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Orbits of linear operators and Banach space geometry

by

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Abstract. Let T be a bounded linear operator on a (real or complex) Banach space X. If (a_n) is a sequence of non-negative numbers tending to 0, then the set of $x \in X$ such that $\|T^n x\| \geq a_n \|T^n\|$ for infinitely many n's has a complement which is both σ-porous and Haar-null. We also compute (for some classical Banach space) optimal exponents $q > 0$ such that for every non-nilpotent operator T, there exists $x \in X$ such that $(\|T^n x\|/\|T^n\|) \notin \ell^q(\mathbb{N})$, using techniques which involve the modulus of asymptotic uniform smoothness of X.

1. Introduction. Let X be a (real or complex) Banach space, and let T be a bounded linear operator on X. For $x \in X$, let

$$O_T(x) = \{T^n x : n \geq 0\}$$

be the orbit of x under the action of T. The study of orbits is connected with the famous invariant subset problem which asks if there exists an operator on X with non-trivial invariant subset. Indeed, T does not have any trivial invariant subset if and only if for each $x \neq 0$, $O_T(x)$ is dense in X. Such an operator was constructed by Read [R] in the space ℓ^1, but in the Hilbert space, the problem is still open. If at least one orbit is dense, the operator is called *hypercyclic* (and the corresponding vector a *hypercyclic vector*). This class of operators has received much attention during the last two decades (see [BM] for much information on this topic). In this paper, we will, however, study some more regular orbits. Müller [Mu] showed the following result, which roughly says that there are many points with large orbits for many powers.

THEOREM 1.1. *Let T be a bounded linear operator on X, and let (a_n) be a sequence of non-negative numbers such that $a_n \to 0$. Then the set*

$$\{x \in X : \|T^n x\| \geq a_n \|T^n\| \text{ for infinitely many } n\text{'s}\}$$

is residual in X.

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A quick glance at the proof shows that the powers can be replaced by a sequence (T_n) of bounded linear operators. It may also be worth emphasizing that this is stronger than the uniform boundedness principle. Indeed, suppose that $\sup\|T_n\| = \infty$ and find a sequence (n_j) such that $\|T_{n_j}\| \to \infty$. Put $a_j = 1/\sqrt{\|T_{n_j}}$ and apply the above result to $S_j = T_{n_j}$ to deduce that there is a residual set of points $x \in X$ such that $\|T_{n_j}x\| \geq \sqrt{\|T_{n_j}}$ for infinitely many j's, and in particular $\sup\|T_n x\| = \infty$. Equivalently, Theorem 1.1 says that the complement of the set in question is of the first category.

Now, there are several other notions of smallness in analysis. In this paper, we consider two of them: σ-porosity, which is a stronger form of smallness than being of first category, and Haar-negligibility, which is an extension of having Lebesgue measure 0 in infinite dimensions (see next section for definitions). These two notions are actually not comparable: Preiss and Tišer (see [BL, Chapter 6]) showed that any real separable Banach space of infinite dimension can be decomposed as the disjoint union of two sets, one of which is σ-porous and the other Haar-null.

In Section 2, we generalize Theorem 1.1 as follows:

THEOREM 1.2. *Let X be a Banach space* (*real or complex*) *and (T_n) be a sequence of bounded linear operators on X. Let also (a_n) be a sequence of non-negative numbers such that $a_n \to 0$. Then the complement of the set*

$$\{x \in X : \|T_n x\| \geq a_n\|T_n\| \text{ for infinitely many } n\text{'s}\}$$

is σ-porous. If X is separable, then this complement is also Haar-null.

These notions of smallness, together with linear dynamics, have also been studied (for different problems) in [Bay1] and [BMM]. The example $T = B$ where B is the unweighted backward shift defined on $\ell^1(\mathbb{N})$ by $Be_1 = 0$ and $Be_k = e_{k-1}$ for $k \geq 2$ (where (e_k) is the canonical basis of ℓ^1) satisfies $\|T^n x\|/\|T^n\| \to 0$ for each x, because $\|T^n\| = 1$ and $\|T^n x\| = \sum_{k=n+1}^{\infty}|x_k|$ for $x = \sum_{k=1}^{\infty} x_k e_k$. Thus, in general, the condition "$a_n \to 0$" cannot be improved.

In Section 3, we study some cases involving compact operators to get better estimates. We also give some examples to discuss the limitations of our results. Section 4 contains our main result. As should be clear from what we said, the underlying theme of this paper is to look for points $x \in X$ such that many powers $\|T^n x\|$ are as close as possible to $\|T^n\|$. Beauzamy [Be] showed that given a bounded linear operator T on a Hilbert space H, the set

$$\left\{x \in H : \sum_{n=1}^{\infty} \frac{\|T^n x\|}{\|T^n\|} = \infty\right\}$$

is dense in H (which in some sense says that $\|T^n x\|$ is not too far from $\|T^n\|$ for many powers). Using an alternative proof, Müller [Mu] showed that for each $q < 2$, the set

$$\left\{x \in H : \sum_{n=1}^{\infty} \left(\frac{\|T^n x\|}{\|T^n\|}\right)^q = \infty\right\}$$

is dense in H, and also showed a similar statement for operators on Banach spaces (replacing $q < 2$ by $q < 1$). He also exhibited examples showing that the constants 1 and 2 are optimal for Banach space and Hilbert space operators. These problems also have connections with some famous plank theorems of Ball (see [Bal1] and [Bal2]). We recall those results.

THEOREM 1.3 (K. Ball, [Bal1]). *Let X be a (real or complex) Banach space and $(f_n) \subset X^*$ such that $\|f_n\| = 1$ for each n. Let also $(\alpha_n) \subset \mathbb{R}^+$ be such that $\sum_{n=1}^{\infty} \alpha_n < 1$. Then there is a point x with $\|x\| = 1$ such that $|\langle f_n, x\rangle| \geq \alpha_n$ for each n.*

In [Bal2], the condition $\sum_{n=1}^{\infty} \alpha_n < 1$ is improved to $\sum_{n=1}^{\infty} \alpha_n^2 < 1$ for complex Hilbert spaces. Now, considering the adjoint of T^n, one can show that a similar statement holds for sequences of operators (see [MV] for details). This gives a direct proof of the above results. Anyway, the previous exponents suggest that for a Banach space X, the quantity

$$q_X = \sup\Big\{q > 0 : \text{for every non-nilpotent and bounded linear operator } T,$$
$$\sum_{n=1}^{\infty} \left(\frac{\|T^n x\|}{\|T^n\|}\right)^q = \infty \text{ for some } x \in X\Big\}$$

should depend on the geometry of X. This will be the case and we will in particular obtain:

THEOREM 1.4. *$q_{\ell^p} = p$, $q_{L^p} = \min(p, 2)$ $(1 \leq p < \infty)$, $q_{c_0} = \infty$.*

To unify those results, we will use the modulus of asymptotic uniform smoothness of X, which is a tool from Banach space geometry that has been used for several problems of nonlinear functional analysis. We refer the reader to Section 4 for definitions. Note that a similar discussion, involving weakly closed sequences and type of the space, can be found in [Bay2] and [BM, Chapter 10]. Throughout, we shall denote by $\mathcal{L}(X)$ the set of bounded linear operators on a Banach space X and by $B(x, r)$ the open ball of center x and radius r $(x \in X, r > 0)$.

2. σ-porosity and Haar-negligibility: proof of Theorem 1.2. Let us first recall the definitions of σ-porous and Haar-null sets. The notion of porosity quantifies the fact that a set has empty interior. Porosity was introduced by E. P. Dolženko [D], and has been studied in detail since then (see [Z]). It appears for example in the study of differentiability properties of real-valued convex functions defined on a separable Banach space X.

Definition 2.1. A Borel subset E of a Banach space X is called *porous* if there exists $\lambda \in]0,1[$ such that the following is true: for every $x \in E$ and every $\epsilon > 0$, there exists a point $y \in X$ such that $0 < \|y - x\| < \epsilon$ and $E \cap B(y, \lambda\|x - y\|)$ is empty. A countable union of porous sets is said to be *σ-porous.*

Haar-null sets were introduced by Christensen [C]. They appear in the study of differentiability of Lipschitz functions defined on a Banach space. Note that the definition of Haar-null sets makes sense in any Polish abelian group G. Here, we restrict ourselves to the Banach space setting.

Definition 2.2. Let X be a separable Banach space. A set $E \subset X$ is said to be *Haar-null* if there exists a Borel probability measure m on X such that for every $x \in X$, the translate $x + E$ has m-measure 0.

With those definitions in mind, we can now start the proof of Theorem 1.2. Let $(T_n) \subset \mathcal{L}(X)$ and $(a_n) \subset \mathbb{R}^+$ be such that $a_n \to 0$. If infinitely many T_n's are 0, then the result is obvious. We may assume, without loss of generality, that $T_n \neq 0$ for every n. Let us first prove the assertion about σ-porosity. We can write the complement in the form $\bigcup_{N=1}^{\infty} E_N$ with

$$E_N = \{x : \forall n \geq N,\ \|T_n x\| < a_n \|T_n\|\}.$$

We shall see that for fixed $N \geq 1$, E_N is porous with the constant $\lambda = 1/4$ (but, actually, any fixed $\lambda \in]0,1[$ works, by adjusting the computations in what follows). Consider now $x \in E_N$ and $\epsilon > 0$. Fix $n \geq N$ and y_0 with $\|y_0\| = 1$ such that

$$a_n \leq \frac{\epsilon}{8} \quad \text{and} \quad \|T_n y_0\| \geq \frac{\|T_n\|}{2}.$$

We have

$$\left\|T_n\left(x + \frac{\epsilon}{2} y_0\right)\right\| + \left\|T_n\left(x - \frac{\epsilon}{2} y_0\right)\right\| \geq \epsilon \|T_n y_0\| \geq \frac{\epsilon}{2}\|T_n\|,$$

so

$$\left\|T_n\left(x + \frac{\epsilon}{2} y_0\right)\right\| \geq \frac{\epsilon}{4}\|T_n\| \quad \text{or} \quad \left\|T_n\left(x - \frac{\epsilon}{2} y_0\right)\right\| \geq \frac{\epsilon}{4}\|T_n\|.$$

Replacing possibly y_0 by $-y_0$, we can assume that $\|T_n(x + \frac{\epsilon}{2} y_0)\| \geq \frac{\epsilon}{4}\|T_n\|$. Put $y = x + \frac{\epsilon}{2} y_0$, so $\|y - x\| = \epsilon/2 < \epsilon$. To conclude, it is enough to show that

$$B\left(y, \frac{1}{4}\|x - y\|\right) \cap E_N = B\left(y, \frac{\epsilon}{8}\right) \cap E_N = \emptyset.$$

Let $z \in B(y, \epsilon/8)$. We have

$$\begin{aligned} \|T_n z\| \geq \|T_n y\| - \|T_n(z - y)\| &\geq \frac{\epsilon}{4}\|T_n\| - \frac{\epsilon}{8}\|T_n\| \\ &\geq \frac{\epsilon}{8}\|T_n\| \geq a_n \|T_n\| \end{aligned}$$

and $z \notin E_N$, as announced.

Let us now prove the second assertion. Considering only the real linear structure of X, we may assume that X is a real Banach space. Keeping the same notation as above, it is enough to show that each E_N is Haar-null since a countable union of Haar-null sets is Haar-null. An efficient way to prove that a Borel subset $E \subset X$ is Haar-null is to find a subspace $V \subset X$ of finite dimension such that

$$\text{for all } x \in X \text{ and almost every } v \in V, \qquad x + v \notin E$$

(here "almost every" refers to the Lebesgue measure on V). We can find $u \in X$ such that $\|T_n u\| \geq \sqrt{a_n}\|T_n\|$ for infinitely many n's (by the first part of the theorem); we will show that $V = \mathbb{R}u$ is the subspace we are looking for. Fix $x \in X$ and put

$$\Lambda = \{\lambda \in \mathbb{R} : x + \lambda u \in E_N\}.$$

It remains to check that Λ has Lebesgue measure 0. Let $\lambda \in \Lambda$. Then

$$\|T_n(x + \lambda u)\| \leq a_n \|T_n\| \qquad (n \geq N).$$

Hence, we get

$$\left| |\lambda| - \frac{\|T_n x\|}{\|T_n u\|} \right| \leq a_n \frac{\|T_n\|}{\|T_n u\|} \qquad (n \geq N).$$

Put $b_n = \|T_n x\|/\|T_n u\|$. The above inequality shows that $\lambda \in E_+ \cup E_-$, where

$$E_+ = \bigcap_{n \geq N} \left[b_n - a_n \frac{\|T_n\|}{\|T_n u\|}, b_n + a_n \frac{\|T_n\|}{\|T_n u\|} \right],$$

$$E_- = \bigcap_{n \geq N} \left[-b_n - a_n \frac{\|T_n\|}{\|T_n u\|}, -b_n + a_n \frac{\|T_n\|}{\|T_n u\|} \right].$$

This implies that the Lebesgue measure of Λ is not greater than

$$4 \inf_{n \geq N} a_n \frac{\|T_n\|}{\|T_n u\|},$$

which is 0, because for infinitely many n's,

$$a_n \frac{\|T_n\|}{\|T_n u\|} \leq \sqrt{a_n} \to 0 \qquad (n \to \infty).$$

This completes the proof.

Let X be a complex Banach space and $T \in \mathcal{L}(X)$. Let $r(T)$ be the spectral radius of T and $r_x(T)$ be its local spectral radius defined by $r_x(T) = \overline{\lim} \|T^n x\|^{1/n}$. We obtain:

Corollary 2.3. *If X is a complex Banach space and $T \in \mathcal{L}(X)$, then the complement of the set of x such that $r_x(T) = r(T)$ is σ-porous, and Haar-null if X is separable.*

Proof. Apply the above result to $a_n = 1/n$ and use the spectral radius formula: $\lim \|T^n\|^{1/n} = r(T)$. ■

3. Compact case. In this section, we move away from Haar-negligibility and σ-porosity and try to improve the condition $a_n \to 0$. Note that in contrast to Theorem 1.2, the proof of the next proposition really uses the powers of the operator T.

PROPOSITION 3.1. *Let X be a real or complex Banach space and let $T \in \mathcal{L}(X)$ be such that:*

(i) *T is compact.*
(ii) *$(\|T^n\|)$ is non-decreasing.*

Then, for each $\epsilon > 0$, there exists $x \in X$ with $\|x\| \le 1$ such that for infinitely many n's, we have

$$\|T^n x\| \ge (1-\epsilon)\|T^n\|.$$

Furthermore,

$$\left\{x \in X : \overline{\lim} \frac{\|T^n x\|}{\|T^n\|} > 0\right\}$$

is a dense subset of X.

Proof. There exists $(x_n) \subset X$ with $\|x_n\| = 1$ such that $\|T^n x_n\| \ge (1-\epsilon/2)\|T^n\|$. By the compactness of T, one can extract from (Tx_n) a norm convergent subsequence (Tx_{n_k}). So, we can find N such that for $k \ge N$,

$$\|Tx_{n_k} - Tx_N\| \le \epsilon/2.$$

Put $x = x_N$. For $k \ge N$, we get

$$\begin{aligned}
\|T^{n_k} x\| &\ge \|T^{n_k} x_{n_k}\| - \|T^{n_k}(x - x_{n_k})\| \\
&\ge (1-\epsilon/2)\|T^{n_k}\| - \|T(x - x_{n_k})\|\,\|T^{n_k-1}\| \\
&\ge (1-\epsilon/2 - \|T(x-x_{n_k})\|)\|T^{n_k}\| \quad \text{as } (\|T^n\|) \text{ is non-decreasing} \\
&\ge (1-\epsilon)\|T^{n_k}\|.
\end{aligned}$$

Let us see the density. Take $\eta > 0$ and $x \in X$. By the above, we can find a point x_0 with $\|x_0\| \le 1$ such that for infinitely many n's,

$$\|T^n x_0\| \ge \frac{1}{2}\|T^n\|.$$

For those n, we have

$$\|T^n(x+\eta x_0)\| + \|T^n(x-\eta x_0)\| \ge 2\eta\|T^n x_0\| \ge \eta\|T^n\|.$$

Hence

$$\overline{\lim} \frac{\|T^n(x+\eta x_0)\|}{\|T^n\|} > 0 \quad \text{or} \quad \overline{\lim} \frac{\|T^n(x-\eta x_0)\|}{\|T^n\|} > 0,$$

and since $\|x - (x \pm \eta x_0)\| \le \eta$, we get the density. ■

The following two examples show that we cannot remove assumptions (i) or (ii).

EXAMPLE 3.2.

(i′) *There exists* $T \in \mathcal{L}(\ell^p(\mathbb{N}))$ *such that* $(\|T^n\|)$ *is non-decreasing and for all* $x \in \ell^p(\mathbb{N})$,

$$\frac{\|T^n x\|}{\|T^n\|} \to 0.$$

(ii′) *There exists a compact operator* $T \in \mathcal{L}(\ell^p(\mathbb{N}))$ *such that for every* $x \in \ell^p(\mathbb{N})$,

$$\frac{\|T^n x\|}{\|T^n\|} \to 0.$$

Proof. For (i′), it is enough to take $T = B$ where B is the unweighted backward shift (see introduction). For (ii′), we let T be the weighted backward shift defined on $\ell^p(\mathbb{N})$ with its natural norm and canonical basis (e_k) by

$$Tx = \sum_{k=2}^{\infty} w_k x_k e_{k-1}$$

where (w_k) is a sequence decreasing to zero; it is easy to see that T is compact as a norm limit of finite rank operators. For $n \in \mathbb{N}$, we have

$$T^n x = \sum_{k=n+1}^{\infty} x_k w_k w_{k-1} \cdots w_{k-n+1} e_{k-n},$$

which implies

$$\|T^n x\|^p = \sum_{k=n+1}^{\infty} |x_k w_k w_{k-1} \cdots w_{k-n+1}|^p \le \Big(\prod_{k=2}^{n+1} w_k\Big)^p \sum_{k=n+1}^{\infty} |x_k|^p.$$

Considering $\|T^n e_{n+1}\|$, we obtain exactly $\|T^n\|^p = (\prod_{k=2}^{n+1} w_k)^p$ and hence

$$\frac{\|T^n x\|^p}{\|T^n\|^p} \le \sum_{k=n+1}^{\infty} |x_k|^p \to 0. \ ■$$

If the space X is reflexive, we can slightly improve the previous result.

PROPOSITION 3.3. *Let X be a real or complex reflexive Banach space and suppose $T \in \mathcal{L}(X)$ satisfies assumptions* (i) *and* (ii) *of Proposition* 3.1. *Then there exists $x \in X$ with $\|x\| \le 1$ such that*

$$\overline{\lim}\, \frac{\|T^n x\|}{\|T^n\|} = 1.$$

Proof. A compact operator always attains its norm on a reflexive space (although this fact is not strictly necessary here). Write $\|T^n\| = \|T^n x_n\|$

with $(x_n) \subset B_X$. From (x_n), we can by reflexivity extract a subsequence (x_{n_k}) which converges weakly to some point x with $\|x\| \leq 1$. Estimating $\|T^{n_k}x\|$ as above, we get

$$\|T^{n_k}x\| \geq (1 - \|T(x - x_{n_k})\|)\|T^{n_k}\|.$$

Then we use the (well-known) fact that a compact operator maps weakly convergent sequences into norm convergent ones. Hence $\|T(x - x_{n_k})\|$ goes to 0 and

$$\overline{\lim} \frac{\|T^n x\|}{\|T^n\|} \geq 1,$$

which concludes the proof. ∎

Our last example shows that the reflexivity cannot be omitted.

Example 3.4. *There exists a compact operator T on c_0 (the space of sequences converging to zero with the usual norm) such that $(\|T^n\|)$ is non-decreasing and for each $x \in c_0$ with $\|x\| \leq 1$,*

$$\overline{\lim} \frac{\|T^n x\|}{\|T^n\|} < 1.$$

Proof. We consider this time the operator T on c_0 defined by

$$Tx = \sum_{k=1}^{\infty} w_k x_k e_{k-1} + x_0 e_0$$

where (w_n) is a sequence decreasing to zero with $w_0 = 1$. Then T is compact (same argument as in case (ii′)). From the formulas

$$\begin{cases} Te_0 = e_0, \\ Te_k = w_k e_{k-1} \quad (k \geq 1), \end{cases}$$

we deduce by induction that

$$T^n x = \Big(\sum_{k=0}^{n} W_k x_k\Big) e_0 + \sum_{k=1}^{\infty} w_{k+n} w_{k+n-1} \cdots w_{k+1} x_{k+n} e_k$$

where we put $W_n = \prod_{i=0}^{n} w_i$. Since (w_n) decreases to 0, (W_n) decreases to zero faster than any geometric sequence, so $W = \sum_{k=0}^{\infty} W_k < \infty$. On the other hand, for $\|x\| \leq 1$ we have

$$\begin{aligned} \|T^n x\| &= \max\Big(\Big|\sum_{k=0}^{n} W_k x_k\Big|, \sup_{k \geq 1} w_{k+n} w_{k+n-1} \cdots w_{k+1} |x_{k+n}|\Big) \\ &\leq \sum_{k=0}^{n} W_k, \end{aligned}$$

because $0 \leq w_i \leq 1$. Considering the vector $e_0 + \cdots + e_n$, we get $\|T^n\| = \sum_{k=0}^{n} W_k$, so $(\|T^n\|)$ is indeed non-decreasing. Suppose now that there exists a point $x \in c_0$ with $\|x\| \le 1$ such that

$$\overline{\lim} \frac{\|T^n x\|}{\|T^n\|} = 1.$$

We see that there exists a non-decreasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$\Big| \sum_{k=0}^{\varphi(n)} W_k x_k \Big| \to W \qquad (n \to \infty).$$

Let N be an integer such that $|x_k| \le 1/2$ for $k \ge N$. Then

$$\Big| \sum_{k=0}^{\varphi(n)} W_k x_k \Big| \le \sum_{k=0}^{\infty} |x_k| W_k \le \sum_{k=0}^{N} W_k + \frac{1}{2} \sum_{k=N+1}^{\infty} W_k.$$

Letting $n \to \infty$ in the above inequality yields

$$W \le \sum_{k=0}^{N} W_k + \frac{1}{2} \sum_{k=N+1}^{\infty} W_k < W,$$

a contradiction. ∎

REMARK 3.5. In the last example and in (i′), one can also require that $\|T^n\| \to \infty$ (replace T by $2T$).

Finally, the last proposition of this section can be seen as a variation of the compact case (see the remark after the proof). It will also be useful in Section 4.

PROPOSITION 3.6. *Let X be a real or complex Banach space and $T \in \mathcal{L}(X)$ be a non-nilpotent operator. Assume there exists $a \in \,]0,1[$ and a subspace M of finite codimension such that $\|T^n_{|M}\| \le a \|T^n\|$ for infinitely many n's. Then*

$$\Big\{ x \in X : \overline{\lim} \frac{\|T^n x\|}{\|T^n\|} > 0 \Big\}$$

is a dense subset of X.

Proof. By a previous argument (Proposition 3.1), it is enough to find only one point to have automatic density. Replacing M by its closure, we may assume that M is closed in X. Write $X = F \oplus M$ where F is a subspace of finite dimension. Let $(f_1, \dots, f_r)$ be a normalized basis of F, and $(f_1^*, \dots, f_r^*)$ its dual basis (in F). For $1 \le i \le r$, extend each f_i^* to X requiring that the restriction of f_i^* to M is 0. If we denote by P the continuous projection onto F (with respect to the previous decomposition), we easily see that f_i^* is continuous with $\|f_i^*\| \le \|f_{i\,|F}^*\|\,\|P\|$. Set $C = \sup_i \|f_i^*\|$ and choose $a_r > 0$ such that $Cra_r + a < 1$. Now, set

$$A = \{n \in \mathbb{N} : \|T^n_{|M}\| \le a\|T^n\|\},$$
$$A_i = \{n \in A : \|T^n f_i\| \ge a_r\|T^n\|\}.$$

We will show that $A = \bigcup_{i=1}^r A_i$; since A is infinite by hypothesis, so will be one of the A_i, and this will give the conclusion.

Suppose on the contrary that there exists $n \in A \setminus \bigcup_{i=1}^r A_i$. Fix $\alpha_r \in]Cra_r + a, 1[$ and let $x \in X$ with $\|x\| = 1$ be such that $\|T^n x\| \ge \alpha_r\|T^n\|$. Write $x = \sum_{i=1}^r x_i f_i + u$ where $u \in M$. By construction of the f_i^*, $x_i = f_i^*(x)$, hence $|x_i| \le C$ and we get

$$\begin{aligned}\|T^n x\| &\le \sum_{i=1}^r |x_i|\,\|T^n f_i\| + \|T^n u\| \\ &\le (Cra_r + a)\|T^n\| < \alpha_r\|T^n\|,\end{aligned}$$

a contradiction. ■

Remark 3.7. For $T \in \mathcal{L}(X)$, define

$$\|T\|_\mu = \inf\{\|T_{|M}\| : M \subset X, \operatorname{codim} M < \infty\}.$$

This quantity measures the degree of non-compactness of T since $\|T\|_\mu = 0$ if and only if T is compact (see [LS] for details). The above result roughly says that if $\|T^n\|_\mu$ is not too large "uniformly" (that is, the same M works for infinitely many n's), then we have the same conclusion as in the compact case.

4. Modulus of asymptotic uniform smoothness and optimal exponents. Our goal in this section is to compute the value of

$$q_X = \sup\Big\{q > 0 : \text{for every non-nilpotent and bounded linear operator } T,\ \sum_{n=1}^\infty \Big(\frac{\|T^n x\|}{\|T^n\|}\Big)^q = \infty \text{ for some } x \in X\Big\}$$

for some classical Banach spaces. From the results of Müller and Beauzamy quoted in the Introduction, we know that that q_X is well-defined (i.e. the set over which we take the supremum is not empty) and $q_X \ge 1$. Furthermore, the known values [Mu] are $q_{\ell^1} = 1$ and $q_H = 2$ if H is a Hilbert space. We will compute q_{c_0}, q_{ℓ^p} and $q_{L^p(0,1)}$ for $1 \le p < \infty$. Observe that we actually have $q_X = q'_X$ where

$$q'_X = \sup\Big\{q > 0 : \text{for every non-nilpotent and bounded linear operator } T,\ \text{the set } \Big\{x \in X : \sum_{n=1}^{\infty} \Big(\frac{\|T^n x\|}{\|T^n\|}\Big)^q = \infty\Big\} \text{ is dense in } X\Big\}.$$

Indeed, this follows from the following observation which is a simple consequence of the Baire category theorem.

Proposition 4.1. *Let $T \in \mathcal{L}(X)$ be non-nilpotent, $q_0 > 0$ and assume that for every $q < q_0$, there exists $x_0 \in X$ such that*

$$\sum_{n=1}^{\infty} \Big(\frac{\|T^n x_0\|}{\|T^n\|}\Big)^q = \infty.$$

Then the set

$$A = \Big\{x \in X : \forall q < q_0,\ \sum_{n=1}^{\infty} \Big(\frac{\|T^n x\|}{\|T^n\|}\Big)^q = \infty\Big\}$$

is a dense G_δ subset of X.

Proof. Considering a non-decreasing sequence (s_k) such that $s_k < q_0$ for each k and $s_k \to q_0$ as $k \to \infty$, we see that it is enough to show that for each $q < q_0$, $\widetilde{A}$ is a dense G_δ subset of X where

$$\widetilde{A} = \Big\{x \in X : \sum_{n=1}^{\infty} \Big(\frac{\|T^n x\|}{\|T^n\|}\Big)^q = \infty\Big\}.$$

Write $\widetilde{A} = \bigcap_{N=1}^{\infty} \Omega_N$, where

$$\Omega_N = \Big\{x \in X : \sum_{n=1}^{\infty} \Big(\frac{\|T^n x\|}{\|T^n\|}\Big)^q > N\Big\}.$$

Using for example the Fatou lemma, it is easy to see that $X \setminus \Omega_N$ is a closed subset of X for each N. By the Baire category theorem, we need to check that each Ω_N is dense in X. Fix $\epsilon > 0$ and $x \in X$. Replacing x_0 by λx_0 for some $\lambda > 0$, we can assume that $\|x_0\| = \epsilon$. Now, there exists a constant $C > 0$ such that $(x+y)^q \le C(x^q + y^q)$ for every $x, y \ge 0$. From this and the triangle inequality, we get

$$\sum_{n=1}^{\infty} \Big(\frac{\|T^n(x - x_0)\|}{\|T^n\|}\Big)^q + \sum_{n=1}^{\infty} \Big(\frac{\|T^n(x + x_0)\|}{\|T^n\|}\Big)^q \ge \frac{2^q}{C} \sum_{n=1}^{\infty} \Big(\frac{\|T^n x_0\|}{\|T^n\|}\Big)^q = \infty.$$

So $x - x_0$ or $x + x_0$ belongs to $\widetilde{A}$ and since $\|x - (x \pm x_0)\| = \epsilon$, this concludes the proof. ∎

Remark 4.2. If $q \geq 1$ and X is reflexive, under the assumptions of Proposition 4.1, the set A is Haar-null. Indeed, it is enough to show that for each N, $X \setminus \Omega_N$ is Haar-null, and this follows directly from the theorem of Matoušková [Ma]: a closed and convex set with empty interior in a reflexive Banach space is Haar-null.

To compute q_X, we introduce the modulus of asymptotic uniform smoothness of X which is a useful tool from Banach space geometry (but probably not much used in linear dynamics). This quantity has been introduced for the first time by Milman [Mi] under some different names. We follow here the more recent terminology which can be found in [JLPS].

Definition 4.3. Let X be a real or complex Banach space. The *modulus of asymptotic uniform smoothness* of X is the function $\overline{\rho}_X(t)$ defined by

$$\overline{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\dim(X/Y)<\infty} \sup_{y\in Y,\, \|y\|=1} (\|x+ty\|-1) \qquad (t \geq 0).$$

$\overline{\rho}_X$ is a 1-Lipschitz, convex and non-decreasing map such that $\overline{\rho}_X(0) = 0$ and $\overline{\rho}_X(t) \leq t$ for $t \geq 0$. We can now state:

Theorem 4.4. *Let X be a Banach space and assume that its modulus of asymptotic uniform smoothness satisfies $\overline{\rho}_X(2t) = O(\overline{\rho}_X(t))$ as $t \to 0$. Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing map such that $\rho(t) > 0$ whenever $t > 0$, and*

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{\rho(t)} = 0.$$

Then there exists a point $x \in X$ such that

$$\sum_{n=1}^{\infty} \rho\biggl(\frac{\|T^n x\|}{\|T^n\|}\biggr) = \infty.$$

As a consequence, we obtain the results claimed in the Introduction. More precisely:

Theorem 4.5. *Let $T \in \mathcal{L}(X)$ ($X = \ell^p(\mathbb{N})$, $X = L^p(0,1)$ $(1 \leq p < \infty)$ or $X = c_0(\mathbb{N})$) be a non-nilpotent operator. Then:*

(a) *If $X = \ell^p$, the set*

$$\biggl\{x \in X : \forall q < p,\ \biggl(\frac{\|T^n x\|}{\|T^n\|}\biggr)_n \notin \ell^q\biggr\}$$

is a dense G_δ subset of X. On the other hand, there exists $S \in \mathcal{L}(X)$ (non-nilpotent) such that for each x,

$$\sum_{n=1}^{\infty} \biggl(\frac{\|S^n x\|}{\|S^n\|}\biggr)^p < \infty.$$

(b) *If $X = L^p$, the set*

$$\left\{x \in X : \forall q < \min(p,2), \left(\frac{\|T^n x\|}{\|T^n\|}\right)_n \notin \ell^q\right\}$$

is a dense G_δ subset of X. On the other hand, there exists $R \in \mathcal{L}(X)$ (non-nilpotent) such that for each x,

$$\sum_{n=1}^{\infty}\left(\frac{\|R^n x\|}{\|R^n\|}\right)^{\min(p,2)} < \infty.$$

(c) *If $X = c_0$, the set*

$$\left\{x \in X : \forall q > 0, \left(\frac{\|T^n x\|}{\|T^n\|}\right)_n \notin \ell^q\right\}$$

is a dense G_δ subset of X. In particular,

$$q_{\ell^p} = p, \qquad q_{L^p} = \min(p,2) \quad (1 \le p < \infty), \qquad q_{c_0} = \infty.$$

Proof. It is known (and easy to see) that $\overline{\rho}_{c_0}(t) = 0$ for $0 \le t \le 1$, and also for $t \ge 0$,

$$\overline{\rho}_{\ell^p}(t) = (1+t^p)^{1/p} - 1 \sim \frac{t^p}{p} \quad (t \to 0).$$

For $L^p = L^p(0,1)$, Milman [Mi] obtained the following estimates. For L^1, $\overline{\rho}_{L^1}(t) = t$. For $1 < p < 2$,

$$\frac{1}{p}t^p \le \overline{\rho}_{L^p}(t) \le \frac{2}{p}t^p \quad (t \to 0).$$

For $2 < p < \infty$, there exists a constant $C_p > 0$ such that

$$(p-1)t^2 \le \overline{\rho}_{L^p}(t) \le C_p t^2 \quad (t \to 0).$$

For $p = 2$, $\overline{\rho}_{L^2}(t) = (1+t^2)^{1/2} - 1$ since L^2 and ℓ^2 are isometric. Hence the first statement of (a) and (b), and statement (c), are straightforward consequences of Theorem 4.4 and Proposition 4.1.

We now turn to the examples. (a) can be found in [Mu]. The job there is done for $p = 2$, but the general case is almost the same. We include the example anyway for the sake of completeness. Let (e_i) be the usual canonical basis of ℓ^p and set $e_{1,0} = e_1$, $e_{1,1} = e_2$, $e_{2,0} = e_3$, $e_{2,1} = e_4$, $e_{2,2} = e_5, \ldots$. In this way, we can write $\ell^p = \bigoplus_{k=1}^{\infty} X_k$ where X_k is the $(k+1)$-dimensional ℓ^p space with basis $e_{k,0}, \ldots, e_{k,k}$. Then S is defined by $S = \bigoplus_{k=1}^{\infty} 2^{-k} B_k$ where B_k is the usual backward shift on $\mathcal{L}(X_k)$, i.e. $B_k(e_{k,j}) = e_{k,j-1}$ for $j \ge 1$ and $B_k e_{k,0} = 0$. For $n \ge 1$, $S^n(e_{n,n}) = 2^{-n^2} e_{n,0}$ so $\|S^n\| \ge 2^{-n^2}$. Let $x_k = \sum_{j=0}^{k} \alpha_j e_{k,j} \in X_k$. We have

$$\sum_{n=1}^{\infty}\left(\frac{\|S^n x_k\|}{\|S^n\|}\right)^p \le \sum_{n=1}^{k}\left(\frac{2^{n^2}}{2^{nk}}\left(\sum_{j=n}^{k}|\alpha_j|^p\right)^{1/p}\right)^p \le \sum_{n=1}^{k}\frac{1}{2^{n(k-n)}}\|x_k\|^p \le 2\|x_k\|^p.$$

It follows that for every $x = \sum_{k=1}^{\infty} x_k$, we have

$$\sum_{n=1}^{\infty} \left(\frac{\|S^n x\|}{\|S^n\|} \right)^p \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{\|S^n x_k\|}{\|S^n\|} \right)^p \leq \sum_{k=1}^{\infty} 2\|x_k\|^p < \infty.$$

Now for (b), recall that for every $1 \leq p < \infty$, ℓ^p is isomorphic to a complemented subspace $E \subset L^p$. So write $L^p = E \oplus F$ where F is a closed subspace of L^p. If $Q : \ell^p \to E$ is an isomorphism, then clearly $S_0 = QSQ^{-1}$ is a bounded operator on E such that for every $x \in E$,

$$\sum_{n=1}^{\infty} \left(\frac{\|S_0^n x\|}{\|S_0^n\|} \right)^p < \infty.$$

Let P be the projection onto E with respect to the decomposition $L^p = E \oplus F$ and put $R = S_0 P$, which is bounded on L^p. Then for every n, $R^n = S_0^n P$. Since P is a projection onto E, we have $\|R^n\| \geq \|S_0^n\|$. Hence

$$\sum_{n=1}^{\infty} \left(\frac{\|R^n x\|}{\|R^n\|} \right)^p \leq \sum_{n=1}^{\infty} \left(\frac{\|S_0^n (Px)\|}{\|S_0^n\|} \right)^p < \infty.$$

This shows (b) for $p \leq 2$. For $p \geq 2$, use a similar argument and the fact that ℓ^2 is isomorphic to a complemented subspace of L^p. ∎

It remains to prove Theorem 4.4. Before going into the proof, we will need the following elementary lemma.

LEMMA 4.6. *Let f, g be two maps from $\mathbb{R}^+$ to $\mathbb{R}^+$. Assume that*

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0 \quad \textit{and} \quad \lim_{x \to 0} f(x) = 0.$$

Then there exists a sequence $(\alpha_i) \subset \mathbb{R}^+$ with $\alpha_i \to 0$ (as $i \to \infty$) such that

$$\sum_{i=1}^{\infty} f(\alpha_i) < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} g(\alpha_i) = \infty.$$

Proof. If $g(x) \nrightarrow 0$ as $x \to 0$, there exist $\epsilon > 0$ and $(\alpha_i) \subset \mathbb{R}^+$ with $\alpha_i \to 0$ such that $g(\alpha_i) \geq \epsilon$ for each $i \geq 1$. By passing to a subsequence, one can assume that $\sum_{i=1}^{\infty} f(\alpha_i) < \infty$ (because $f(x) \to 0$) and since $g(\alpha_i) \geq \epsilon$ for each i, one also gets $\sum_{i=1}^{\infty} g(\alpha_i) = \infty$.

Now, if $g(x) \to 0$ as $x \to 0$, fix $(\epsilon_i) \subset \mathbb{R}^+$ with $\epsilon_i > 0$ for each i such that $\sum_{i=1}^{\infty} \epsilon_i < \infty$. Choose, for each $i \geq 1$, $x_i \in \,]0, 1/i[$ such that $f(x_i) \leq \epsilon_i g(x_i)$ and $g(x_i) \leq 1/2$. Put $n_1 = 1$ and $n_{i+1} = n_i + [1/g(x_i)]$ where $[x]$ denotes the integer part of x. If k is an integer with $n_i \leq k \leq n_{i+1} - 1$, put $\alpha_k = x_i$.

Then $\alpha_k \to 0$ because $x_i \to 0$. We have

$$\sum_{j=1}^{\infty} f(\alpha_j) = \sum_{i=1}^{\infty} \sum_{k=n_i}^{n_{i+1}-1} f(\alpha_k) = \sum_{i=1}^{\infty} f(x_i)(n_{i+1} - n_i)$$
$$\leq \sum_{i=1}^{\infty} \epsilon_i g(x_i)(n_{i+1} - n_i) \leq \sum_{i=1}^{\infty} \epsilon_i < \infty.$$

On the other hand,

$$\sum_{j=1}^{\infty} g(\alpha_j) = \sum_{i=1}^{\infty} g(x_i)(n_{i+1} - n_i)$$
$$\geq \sum_{i=1}^{\infty}(1 - g(x_i)) \geq \sum_{i=1}^{\infty} \frac{1}{2} = \infty,$$

and this concludes the proof of the lemma. ∎

We are now ready for the proof of Theorem 4.4. We will combine some techniques from [Mu] and [L].

Proof of Theorem 4.4. We can make the following assumption:

($*$) for every subspace M of finite codimension, $\|T^n_{|M}\| > \frac{1}{2}\|T^n\|$ for all but a finite number of n's.

Indeed, if this is not true, then by Proposition 3.6, there exists $\epsilon > 0$, a point $x \in X$ and a non-decreasing sequence (n_j) such that for each j,

$$\frac{\|T^{n_j}x\|}{\|T^{n_j}\|} \geq \epsilon.$$

Since $\rho(\epsilon) > 0$, this obviously implies that

$$\sum_{n=1}^{\infty} \rho\left(\frac{\|T^n x\|}{\|T^n\|}\right) = \infty.$$

Hence we can suppose that ($*$) holds. By Lemma 4.6, there exists a sequence $(\tilde{\alpha_i})$ with $\tilde{\alpha_i} \to 0$ such that

$$\sum_{i=1}^{\infty} \overline{\rho}_X(\tilde{\alpha_i}) < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \rho(\tilde{\alpha_i}) = \infty.$$

Since ρ is non-decreasing and $\overline{\rho}_X(2t) = O(\overline{\rho}_X(t))$, we see that

$$\sum_{i=1}^{\infty} \overline{\rho}_X(\alpha_i) < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \rho\left(\frac{\alpha_i}{2}\right) = \infty,$$

where we have put $\alpha_i = 2\tilde{\alpha_i}$. Now, using again the hypothesis $\overline{\rho}_X(2t) = O(\overline{\rho}_X(t))$, we see that $\sum_{i=1}^{\infty} \overline{\rho}_X(2^k \alpha_i) < \infty$ for each k, and thus

$\prod_{i=1}^{\infty}(1+\overline{\rho}_X(2^k\alpha_i))$ converges. From this and the fact that $\alpha_i \to 0$, we can find an increasing sequence (m_k) of integers such that

$$\alpha_i \le 2^{-k} \quad (i \ge m_k) \quad \text{and} \quad \prod_{i=m_k}^{\infty}(1+\overline{\rho}_X(2^k\alpha_i)) \le 2.$$

Without loss of generality, we may assume that $m_1 = 1$. Let us also fix once for all a sequence (β_i) with $\beta_i > 0$ such that $\prod_{i=1}^{\infty}(1+\beta_i)$ converges. Next, we are going to construct by induction two sequences (n_j) and (u_i) such that $n_1 < n_2 < \cdots$ and $\|u_i\| = 1$. Further, these sequences will have two properties; First, for each $l \ge 1$ and $j \le l$,

$$\Big\|T^{n_j}\Big(\sum_{i=1}^{l}\alpha_i u_i\Big)\Big\| \ge \frac{\alpha_j}{2}\|T^{n_j}\|. \tag{1}$$

Secondly, for each $k \ge 1$, and $m_k \le l \le m_{k+1}-1$,

$$\Big\|\sum_{i=m_k}^{l}\alpha_i u_i\Big\| \le 2^{1-k}\Big(\prod_{i=m_k}^{l}(1+\beta_i)\Big)\Big(\prod_{i=m_k}^{l}(1+\overline{\rho}_X(2^k\alpha_i))\Big). \tag{2}$$

Once this is done, by (2) and since $\prod_{i=m_k}^{\infty}(1+\overline{\rho}_X(2^k\alpha_i)) \le 2$, we see that $(\sum_{i=1}^{l}\alpha_i u_i)_l$ is Cauchy, and thus converges to some x. By (1), for fixed j, we get in the limit

$$\|T^{n_j}x\| \ge \frac{\alpha_j}{2}\|T^{n_j}\|,$$

whence

$$\sum_{n=1}^{\infty}\rho\Big(\frac{\|T^n x\|}{\|T^n\|}\Big) \ge \sum_{j=1}^{\infty}\rho\Big(\frac{\|T^{n_j}x\|}{\|T^{n_j}\|}\Big) \ge \sum_{j=1}^{\infty}\rho\Big(\frac{\alpha_j}{2}\Big) = \infty,$$

and this is the desired conclusion.

Now, we turn to the inductive construction. Set $n_1 = 1$. There exists $u_1 \in X$ with $\|u_1\| = 1$ and $\|Tu_1\| \ge \alpha_1/2$. Let $k \ge 1$ and assume the construction has been carried out up to $m_k \le l \le m_{k+1}-1$. If $l+1 = m_{k+1}$, then

$$\|\alpha_{l+1}u_{l+1}\| \le 2^{-(k+1)} \le 2^{1-(k+1)}(1+\beta_{l+1})(1+\overline{\rho}_X(2^{k+1}\alpha_{l+1})),$$

so (2) is automatically satisfied for $l+1$. Arguments to get (1) will be detailed later. Suppose now that $l+1 \le m_{k+1}-1$. For convenience until the end of the proof, we put $s_l = \sum_{i=m_k}^{l}\alpha_i u_i$ and $x_l = \sum_{i=1}^{l}\alpha_i u_i$.

We distinguish two cases. Suppose first that $\|s_l\| \le 2^{-k}$. Then for any choice of u_{l+1} such that $\|u_{l+1}\| = 1$,

$$\Big\|\sum_{i=m_k}^{l+1}\alpha_i u_i\Big\| \le \|s_l\| + \alpha_{l+1} \le 2^{1-k}.$$

Hence (2) is again always satisfied.

We now indicate how to get (1). For each $j \le l$, select a linear functional f_j such that $\|f_j\| = 1$ and $\|f_j(T^{n_j} x_l)\| = \|T^{n_j} x_l\|$ and put

$$M = \bigcap_{j=1}^{l} \ker(f_j T^{n_j}),$$

which is clearly of finite codimension. Applying $(*)$, we can find $n_{l+1} > n_l$ and $v \in M$ with $\|v\| = 1$ such that

$$\|T^{n_{l+1}} v\| \ge \frac{1}{2} \|T^{n_{l+1}}\|.$$

Since

$$\begin{aligned}\|T^{n_{l+1}}(x_l + \alpha_{l+1} v)\| + \|T^{n_{l+1}}(x_l - \alpha_{l+1} v)\| \\ \ge 2\|T^{n_{l+1}} v\| \alpha_{l+1} \ge \|T^{n_{l+1}}\| \alpha_{l+1},\end{aligned}$$

there exists $\epsilon = \pm 1$ such that

$$\|T^{n_{l+1}}(x_l + \epsilon \alpha_{l+1} v)\| \ge \frac{\alpha_{l+1}}{2} \|T^{n_{l+1}}\|.$$

Putting $u_{l+1} = \epsilon v$, we have proved (1) for $j = l+1$. For $j \le l$, we obtain

$$\begin{aligned}\|T^{n_j}(x_l + \alpha_{l+1} u_{l+1})\| \ge \|f_j T^{n_j}(x_l + \alpha_{l+1} u_{l+1})\| = \|f_j T^{n_j} x_l\| = \|T^{n_j} x_l\| \\ \ge \frac{\alpha_j}{2} \|T^{n_j}\|,\end{aligned}$$

where the last inequality follows from the induction hypothesis of (1).

Suppose now that $\|s_l\| \ge 2^{-k}$. From the definition of the modulus of asymptotic smoothness, we can find a subspace $Y \subset X$ of finite codimension such that for all $y \in Y$ with $\|y\| = 1$,

$$\left\| \frac{s_l}{\|s_l\|} + 2^k \alpha_{l+1} y \right\| \le (1 + \beta_{l+1})(1 + \overline{\rho}_X(2^k \alpha_{l+1})). \tag{3}$$

This time, we set

$$M = \bigcap_{j=1}^{l} \ker(f_j T^{n_j}) \cap Y$$

where the f_j are constructed exactly as in the previous case. We also construct in the same way n_{l+1} and $u_{l+1} \in M$ with $\|u_{l+1}\| = 1$ so that (1) holds for $l+1$. Now (3) with $y = u_{l+1} \in M \subset Y$ yields

$$\big\| s_l + 2^k \|s_l\| \alpha_{l+1} u_{l+1} \big\| \le \|s_l\| (1 + \beta_{l+1})(1 + \overline{\rho}_X(2^k \alpha_{l+1})). \tag{4}$$

The condition $\|s_l\| \ge 2^{-k}$ implies that $s_l + \alpha_{l+1} u_{l+1}$ lies on the segment joining s_l to $s_l + 2^k\|s_l\|\alpha_{l+1}u_{l+1}$. Hence, from (4) we get

$$\Big\|\sum_{i=m_k}^{l+1}\alpha_i u_i\Big\| = \|s_l + \alpha_{l+1}u_{l+1}\| \le \|s_l\|(1+\beta_{l+1})(1+\overline{\rho}_X(2^k\alpha_{l+1}))$$
$$\le 2^{1-k}\Big(\prod_{i=m_k}^{l+1}(1+\beta_i)\Big)\Big(\prod_{i=m_k}^{l+1}(1+\overline{\rho}_X(2^k\alpha_i))\Big),$$

where the last inequality follows from the induction hypothesis. We see that (2) is satisfied for $l+1$, and this ends the construction. ∎

REMARK 4.7. Lindenstrauss [L] showed the following result: if $(x_i) \subset X$ is such that $\sum_{i=1}^{\infty}\epsilon_i x_i$ diverges for every choice of signs $(\epsilon_i) \subset \{-1,1\}$, then $\sum_{i=1}^{\infty}\rho_X(\|x_i\|) = \infty$ where ρ_X is the usual modulus of smoothness defined for $t \ge 0$ by

$$\rho_X(t) = \frac{1}{2}\sup_{\|x\|=\|y\|=1}(\|x+ty\| + \|x-ty\| - 2).$$

REMARK 4.8. We do not know if the assumption $\overline{\rho}_X(2t) = O(\overline{\rho}_X(t))$ is really necessary, although it seems to us that a "bad" Orlicz space may not have this property.

References

[Bal1] K. M. Ball, *The plank problem for symmetric bodies*, Invent. Math. 10 (1991), 535–543.

[Bal2] K. M. Ball, *The complex plank problem*, Bull. London Math. Soc. 33 (2001), 433–442.

[Bay1] F. Bayart, *Porosity and hypercyclic operators*, Proc. Amer. Math. Soc. 133 (2005), 3309–3316.

[Bay2] F. Bayart, *Weak-closure and polarization constant by Gaussian meausure*, Math. Z. 264 (2010), 459–468.

[Be] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland Math. Library 42, North-Holland, Amsterdam, 1988.

[BM] F. Bayart and E. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Math. 179, Cambridge Univ. Press, 2009.

[BMM] F. Bayart, E. Matheron and P. Moreau, *Small sets and hypercyclic vectors*, Comment. Math. Univ. Carolin. 49 (2008), 53–65.

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Colloq. Publ. 48, Amer. Math. Soc., 2000.

[C] J. P. R. Christensen, *On sets of Haar measure zero in Polish abelian groups*, Israel J. Math. 13 (1972), 255–260.

[D] E. P. Dolženko, *Boundary properties of arbitrary functions*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 3–14 (in Russian).

[JLPS] W. B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman, *Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces*, Proc. London Math. Soc. (3) 84 (2002), 711–746.

[LS] A. Lebow and M. Schechter, *Semigroups of operators and measures of noncompactness*, J. Funct. Anal. 7 (1971), 1–26.

[L] J. Lindenstrauss, *On the modulus of smoothness and divergent series in Banach spaces*, Michigan Math. J. 10 (1963), 241–252.

[Ma] E. Matoušková, *Translating finite sets into convex sets*, Bull. London Math. Soc. 33 (2001), 711–714.

[Mi] V. D. Milman, *Geometric theory of Banach spaces. II. Geometry of the unit ball*, Uspekhi Mat. Nauk 26 (1971) no. 6, 73–149 (in Russian); English transl.: Russian Math. Surveys 26 (1971), no. 6, 79–163.

[Mu] V. Müller, *Orbits, weak orbits and local capacity of operators*, Integral Equations Operator Theory 41 (2001), 230–253.

[MV] V. Müller and J. Vršovský, *On orbit-reflexive operators*, J. London Math. Soc. 79 (2009), 497–510.

[R] C. Read, *The invariant subspace problem for a class of Banach spaces. II. Hypercyclic operators*, Israel J. Math. 63 (1988), 1–40.

[Z] L. Zajíček, *Porosity and σ-porosity*, Real Anal. Exchange 13 (1987-1988), 314–350.

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