcal{U}} \operatorname{dist}(T^{n+1}y, T^n(Y)) &= \lim_{\mathcal{U}} \operatorname{dist}(T^{n+1}y_n, T^n(Y)) \\ &\geq \varepsilon \lim_{\mathcal{U}} \|T^{n+1}y_n\| = \varepsilon \lim_{\mathcal{U}} \|T^{n+1}y\| > 0,\end{aligned}$$

since T is power bounded and since $y \notin X_0 = \ker \mathcal{T}$. We define $\widehat{T}$ on $X_{\mathcal{U}}$ by $\widehat{T}\widehat{(x_n)} = \widehat{(Tx_n)}$ and obtain

$$\operatorname{dist}(\widehat{T}\mathcal{T}y, \mathcal{T}(Y)) \geq \lim_{\mathcal{U}} \operatorname{dist}(T^{n+1}y, T^n(Y)) > 0.$$

But $\widehat{T}\mathcal{T}y = \widehat{(T^{n+1}y)} = \mathcal{T}Ty \in \mathcal{T}(X) = \mathcal{T}(Y)$, a contradiction. ∎

Let us give some easy examples of power bounded quasi-constricted operators which are not constricted:

Example 1. Let $X := C[0,1]$. Define $T : X \to X$ by $Tf(t) := tf(t)$. Then $X_0 := \{f \in C[0,1] : \lim_{n\to\infty} \|T^n f\| = 0\} = \{f \in C[0,1] : f(1) = 0\}$ has codimension 1. So, T is quasi-constricted but not constricted, since it has no nontrivial eigenvectors.

Example 2 ([EW, p. 217]). Let $X = c_0$ with sup-norm $\|\cdot\|$. Denote by e_k the element of X whose kth coordinate is 1, and all other coordinates are zero. Fix a real $\alpha \geq 0$ and define the operator $S_\alpha : X \to X$ by

$$S_\alpha(e_k) = \begin{cases} e_1 + \alpha e_2 & \text{if } k = 1, \\ e_{k+1} & \text{else,} \end{cases}$$

and let $T_\alpha := (I + S_\alpha)/2$. It was shown in [EW, Lemma 2.3] that $X_0 = \{x \in X : \lim_{n\to\infty} \|T_\alpha^n x\| = 0\} = \{x \in c_0 : x_1 = 0\}$, and consequently, $\operatorname{codim}(X_0) = 1$. So, the operator T_α is quasi-constricted. More exactly, T_α has an attractor $A_\alpha := [-e_1, e_1] + \alpha B_X$ such that $\chi(A_\alpha) = \alpha$, and for every $A \in \operatorname{Attr}_{\|\cdot\|}(T_\alpha)$ we have $\chi(A) \geq \alpha$, since the sequence $T_\alpha^n(e_1)$ is increasing and its supremum in ℓ^∞ is easily determined as $(1, \alpha, \alpha, \ldots)$. This also implies that when $\alpha > 0$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$, then the operator λT_α is mean ergodic if and only if $\lambda \notin \sigma_\pi(T_\alpha) = \{1\}$. In particular, the operator T_α is not constricted for $\alpha > 0$.

Take some real $\beta > 0$ and consider the equivalent norm $\|\cdot\|_\beta$ on c_0 defined by the formula

$$\|x\|_\beta := \sup\{|x_1|, \beta\|x - x_1 e_1\|\}.$$

It is easy to see that $\chi_{\|\cdot\|_\beta}(A) \geq \alpha\beta$ for every $A \in \operatorname{Attr}_{\|\cdot\|_\beta}(T_\alpha)$. In particular, for $\beta = 1/\alpha$ the operator T_α has no attractor A satisfying $\chi_{\|\cdot\|_\beta}(A) < 1$. It should be noted that the operator T_α is a contraction with respect to the norm $\|\cdot\|_\beta$ whenever $\beta \leq 1/\alpha$.

Example 3. The same example can also be considered on $\ell^2(\mathbb{N})$ where it is an example of a quasi-constricted operator which is neither power bounded nor constricted. As we shall see later on, as a consequence of Theorem 7 quasi-constricted power bounded operators on a Hilbert space are constricted.

EXAMPLE 4. Here we consider a generalization of Example 2. Let $0 \neq S \in L(X)$ be a power bounded operator such that $\operatorname{codim} \overline{(I-S)X} < \infty$. Without any restriction we may assume that $\|S\| = 1$. Take some real $\alpha \in (0,1)$ and consider the operator $T_\alpha := \alpha I + (1-\alpha)S$. The result from Foguel and Weiss [FW, Lemma 2.1] implies

$$\|T_\alpha^{n+1} - T_\alpha^n\| = \|T_\alpha^n(T_\alpha - I)\| = \|(1-\alpha)T_\alpha^n(I-S)\| \to 0.$$

Then $\overline{(I-S)X} \subseteq X_0 := \{x \in X : \lim_{n\to\infty} \|T_\alpha^n x\| = 0\}$. So, $\operatorname{codim}(X_0) \leq \operatorname{codim} \overline{(I-S)X} < \infty$. Consequently, T_α is quasi-constricted.

3. On the spectrum of quasi-constricted operators. Let T be a quasi-constricted operator. Let $X = X_0 \oplus Y$ where Y is an arbitrary finite-dimensional complement of X_0. Let P be the projection from X onto Y with kernel X_0 and define $Q = I - P$, $U = QTQ$, $V = QTP$, and $W = PTP$. Since X_0 is invariant under T, we have $PTQ = 0$, hence $T = U + V + W$. More precisely, identifying X with $X_0 \times Y$ we obtain the following matrix representation of T:

$$T = \begin{pmatrix} U & V \\ 0 & W \end{pmatrix}. \tag{1}$$

Then T^n is represented by the matrix

$$T^n = \begin{pmatrix} U^n & \sum_{k=0}^{n-1} U^k V W^{n-1-k} \\ 0 & W^n \end{pmatrix}. \tag{2}$$

Let $\varrho(S)$ denote the resolvent set $\mathbb{C} \setminus \sigma(S)$ of the operator S. Then an easy calculation shows that for $\lambda \in \varrho(U) \cap \varrho(W)$,

$$(\lambda - T)^{-1} = \begin{pmatrix} (\lambda - U)^{-1} & (\lambda - U)^{-1} V (\lambda - W)^{-1} \\ 0 & (\lambda - W)^{-1} \end{pmatrix}.$$

In particular, $\sigma(T) \subset \sigma(U) \cup \sigma(W)$.

The following elementary proposition shows a close connection between the essential spectral radius $r_{\mathrm{ess}}(T)$ and the property of being quasi-constricted.

PROPOSITION 2. *Let $T \in L(X)$.*

(i) *If $r_{\mathrm{ess}}(T) < 1$ then T is constricted.*
(ii) *If T is quasi-constricted then $r_{\mathrm{ess}}(T) \leq 1$.*

Proof. (i) Let $r_{\mathrm{ess}}(T) < 1$. Let $\beta = \inf\{|\lambda| : \lambda \in \sigma(T),\ |\lambda| \geq 1\}$, $\delta = \sup\{|\lambda| : \lambda \in \sigma(T),\ |\lambda| < 1\}$, and

$$\gamma = \begin{cases} 1 & \text{if } \beta > 1, \\ (1+\delta)/2 & \text{otherwise.} \end{cases}$$

Then the circle $K := \{z : |z| = \gamma\}$ is in the resolvent set of T and the projection $P = I - (2\pi i)^{-1} \int_K (z-T)^{-1}\, dz$ maps X onto $P(X)$ which is invariant under T and finite-dimensional because $\sigma(T) \cap \{\lambda : |\lambda| > \delta\}$ consists only of finitely many Riesz points of $\sigma(T)$. Moreover, $\sigma(T|_{P(X)}) \subset \{\lambda : |\lambda| \geq 1\}$. The space $(I-P)(X) =: Z$ is also invariant under T and by construction $r(T|_Z) < 1$ hence $Z = X_0$, and (i) is proved.

(ii) Using the notation of (1) we have $U^n \to 0$ strongly. Therefore, by the Uniform Boundedness Principle, U is power bounded, hence $r(U) \leq 1$. So if $\lambda \in \sigma(T)$ satisfies $|\lambda| > 1$ then $\lambda \in \sigma(W)$, and since the subspace Y on which W acts is finite-dimensional, λ is an isolated point of $\sigma(T)$. Let L be the spectral projection corresponding to the spectral set $\{\lambda\}$ and set $Z = L(X)$. Then Z is T-invariant and $\sigma(T|_Z) = \{\lambda\}$. This in turn yields $Z \cap X_0 = \{0\}$, so $\dim(Z) < \infty$ and hence λ is a Riesz point of T. Therefore $r_{\text{ess}}(T) \leq 1$. ∎

Now we use this result in order to give a new characterization of the essential spectral radius of an arbitrary bounded operator T on the Banach space X. For $T \in L(X)$ we let

$$\alpha_{\text{c}}(T) := \inf\{\alpha \in \mathbb{R}_+ : \alpha^{-1}T \text{ is constricted}\},$$
$$\alpha_{\text{q}}(T) := \inf\{\alpha \in \mathbb{R}_+ : \alpha^{-1}T \text{ is quasi-constricted}\}.$$

Then our proposition yields the following characterization of the essential spectral radius of the operator T.

Corollary 3. *Let $T \in L(X)$. Then $\alpha_{\text{c}}(T) = \alpha_{\text{q}}(T) = r_{\text{ess}}(T)$.*

Proof. Obviously, $\alpha_{\text{q}}(T) \leq \alpha_{\text{c}}(T)$. To finish the proof we show that:

(I) $r_{\text{ess}}(T) \leq \alpha_{\text{q}}(T)$, and
(II) $\alpha_{\text{c}}(T) \leq r_{\text{ess}}(T)$.

(I) Let $\varepsilon > 0$. Then $T/(\alpha_{\text{q}}(T)+\varepsilon)$ is quasi-constricted, hence its essential spectral radius is less than or equal to 1 by the proposition. This in turn yields $r_{\text{ess}}(T) \leq \alpha_{\text{q}}(T) + \varepsilon$ and the assertion follows.

(II) Assume $r_{\text{ess}}(T) < \alpha_{\text{c}}(T)$ and choose $r_{\text{ess}}(T) < \beta < \alpha_{\text{c}}(T)$. Then $r_{\text{ess}}(T/\beta) < 1$. By the proposition T/β is constricted, a contradiction. ∎

Now we turn to another property of quasi-constricted operators which is a corollary of the next proposition about the spectrum of the operator W (see (1)).

Proposition 4. *Let T be quasi-constricted and let $X = X_0 \oplus Y$ be an arbitrary decomposition. For W as in the matrix representation* (1) *above, $\sigma(W) \subset \{\lambda : |\lambda| \geq 1\}$.*

Proof. Suppose that the assertion fails. Since Y is finite-dimensional there exists an eigenvalue λ of W of absolute value strictly less than 1. Let e be a corresponding eigenvector. Then by (2) we obtain

$$T^n(e) = \lambda^n e + \sum_{k=0}^{n-1} \lambda^{n-1-k} U^k V e.$$

Now $Ve \in X_0$ implies that for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $\|U^n Ve\| < \varepsilon$ for all $n \geq n(\varepsilon)$. Moreover, U is power bounded on X_0 by the Uniform Boundedness Principle. So set $M = \sup\{\|U^n\| : n \in \mathbb{N}\}$. Set $N = n(\varepsilon)$ and let $p \geq 1$. Then

$$\sum_{k=0}^{N+p-1} \lambda^{N+p-1-k} U^k Ve = \sum_{k=0}^{N-1} \lambda^{N+p-1-k} U^k Ve + \sum_{k=1}^{p} \lambda^{p-1-k} U^k Ve$$

implies

$$\begin{aligned}\Big\| \sum_{k=0}^{N+p-1} \lambda^{N+p-1-k} U^k Ve \Big\| & \\ &\leq \sum_{k=0}^{N-1} |\lambda|^{N+p-1-k} \|U^k Ve\| + \sum_{k=1}^{p} |\lambda|^{p-1-k} \|U^{N+k} Ve\| \\ &\leq |\lambda|^p \frac{M\|Ve\|}{1-|\lambda|} + \frac{\varepsilon}{|\lambda|(1-|\lambda|)},\end{aligned}$$

which yields $\lim_{n\to\infty} T^n e = 0$, i.e. $e \in X_0$, a contradiction. ∎

Corollary 5. *Let T be a quasi-constricted operator on the Banach space X and let Z be the quotient space X/X_0. Then the operator $\widetilde{T}$ induced by T on Z satisfies $\sigma(\widetilde{T}) \subset \{\lambda : |\lambda| \geq 1\}$. Moreover, let $X_0^\perp$ denote the polar of X_0 in the dual space X^*. Then $X_0^\perp$ is T^*-invariant, isomorphic to the dual space Z^* of Z and $\sigma(T^*|_{X_0^\perp}) = \sigma(\widetilde{T})$.*

Proof. We use the notation which led to (1). Suppose that the assertion fails. Then there exists an eigenvector $\widetilde{e}$ corresponding to an eigenvalue λ of absolute value strictly less than 1. Since $\widetilde{e} \neq 0$ there exists $e \in Y$ such that $\widetilde{e} = e + X_0$. But then $We = \lambda e$, a contradiction to the proposition above. The remainder is an elementary exercise on polars (see e.g. [Sch2], Chapt. IV, Sect. 1 and 2). ∎

Corollary 6. *Let T be a quasi-constricted operator on the Banach space X. Then $\lim_{n\to\infty} \operatorname{dist}(T^n x, X_0) = 0$ always implies $x \in X_0$.*

Proof. The hypothesis implies $\lim_{n\to\infty} \widetilde{T}^n \widetilde{x} = 0$ where the $\widetilde{\ }$ denotes the corresponding objects in $Z = X/X_0$ (see the foregoing corollary). Since $\sigma(\widetilde{T}) \subset \{\lambda : |\lambda| \geq 1\}$ and $\dim(Z) < \infty$ we obtain $\widetilde{x} = 0$. ∎

Quasi-constricted operators need not be power bounded (see Example 3). However, it is easy to see that for the result of Corollary 6 to hold the operator T need not be quasi-constricted if it is power bounded. For, X_0 is then obviously closed. Moreover, $\lim_{n\to\infty} \operatorname{dist}(T^n x, X_0) = 0$ implies the existence of a sequence $(y_n)_n$ in X_0 satisfying $\lim_{n\to\infty} \|T^n x - y_n\| = 0$. The inequality

$$\|T^{k+n}x\| \leq \|T^k\| \cdot \|T^n x - y_n\| + \|T^k y_n\|$$

yields $\limsup_k \|T^k x\| \leq M\|T^n x - y_n\|$ for fixed n since y_n is in X_0. So $x \in X_0$.

On the other hand, let T be an arbitrary bounded linear operator on the Banach space X such that $X_0 = \{x : \lim_{n\to\infty} T^n x = 0\}$ is closed. Then $\lim_{n\to\infty} \operatorname{dist}(T^n x, X_0) = 0$ does not in general imply $\lim_{n\to\infty} T^n x = 0$ as the following example shows.

Example 5. Let H be a separable Hilbert space. Define an operator $T \in L(H)$ as follows:

$$T(e_{i,j}) = \begin{cases} \alpha_i e_{i+1,1} + e_{i,2}, & j = 1, \\ (e_{i,j} + e_{i,j+1})/2, & j > 1, \end{cases}$$

where $\{e_{i,j}\}_{i,j=1}^{\infty}$ is an orthonormal basis of H and

$$\alpha_i = \Big(\sum_{l=1}^{i} l^{-1}\Big)^{1/3} \Big(\sum_{l=1}^{i+1} l^{-1}\Big)^{-1/3}$$

for all $i \in \mathbb{N}$. By use of the inequality

$$2^{-2l} \sum_{k=0}^{l} \binom{l}{k}^2 \geq \frac{1}{4l},$$

which holds for each $l \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|T^{n+1}(e_{i,1})\| &\geq \Big\| \Big(\prod_{\xi=i}^{n+i-2} \alpha_\xi\Big) \sum_{l=1}^{n-1} 2^{-l} \sum_{k=0}^{l} \binom{l}{k} e_{n+i-l-1,2+k} \Big\| \\
&\geq \Big(\sum_{l=1}^{n-1} 2^{-2l} \sum_{k=0}^{l} \binom{l}{k}^2\Big)^{1/2} \prod_{\xi=i}^{n+i-2} \alpha_\xi \geq \Big(\frac{1}{4}\sum_{l=1}^{n-1} l^{-1}\Big)^{1/2} \prod_{\xi=1}^{n-1} \alpha_\xi \\
&= \frac{1}{2}\Big(\sum_{l=1}^{n-1} l^{-1}\Big)^{1/2} \Big(\sum_{l=1}^{n} l^{-1}\Big)^{-1/3} \\
&\geq \frac{1}{4}\Big(\sum_{l=1}^{n} l^{-1}\Big)^{1/6}
\end{aligned}$$

for all $n \geq 2$ and $i \geq 1$. Since $\|T^n x\| \geq |x_{i,1}| \cdot \|T^n(e_{i,1})\|$ for $i \in \mathbb{N}$, we have $\lim_{n\to\infty} \|T^n x\| = \infty$ whenever $x_{i,1} \neq 0$ for some $i \in \mathbb{N}$. On the other hand,

it can be shown that
$$(\forall x \in H)[[(\forall i \in \mathbb{N})[x_{i,1} = 0]] \Rightarrow \lim_{n\to\infty} \|T^n x\| = 0].$$
Then $H_0 := \{x \in H : \lim_{n\to\infty} \|T^n x\| = 0\} = \{x \in H : (\forall i \in \mathbb{N})[x_{i,1} = 0]\}$ is a closed subspace of H. At the same time
$$\lim_{n\to\infty} \operatorname{dist}(T^n x, H_0) = \lim_{n\to\infty} \Big(\sum_{p=1}^{\infty} \Big(\prod_{i=p}^{p+n-1} \alpha_i\Big)|x_{p,1}|^2\Big)^{1/2} = 0$$
for all $x \in H$.

4. When are quasi-constricted operators constricted? Our last result presents a sufficient condition for quasi-constricted operators to be constricted. Let us remark that, for example, a normal quasi-constricted operator on a Hilbert space is always constricted, as is easily seen by use of the spectral measure of the operator.

THEOREM 7. *Let X be a Banach space and let $T \in L(X)$ be such that $\overline{\lambda}T$ is mean ergodic for all $\lambda \in \sigma_\pi(T)$. Then the following conditions are equivalent*:

(i) *T is quasi-constricted.*

(ii) *T is constricted and power bounded.*

(iii) *T has an attractor A that satisfies $\chi_{\|\cdot\|_1}(A) < 1$ for some equivalent norm $\|\cdot\|_1$ on X.*

Proof. Let us recall that whenever the bounded operator T is mean ergodic then $(T^n/n)_n$ converges strongly to 0 (see [Kr], p. 72). Then by the Uniform Boundedness Principle the sequence is uniformly bounded, which yields in turn $r(T) \le 1$. In fact, for $r(T) < 1$ the assertion of the theorem is trivial. So without loss of generality we assume $r(T) = 1$.

(i)⇒(ii). A unimodular eigenvalue for T is obviously also an eigenvalue for $\widetilde{T}$ on the space Z defined in Corollary 5. Hence there are only finitely many unimodular eigenvalues $\{\lambda_i\}_{i=1}^p$. Let $F_{\lambda_i}(T)$ be the subspace of all eigenvectors corresponding to λ_i. Finally, let $E(T)$ be the linear span of all $F_{\lambda_i}(T)$. Since X_0 is closed and has finite codimension and since $F_{\lambda_i}(T) \cap X_0 = \{0\}$, the space $E(T)$ is finite-dimensional, hence closed, and $E(T) \cap X_0 = \{0\}$. Moreover, each $F_{\lambda_i}(T)$ is the space of fixed vectors of the operator $\overline{\lambda_i}T$.

Now let $G = X_0^\perp$. Since T^* is weak*-mean ergodic and G is T^*-invariant and finite-dimensional, $T^*|_G$ is uniformly mean ergodic, which by Corollary 5 implies that G is the span of all eigenspaces corresponding to unimodular eigenvalues of T^* (see e.g. exercise 6 on p. 43 of [Sch1]), or in other terms, that $G = E(T^*)$ (see the preceding paragraph). By Sine's famous ergodic theorem (see [Kr], p. 74), $F_{\lambda_i}(T)$ separates points in $F_{\lambda_i}(T^*)$, hence $E(T)$ separates points in $E(T^*)$, which in turn implies $\dim(E(T)) \geq \dim(G) = \operatorname{codim}(X_0)$. So we obtain $X = X_0 \oplus E(T)$ and (ii) follows.

(ii)⇒(iii) and (iii)⇒(i) follow from Theorem 1. ∎

COROLLARY 8. *Let T be a bounded linear operator on the Banach space X and assume that the semigroup $\{T^n\}_{n=0}^{\infty}$ is weakly almost periodic. The following assertions are equivalent*:

(i) *T is constricted and power bounded.*

(ii) *T has a compact attractor.*

(iii) *T has an attractor A which satisfies $\chi_{\|\cdot\|_1}(A) < 1$ for some equivalent norm $\|\cdot\|_1$.*

(iv) *T is quasi-constricted.*

Proof. One only needs to notice that the semigroup $\{(\lambda T)^n\}_{n=0}^{\infty}$ is weakly almost periodic for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then λT is mean ergodic by the Eberlein theorem [Kr, p. 76] for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$. The corollary follows immediately from Theorem 7. ∎

COROLLARY 9. *Let T be a power bounded operator on the Banach space X and assume that X is reflexive. Then T is quasi-constricted if and only if T is constricted.*

Proof. The semigroup generated by T is weakly almost periodic since the unit ball of X is weakly compact. ∎

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