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On the Bernstein–Walsh–Siciak theorem

by

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Abstract. By the Oka–Weil theorem, each holomorphic function f in a neighbourhood of a compact polynomially convex set $K \subset \mathbb{C}^N$ can be approximated uniformly on K by complex polynomials. The famous Bernstein–Walsh–Siciak theorem specifies the Oka–Weil result: it states that the distance (in the supremum norm on K) of f to the space of complex polynomials of degree at most n tends to zero not slower than the sequence $M(f)\rho(f)^n$ for some $M(f) > 0$ and $\rho(f) \in (0, 1)$. The aim of this note is to deduce the uniform version, sometimes called family version, of the Bernstein–Walsh–Siciak theorem, which is due to Pleśniak, directly from its classical (weak) form. Our method, involving the Baire category theorem in Banach spaces, appears to be useful also in a completely different context, concerning Łojasiewicz's inequality.

1. Introduction. For a nonempty set $A \subset \mathbb{C}^N$ and $h : A \to \mathbb{C}^{N'}$, we put $\|h\|_A := \sup_{z\in A} |h(z)|$, where $|\ |$ denotes the Euclidean norm in $\mathbb{C}^{N'}$. If $\emptyset \neq A \subset B \subset \mathbb{C}^N$ and $\xi : B \to \mathbb{C}$, then for each $n \in \mathbb{N}$, put

$$E_n(\xi; A) := \inf\{\|\xi - Q\|_A : Q \in \mathbb{C}[Z],\ \deg Q \le n\}.$$

Throughout the paper $\mathbb{N} := \{1, 2, 3, \ldots\}$.

Recall one of the most important results in complex approximation.

THEOREM 1.1 (Oka–Weil). *Let f be a holomorphic function in a neighbourhood of a (nonempty) compact polynomially convex set $K \subset \mathbb{C}^N$* [1]*. Then there is a sequence of complex polynomials $P_n \in \mathbb{C}[Z] = \mathbb{C}[Z_1, \ldots, Z_N]$ such that $\|f - P_n\|_K \to 0$.*

This is a generalization of the classical result of Runge (cf. [L]). The proof of the Oka–Weil theorem can be found in most of the books on complex analysis (see for example [H, p. 55]).

The next result is a significant improvement on the Oka–Weil theorem. It is due to Siciak (cf. [S1, S2]), but because of the contributions made in

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[1] We say that a compact set $K \subset \mathbb{C}^N$ is *polynomially convex* if $K = \hat{K} := \{z \in \mathbb{C}^N : |P(z)| \le \|P\|_K$ for all polynomials $P \in \mathbb{C}[Z]\}$.

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one variable, i.e. for $N = 1$, by Bernstein and Walsh (cf. [B, W]), it is called the *Bernstein–Walsh–Siciak theorem*.

Theorem 1.2 (Siciak). *Let f be a holomorphic function in a neighbourhood of a (nonempty) compact polynomially convex set $K \subset \mathbb{C}^N$. Then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(f;K)} < 1.$$

This is a weak version of Siciak's result. The full version is actually much stronger. Let us mention, moreover, that the problems of this type, but in the space $\mathbb{R}^N$, were deeply investigated by Baouendi and Goulaouic (cf. [BG1, BG2]).

In 1972, a very precise (uniform) version of Theorem 1.2 was proved by Pleśniak. Suppose that $U \subset \mathbb{C}^N$ is a nonempty open set. We will denote by $H^\infty(U)$ the Banach space of all bounded and holomorphic functions in U (with the norm $\| \ \|_U$). Let a nonempty set $K \subset U$ be compact and polynomially convex.

Theorem 1.3 (Pleśniak). *There exist constants $M > 0$ and $\rho \in (0,1)$ such that, for all $f \in H^\infty(U)$ and $n \in \mathbb{N}$,*

$$E_n(f;K) \le M\|f\|_U\,\rho^n.$$

The original argument of Pleśniak (see [P1, P2]) relied heavily on Siciak's difficult and deep proof of Theorem 1.2. A more elementary proof, inspired by an idea of Baouendi and Goulaouic [BG2], was given in [P3] (see also [P4]). However, it was essentially based on a more precise version of Theorem 1.2 involving the so-called Siciak extremal function (cf. [S1, S2]). This version along with the theory of the Siciak extremal function allowed Pleśniak to notice first the existence of $\rho \in (0,1)$ independent of f such that the estimate in Theorem 1.3 holds with some $M = M(f) > 0$ possibly dependent on f.

2. Proof that Theorem 1.2 implies Theorem 1.3. Our first purpose is to deduce Theorem 1.3 directly from Theorem 1.2. We need two elementary lemmata.

Lemma 2.1. *Suppose that $\emptyset \ne A \subset B \subset \mathbb{C}^N$ and denote by $\mathcal{B}(B;\mathbb{C})$ the Banach space of all bounded functions $\xi : B \to \mathbb{C}$ (with the norm $\| \ \|_B$). Let $n \in \mathbb{N}$. Then*

(1) *For each $\xi \in \mathcal{B}(B;\mathbb{C})$ and $\alpha \in \mathbb{C}$,*

$$E_n(\alpha\xi;A) = |\alpha|E_n(\xi;A).$$

(2) *For all $\xi_1, \xi_2 \in \mathcal{B}(B;\mathbb{C})$,*

$$|E_n(\xi_1;A) - E_n(\xi_2;A)| \le E_n(\xi_1 - \xi_2;A) \le \|\xi_1 - \xi_2\|_B.$$

In particular, the function $\mathcal{B}(B;\mathbb{C}) \ni \xi \mapsto E_n(\xi;A) \in \mathbb{R}$ is continuous.

Proof. The equality in (1) and the second inequality in (2) are trivial. By symmetry, it is enough therefore to prove that

$$E_n(\xi_1; A) \le E_n(\xi_2; A) + E_n(\xi_1 - \xi_2; A).$$

Suppose that $P, Q \in \mathbb{C}[Z]$ are polynomials of degree $\le n$. Clearly,

$$E_n(\xi_1; A) \le \|\xi_1 - (P+Q)\|_A \le \|\xi_2 - P\|_A + \|\xi_1 - \xi_2 - Q\|_A.$$

As P, Q are arbitrary, our assertion follows. ∎

As in Theorem 1.3, suppose that $U \subset \mathbb{C}^N$ is open and a nonempty set $K \subset U$ is compact and polynomially convex. For $M > 0$ and $\rho \in (0,1)$, put

$$V(M,\rho) := \{f \in H^\infty(U) : \forall n \in \mathbb{N},\ E_n(f;K) \le M\|f\|_U\, \rho^n\}.$$

Lemma 2.2. *The set $V(M,\rho)$ is closed in $H^\infty(U)$* (2).

Proof. Fix $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$. Since the intersection of closed sets is closed, it is enough to prove that the function

$$H^\infty(U) \ni f \mapsto E_n(f;K) - \theta\|f\|_U \in \mathbb{R}$$

is continuous. But this follows immediately from Lemma 2.1. ∎

Note that an equivalent formulation of Theorem 1.3 is the following: there exist $M > 0$ and $\rho \in (0,1)$ such that $H^\infty(U) = V(M,\rho)$.

Proof that Theorem 1.2 implies Theorem 1.3. It follows from Theorem 1.2 that

$$H^\infty(U) = \bigcup_{k\in\mathbb{N}} V(k, 1-(k+1)^{-1}).$$

Baire's theorem, via Lemma 2.2, implies that for some $l \in \mathbb{N}$ the set $\mathrm{Int}(V(l,\rho))$ is nonempty, where $\rho := 1-(l+1)^{-1}$. Take $f_0 \in H^\infty(U)$ and $r > 0$ such that $\{f \in H^\infty(U) : \|f - f_0\|_U \le r\} \subset V(l,\rho)$. Put $M := l(1 + 2r^{-1}\|f_0\|_U)$.

Claim. $\{g \in H^\infty(U) : \|g\|_U = r\} \subset V(M,\rho)$.

Note that the claim completes the proof, because

$$H^\infty(U) = [0,\infty)\cdot\{g \in H^\infty(U) : \|g\|_U = r\} \subset [0,\infty)\cdot V(M,\rho) = V(M,\rho).$$

Take therefore any $g \in H^\infty(U)$ such that $\|g\|_U = r$. Clearly, f_0 and $g + f_0$ belong to $V(l,\rho)$. Note that $l(\|f_0\|_U + \|g+f_0\|_U) \le M\|g\|_U$. Combine this with the inequality $E_n(g;K) \le E_n(f_0;K) + E_n(g+f_0;K)$ (cf. Lemma 2.1) to conclude that $g \in V(M,\rho)$. ∎

The argument presented above allows us to abstract the following lemma.

(2) Clearly, the polynomial convexity assumption on K is not necessary here.

Lemma 2.3. *Let X be a Banach space over $\mathbb{C}$ or $\mathbb{R}$. Suppose that a sequence of sets $V_k \subset X$ ($k \in \mathbb{N}$) satisfies the following conditions:*

(1) $\operatorname{Int}(\bigcup_{k\in\mathbb{N}} V_k) \neq \emptyset$.
(2) *For each $k \in \mathbb{N}$ there exist $j_1, j_2 \in \mathbb{N}$ satisfying $\overline{V}_k \subset V_{j_1}$ and $[0,\infty) \cdot V_k \subset V_{j_2}$.*
(3) *For each $j \in \mathbb{N}$, $x_0 \in V_j$ and $r > 0$ there exists $\mu = \mu(j, x_0, r) \in \mathbb{N}$ such that*
$$(V_j - x_0) \cap \{x \in X : \|x\| = r\} \subset V_\mu.$$

Then $X = V_{k_0}$ for some $k_0 \in \mathbb{N}$.

Proof. We may assume that $\#X > 1$, i.e. $X \neq \{0\}$. By Baire's theorem, for some $m \in \mathbb{N}$, $\operatorname{Int} \overline{V}_m \neq \emptyset$. The assumption (2) implies that there is $j_0 \in \mathbb{N}$ such that $\operatorname{Int} V_{j_0} \neq \emptyset$. Therefore $\{y \in X : \|y - x_0\| \leq r\} \subset V_{j_0}$ for some $x_0 \in V_{j_0}$ and $r > 0$. By the assumption (3), we can conclude that there exists $\mu \in \mathbb{N}$ such that
$$\{x \in X : \|x\| = r\} = (V_{j_0} - x_0) \cap \{x \in X : \|x\| = r\} \subset V_\mu.$$
Since $X = [0,\infty) \cdot \{x \in X : \|x\| = r\} \subset [0,\infty) \cdot V_\mu$, it follows by (2) that $X = V_{k_0}$ for some $k_0 \in \mathbb{N}$. ∎

We decided to state the above abstract lemma, because it also finds its application in the next section.

3. A version of Łojasiewicz's inequality. The subject of this section seems to be completely different from the prior part of the article. However, a strong link between these two parts is Lemma 2.3, which gives a uniform estimate in both contexts.

The classical Łojasiewicz inequality (recalled below) is a powerful tool in geometry and analysis. Lemma 2.3 will be used to prove a (uniform) version of this inequality (Proposition 3.1).

We need some definitions and facts from subanalytic geometry (cf. [BM, DS, Hi]). A subset $A \subset \mathbb{R}^N$ is said to be *semianalytic* if each point in $\mathbb{R}^N$ has a neighbourhood U such that $A \cap U$ is a finite union of sets of the form
$$\{x \in U : \xi(x) = 0,\ \xi_1(x) > 0, \ldots, \xi_q(x) > 0\},$$
where $\xi, \xi_1, \ldots, \xi_q$ are real analytic functions in U (cf. [Ł]). A set $A \subset \mathbb{R}^N$ is called *subanalytic* if each point in $\mathbb{R}^N$ has a neighbourhood U such that $A \cap U$ is the projection of some relatively compact semianalytic set in $\mathbb{R}^{N+N'} = \mathbb{R}^N \times \mathbb{R}^{N'}$ (cf. [BM, DS]). In a similar way we can define semianalytic and subanalytic subsets of any real analytic manifold.

In this paper, we are interested in *globally subanalytic* subsets of $\mathbb{R}^N$, that is, subanalytic subsets of $\mathbb{R}^N$ that are also subanalytic as subsets of the projective space $\mathbb{P}^N(\mathbb{R})$. Recall that the two notions (subanalytic in $\mathbb{R}^N$ and globally subanalytic in $\mathbb{R}^N$) coincide for bounded sets. From now on, we will omit the word "globally", and saying "subanalytic in $\mathbb{R}^N$" we will always mean "globally subanalytic in $\mathbb{R}^N$".

Recall the most important (from the point of view of further arguments) properties of subanalytic sets and maps:

- Any interval in $\mathbb{R}$ is subanalytic. A finite union or intersection of subanalytic sets is subanalytic. The Cartesian product of subanalytic sets is subanalytic. If $A \subset \mathbb{R}^m$ is subanalytic, then so are $\mathbb{R}^m \setminus A$ and $\pi(A)$, where $\pi : \mathbb{R}^m \to \mathbb{R}^{m'}$ denotes the natural projection ($m' \leq m$). Moreover $\overline{A}$ and $\operatorname{Int} A$ are subanalytic.
- If $\xi : A \to \mathbb{R}^q$ is subanalytic ([3]), where $A \subset \mathbb{R}^m$, then A and $\{\xi = 0\}$ are subanalytic. If additionally $q = 1$, then $\{\xi > 0\}$ and $\{\xi < 0\}$ are subanalytic.
- If $\xi_\nu : A \to \mathbb{R}$, where $A \subset \mathbb{R}^m$, are subanalytic (for $\nu = 1, 2$), then so are $\xi_1 + \xi_2$ and $\xi_1 \cdot \xi_2$. Moreover, ξ_1/ξ_2 is subanalytic on $A \setminus \{\xi_2 = 0\}$. The same is true if $\xi_\nu : A \to \mathbb{C}$ ($\nu = 1, 2$) ([4]).
- Polynomials (real and complex) are subanalytic.
- Let $f = (f^1, \ldots, f^p) : \Omega \to \mathbb{C}^p$ be a holomorphic mapping, where $\Omega \subset \mathbb{C}^N$. Suppose that $B \subset \mathbb{C}^N$ is bounded, subanalytic and $\overline{B} \subset \Omega$. Then the maps $f|_B$ and $B \ni z \mapsto |f(z)| \in [0, \infty)$ are subanalytic.
- *Łojasiewicz's inequality* (cf. [Ł, BM, DS]). Let $\varphi, \phi : E \to \mathbb{R}$ be continuous and subanalytic functions, where $E \subset \mathbb{R}^m$ is compact. Assume that $\{\phi = 0\} \subset \{\varphi = 0\}$. Then $|\varphi(x)| \leq \eta |\phi(x)|^\alpha$ in E for some $\eta, \alpha > 0$.

Fix a nonempty open set $\Omega \subset \mathbb{C}^N$. For any set $S \subset \Omega$ and $p \in \mathbb{N}$, let

$$\mathcal{I}_\Omega(p; S) := \{f = (f^1, \ldots, f^p) : \Omega \to \mathbb{C}^p : f^\nu \in H^\infty(\Omega) \text{ and } f^\nu = 0 \text{ on } S, \text{ for each } \nu \leq p\}.$$

Since $\mathcal{I}_\Omega(p; S)$ is a closed linear subspace of the Banach space $\mathcal{B}(\Omega; \mathbb{C}^p)$ of all bounded mappings $f : \Omega \to \mathbb{C}^p$ (with the norm $\| \ \|_\Omega$), it follows that $\mathcal{I}_\Omega(p; S)$ is a Banach space with the induced norm. We will write $\mathcal{I}_\Omega(S)$ instead of $\mathcal{I}_\Omega(1; S)$.

PROPOSITION 3.1. *Assume that $\phi : E \to [0, \infty)$ is a continuous subanalytic function, where $E \subset \Omega$ is compact, and let $p \in \mathbb{N}$. Put $S := \{\phi = 0\}$. Then there are $M > 0$ and $l \in \mathbb{N}$ such that for any $f \in \mathcal{I}_\Omega(p; S)$,*

$$|f(z)| \leq M \|f\|_\Omega \phi(z)^{1/l}$$

whenever $z \in E$.

([3]) That is, its graph $\Gamma(\xi) \subset \mathbb{R}^{m+q}$ is subanalytic.

([4]) We identify $\mathbb{C}^k$ with $\mathbb{R}^{2k}$ via the map

$$\mathbb{C}^k \ni z = (z_1, \ldots, z_k) \mapsto (\operatorname{Re}(z_1), \operatorname{Im}(z_1), \ldots, \operatorname{Re}(z_k), \operatorname{Im}(z_k)) \in \mathbb{R}^{2k}.$$

Proof. To shorten notation we put $X := \mathcal{I}_\Omega(p; S)$. For each $k \in \mathbb{N}$, let

$$V_k := \{f \in X : |f(z)| \le k\|f\|_\Omega \phi(z)^{1/k} \text{ for all } z \in E\}.$$

Obviously, V_k is closed in X and $\mathbb{C} \cdot V_k = V_k$. By Lemma 2.3, it is enough to prove that:

(1) $\bigcup_{k\in\mathbb{N}} V_k = X$.
(2) For each $j \in \mathbb{N}$, $f_0 \in V_j$ and $r > 0$ there exists $\mu = \mu(j, f_0, r) \in \mathbb{N}$ such that

$$(V_j - f_0) \cap \{g \in X : \|g\|_\Omega = r\} \subset V_\mu.$$

Proof of (1). Take any $f \in X$. So $f : \Omega \to \mathbb{C}^p$ is a bounded holomorphic mapping such that $S = \{\phi = 0\} \subset \{f^1 = 0, \dots, f^p = 0\}$. We use Łojasiewicz's inequality along with the fact that the map $E \ni z \mapsto |f(z)| \in [0, \infty)$ is subanalytic to conclude that, for some $\eta, \alpha > 0$, $|f(z)| \le \eta\phi(z)^\alpha$ for all $z \in E$. We may assume that $\|f\|_\Omega > 0$, because otherwise $f \equiv 0$ and then the situation is trivial. Note that $\eta\phi^\alpha \le k\|f\|_\Omega \phi^{1/k}$ on E whenever $k \in \mathbb{N}$ is sufficiently large. Consequently, $f \in V_k$.

Proof of (2). Fix $j \in \mathbb{N}$, $f_0 \in V_j$ and $r > 0$. We need to show the existence of $\mu \in \mathbb{N}$ such that

$$(f \in V_j,\ \|f - f_0\|_\Omega = r) \ \Rightarrow\ f - f_0 \in V_\mu.$$

Assume therefore that $f \in V_j$ and $\|f - f_0\|_\Omega = r$. We have

- $|f_0(z)| \le j\|f_0\|_\Omega \phi(z)^{1/j}$ for all $z \in E$,
- $|f(z)| \le j\|f\|_\Omega \phi(z)^{1/j}$ for all $z \in E$.

Put $\theta := 1 + 2r^{-1}\|f_0\|_\Omega$. For any $z \in E$,

$$\begin{aligned}|f(z) - f_0(z)| &\le |f_0(z)| + |f(z)| \le j\big(\|f_0\|_\Omega + \|f\|_\Omega\big)\phi(z)^{1/j} \\ &\le j(2\|f_0\|_\Omega + \|f - f_0\|_\Omega)\phi(z)^{1/j} = j\theta\|f - f_0\|_\Omega \phi(z)^{1/j}.\end{aligned}$$

By the above estimates, we see easily that, for $\mu \in \mathbb{N}$ large enough (depending only on j, f_0, r and on ϕ), $f - f_0 \in V_\mu$. ∎

COROLLARY 3.2. *Assume that $h : \Omega \to \mathbb{C}^m$ is a holomorphic mapping. Let $K \subset \Omega$ be compact and $p \in \mathbb{N}$. Put $S := \{h = 0\}$. Then there are $M > 0$ and $l \in \mathbb{N}$ such that for any $f \in \mathcal{I}_\Omega(p; S)$,*

$$|f(z)| \le M\|f\|_\Omega |h(z)|^{1/l}$$

whenever $z \in K$. If moreover K is subanalytic, then we can choose $M > 0$ and $l \in \mathbb{N}$ so that the above inequality holds for all $f \in \mathcal{I}_\Omega(p; S \cap K)$.

Proof. Let E be a compact subanalytic set (for example, a finite union of compact boxes) such that $K \subset E \subset \Omega$. Since the map $E \ni z \mapsto$

$|h(z)| \in [0, \infty)$ is subanalytic, Proposition 3.1 yields the first part of the corollary. The second part is proved in the same manner by taking simply $E := K$. ∎

REMARK 3.3. Without the assumption that K is subanalytic the conclusion in the second part of Corollary 3.2 is no longer true.

EXAMPLE. Put

$$K := \{(t, \exp(-t^{-1})) : t \in (0,1]\} \cup \{(0,0)\}, \qquad \Omega := \{|z| < 2\} \subset \mathbb{C}^2.$$

Let $h : \Omega \ni z \mapsto z_1 z_2 \in \mathbb{C}$. Clearly, $S \cap K = \{(0,0)\}$. Let $f : \Omega \ni z \mapsto z_1 \in \mathbb{C}$. Although $f \in \mathcal{I}_\Omega(S \cap K)$, there are no $M > 0$, $l \in \mathbb{N}$ such that $|f(z)| \le M\|f\|_\Omega |h(z)|^{1/l}$ for all $z \in K$. ∎

The next result should be regarded, first and foremost, as another illustration of the usefulness of our method ([5]).

COROLLARY 3.4. *Assume that* $g : \Omega \to \mathbb{C}$ *is a holomorphic function. Let* $\Omega_0 \subset \mathbb{C}^N$ *be an open, bounded and subanalytic set such that* $\overline{\Omega}_0 \subset \Omega$. *Put* $S := \{g = 0\}$. *Then there are* $\theta > 0$ *and* $l \in \mathbb{N}$ *such that, for any* $f \in \mathcal{I}_\Omega(S \cap \Omega_0)$, *we can find a holomorphic function* $\tau : \Omega_0 \to \mathbb{C}$ *satisfying the following conditions:*

(1) $f^l = \tau g$ *in* Ω_0.
(2) τ *is subanalytic.*
(3) $\|\tau\|_{\Omega_0} \le \theta \|f\|_\Omega^l$ *if* $f = 0$ *on* $S \cap \partial\Omega_0$ (*i.e.* $f \in \mathcal{I}_\Omega(S \cap \overline{\Omega}_0)$).

Proof. We will consider two cases depending on whether or not Ω_0 is connected.

CASE 1: Ω_0 *is connected.* If $g \equiv 0$ in Ω_0, the situation is trivial (take $\tau \equiv 0$). Assume then that $S \cap \Omega_0$ is nowhere dense in $\mathbb{C}^N$. For each $x \in S \cap \Omega_0$, take $r(x) > 0$ such that $K(x, 2r(x)) \subset \Omega_0$ ([6]). Since Ω_0 and $S \cap \Omega_0$ are subanalytic, we may assume, by the theorem on subanalytic choice (cf. [DS, p. 78]), that the map $S \cap \Omega_0 \ni x \mapsto r(x) \in (0, \infty)$ is subanalytic. Put $B := \bigcup_{x \in S \cap \Omega_0} K(x, r(x))$. Note that B is the image of the set $\{(y, x) \in \mathbb{C}^N \times (S \cap \Omega_0) : |y - x| - r(x) < 0\}$ under the projection $(y, x) \mapsto y$. Consequently, B and its closure are subanalytic. We will now prove that $\overline{B} \subset \Omega_0 \cup \overline{S \cap \Omega_0}$.

Take $b \in \overline{B} \cap \partial\Omega_0$. We need to show that $b \in \overline{S \cap \Omega_0}$. There are sequences (b_n), (c_n) such that $b_n \to b$, $b_n \in K(c_n, r(c_n))$ and $c_n \in S \cap \Omega_0$. Note that

$$|b_n - b| \ge |c_n - b| - |c_n - b_n| > 2r(c_n) - r(c_n) = r(c_n).$$

([5]) Especially as part of this corollary can be obtained by using completely different tools—see [Wh].

([6]) $K(x, r) := \{z \in \mathbb{C}^N : |z - x| < r\}$.

Consequently, $r(c_n) \to 0$. Since $|c_n - b| < r(c_n) + |b_n - b|$, it follows that $c_n \to b$. So $b \in \overline{S \cap \Omega_0}$, as desired.

First we will prove (1) and (2). Put $K := \overline{B}$. Note that $\mathcal{I}_\Omega(S \cap \Omega_0) = \mathcal{I}_\Omega(\overline{S \cap \Omega_0}) = \mathcal{I}_\Omega(S \cap K)$. Take $l \in \mathbb{N}$ as in the second part of Corollary 3.2. Assume that $f \in \mathcal{I}_\Omega(S \cap \Omega_0)$. Let $\tau_0 : \Omega_0 \setminus S \ni z \mapsto f(z)^l/g(z) \in \mathbb{C}$. Clearly, τ_0 is holomorphic and, by Corollary 3.2, locally bounded in Ω_0 (use the fact that $S \cap \Omega_0 \subset \operatorname{Int} K$). By Riemann's removable singularity theorem, τ_0 extends to a holomorphic function τ in Ω_0. Note that τ_0 is subanalytic. Since $\Gamma(\tau) = \overline{\Gamma}(\tau_0) \cap (\Omega_0 \times \mathbb{C})$, it follows that τ is subanalytic as well ([7]).

To obtain (3) we similarly apply the second part of Corollary 3.2 (along with Riemann's removable singularity theorem), but this time we put $K := \overline{\Omega}_0$.

Case 2: *Ω_0 is not connected.* Then Ω_0 (being subanalytic) has only finitely many connected components $\Omega_0^1, \ldots, \Omega_0^k$ and each of them is subanalytic. By applying Case 1 to Ω_0^ν we obtain $\theta_\nu > 0$ and $l_\nu \in \mathbb{N}$ ($\nu = 1, \ldots, k$). Put $\theta := \max \theta_\nu$ and $l := \max l_\nu$. A straightforward argument proves that the constants θ, l have the required properties. ■

Remark 3.5. In the above corollary, to obtain (1), without any additional assumptions on g, we at least should assume that $\overline{S \cap \Omega_0} \cap \partial\Omega = \emptyset$. Clearly, the assumptions in Corollary 3.4 are stronger, but we need them to obtain also (2) and (3).

Example. For each $\nu \in \mathbb{N}$, put $a_\nu := 1 - 2^{-\nu}$. Let $\Omega := \{|z| < 1\} \subset \mathbb{C}$. Put

$$g : \Omega \ni z \mapsto \prod_{\nu \in \mathbb{N}} (a_\nu z - a_\nu^2)^\nu (a_\nu z - 1)^{-\nu} \in \mathbb{C},$$

$$f : \Omega \ni z \mapsto \prod_{\nu \in \mathbb{N}} (a_\nu z - a_\nu^2)(a_\nu z - 1)^{-1} \in \mathbb{C}.$$

It is easy to check that g and f are bounded holomorphic functions in Ω. Moreover, $S = \{1 - 2^{-\nu} : \nu \in \mathbb{N}\}$ and $f \in \mathcal{I}_\Omega(S)$. Suppose that $\Omega_0 \subset \Omega$ is an open set such that $\overline{S \cap \Omega_0} \cap \partial\Omega \neq \emptyset$. This means that $1 \in \overline{S \cap \Omega_0}$. Obviously, there is no $l \in \mathbb{N}$ such that $f^l = \tau g$ in Ω_0 for any holomorphic function $\tau : \Omega_0 \to \mathbb{C}$. ■

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([7]) As before, $\Gamma(\xi)$ denotes the graph of ξ.

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