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Absolutely (r,p,q)-summing inclusions

by

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Abstract. As a continuation of the work of Bennett and Carl for the case $q=\infty$, we consider absolutely (r,p,q)-summing inclusion maps between Minkowski sequence spaces, $1\le p,q\le 2$. Using these results we deduce parts of the limit orders of the corresponding operator ideals and an inclusion theorem between the ideals of (u,s,t)-nuclear and of absolutely (r,p,q)-summing operators, which gives a new proof of a result of Carl on Schatten class operators. Furthermore, we also consider inclusions between arbitrary Banach sequence spaces and inclusions between finite-dimensional Schatten classes. Finally, applications to Hilbert numbers of inclusions are given.

1. Introduction and basic tools. Let $1\le r,p,q\le\infty$ be such that $1/p+1/q\ge 1/r$. According to Pietsch [31, 17.1.1], an operator $T:X\to Y$ between Banach spaces X and Y is called *absolutely* (r,p,q)-*summing* if there exists a constant $C>0$ such that for any choice of $x_1,\dots,x_n\in X$ and $y_1',\dots,y_n'\in Y'$, the inequality

$$\Big(\sum_{k=1}^n |y_k'(Tx_k)|^r\Big)^{1/r}\le C\sup_{x'\in B_{X'}}\Big(\sum_{k=1}^n|x'(x_k)|^p\Big)^{1/p}\sup_{y\in B_Y}\Big(\sum_{k=1}^n|y_k'(y)|^q\Big)^{1/q}$$

holds. We put $\pi_{r,p,q}(T):=\inf C$ with C as above. In this way, we obtain the maximal Banach operator ideal $(\mathit{\Pi}_{r,p,q},\pi_{r,p,q})$. Let us list the most prominent special cases which have been dealt with in the literature so far:

- $\mathit{\Pi}_{r,p}:=\mathit{\Pi}_{r,p,\infty}$, the ideal of all *absolutely* (r,p)-*summing* operators;
- $\mathit{\Pi}_p:=\mathit{\Pi}_{p,p}=\mathit{\Pi}_{p,p,\infty}$, the ideal of all *absolutely* p-*summing* operators;
- $\mathcal{D}_{p,q}:=\mathit{\Pi}_{r,p,q}$ with r,p,q such that $1/r=1/p+1/q$, the ideal of all (p,q)-*dominated* operators;
- $\mathcal{D}_p:=\mathcal{D}_{p,p}=\mathit{\Pi}_{1,p,p'}$, the ideal of all p-*dominated* operators.

In this paper we deal with the ideal $\mathit{\Pi}_{r,p,q}$ where $1\le p,q\le 2$. For the special case $r=p=1$ and $q=2$, this ideal has become of interest recently in an article of Bu [4], where the author has shown that a Banach

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space X of cotype 2 is a G.T. space (i.e., $\mathcal{L}(X,\ell_2)=\Pi_1(X,\ell_2)$) if and only if $X\mathbin{\widetilde{\otimes}_\varepsilon}\ell_1\subseteq X\mathbin{\widetilde{\otimes}_\pi}\ell_2$, i.e., the identity map on X is absolutely $(1,1,2)$-summing.

Bennett [2] and Carl [6] independently successfully investigated under which assumptions on the indices involved the inclusion mapping $\mathrm{id}:\ell_u\hookrightarrow\ell_v$ is in $\Pi_{q,p}$. In this article, we present analogs of their results for the Banach operator ideal $\Pi_{r,p,q}$ and then derive in parts the limit order of these ideals. This can also be used to give an alternative proof of a result due to Carl [5] on Schatten class operators. Furthermore, in the spirit of [13] and [16], we also consider inclusions $E\hookrightarrow F$, E and F arbitrary Banach sequence spaces, and $\mathcal{S}_u^n\hookrightarrow\mathcal{S}_v^n$. Finally, we give applications to Hilbert numbers of inclusions.

We start with some basic notations. For $1\le p\le\infty$, its conjugate number p' is defined by $1/p+1/p'=1$. For two real sequences (a_n) and (b_n) we mean by $a_n\prec b_n$ that there exists $C>0$ such that $a_n\le Cb_n$ for all $n\in\mathbb{N}$, and by $a_n\succ b_n$ that $b_n\prec a_n$. If $a_n\prec b_n$ and $a_n\succ b_n$ simultaneously, then we write $a_n\asymp b_n$.

We shall use standard notation and notions from Banach space theory, as presented e.g. in [10, 19]. If X is a Banach space, then B_X is its (closed) unit ball and X' its dual. As usual $\mathcal{L}(X,Y)$ denotes the Banach space of all (bounded linear) operators from X into Y endowed with the operator norm $\|\cdot\|$, where X and Y are Banach spaces, and $\mathcal{N}(X,Y)$ the Banach space of all nuclear operators endowed with the nuclear norm $\mathrm{N}(\cdot)$. By $X\otimes_\varepsilon Y$ and $X\otimes_\pi Y$ we denote their injective and projective tensor products, respectively, and by $X\mathbin{\widetilde{\otimes}_\varepsilon}Y$ and $X\mathbin{\widetilde{\otimes}_\pi}Y$ the respective completions. If one of the spaces involved is finite-dimensional, we can identify $\mathcal{L}(X,Y)=X'\otimes_\varepsilon Y$ and $\mathcal{N}(X,Y)=X'\otimes_\pi Y$ isometrically. Furthermore, if X is finite-dimensional, the tensor norm Δ_2 on $\ell_2\otimes X$ is given by the identification $\ell_2\otimes_{\Delta_2}X=\ell_2(X)$, where the latter is as usual the corresponding Köthe–Bochner space. For this and more information on tensor products of Banach spaces we refer to [10].

For $1\le r<\infty$ we denote by $\mathcal{S}_r$ the Banach space of all compact operators $T:\ell_2\to\ell_2$ for which the sequence of singular numbers is in ℓ_r. We put $\mathcal{S}_\infty:=\mathcal{L}(\ell_2)$.

Standard techniques (see, e.g., [19] or [10]) allow us to formulate the following useful characterization.

Proposition 1.1. *Let $1\le p,q,r\le\infty$ such that $1/r\le 1/p+1/q$. Then for an operator $T:X\to Y$ between Banach spaces X and Y, the following are equivalent*:

(i) *T is absolutely (r,p,q)-summing*;

(ii) *the bilinear mappings* $\varphi_n : X \otimes_\varepsilon \ell_p^n \times Y' \otimes_\varepsilon \ell_q^n \to \ell_r^n$, *defined by* $\varphi_n((x_1,\ldots,x_n),(y_1',\ldots,y_n')) := (y_1'(Tx_1),\ldots,y_n'(Tx_n))$, *are uniformly bounded.*

In this case, $\pi_{r,p,q}(T) = \sup_n \|\varphi_n\|$. *If* $r=1$, *then the above is equivalent to*

(iii) *the mappings* $T \otimes \mathrm{id}_{pq'}^n : X \otimes_\varepsilon \ell_p^n \to Y \otimes_\pi \ell_{q'}^n$ *are uniformly bounded.*

In this case, $\pi_{1,p,q}(T) = \sup_n \|T \otimes \mathrm{id}_{pq'}^n\|$.

The following inclusion can be found in [31, 17.1.4].

PROPOSITION 1.2. *Let* $r_0 \le r_1$, $p_0 \le p_1$ *and* $q_0 \le q_1$. *Suppose that* $0 \le 1/p_0 + 1/q_0 - 1/r_0 \le 1/p_1 + 1/q_1 - 1/r_1$. *Then* $\Pi_{r_0,p_0,q_0} \subseteq \Pi_{r_1,p_1,q_1}$.

2. Interpolation of summing norms. Basics on interpolation theory of Banach spaces can be found in [3]. Let us just introduce our notation for the complex interpolation method. For a given compatible Banach couple (X_0,X_1) and $0<\theta<1$, we denote by $[X_0,X_1]_\theta$ the resulting complex interpolation space. We will frequently use the fact that for $1 \le p_0,p_1 \le \infty$ and $0<\theta<1$ one has

$$[\ell_{p_0},\ell_{p_1}]_\theta = \ell_p, \tag{2.1}$$

where p is determined by $1/p = (1-\theta)/p_0 + \theta/p_1$. The following crucial interpolation tool is due to Kouba [25].

PROPOSITION 2.1. *Let* $1 \le p_0,p_1,q_0,q_1 \le 2$, $0<\theta<1$ *and* $1 \le p,q \le 2$ *with* $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$. *Then* $[\ell_{p_0} \widetilde{\otimes}_\varepsilon \ell_{q_0}, \ell_{p_1} \widetilde{\otimes}_\varepsilon \ell_{q_1}]_\theta = \ell_p \widetilde{\otimes}_\varepsilon \ell_q$. *In particular*,

$$\sup_{n,m} \|\ell_p^m \otimes_\varepsilon \ell_q^n \hookrightarrow [\ell_{p_0}^m \otimes_\varepsilon \ell_{q_0}^n, \ell_{p_1}^m \otimes_\varepsilon \ell_{q_1}^n]_\theta\| < \infty.$$

The following one-sided interpolation formula can be found in [29].

PROPOSITION 2.2. *Let* $1 \le p_0 < p_1 < 2$, $1 \le r < p_1'$, $0<\theta<1$ *and* $1 \le p < p_1$ *with* $1/p = (1-\theta)/p_0 + \theta/p_1$. *Then* $[\ell_r \widetilde{\otimes}_\varepsilon \ell_{p_0}, \ell_r \widetilde{\otimes}_\varepsilon \ell_{p_1}]_\theta = \ell_r \widetilde{\otimes}_\varepsilon \ell_p$. *In particular*,

$$\sup_{n,m} \|\ell_r^m \widetilde{\otimes}_\varepsilon \ell_p^n \hookrightarrow [\ell_r^m \widetilde{\otimes}_\varepsilon \ell_{p_0}^n, \ell_r^m \widetilde{\otimes}_\varepsilon \ell_{p_1}^n]_\theta\| < \infty.$$

To simplify our statements, we denote for $1 \le u \le v \le \infty$ the inclusion map $\mathrm{id} : \ell_u \hookrightarrow \ell_v$ by id_{uv}, and by id_{uv}^n the finite-dimensional inclusion map $\mathrm{id} : \ell_u^n \hookrightarrow \ell_v^n$ (for u and v not necessarily ordered as above).

LEMMA 2.3. *Let* $1 \le u_0,u_1 \le 2$, $u_0 \le v_0 \le \infty$, $u_1 \le v_1 \le \infty$, $1 \le r_0,r_1 \le \infty$ *and* $1 \le s_0,s_1,t_0,t_1 \le 2$. *Then for all* $0<\theta<1$, *there exists* $C>0$ *such that*

$$\pi_{r,s,t}(\mathrm{id}_{uv}^n) \le C \pi_{r_0,s_0,t_0}(\mathrm{id}_{u_0v_0}^n)^{1-\theta} \pi_{r_1,s_1,t_1}(\mathrm{id}_{u_1v_1}^n)^\theta,$$

where $1/u=(1-\theta)/u_0+\theta/u_1$, $1/v=(1-\theta)/v_0+\theta/v_1$, $1/r=(1-\theta)/r_0+\theta/r_1$, $1/s=(1-\theta)/s_0+\theta/s_1$ *and* $1/t=(1-\theta)/t_0+\theta/t_1$, *in each of the following cases*:

(i) $2 \le v_0, v_1 \le \infty$;
(ii) $v=v_0=v_1$ *and* $t=t_0=t_1$;
(iii) $\max(t_0,t_1) < v = v_0 = v_1$.

In particular, *in all of these cases*, $\mathrm{id}_{u_0v_0} \in \Pi_{r_0,s_0,t_0}$ *and* $\mathrm{id}_{u_1v_1} \in \Pi_{r_1,s_1,t_1}$ *imply* $\mathrm{id}_{uv} \in \Pi_{r,s,t}$.

Proof. We define the bilinear mappings

$$\psi_{n,m} : \mathbb{K}^n \otimes \mathbb{K}^m \times \mathbb{K}^n \otimes \mathbb{K}^m \to \mathbb{K}^m, \qquad ((x_k),(y_k')) \mapsto (y_k'(x_k)).$$

Then the mappings

$$\psi_{n,m} : \ell_{u_i}^n \otimes_\varepsilon \ell_{s_i}^m \times \ell_{v_i'}^n \otimes_\varepsilon \ell_{t_i}^m \to \ell_{r_i}^m$$

are bounded from above by $\pi_{r_i,s_i,t_i}(\mathrm{id}_{u_iv_i}^n)$, $i=0,1$. Thus, by bilinear interpolation (see, e.g., [3, 4.4.1]), the mappings

$$\psi_{n,m} : [\ell_{u_0}^n \otimes_\varepsilon \ell_{s_0}^m, \ell_{u_1}^n \otimes_\varepsilon \ell_{s_1}^m]_\theta \times [\ell_{v_0'}^n \otimes_\varepsilon \ell_{t_0}^m, \ell_{v_1'}^n \otimes_\varepsilon \ell_{t_1}^m]_\theta \to [\ell_{r_0}^m, \ell_{r_1}^m]_\theta$$

are bounded from above by $\pi_{r_0,s_0,t_0}(\mathrm{id}_{u_0v_0}^n)^{1-\theta}\pi_{r_1,s_1,t_1}(\mathrm{id}_{u_1v_1}^n)^\theta$. Clearly, by (2.1) we have $[\ell_{r_0}^m, \ell_{r_1}^m]_\theta = \ell_r^m$ isometrically. By Proposition 2.1,

$$\sup_{n,m} \|\mathrm{id} : \ell_u^n \otimes_\varepsilon \ell_s^m \hookrightarrow [\ell_{u_0}^n \otimes_\varepsilon \ell_{s_0}^m, \ell_{u_1}^n \otimes_\varepsilon \ell_{s_1}^m]_\theta\| < \infty$$

and by [3, 4.5.2] together with Proposition 2.1 in case (i), [3, 4.2.1(c)] in case (ii), and Proposition 2.2 in case (iii),

$$\sup_{n,m} \|\mathrm{id} : \ell_{v'}^n \otimes_\varepsilon \ell_t^m \hookrightarrow [\ell_{v_0'}^n \otimes_\varepsilon \ell_{t_0}^m, \ell_{v_1'}^n \otimes_\varepsilon \ell_{t_1}^m]_\theta\| < \infty.$$

Thus, the mappings

$$\psi_{n,m} : \ell_u^n \otimes_\varepsilon \ell_s^m \times \ell_{v'}^n \otimes_\varepsilon \ell_t^m \to \ell_r^m$$

have norm less than or equal to $C\pi_{r_0,s_0,t_0}(\mathrm{id}_{u_0v_0}^n)^{1-\theta}\pi_{r_1,s_1,t_1}(\mathrm{id}_{u_1v_1}^n)^\theta$ for some $C>0$ not depending on m and n. The final assertion then follows by the maximality of the operator ideal $\Pi_{r,s,t}$ and by density. ∎

3. Absolutely summing inclusion maps. To apply the above lemma, we need some extreme cases.

LEMMA 3.1. *The following hold true*:

(i) $\mathrm{id}_{22} \in \Pi_{1,1,1}$;
(ii) $\mathrm{id}_{1\infty} \in \Pi_{1,2,2}$;
(iii) $\mathrm{id}_{12} \in \Pi_{1,1,2} \cap \Pi_{1,2,1}$;
(iv) $\mathrm{id}_{2\infty} \in \Pi_{1,1,2} \cap \Pi_{1,2,1}$.

Proof. (i) is clear as $\Pi_{1,1,1} = \mathcal{L}$. This means that

$$\sup_{n,m} \|\mathrm{id} \otimes \mathrm{id} : \ell_2^m \otimes_\varepsilon \ell_1^n \to \ell_2^m \otimes_\pi \ell_\infty^n\| = 1,$$

which also gives $\mathrm{id}_{1\infty} \in \Pi_{1,2,2}$. Furthermore, by Grothendieck, $\mathrm{id}_{12} \in \Pi_2$ implies $\mathrm{id}_{12} \in \Pi_1$, i.e.,

$$\sup_{n,m} \|\mathrm{id} \otimes \mathrm{id} : \ell_1^n \otimes_\varepsilon \ell_1^m \to \ell_2^n \otimes_\pi \ell_1^m\| < \infty.$$

Thus, by factorization,

$$\sup_{n,m} \|\mathrm{id} \otimes \mathrm{id} : \ell_1^n \otimes_\varepsilon \ell_1^m \to \ell_2^n \otimes_\pi \ell_2^m\| < \infty,$$

which yields $\mathrm{id}_{12} \in \Pi_{1,1,2}$ and $\mathrm{id}_{2\infty} \in \Pi_{1,2,1}$ (by duality).

By [31, 22.4.8] it is known that $\pi_2(\mathrm{id}_{12}^n) = \pi_{2,2,\infty}(\mathrm{id}_{12}^n) = 1$, i.e.,

$$\sup_{n,m} \|\mathrm{id} \otimes \mathrm{id} : \ell_2^m \otimes_\varepsilon \ell_1^n \to \ell_2^m \otimes_{\Delta_2} \ell_2^n\| = 1.$$

Since $\varepsilon^t = \varepsilon$, and $\Delta_2^t = \Delta_2$ on the tensor product of two Hilbert spaces, it also follows that

$$\sup_{n,m} \|\mathrm{id} \otimes \mathrm{id} : \ell_1^m \otimes_\varepsilon \ell_2^n \to \ell_2^m \otimes_{\Delta_2} \ell_2^n\| = 1.$$

Furthermore, by duality,

$$\sup_{n,m} \|\mathrm{id} \otimes \mathrm{id} : \ell_2^m \otimes_{\Delta_2} \ell_2^n \to \ell_2^m \otimes_\pi \ell_\infty^n\| = 1.$$

Thus, by factorization,

$$\sup_{n,m} \|\mathrm{id} \otimes \mathrm{id} : \ell_1^m \otimes_\varepsilon \ell_2^n \to \ell_2^m \otimes_\pi \ell_\infty^n\| = 1,$$

which gives $\mathrm{id}_{12} \in \Pi_{1,2,1}$ and $\mathrm{id}_{2\infty} \in \Pi_{1,1,2}$. ■

THEOREM 3.2. *Let $1 \le p, q \le 2$ and $1 \le u \le 2 \le v \le \infty$ with $1/u - 1/v \ge 2 - 1/p - 1/q$. Then $\mathrm{id}_{uv} \in \Pi_{1,p,q}$.*

Proof. We start by applying Lemma 2.3(i) for $r = r_0 = r_1$ and using Lemma 3.1 to obtain more extreme cases. Fix $1 \le q \le 2$ and define $1 \le \tilde{q} \le 2 \le \overline{q} \le \infty$ by $1/\tilde{q} = 3/2 - 1/q$ and $1/\overline{q} = 1/q - 1/2$. Then we obtain the following (taking $\theta := 2/\overline{q}$ in Lemma 2.3(i) whenever $1 < q < 2$):

u_0	v_0	u_1	v_1	s_0	t_0	s_1	t_1		
1	2	1	2	1	2	2	1	$\mathrm{id}_{12} \in \Pi_{1,\tilde{q},q}$	(i)
2	∞	2	∞	1	2	2	1	$\mathrm{id}_{2\infty} \in \Pi_{1,\tilde{q},q}$	(ii)
1	2	2	2	1	2	1	1	$\mathrm{id}_{\tilde{q}2} \in \Pi_{1,1,q}$	(iii)
2	∞	2	2	1	2	1	1	$\mathrm{id}_{2\overline{q}} \in \Pi_{1,1,q}$	(iv)
1	∞	2	∞	2	2	2	1	$\mathrm{id}_{\tilde{q}\infty} \in \Pi_{1,2,q}$	(v)
1	∞	1	2	2	2	2	1	$\mathrm{id}_{1\overline{q}} \in \Pi_{1,2,q}$	(vi)

Coming to the parameter p, we have to consider two cases, where we use again Lemma 2.3(i) (taking $\theta = \tilde{q}'/u_0'$, $\theta = \overline{q}/v_0$, $\theta = q'(1/u_1 - 1/2)$ and $\theta = q'(1/2 - 1/v_1)$, respectively) together with the results from the above table:

- $1 \le p < \tilde{q}$: (i) and (iii) give

$$\mathrm{id}_{u_0 2} \in \Pi_{1,p,q}, \quad \text{where } \ 1/u_0 = 5/2 - 1/p - 1/q,$$

  and (ii) and (iv) give

$$\mathrm{id}_{2 v_0} \in \Pi_{1,p,q}, \quad \text{where } \ 1/v_0 = 1/p + 1/q - 3/2.$$

- $\tilde{q} \le p \le 2$: (ii) and (v) give

$$\mathrm{id}_{u_1 \infty} \in \Pi_{1,p,q}, \quad \text{where } \ 1/u_1 = 2 - 1/p - 1/q,$$

  and (i) and (vi) give

$$\mathrm{id}_{1 v_1} \in \Pi_{1,p,q}, \quad \text{where } \ 1/v_1 = 1/p + 1/q - 1.$$

If we take $s_0 = s_1 = p$ and $t_0 = t_1 = q$, Lemma 2.3(i) applied to these cases for all $0 < \theta < 1$ gives $\mathrm{id}_{uv} \in \Pi_{1,p,q}$ for all those $1 \le u \le 2 \le v \le \infty$ such that $1/u - 1/v = 2 - 1/p - 1/q$. The rest is clear by factorization. ∎

The above can be extended to the general case of absolutely (r, p, q)-summing operators. When considering limit orders in the next section, we will see that the assumption $1/u - 1/v \ge 1 + 1/r - 1/p - 1/q$ in the corollary below cannot be weakened.

COROLLARY 3.3. *Let* $1 \le p, q \le 2$, $1 \le r \le \infty$ *and* $1 \le u \le 2 \le v \le \infty$ *with* $1/u - 1/v \ge 1 + 1/r - 1/p - 1/q \ge 0$. *Then* $\mathrm{id}_{uv} \in \Pi_{r,p,q}$.

Proof. The idea is to find $p_0 \le p$ and $q_0 \le q$ such that $1/p_0 + 1/q_0 - 1 = 1/p + 1/q - 1/r$. Then $2 - 1/p_0 - 1/q_0 = 1/u$, so that the above theorem together with Proposition 1.2 gives

$$\mathrm{id}_{uv} \in \Pi_{1,p_0,q_0} \subseteq \Pi_{r,p,q}.$$

We have to consider several cases, for which we simply list our choices of p_0, q_0 and leave the verification to the reader:

- $r \ge \max(p, q)$: $q_0 = 1$ and p_0 such that $1/p_0 = 1/p + 1/q - 1/r$;
- $r \le q$: $p_0 = p$ and q_0 such that $1/q_0 = 1 - 1/r + 1/q$;
- $r \le p$: $q_0 = q$ and p_0 such that $1/p_0 = 1 - 1/r + 1/p$. ∎

The situation in the cases other than $1 \le u \le 2 \le v \le \infty$ seems to be more complicated. We can give the following partial result for $1 \le u \le v \le 2$. It will turn out later on that (i) is optimal in the case $q = 2$ and that (ii) is almost optimal in the case $p = 1$ (that is, the only improvement possible in this case is to replace "$>$" by "$\ge$"). However, (iii) shows that (i) is not optimal in general.

PROPOSITION 3.4. *Let $1 \le p,q \le 2$, $1 \le r \le \infty$ and $1 \le u < v \le 2$.*

(i) $\mathrm{id}_{uv} \in \Pi_{r,p,q}$ *whenever* $1/r \le 1/p - v'/u'q'$;
(ii) $\mathrm{id}_{uv} \in \Pi_{p,p,q}$ *whenever* $1/u - 1/v > 1/q'$;
(iii) $\mathrm{id}_{uv} \in \Pi_{r,p,q}$ *whenever* $v > q$ *and* $1/r < 1/p - (v'/q')(1/q' + 1/v - 1/u)$.

Proof. (i) Since $\mathrm{id}_{11} \in \Pi_{1,1,2}$, it follows that $\mathrm{id}_{1v} \in \Pi_{1,1,2} \subseteq \Pi_{1,1,q} \subseteq \Pi_{p,p,q}$. Furthermore, $\mathrm{id}_{vv} \in \Pi_{\tilde r,p,q}$, where $1/\tilde r = 1/p + 1/q - 1$. Set $\theta := v'/u'$. Then $1/u = (1-\theta)/1 + \theta/v$. Now Lemma 2.3(ii) implies $\mathrm{id}_{uv} \in \Pi_{r,p,q}$, where

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{p} + \frac{\theta}{q} - \theta = \frac{1}{p} - \frac{\theta}{q'} = \frac{1}{p} - \frac{v'}{u'q'}.$$

(ii) Let $t < v$ be arbitrary. Then as above, $\mathrm{id}_{1v} \in \Pi_{1,1,t}$ and $\mathrm{id}_{tv} \in \Pi_{1,1,1}$. Set $\theta := t'/u'$; then $1/u = (1-\theta)/1 + \theta/t$. Now Lemma 2.3(iii) implies $\mathrm{id}_{uv} \in \Pi_{1,1,q}$, where

$$\frac{1}{q} = \frac{1-\theta}{t} + \frac{\theta}{1} = \frac{1}{t} + \frac{\theta}{t'} = \frac{1}{t} + \frac{1}{u'} > \frac{1}{v} + \frac{1}{u'}.$$

The claim for p arbitrary follows by Proposition 1.2.

(iii) Let $\tilde u < u$ be such that $1/\tilde u > 1/q' + 1/v$. Then (ii) implies $\mathrm{id}_{\tilde u v} \in \Pi_{p,p,q}$, and once again $\mathrm{id}_{vv} \in \Pi_{\tilde r,p,q}$ with $1/\tilde r = 1/p + 1/q - 1$. Set $\theta = v'(1/\tilde u - 1/u)$; then $1/u = (1-\theta)/\tilde u + \theta/v$. Now Lemma 2.3(ii) implies $\mathrm{id}_{uv} \in \Pi_{r,p,q}$, where

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{p} + \frac{\theta}{q} - \theta = \frac{1}{p} - \frac{v'}{q'}\left(\frac{1}{\tilde u} - \frac{1}{u}\right) < \frac{1}{p} - \frac{v'}{q'}\left(\frac{1}{q'} + \frac{1}{v} - \frac{1}{u}\right),$$

which gives the claim. ∎

4. Limit orders. We continue with a result on limit orders. For the definition and basic facts mentioned subsequently, we refer to [31, 14.4].

Let $1 \le u, v \le \infty$ and $\sigma = (\sigma_n) \in \ell_\infty$ be such that the diagonal operator $D_\sigma : \ell_u \to \ell_v$, $(x_n) \mapsto (\sigma_n x_n)$, is defined (and continuous). Then for a Banach operator ideal $(\mathcal{A}, \mathrm{A})$, its *limit order* $\lambda(\mathcal{A}, u, v)$ is defined by

$$\lambda(\mathcal{A}, u, v) := \inf\{1/r \ge 0;\ D_\sigma \in \mathcal{A}(\ell_u, \ell_v) \text{ for all } \sigma \in \ell_r\}.$$

Very useful in computing special limit orders is the following formula:

$$\lambda(\mathcal{A}, u, v) = \inf\{\lambda \ge 0; \exists \varrho \ge 0 \text{ such that } \mathrm{A}(\mathrm{id}^n_{uv}) \le \varrho n^\lambda\}.$$

König [24] in a famous paper proved an important connection to embedding maps of Sobolev spaces and weakly singular integral operators (see also [31, 22.7]).

Before stating our partial result for the limit order of $\Pi_{r,p,q}$, we prove a minor lemma first.

LEMMA 4.1. *Let $1 \le r \le p \le 2$. Then* $\pi_{r,p,2}(\mathrm{id}^n_{11}) \prec n^{1/r-1/p}$.

Proof. We start with the case $r = 1$. Since $\mathcal{L}(\ell_1, \ell_2) = \Pi_1(\ell_1, \ell_2)$ by Grothendieck (see, e.g., [31, 22.4.4]), we have $\mathrm{id}_{12} \in \Pi_1$, hence

$$\|\mathrm{id} \otimes \mathrm{id} : \ell_1 \otimes_\varepsilon \ell_1^n \to \ell_2 \otimes_\pi \ell_1^n\| \leq \pi_1(\mathrm{id}_{12}) \prec 1,$$

which gives $\mathrm{id}_{11} \in \Pi_{1,1,2}$. Furthermore, $\pi_2(\mathrm{id}_{12}^n) = 1$, and $\pi_2(\mathrm{id}_{\infty 2}^n) = n^{1/2}$ (see, e.g., [31, 22.4.9]), hence,

$$\|\mathrm{id} \otimes \mathrm{id} : \ell_2 \otimes_\varepsilon \ell_1^n \to \ell_2 \otimes_{\Delta_2} \ell_2^n\| = 1$$

and (by duality)

$$\|\mathrm{id} \otimes \mathrm{id} : \ell_2 \otimes_{\Delta_2} \ell_2^n \to \ell_2 \otimes_\pi \ell_1^n\| \leq n^{1/2}.$$

Thus, by factorization,

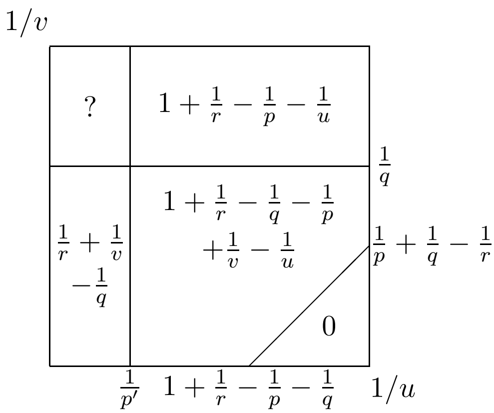
$$\|\mathrm{id} \otimes \mathrm{id} : \ell_2 \otimes_\varepsilon \ell_1^n \to \ell_2 \otimes_\pi \ell_1^n\| \leq n^{1/2},$$

which gives $\pi_{1,2,2}(\mathrm{id}_{11}^n) \leq n^{1/2}$. The claim for $1 < p < 2$ now follows by Lemma 2.3(ii) with $r_0 = r_1 = 1$, $s_0 = 1, s_1 = 2, s = p$ and $\theta = 2/p'$.

Coming to the general case $r \leq p$, we observe that $\Pi_{1,1,2} \subseteq \Pi_{p,p,2}$ by Proposition 1.2. Hence, $\mathrm{id}_{11} \in \Pi_{p,p,2}$, which implies $\pi_{p,p,2}(\mathrm{id}_{11}^n) \prec 1$. By the above, $\pi_{1,p,2}(\mathrm{id}_{11}^n) \prec n^{1-1/p}$. Thus, Lemma 2.3(ii) with $r_0 = 1, r_1 = p$, $s_0 = s_1 = p$ and $\theta = p'/r'$ gives the claim. ∎

We now get the following partial result for the limit order of $\Pi_{r,p,q}$:

THEOREM 4.2. *Let $1 \leq r \leq p, q \leq 2$ with $1/p + 1/q \leq 1/2 + 1/r$. Then $\Pi_{r,p,q}(\mathrm{id}_{uv}^n) \asymp n^{\alpha_{r,p,q}(u,v)}$, where $\alpha_{r,p,q}(u,v)$ is given by the following (incomplete) diagram*:



In particular, $\lambda(\Pi_{r,p,q}, u, v) = \alpha_{r,p,q}(u,v)$. In the case $r = 1$, the diagram can be completed by substituting "1" *for* "?" *in the upper left quadrant. In particular*,

$$\pi_{1,p,q}(\mathrm{id}_{uv}^n) \asymp \|\mathrm{id} \otimes \mathrm{id} : \ell_u^n \otimes_\varepsilon \ell_p^n \to \ell_v^n \otimes_\pi \ell_{q'}^n\| \asymp \max\left(1, \frac{\mathrm{N}(\mathrm{id}_{v'q'}^n)}{\|\mathrm{id}_{u'p}^n\|}\right)$$

for all $1 \leq u, v \leq \infty$ whenever $1 \leq p, q \leq 2$ are such that $1/p + 1/q \leq 3/2$.

Proof. Consider the standard unit vectors $e_1,\dots,e_n$ in ℓ_u^n and $\ell_{v'}^n$, respectively. Then

$$\sup_{x'\in B_{\ell'_u}}\Big(\sum_{k=1}^n |x'(e_k)|^p\Big)^{1/p} = \|\mathrm{id}^n_{u'p}\|,\qquad \sup_{y\in B_{\ell_v}}\Big(\sum_{k=1}^n |e_k(y)|^q\Big)^{1/q} = \|\mathrm{id}^n_{q'v'}\|$$

and

$$\Big(\sum_{k=1}^n |e_k(\mathrm{id}_{uv}(e_k))|^r\Big)^{1/r} = n^{1/r}.$$

Thus,

$$\pi_{r,p,q}(\mathrm{id}^n_{uv}) \ge \max\Big(1, \frac{n^{1/r}}{\|\mathrm{id}^n_{u'p}\|\,\|\mathrm{id}^n_{q'v'}\|}\Big),$$

which gives the lower estimates.

Concerning the upper ones, Corollary 3.3 gives the part of the diagram where we have a "0", so we are left with the remaining four parts. Since $(\mathcal{N},\mathrm{N})$ is the smallest operator ideal, we have $\Pi_{r,p,q}(\mathrm{id}^n_{uv}) \le \mathrm{N}(\mathrm{id}^n_{uv}) \le n$, which gives the exact estimate in the case $r=1$ for the "?"-part. The upper estimates for the part above the "0" follow by factorization from Corollary 3.3. For the last two parts, observe that by Lemma 4.1,

$$\pi_{r,p,q}(\mathrm{id}^n_{11}) \le \pi_{r,p,2}(\mathrm{id}^n_{11}) \prec n^{1/r-1/p},$$

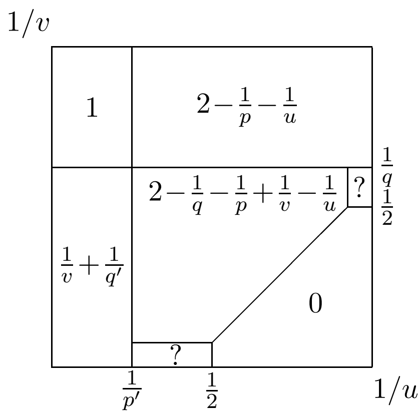
and by duality,

$$\pi_{r,p,q}(\mathrm{id}^n_{\infty\infty}) = \pi_{r,q,p}(\mathrm{id}^n_{11}) \prec n^{1/r-1/q}.$$

The remaining estimates then follow by factorization from these corner cases. ∎

Remark 4.3.

(a) For the case $1\le p,q\le 2$ and $1/p+1/q>3/2$, the same techniques give the following incomplete picture for the limit order of $\Pi_{1,p,q}$:



(b) Apart from the gaps in the above (which, as we conjecture, may be filled according to the first diagram), there is not much hope that the "easy" formula

$$\pi_{1,p,q}(\ell_u^n \hookrightarrow \ell_v^n) \asymp \max(1, \mathrm{N}(\mathrm{id}^n_{v'q'})/\|\mathrm{id}^n_{u'p}\|)$$

or the (weaker) formula for the limit order hold whenever one of the indices p and q is strictly greater than 2; e.g., they do not hold for the ideals $\mathcal{D}_p = \varPi_{1,p,p'}$ whenever $p \neq 2$ (see, e.g., [31, 22.5]).

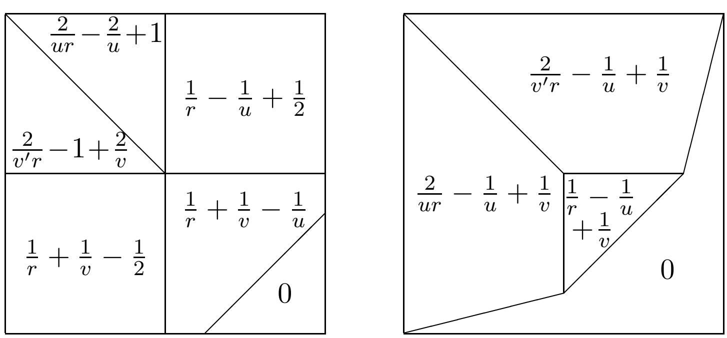
An immediate consequence of the above considerations (the case $r = 1$) for the norms of tensor product identities is the following:

Corollary 4.4. *Let $1 \le p, q \le 2 \le r, s \le \infty$. Then*

$$\|\mathrm{id} \otimes \mathrm{id} : \ell_p^n \otimes_\varepsilon \ell_q^m \to \ell_r^n \otimes_\pi \ell_s^m\| \asymp \min(n, m)^{\max(0, 1-1/p-1/q+1/r+1/s)}.$$

Let $\mathfrak{H}_r$ and $\mathfrak{H}_{r,\infty}$ denote the operator ideals of all operators T with $(h_n(T)) \in \ell_r$ and $(h_n(T)) \in \ell_{r,\infty}$, respectively (for the definition of Hilbert numbers and the facts mentioned here we refer to the very last section). By the fact that $\varPi_{r,2,2}$ is the largest extension of $\mathcal{S}_r$ and by (8.1) we know that $\mathfrak{H}_r \subseteq \varPi_{r,2,2} \subseteq \mathfrak{H}_{r,\infty}$, hence for all $1 \le u, v, r \le \infty$ it follows that $\lambda(\mathfrak{H}_r, u, v) = \lambda(\varPi_{r,2,2}, u, v)$. Thus, the diagrams for the limit order of $\mathfrak{H}_r$ given in [20] give the following ones for $\varPi_{r,2,2}$:

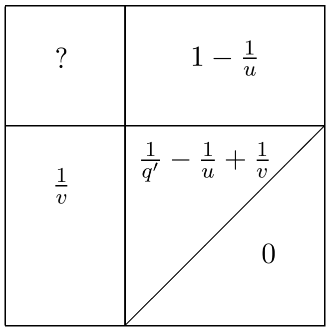
$1 \le r \le 2$ $\qquad$ $\varPi_{r,2,2}$ $\qquad$ $2 < r < \infty$

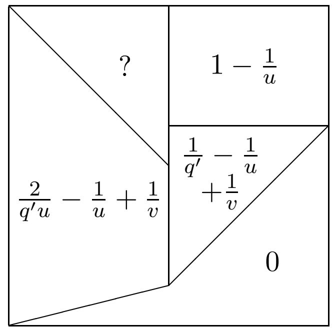


Note that the special case $\mathrm{id}_{uv} \in \varPi_{r,2,2}$ whenever $1 \le u < v \le 2$ and $1/r \le (v'/2)(1/u - 1/v)$ has also been proved in Proposition 3.4(i).

Proposition 3.4 can be used to give more results for the limit order of $\varPi_{r,p,q}$; as an example we will validate the following two diagrams, which may give some impression how diverse the limit orders of $\varPi_{r,p,q}$ may be:

$\Pi_{q,q,q}$ $\quad 1 < q < 2 \quad$ $\Pi_{2,2,q}$





By what has been done before, we can exclude the case $1 \le u \le 2 \le v \le \infty$. Now let $1 \le u < v \le 2$. Then by Proposition 3.4(ii) we know that $\mathrm{id}_{uv} \in \Pi_{p,p,q}$ whenever $1/u - 1/v > 1/q'$. This gives, by symmetry, the diagram for $\Pi_{q,q,q}$, and the right-hand side of the diagram for $\Pi_{2,2,q}$. For the left-hand side, consider the ideal $\Pi_{2,q,2}$. By Proposition 3.4(i) we have $\mathrm{id}_{uv} \in \Pi_{2,q,2}$ whenever $1 \le u < v \le 2$ and $1/q' \le (v'/2)(1/u - 1/v)$. By duality, this gives $\mathrm{id}_{uv} \in \Pi_{2,2,q}$ whenever $2 \le u < v \le \infty$ and $1/q' \le (u/2)(1/u - 1/v)$. Factorization now gives the upper estimates for the left-hand side of the diagram for $\Pi_{2,2,q}$, and the lower ones follow from the diagram for $\Pi_{q',2,2}$, since $\Pi_{2,2,q} \subset \Pi_{q',2,2}$.

5. Connections to nuclear operators and Schatten classes. Going back to the definition, it is not clear (and very often false) whether for a given Banach operator ideal $(\mathcal{A}, \mathrm{A})$ its limit order is attained, i.e., whether $D_\sigma \in \mathcal{A}(\ell_u, \ell_v)$ for all $\sigma \in \ell_{1/\lambda(\mathcal{A},u,v)}$. For special choices of the indices involved, we can confirm this. The proof goes along similar lines to the one of [11, Lemma 3], but we give the details for the convenience of the reader. Let us first recall a result of [7, 1.4.3], for which we introduce the following temporary notation: Let $x_1, \ldots, x_n \in \ell_u^m$. Then for $1 \le p \le \infty$ we set

$$w_p(x_i) := \sup_{x' \in B_{\ell_{u'}^m}} \Big(\sum_{i=1}^n |x'(x_i)|^p \Big)^{1/p}.$$

LEMMA 5.1. *Let $1 \le p, u \le \infty$. Then there exists a constant $C > 0$ such that for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \ell_u^m$,*

$$w_p(x_i \otimes y_j; \ell_u^{m^2}) \le C w_p(x_i; \ell_u^m) w_p(y_j; \ell_u^m)$$

whenever either $1 \le p' \le u \le \infty$, $1 \le u \le p' = 2$, or $1 \le u \le 2 < p' \le \infty$. In particular, such a constant exists for all u whenever $p \in \{2, \infty\}$.

PROPOSITION 5.2. *Let $1 \le p, q, r \le \infty$ with $1/p + 1/q \ge 1/r$. Then for $1 \le u, v \le \infty$, the limit order $\lambda(\Pi_{r,p,q}, u, v)$ is attained whenever $1 \le u, p \le 2$*

or $1 \le p' \le u \le \infty$, *and* $1 \le q \le 2 \le v \le \infty$ *or* $1 \le v \le q \le \infty$. *In particular, it is attained for all* u, v *whenever* $p, q \in \{2, \infty\}$.

Proof. First we show that under the given assumptions, there exists a constant $C > 0$ such that for all $\sigma_1, \ldots, \sigma_m \in \mathbb{K}$,

$$\pi_{r,p,q}(D_\sigma : \ell_u^m \to \ell_v^m)^2 \le C\pi_{r,p,q}(D_\sigma \otimes D_\sigma : \ell_u^{m^2} \to \ell_v^{m^2}). \tag{5.1}$$

Let $x_1, \ldots, x_n \in \ell_u^m$ and $y_1, \ldots, y_n \in \ell_{v'}^m$. Then

$$\begin{aligned}
\Big(\Big(\sum_{k=1}^n |y_k'(D_\sigma x_k)|^r\Big)^{1/r}\Big)^2 &= \Big(\Big(\sum_{k=1}^n |y_k'(D_\sigma x_k)|^r\Big)^2\Big)^{1/r} \\
&= \Big(\sum_{k,l=1}^n (|y_k'(D_\sigma x_k)|\,|y_l'(D_\sigma x_l)|)^r\Big)^{1/r} \\
&= \Big(\sum_{k,l=1}^n |(y_k' \otimes y_\ell')(D_\sigma \otimes D_\sigma)(x_k \otimes x_l)|^r\Big)^{1/r} \\
&\le \pi_{r,p,q}(D_\sigma \otimes D_\sigma) w_p(x_k \otimes x_l; \ell_u^{m^2}) w_q(y_k' \otimes y_l'; \ell_{v'}^{m^2}).
\end{aligned}$$

Thus, the assumptions together with the lemma above give (5.1).

Now set $\lambda := \lambda(\Pi_{r,p,q}, u, v)$. Then for all $\varepsilon > 0$ sufficiently small and all $\sigma \in \ell_{(\lambda+\varepsilon)^{-1}}$ we have $D_\sigma \in \Pi_{r,p,q}(\ell_u, \ell_v)$, i.e.,

$$\pi_{r,p,q}(D_\sigma : \ell_u \to \ell_v) \le c(\varepsilon)\|\sigma\|_{(\lambda+\varepsilon)^{-1}}.$$

Denote by $\mathcal{D}$ the set of all finite-dimensional diagonal operators $D_\sigma : \mathbb{K}^m \to \mathbb{K}^m$, m arbitrary. Obviously, $D_\sigma \otimes D_\sigma \in \mathcal{D}$ for all $\sigma \in \mathbb{K}^m$. Define on $\mathcal{D}$ two functions A and B by

$$\begin{aligned}
A(D_\sigma : \mathbb{K}^m \to \mathbb{K}^m) &:= \pi_{r,p,q}(D_\sigma : \ell_u^m \to \ell_v^m), \\
B(D_\sigma : \mathbb{K}^m \to \mathbb{K}^m) &:= \|\sigma\|_{\ell_{\lambda^{-1}}}.
\end{aligned}$$

Then it follows from the above that for all $\sigma \in \mathbb{K}^m$ and $\varepsilon > 0$ sufficiently small,

$$A(D_\sigma) \le c(\varepsilon)\|\sigma\|_{(\lambda+\varepsilon)^{-1}} \le \tilde{c}(\varepsilon) m^\varepsilon \|\sigma\|_{\lambda^{-1}}.$$

Clearly, $B(D_\sigma \otimes D_\sigma) = B(D_\sigma)^2$ and, by (5.1), $A(D_\sigma)^2 \le CA(D_\sigma \otimes D_\sigma)$. Hence, an application of [7, 1.3.1] yields, for all $\sigma \in \mathbb{K}^m$,

$$\pi_{r,p,q}(D_\sigma) = A(D_\sigma) \le CB(D_\sigma) = C\|\sigma\|_{\lambda^{-1}},$$

which by an obvious continuity argument gives the claim. ∎

Corollary 5.3. *Let* $1 \le r \le \infty$ *and* $1 \le p, q, u \le 2 \le v \le \infty$, *and define* $1 \le s \le \infty$ *by* $1/s := \max(0, 1 + 1/r - 1/q - 1/p + 1/v - 1/u)$. *Then*

$$D_\sigma \in \Pi_{r,p,q}(\ell_u, \ell_v) \quad \textit{for all } \sigma \in \ell_s.$$

Proof. This is now a direct consequence of Proposition 5.2 and our diagrams in the previous section. ∎

The above result for diagonal operators has deep consequences for the connection to nuclear operators. Let $0 < u \le \infty$ and $1 \le s,t \le \infty$ with $1+1/u \ge 1/s+1/t$. Then an operator $T : X \to Y$ between Banach spaces X and Y is called (u,s,t)-*nuclear* (shorthand: $T \in \mathcal{N}_{u,s,t}(X,Y)$) if T factorizes through a diagonal operator $D_\sigma : \ell_{t'} \to \ell_s$ with $\sigma \in \ell_u$ if $u < \infty$, and $\sigma \in c_0$ if $u = \infty$ (see, e.g., [31, 18.1]). We start by recalling a useful inclusion result related to Proposition 1.2 (see, e.g., [31, 18.1.5]).

PROPOSITION 5.4. *Let $0 < u_0 \le u_1 \le \infty$, $1 \le s_0 \le s_1 \le \infty$ and $1 \le t_0 \le t_1 \le \infty$ with $1/s_0 + 1/t_0 - 1/u_0 \le 1/s_1 + 1/t_1 - 1/u_1 \le 1$. Then $\mathcal{N}_{u_0,s_0,t_0} \subseteq \mathcal{N}_{u_1,s_1,t_1}$.*

PROPOSITION 5.5. *Let $1 \le u,r \le \infty$, and either $1 \le s,t \le 2$ or $2 \le s,t \le \infty$ or $1/\min(s,t) - 1/u \le 1/2$, and either $1 \le p,q \le 2$ or $2 \le p,q \le \infty$ or $1/r - 1/\max(p,q) \le 1/2$, and $0 \le 1/p + 1/q - 1/r \le 1$. Then*

$$\mathcal{N}_{u,s,t} \subseteq \Pi_{r,p,q}$$

whenever $1/s + 1/t - 1/u \le 1/p + 1/q - 1/r$.

Proof. The case $1 \le p,q \le 2 \le s,t \le \infty$ follows from the corollary above by definition.

Now let $1 \le s,t \le 2$. Choose $u \le u_0 \le \infty$ such that $1/u_0 = 1 + 1/u - 1/s - 1/t$, i.e., $1/s + 1/t - 1/u = 1/2 + 1/2 - 1/u_0$. Then by Proposition 5.4 we have $\mathcal{N}_{u,s,t} \subseteq \mathcal{N}_{u_0,2,2}$. If $1 \le s \le 2 \le \infty$ and $1/s - 1/u \le 1/2$, then define $u \le u_0$ by $1/u_0 = 1/u + 1/2 - 1/s$. Proposition 5.4 then gives $\mathcal{N}_{u,s,t} \subseteq \mathcal{N}_{u_0,2,t}$. The case $1 \le t \le 2 \le s \le \infty$ goes similarly.

For $2 \le p \le q \le \infty$, we let $r \le r_0 \le \infty$ be defined by $1/r_0 = 1 + 1/r - 1/p - 1/q$, i.e., $1/2 + 1/2 - 1/r_0 = 1/p + 1/q - 1/r$. Thus, Proposition 1.2 gives $\Pi_{r_0,2,2} \subseteq \Pi_{r,p,q}$. Now, if $1 \le p \le 2 \le q \le \infty$ and $1/r - 1/q \le 1/2$, then we define $r_0 \le r$ by $1/r_0 = 1/r + 1/2 - 1/q$. Proposition 1.2 then gives $\Pi_{r_0,p,2} \subseteq \Pi_{r,p,q}$. The case $1 \le q \le 2 \le p \le \infty$ and $1/r - 1/p \le 1/2$ goes similarly.

Combining all these observations with the case $1 \le p,q \le 2 \le s,t \le \infty$ gives the claim. ∎

Our exposition now culminates in the confirmation of a result of [5] for absolutely (r,p,q)-summing operators on ℓ_2 as well as of a related result for (u,s,t)-nuclear operators on ℓ_2 (see also [22, 2.7]). It also shows that the above inclusion result in the case $1 \le p,q \le 2 \le s,t \le \infty$ is optimal.

COROLLARY 5.6. *Let $1 \le u,r \le \infty$ and $1 \le p,q \le 2 \le s,t \le \infty$ with $1/s + 1/t - 1/u = 1/p + 1/q - 1/r < 1$. Then*

$$\mathcal{N}_{u,s,t}(\ell_2) = \Pi_{r,p,q}(\ell_2) = \mathcal{S}_v,$$

where $1/v = 1 + 1/u - 1/s - 1/t = 1 + 1/r - 1/p - 1/q$.

Proof. By Propositions 5.4, 1.2 and 5.5,

$$\mathcal{N}_{v,2,2} \subseteq \mathcal{N}_{u,s,t} \subseteq \Pi_{r,p,q} \subseteq \Pi_{v,2,2}.$$

Thus, by [31, 17.5.2, 18.5.4],

$$\mathcal{S}_v = \mathcal{N}_{v,2,2}(\ell_2) \subseteq \mathcal{N}_{u,s,t}(\ell_2) \subseteq \Pi_{r,p,q}(\ell_2) \subseteq \Pi_{v,2,2}(\ell_2) = \mathcal{S}_v,$$

which gives the claim. ∎

6. Inclusions between arbitrary sequence spaces. In this section we need to extend the definition of absolutely (r,p,q)-summing operators. For technical reasons we will only consider the case $p = q = 2$, and the r-norm replaced by a sequence space norm.

We refer to [26] for all notation and information on symmetric Banach sequence spaces and recall only briefly the notions needed here. For a symmetric Banach sequence space E, its *fundamental sequence* $(\lambda_E(n))$ is defined by $\lambda_E(n) := \|\sum_{i=1}^n e_i\|_E$, where e_i is the ith standard unit vector. The span of the first n standard unit vectors, equipped with the norm induced by E, is denoted by E^n. If $E^\times$ denotes the Köthe dual of E, then $\lambda_{E^\times}(n) = n/\lambda_E(n)$. For two symmetric Banach sequence spaces E and F, we define the *space of multipliers* $M(E,F)$ by

$$M(E,F) := \{\lambda \in \ell_\infty;\ \lambda\mu \in F \text{ for all } \mu \in E\},$$

equipped with the norm $\|\lambda\|_{M(E,F)} := \sup_{\|\mu\|_E \le 1} \|\lambda\mu\|_F$. If E is 2-concave and F is 2-convex (for these notions, we refer to [26]), then the following hold (see, e.g., [18, 2.1]):

$$\|\mathrm{id} : \ell_2^n \hookrightarrow E^n\| \asymp \lambda_E(n)/\sqrt{n}; \tag{6.1}$$

$$\|\mathrm{id} : F^n \hookrightarrow \ell_2^n\| \asymp \sqrt{n}/\lambda_F(n); \tag{6.2}$$

$$\lambda_{M(F,E)}(n) \asymp \lambda_E(n)/\lambda_F(n). \tag{6.3}$$

For a symmetric Banach sequence space E we denote by $\mathcal{S}_E$ the Banach space of all compact operators $T : \ell_2 \to \ell_2$ for which the sequence of singular numbers is contained in E, equipped with the norm $\|T\|_{\mathcal{S}_E} := \|\sum_{i=1}^\infty s_i(T)e_i\|_E$. By $\mathcal{S}_E^n$ we denote the space $\mathcal{L}(\ell_2^n)$ equipped with the norm $\|T\|_{\mathcal{S}_E^n} := \|\sum_{i=1}^n s_i(T)e_i\|_E$.

Let E be a maximal symmetric Banach sequence space. We call an operator $T : X \to Y$ between Banach spaces X and Y *absolutely* $(E,2,2)$*-summing* if there exists a constant $C > 0$ such that for any choice of $x_1,\dots,x_n \in X$

and $y_1',\dots,y_n'\in Y'$, the inequality

$$\Big\|\sum_{k=1}^{n} y_k'(Tx_k)e_k\Big\|_E \le C \sup_{x'\in B_{X'}}\Big(\sum_{k=1}^{n}|x'(x_k)|^2\Big)^{1/2}\sup_{y\in B_Y}\Big(\sum_{k=1}^{n}|y_k'(y)|^2\Big)^{1/2}$$

holds. We put $\pi_{E,2,2}(T):=\inf C$ with C as above. In this way, we obtain the maximal Banach operator ideal $(\varPi_{E,2,2},\pi_{E,2,2})$.

LEMMA 6.1. *Let E_0 and E_1 be symmetric Banach sequence spaces and $\mathcal{F}$ an exact interpolation functor. Then*

$$\mathcal{F}(\varPi_{E_0,2,2}(X,Y),\varPi_{E_1,2,2}(X,Y))\subseteq \varPi_{\mathcal{F}(E_0,E_1),2,2}(X,Y)$$

for any fixed pair of Banach spaces X and Y.

Proof. Fix $x_1,\dots,x_n\in X$, $y_1',\dots,y_n'\in Y'$. For $T\in\mathcal{L}(X,Y)$ consider the mapping $\psi_n(T):=(y_1'(Tx_1),\dots y_n'(Tx_n))$. Then by definition

$$\|\psi_n:\varPi_{E_i,2,2}(X,Y)\to E_i\|\le w_2(x_k)w_2(y_k'),\qquad i=0,1.$$

Thus, interpolation and the definition give the claim. ∎

The following result for operators on a Hilbert space is an extension of [31, 17.5.2].

PROPOSITION 6.2. *Let E be a maximal symmetric Banach sequence space such that $E\ne\ell_\infty$. Then $\varPi_{E,2,2}(\ell_2)=\mathcal{S}_E$. Moreover, $\varPi_{E,2,2}$ is the largest Banach operator ideal extending $\mathcal{S}_E$ to the class of all Banach spaces.*

Proof. By Mityagin [30] (see also [24, 1.b.10]) there exists an exact interpolation functor $\mathcal{F}$ such that $E=\mathcal{F}(\ell_1,\ell_\infty)$. Since $\mathcal{S}_1=\varPi_{1,2,2}(\ell_2)$ and $\mathcal{S}_\infty\subseteq\varPi_{\infty,2,2}(\ell_2)$, the above lemma together with [1] yields

$$\mathcal{S}_E=\mathcal{F}(\mathcal{S}_1,\mathcal{S}_\infty)\subseteq\mathcal{F}(\varPi_{1,2,2}(\ell_2),\varPi_{\infty,2,2})\subseteq\varPi_{E,2,2}(\ell_2).$$

Conversely, we have $\pi_{E,2,2}(\mathrm{id}_{22}^n)\ge\lambda_E(n)$. Thus, $\mathrm{id}_{22}\notin\varPi_{E,2,2}$. Now proceed as in [31, 17.5.2] to obtain $\varPi_{E,2,2}(\ell_2)\subseteq\mathcal{S}_E$. For the last part, note that by [31, 15.6] an operator $T:X\to Y$ belongs to the largest extension of $\mathcal{S}_E$ whenever $RTS\in\mathcal{S}_E$ for all $S\in\mathcal{L}(\ell_2,X)$ and $R\in\mathcal{L}(Y,\ell_2)$. By the definition of $\varPi_{E,2,2}$ it follows that such an operator T also belongs to $\varPi_{E,2,2}$. Since $\varPi_{E,2,2}(\ell_2)=\mathcal{S}_E$ by the above, the claim follows. ∎

We now focus again on inclusion maps. As before, we denote for simplicity by id_{EF} the identity map $\mathrm{id}:E\hookrightarrow F$ whenever E and F are symmetric Banach sequence spaces such that E is continuously embedded in F. If $E=\ell_p$ ($F=\ell_p$, respectively), we write id_{pF} (id_{Ep}, respectively) instead of id_{ℓ_pF} ($\mathrm{id}_{E\ell_p}$, respectively).

LEMMA 6.3. *Let E and F be symmetric Banach sequence spaces both not isomorphic to ℓ_2 such that E is 2-concave, and F is maximal and 2-convex. Then $\mathcal{S}_{M(F,\ell_2)}\circ\mathcal{S}_{M(\ell_2,E)}\subseteq\mathcal{S}_{M(F,E)}$.*

Proof. Simply imitate the first part of the proof of [19, 6.3]. ∎

This now gives the following more general result.

PROPOSITION 6.4. *Let E and F be symmetric Banach sequence spaces such that E is 2-concave, and F is maximal and 2-convex. Then* $\mathrm{id}_{EF} \in \Pi_{M(F,E),2,2}$.

Proof. If $F = \ell_2$, then by [13] it is known that $\mathrm{id}_{E2} \in \Pi_{M(\ell_2,E),2} \subseteq \Pi_{M(\ell_2,E),2,2}$. If $E = \ell_2$, then $\mathrm{id}_{F^\times 2} \in \Pi_{M(\ell_2,F^\times),2} \subset \Pi_{M(F,\ell_2),2,2}$, hence by duality also $\mathrm{id}_{2F} \in \Pi_{M(F,\ell_2),2,2}$. Thus assume that both spaces involved are not isomorphic to ℓ_2. The proof is then only a slight modification of one in [15], but we give the details for the convenience of the reader. It is sufficient to show that $R\circ\mathrm{id}_{EF}\circ S \in \mathcal{S}_{M(F,E)}$ whenever $R \in \mathcal{L}(F,\ell_2)$ and $S \in \mathcal{L}(\ell_2,E)$. By [13] it is known that $\mathrm{id}_{E2} \in \Pi_{M(\ell_2,E),2}$, thus $\mathrm{id}_{E2}\circ S \in \Pi_{M(\ell_2,E),2}(\ell_2) = \mathcal{S}_{M(\ell_2,E)}$. Similarly, $\mathrm{id}_{F^\times 2}\circ R' \in \mathcal{S}_{M(\ell_2,F^\times)}$. Hence, $R\circ\mathrm{id}_{2F} \in \mathcal{S}_{M(F,\ell_2)}$. Thus, by the lemma above, $R\circ\mathrm{id}_{EF}\circ S = R\circ\mathrm{id}_{2F}\circ\mathrm{id}_{E2}\circ S \in \mathcal{S}_{M(F,E)}$, which gives the claim. ∎

This result is best possible in the following sense: Let G be a symmetric Banach sequence space such that $\mathrm{id}_{EF} \in \Pi_{G,2,2}$, where E and F are as above. Then $\lambda_G(n) \prec \lambda_{M(F,E)}(n)$. Indeed, as in the proof of Theorem 4.2 and with the help of (6.1)–(6.3), we deduce that

$$\pi_{G,2,2}(\mathrm{id}_{EF}) \ge \frac{\lambda_G(n)}{\|\mathrm{id}^n_{2E}\|\,\|\mathrm{id}^n_{F2}\|} \succ \frac{\lambda_G(n)}{\frac{\lambda_E(n)}{\lambda_F(n)}} \ge \frac{\lambda_G(n)}{\lambda_{M(F,E)}(n)}.$$

Clearly, the above result includes the case id_{uv} with $1 \le u \le 2 \le v \le \infty$. The case $1 \le u < v < 2$ or $2 < u < v \le \infty$ turned out to be more complicated, which is also the case in this more general setting.

PROPOSITION 6.5. *Let E and F be 2-concave symmetric Banach sequence spaces and $\mathcal{F}$ an exact interpolation functor such that*

$$\sup_{n,m} \|\mathcal{L}(\ell_2^m, E^n) \hookrightarrow \mathcal{F}(\mathcal{L}(\ell_2^m,\ell_1^n), \mathcal{L}(\ell_2^m, F^n))\| < \infty.$$

Then $\mathrm{id}_{EF} \in \Pi_{\mathcal{F}(\ell_2,\ell_\infty),2,2}$.

Proof. Fix $y_1',\dots,y_m' \in F^n$ and consider the mappings

$$\psi_{n,m} : \mathbb{K}^m \otimes \mathbb{K}^n \to \mathbb{K}^m, \qquad \psi_{n,m}((x_1,\dots,x_m)) := (y_1'(x_1),\dots,y_m'(x_m)).$$

Since $\mathrm{id}_{11} \in \Pi_{2,2,2}$, it follows that $\mathrm{id}_{1F} \in \Pi_{2,2,2}$, thus

$$\|\psi_{n,m} : \mathcal{L}(\ell_2^m,\ell_1^n) \to \ell_2^m\| \le C w_2(y_i')$$

for some $C > 0$ independent of n and m. Trivially, $\mathrm{id}_{FF} \in \Pi_{\infty,2,2}$ with norm equal to 1, that is,

$$\|\psi_{n,m} : \mathcal{L}(\ell_2^m, F^n) \to \ell_\infty^m\| \le w_2(y_i').$$

Then the assumption and interpolation give

$$\|\psi_{n,m} : \mathcal{L}(\ell_2^m, E^n) \to \mathcal{F}(\ell_2^m, \ell_\infty^m)\| \le \tilde{C} w_2(y_i'),$$

where $\tilde{C} > 0$ is some other constant independent of n and m. This finishes the proof. ■

We refer the reader to [26] for the proper definition of Lorentz and Orlicz sequence spaces.

Corollary 6.6.

(i) *Let $1 \le u < v < 2$. Then* $\mathrm{id}_{uv} \in \Pi_{r,2,2}$, *where* $1/r = (v'/2)(1/u - 1/v)$.

(ii) *Let $1 < p < r < 2$ and $1 \le q, s \le 2$. Then* $\mathrm{id}_{\ell_{p,q}\ell_{r,s}} \in \Pi_{\ell_{t,\tilde{q}},2,2}$, *where* $1/t = (r'/2)(1/p - 1/r)$ *and* $1/\tilde{q} = 1/q - 1/2$.

(iii) *Let φ and ψ be Orlicz functions such that the functions $t \mapsto \varphi(\sqrt{t})$ and $t \mapsto \psi(\sqrt{t})$ are equivalent to concave functions. If $\varphi^{-1}(t) = t\varrho(\psi^{-1}(t)/t)$ for some continuous and concave function $\varrho : [0,\infty) \to [0,\infty)$ which is positive on $(0,\infty)$, then* $\mathrm{id}_{\ell_\varphi \ell_\psi} \in \Pi_{\lambda,2,2}$, *where* $\lambda^{-1}(t) = t^{1/2}\varrho(t^{-1/2})$.

Proof. (i) This is already included in Proposition 3.4(i).

(ii) In [28, 2.1] it was shown that under the assumptions above,

$$\sup_{n,m} \|\mathcal{L}(\ell_2^m, \ell_{p,q}^n) \hookrightarrow (\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_r^n))_{\theta,\tilde{q}}\| < \infty,$$

where $\theta = r'/p'$. A quick inspection of the proof shows that ℓ_r^n can be replaced by $\ell_{r,s}^n$. Thus, the above proposition applies with the interpolation functor $\mathcal{F} = (\cdot,\cdot)_{\theta,\tilde{q}}$. Furthermore, $(\ell_2, \ell_\infty)_{\theta,\tilde{q}} = \ell_{t,\tilde{q}}$, which gives the claim.

(iii) The assumptions on φ and ψ ensure that ℓ_φ and ℓ_ψ are 2-concave (see, e.g., [23]). Let ϱ_ℓ be the lower Ovchinnikov functor associated to ϱ (see, e.g., [14] for more details and references). Then (see, e.g., [27, p. 179]) we have $\varrho_\ell(\ell_1, \ell_\psi) = \ell_\varphi$, and by [14, Proposition 3],

$$\sup_{n,m} \|\mathcal{L}(\ell_2^m, \ell_\varphi^n) \hookrightarrow \varrho_\ell(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_\psi^n))\| < \infty.$$

Thus, the above proposition applies, and $\varrho_\ell(\ell_2, \ell_\infty) = \ell_\lambda$ (see, e.g., [27, p. 178]), which gives the claim. ■

7. Inclusions between finite-dimensional Schatten classes. We finally consider inclusions $\mathrm{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n$, where E and F are symmetric Banach sequence spaces. Since both unitary ideals involved contain ℓ_2^n, it follows that

$$\pi_{r,p,q}(\mathrm{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \ge \pi_{r,p,q}(\mathrm{id}_{22}^n) = n^{\max(0,1+1/r-1/p-1/q)}$$

and

$$\pi_{G,2,2}(\mathrm{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \ge \pi_{G,2,2}(\mathrm{id}_{22}^n) = \lambda_G(n)$$

for all $1 \leq p, q \leq 2$, $1 \leq r \leq \infty$ and every symmetric Banach sequence space G. To give an analogue of Corollary 3.3, we need some more interpolation formulas.

PROPOSITION 7.1. *Let* $1 \leq p_0, p_1, q_0, q_1 \leq 2$, $0 < \theta < 1$ *and* $1 \leq p, q \leq 2$ *with* $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$. *Then*

$$\sup_{n,m} \|\ell_p^m \otimes_\varepsilon \mathcal{S}_q^n \hookrightarrow [\ell_{p_0}^m \otimes_\varepsilon \mathcal{S}_{q_0}^n, \ell_{p_1}^m \otimes_\varepsilon \mathcal{S}_{q_1}^n]_\theta\| < \infty$$

and

$$\sup_{n,m} \|\mathcal{S}_p^n \otimes_\varepsilon \mathcal{S}_q^m \hookrightarrow [\mathcal{S}_{p_0}^n \otimes_\varepsilon \mathcal{S}_{q_0}^m, \mathcal{S}_{p_1}^n \otimes_\varepsilon \mathcal{S}_{q_1}^m]_\theta\| < \infty.$$

Proof. This follows from the cases $q_0 = q_1 = q = 2$ (Proposition 2.1) and $p_0 = p_1 = p = 2$ ([16, 4.3]) by applying [17, Lemma 9] together with Pisier's factorization theorem as in [17, p. 450]. ∎

LEMMA 7.2. *Let* $1 \leq u_0, u_1 \leq 2 \leq v_0, v_1 \leq \infty$, $1 \leq r_0, r_1 \leq \infty$ *and* $1 \leq s_0, s_1, t_0, t_1 \leq 2$. *Then for all* $0 < \theta < 1$,

$$\pi_{r,s,t}(\mathrm{id} : \mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \leq \pi_{r_0,s_0,t_0}(\mathrm{id} : \mathcal{S}_{u_0}^n \hookrightarrow \mathcal{S}_{v_0}^n)^{1-\theta} \pi_{r_1,s_1,t_1}(\mathrm{id} : \mathcal{S}_{u_1}^n \hookrightarrow \mathcal{S}_{v_1}^n)^\theta,$$

where $1/u = (1-\theta)/u_0 + \theta/u_1$, $1/v = (1-\theta)/v_0 + \theta/v_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$, $1/s = (1-\theta)/s_0 + \theta/s_1$ *and* $1/t = (1-\theta)/t_0 + \theta/t_1$.

Proof. The proof goes along similar lines to the one of Lemma 2.3(i), using the above proposition. ∎

As before, we have to verify some extreme cases.

LEMMA 7.3. *The following hold true*:

(i) $\pi_{1,1,1}(\mathrm{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_2^n) = 1$;
(ii) $\pi_{1,2,2}(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_\infty^n) = n$;
(iii) $\pi_{1,2,1}(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) = \pi_{1,1,2}(\mathrm{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n}$;
(iv) $\pi_{1,1,2}(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) = \pi_{1,2,1}(\mathrm{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n}$.

Proof. (i) is clear as $\Pi_{1,1,1} = \mathcal{L}$. Since $\pi_2(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) = \sqrt{n}$ (see, e.g., [16, 5.2]), we have

$$\sup_m \|\mathrm{id} \otimes \mathrm{id} : \mathcal{S}_1^n \otimes_\varepsilon \ell_2^m \to \ell_2^{n^2 m}\| = \sqrt{n}. \tag{7.1}$$

Thus, by duality,

$$\sup_m \|\mathrm{id} \otimes \mathrm{id} : \ell_2^{n^2 m} \to \mathcal{S}_\infty^n \otimes_\pi \ell_2^m\| = \sqrt{n}.$$

Hence, by factorization,

$$\sup_m \|\mathrm{id} \otimes \mathrm{id} : \mathcal{S}_1^n \otimes_\varepsilon \ell_2^m \to \mathcal{S}_\infty^n \otimes_\pi \ell_2^m\| = n,$$

which means $\pi_{1,2,2}(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_\infty^n) = n$. Next, the identity map $\mathrm{id}_{\mathcal{S}_2}$ is absolutely $(2,1)$-summing, that is,

$$\sup_m \|\mathrm{id} \otimes \mathrm{id} : \mathcal{S}_2^n \otimes_\varepsilon \ell_1^m \to \ell_2^{n^2 m}\| < \infty. \tag{7.2}$$

By duality and factorization, this together with (7.1) yields

$$\sup_m \|\mathrm{id}\otimes\mathrm{id} : \mathcal{S}_1^n \otimes_\varepsilon \ell_2^m \to \mathcal{S}_2^n \otimes_\pi \ell_\infty^m\| \asymp \sqrt{n},$$

which gives $\pi_{1,2,1}(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_{1,1,2}(\mathrm{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n}$. Finally, since $\Pi_1(X,Y) = \Pi_2(X,Y)$ whenever X is of cotype 2, we have $\pi_1(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_2(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \sqrt{n}$. Thus,

$$\sup_m \|\mathrm{id}\otimes\mathrm{id} : \mathcal{S}_1^n \otimes_\varepsilon \ell_1^m \to \mathcal{S}_2^n \otimes_\pi \ell_1^m\| \asymp \sqrt{n},$$

which by factorization gives

$$\sup_m \|\mathrm{id}\otimes\mathrm{id} : \mathcal{S}_1^n \otimes_\varepsilon \ell_1^m \to \mathcal{S}_2^n \otimes_\pi \ell_2^m\| \asymp \sqrt{n}.$$

Hence, $\pi_{1,1,2}(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_{1,2,1}(\mathrm{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n}$, which finishes the proof. ∎

Proposition 7.4. *Let $1 \le p,q \le 2$, $1 \le r \le \infty$ and $1 \le u \le 2 \le v \le \infty$ with $1/u - 1/v = 1 + 1/r - 1/p - 1/q$. Then*

$$\pi_{r,p,q}(\mathrm{id} : \mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1/u-1/v}.$$

Proof. As in the proof of Theorem 3.2, we fix $1 \le q \le 2$ and define $\tilde{q}$ and $\overline{q}$ accordingly. Using Lemmas 7.2 and 7.3, we arrive at the following six cases (ordered according to the proof of Theorem 3.2):

- $\pi_{1,\tilde{q},q}(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_{1,\tilde{q},q}(\mathrm{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n}$;
- $\pi_{1,1,q}(\mathrm{id} : \mathcal{S}_{\tilde{q}}^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_{1,1,q}(\mathrm{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_{\overline{q}}^n) \asymp n^{1/q'}$;
- $\pi_{1,2,q}(\mathrm{id} : \mathcal{S}_{\tilde{q}}^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \pi_{1,2,q}(\mathrm{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_{\overline{q}}^n) \asymp n^{1/\tilde{q}}$.

Then proceed by interpolation as in the proof of Theorem 3.2 to obtain the statement in the case $r = 1$. The general case then follows as in the proof of Corollary 3.3. ∎

Corollary 7.5. *Let $1 \le p,q \le 2$ and $1 \le r \le \infty$ with $1/p + 1/q - 1/r \le 1$. Then for all $1 \le u \le 2 \le v \le \infty$,*

$$\pi_{r,p,q}(\mathrm{id} : \mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1+1/r-1/p-1/q}\pi_{r,p,q}(\mathrm{id} : \ell_u^n \hookrightarrow \ell_v^n).$$

Moreover, for $2 \le r,s \le \infty$,

$$\|\mathrm{id}\otimes\mathrm{id} : \mathcal{S}_p^n \otimes_\varepsilon \ell_q^{n^2} \to \mathcal{S}_r^n \otimes_\pi \ell_s^{n^2}\| \asymp n^{2-1/p-1/q+\max(0,2-1/p-1/q+1/r+1/s)}.$$

Proof. Appropriate factorizations give the upper estimates. For the lower estimates observe first that

$$\pi_{r,p,q}(\mathrm{id} : \mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \ge \pi_{r,p,q}(\mathrm{id}_{22}^n) = n^{1/t},$$

where $1/t = 1 + 1/r - 1/p - 1/q$. By (8.1) and the lower estimate from Corollary 8.8 below,

$$\begin{aligned}\pi_{r,p,q}(\mathrm{id}:\mathcal{S}_u^n\hookrightarrow\mathcal{S}_v^n) &\ge \pi_{t,2,2}(\mathrm{id}:\mathcal{S}_u^n\hookrightarrow\mathcal{S}_v^n)\ge \pi_{t,2,2}^{(n^2)}(\mathrm{id}:\mathcal{S}_u^n\hookrightarrow\mathcal{S}_v^n)\\ &\ge n^{2/t}h_{n^2}(\mathrm{id}:\mathcal{S}_u^n\hookrightarrow\mathcal{S}_v^n)\\ &\succ n^{2/t}n^{1/v-1/u}=n^{2+2/r-2/p-2/q+1/v-1/u}.\end{aligned}$$

This calculation also gives the last part of the statement on taking $r=1$. ∎

Mathematical routine lets us formulate and prove an analogue of Corollary 4.4:

PROPOSITION 7.6. *Let $1\le p,q\le 2\le r,s\le\infty$. Then*

$$\|\mathrm{id}\otimes\mathrm{id}:\mathcal{S}_p^n\otimes_\varepsilon\mathcal{S}_q^n\to\mathcal{S}_r^n\otimes_\pi\mathcal{S}_s^n\|\asymp n\|\mathrm{id}\otimes\mathrm{id}:\ell_p^n\otimes_\varepsilon\ell_q^m\to\ell_r^n\otimes_\pi\ell_s^m\|.$$

Proof. We have to show that

$$\|\mathrm{id}\otimes\mathrm{id}:\mathcal{S}_p^n\otimes_\varepsilon\mathcal{S}_q^n\to\mathcal{S}_r^n\otimes_\pi\mathcal{S}_s^n\|\asymp n^{1+\max(0,1-1/p-1/q+1/r+1/s)}.$$

Again, we first establish the cases where the norm is asymptotically equivalent to n—note that this behaviour is best possible, since all spaces involved contain ℓ_2^n.

By [18, 11.4],

$$\|\mathrm{id}\otimes\mathrm{id}:\mathcal{S}_1^n\otimes_\varepsilon\mathcal{S}_2^n\to\mathcal{S}_2^{n^2}\|\asymp\sqrt{n}.$$

Thus, by duality and factorization the following identities have norms asymptotically equivalent to n:

$$\mathrm{id}\otimes\mathrm{id}:\mathcal{S}_1^n\otimes_\varepsilon\mathcal{S}_2^n\to\mathcal{S}_\infty^n\otimes_\pi\mathcal{S}_2^n,\qquad \mathrm{id}\otimes\mathrm{id}:\mathcal{S}_1^n\otimes_\varepsilon\mathcal{S}_2^n\to\mathcal{S}_2^n\otimes_\pi\mathcal{S}_\infty^n.$$

Furthermore, by [18, 11.3] we have

$$\|\mathrm{id}\otimes\mathrm{id}:\mathcal{S}_1^n\otimes_\varepsilon\mathcal{S}_1^n\to\mathcal{S}_2^n\otimes_\pi\mathcal{S}_2^n\|\asymp n$$

and, by duality,

$$\|\mathrm{id}\otimes\mathrm{id}:\mathcal{S}_2^n\otimes_\varepsilon\mathcal{S}_2^n\to\mathcal{S}_\infty^n\otimes_\pi\mathcal{S}_\infty^n\|\asymp n.$$

Now an interpolation strategy similar to the one in the proof of Theorem 3.2 together with Proposition 7.1 establishes

$$\|\mathrm{id}\otimes\mathrm{id}:\mathcal{S}_p^n\otimes_\varepsilon\mathcal{S}_q^n\to\mathcal{S}_r^n\otimes_\pi\mathcal{S}_s^n\|\asymp n$$

whenever $1/p+1/q-1/r-1/s=1$. The upper estimates now follow by appropriate factorizations. For the lower ones, recall that by [10, p. 35] we have $\mathrm{N}(\mathrm{id}_E)=\dim E$ for all finite-dimensional Banach spaces E. Thus,

$$\mathrm{N}(\mathrm{id}:\mathcal{S}_{r'}^n\hookrightarrow\mathcal{S}_s^n)\ge\frac{n^2}{n^{1/r'-1/s}}=n^{1+1/r+1/s}.$$

Hence,

$$\|\mathrm{id}\otimes\mathrm{id}:\mathcal{S}_p^n\otimes_\varepsilon\mathcal{S}_q^n\to\mathcal{S}_r^n\otimes_\pi\mathcal{S}_s^n\|\ge\frac{\mathrm{N}(\mathrm{id}:\mathcal{S}_{r'}^n\hookrightarrow\mathcal{S}_s^n)}{\|\mathrm{id}:\mathcal{S}_{p'}^n\hookrightarrow\mathcal{S}_q^n\|}\ge n^{2-1/p-1/q+1/r+1/s},$$

which together with the general lower bound n gives the claim. ∎

To formulate an analogue of Proposition 6.4 causes some problems. So far, we are only able to state the following; the proof is similar to the one of Proposition 6.4. We leave the details to the reader.

PROPOSITION 7.7. *Let E_0, E_1 be 2-concave symmetric Banach sequence spaces such that*

$$\pi_{M(\ell_2,E_i),2}(\mathrm{id} : \mathcal{S}^n_{E_i} \hookrightarrow \mathcal{S}^n_2) \asymp \frac{\lambda_{E_i}(n)}{\sqrt{n}}, \qquad i = 0,1. \tag{7.3}$$

Then

$$\pi_{M(E_1^\times,E_0),2,2}(\mathrm{id} : \mathcal{S}^n_{E_0} \hookrightarrow \mathcal{S}^n_{E_1^\times}) \asymp \frac{\lambda_{E_0}(n)\lambda_{E_1}(n)}{n}.$$

In [16, 5.3] the following examples of spaces satisfying (7.3) were given:

- ℓ_p, where $1 \le p \le 2$;
- $\ell_{p,q}$, where $1 < p < 2$ and $1 \le q \le 2$;
- ℓ_φ, where $\varphi(t)$ is a submultiplicative Orlicz function not equivalent to t^2 in a neighbourhood of zero, such that the function $t \mapsto \varphi(\sqrt{t})$ is equivalent to a concave function in a neighbourhood of zero.

However, they also gave examples of Lorentz and Orlicz sequence spaces that are 2-concave but do not satisfy (7.3), which makes it impossible to state a more general result in the spirit of Proposition 6.4.

8. Applications to Hilbert numbers. We refer to [24] and [32] for the general theory of s-numbers of operators. For an operator $T : X \to Y$ between Banach spaces X and Y recall the definition of its kth *approximation number*

$$a_k(T) := \inf\{\|T - S\|;\ S \in \mathcal{L}(X,Y) \text{ with } \|S\| \le 1 \text{ and rank } S < k\},$$

and its kth *Hilbert number*

$$h_k(T) := \sup\{a_k(RTS);\ R \in \mathcal{L}(Y,\ell_2), S \in \mathcal{L}(\ell_2,X), \|S\|, \|R\| \le 1\}.$$

It is clear from the definition that $a_1(T) \ge a_2(T) \ge \cdots \ge 0$ and $h_1(T) \ge h_2(T) \ge \cdots \ge 0$. Furthermore, for a compact operator between Hilbert spaces, the sequences of approximation and Hilbert numbers coincide with the sequence of singular numbers.

An important inequality due to König (see, e.g., [24, 2.a.3]) states that $k^{1/r} x_k(T) \le \pi_{r,2}(T)$ for all $T \in \Pi_{r,2}$, where $x_k(T)$ denotes the kth Weyl number of T (see, e.g., [32]). We now provide an analogue for Hilbert numbers and $(E,2,2)$-summing operators.

For an operator T denote by $\pi^{(k)}_{E,2,2}(T)$ the $(E,2,2)$-summing norm of T computed with at most k vectors $x_1, \dots, x_k$ and k vectors $y'_1, \dots, y'_k$.

Proposition 8.1. *Let E be a maximal symmetric sequence space. Then*

$$\lambda_E(k)h_k(T) \le \pi^{(k)}_{E,2,2}(T) \tag{8.1}$$

for all operators $T \in \mathcal{L}$.

Proof. Let $T \in \mathcal{L}(X,Y)$ where X and Y are Banach spaces. By Bauhardt's characterization of Hilbert numbers (see, e.g., [31, 11.4.3]) there exist operators $S : \ell_2^k \to X$ and $R : Y \to \ell_2^k$ such that $\|S\|, \|R\| \le 1$ and

$$RTS = (1+\varepsilon)^{-1}h_k(T)\mathrm{id}^k_{22}.$$

Equivalently, this means that there exist $x_1, \dots, x_k \in X$ and $y'_1, \dots, y'_k \in Y'$ such that $w_2(x_i) \le 1$, $w_2(y'_i) \le 1$ and $y'_i(Tx_i) = (1+\varepsilon)^{-1}h_k(T)$, $i = 1, \dots, k$. Then by the definition of $\pi^{(k)}_{E,2,2}(T)$,

$$(1+\varepsilon)^{-1}h_k(T)\lambda_E(k) = \Big\|\sum_{i=1}^k |y'_i(Tx_i)|e_i\Big\|_E \le \pi^{(k)}_{E,2,2}(T),$$

which gives the claim. ∎

For a symmetric Banach sequence space E, denote by $\lambda(E)$ and $m(E)$ the Lorentz and Marcinkiewicz spaces associated to the fundamental function λ_E of E, respectively, in the sense of [12, p. 59]. Furthermore, for a scale s of s-numbers in the sense of [32] and a symmetric Banach sequence space F, we define $\mathcal{L}^s_F$ to be the class of all operators T between Banach spaces such that $(s_n(T)) \in F$, equipped with the norm $s_F(T) := \|(s_n(T))\|_F$, $T \in \mathcal{L}^s_F$. In [12, 3.1] the authors proved the following:

For every symmetric Banach sequence space E such that $\ell_2 \hookrightarrow E$, we have $\Pi_{E,2} \hookrightarrow \mathcal{L}^x_{m(E)}$. If in addition E is an interpolation space with respect to the couple (ℓ_2, ℓ_∞), then $\mathcal{L}^x_{\lambda(E)} \hookrightarrow \Pi_{E,2}$.

Moreover, for $r > 2$ Pietsch [32, 2.7.5] showed that $\mathcal{L}_r \subseteq \Pi_{r,2} \subseteq \mathcal{L}_{r,\infty}$. The above proposition together with Proposition 6.2 now yields the following analogue for the scale of Hilbert numbers and $(E,2,2)$-summing operators:

Corollary 8.2. *Let $E \ne \ell_\infty$ be a maximal symmetric Banach sequence space. Then*

$$\mathcal{L}^h_E \hookrightarrow \Pi_{E,2,2} \hookrightarrow \mathcal{L}^h_{m(E)}.$$

A first application to inclusion maps is the following:

Proposition 8.3. *Let E and F be symmetric Banach sequence spaces such that E is 2-concave, and F is 2-convex and maximal. Then*

$$h_k(\mathrm{id}_{EF}) \asymp \frac{\lambda_F(k)}{\lambda_E(k)}.$$

Proof. We have $h_k(\mathrm{id}_{22}^k) = 1$, hence by factorization,

$$h_k(\mathrm{id}_{EF}) \ge h_k(\mathrm{id}_{EF}^k) \ge \frac{\|\mathrm{id}_{2E}^k\|}{\|\mathrm{id}_{F2}^k\|} \asymp \frac{\lambda_F(k)}{\lambda_E(k)}.$$

Conversely, by Proposition 6.4 we know that $\mathrm{id}_{EF} \in \Pi_{M(F,E),2,2}$. Thus, the proposition above gives $h_k(\mathrm{id}_{EF}) \prec 1/\lambda_{M(F,E)}(k)$. By [18], we have $\lambda_{M(F,E)}(k) \asymp \lambda_E(k)/\lambda_F(k)$, which gives the claim. ∎

Now the above and the results from the previous section give the following examples. We guess that (i) is already known; however, we have not found a source where it is written up in this form.

Corollary 8.4.

(i) *Let $1 \le u \le v \le \infty$. Then*

$$h_k(\mathrm{id}_{uv}) \asymp \begin{cases} k^{(v'/2)(1/v-1/u)}, & 1 \le u < v < 2; \\ k^{(u/2)(1/v-1/u)}, & 2 < u < v \le \infty; \\ k^{1/v-1/u}, & 1 \le u \le 2 \le v \le \infty; \\ k^{-1/2}, & u = v = 1 \quad \text{or} \quad u = v = \infty; \\ 1, & 1 < u = v < \infty. \end{cases}$$

(ii) *Let $1 < p \le r < \infty$ and $1 \le q, s \le \infty$. Then*

$$h_k(\mathrm{id}_{\ell_{p,q}\ell_{r,s}}) \asymp \begin{cases} k^{(r'/2)(1/r-1/p)}, & 1<p<r<2, p\le q\le 2 \text{ and } 1\le s\le r; \\ k^{(p/2)(1/r-1/p)}, & 2<p<r<\infty, 2\le q\le p \text{ and } r\le s\le \infty; \\ k^{1/r-1/p}, & 1<p<2<r<\infty \text{ and } 1\le q\le 2\le s\le\infty; \\ 1, & p=r \text{ and } 1<q=s<\infty. \end{cases}$$

(iii) *Let $1 < p < 2 \le q < \infty$ and w be a Lorentz sequence such that $nw_n^{2/(2-p)} \asymp \sum_{i=1}^n w_i^{2/(2-p)}$. Then*

$$h_k(\mathrm{id}_{d(w,p)d(w,q)}) \asymp (kw_k)^{1/q-1/p}.$$

(iv) *Let φ and ψ be Orlicz functions such that $t \mapsto \varphi(\sqrt{t})$ and $t \mapsto \sqrt{\psi(t)}$ are equivalent to concave and convex functions, respectively, and ψ satisfies the Δ_2-condition. Then*

$$h_k(\mathrm{id}_{\ell_\varphi\ell_\psi}) \asymp \frac{\varphi^{-1}(1/k)}{\psi^{-1}(1/k)}.$$

(v) *Let φ and ψ be Orlicz functions such that $t \mapsto \varphi(\sqrt{t})$ and $t \mapsto \psi(\sqrt{t})$ are equivalent to concave functions, respectively. If $\varphi^{-1}(t) = t\varrho(\psi^{-1}(t)/t)$ for some continuous and concave function $\varrho : [0,\infty) \to [0,\infty)$ which is positive on $(0,\infty)$, then*

$$h_k(\mathrm{id}_{\ell_\varphi\ell_\psi}) \prec \frac{\varrho(k^{1/2})}{k^{1/2}}.$$

Proof. (i) The case $1 \le u \le 2 \le v \le \infty$ is contained in the above proposition. Now let $1 \le u < v < 2$. Then the upper estimate follows from (8.1) together with Corollary 6.6(i). For the lower estimate, choose $m \in \mathbb{N}$ such that $m^{2/v'}/2 \le k \le m^{2/v'}$. Now [20, Proposition (2)] gives

$$h_k(\mathrm{id}_{uv}) \ge h_k(\mathrm{id}_{uv}^m) \succ m^{1/v-1/u} \ge 2^{-1}k^{(v'/2)(1/v-1/u)},$$

which gives the lower estimate. The case $2 < u < v \le \infty$ then follows by duality. Since $\mathrm{id}_{11} \in \Pi_{2,2,2}$, we have $h_k(\mathrm{id}_{11}) \prec k^{-1/2}$ by (8.1); the lower estimate follows by factorizing id_{22}^k through id_{11} (see also [32, 2.9.19]). The claim for $\mathrm{id}_{\infty\infty}$ then follows by duality. Finally, any K-convex infinite-dimensional Banach space (for this notion see, e.g., [19]) contains a complemented copy of ℓ_2^k (see, e.g., [19, 19.3]). Thus, $h_k(\mathrm{id}_X) \asymp 1$ for any K-convex infinite-dimensional Banach space X, in particular for $X = \ell_u$, $1 < u < \infty$.

(ii) This follows as in (i) together with Corollary 6.6(ii)—note that $\lambda_{\ell_{t,\tilde{q}}}(k) \asymp k^{1/t}$—and the lower estimate for $h_k(\mathrm{id}_{pr})$ in (i).

(iii) The assumption on w implies that $d(w,p)$ is 2-concave (see, e.g., [33]), and $d(w,q)$ for $q \ge 2$ is always 2-convex (and maximal). Thus, the above proposition gives the claim, if we take into account that $\lambda_{d(w,r)}(k) \asymp (kw_k)^{1/r}$ for any $1 < r < \infty$.

(iv) The assumptions ensure that ℓ_φ is 2-concave and that ℓ_ψ is 2-convex and maximal. Hence, the claim follows from the proposition above—note that $\lambda_{\ell_\varphi}(k) \asymp 1/\varphi^{-1}(1/k)$ for any Orlicz sequence space ℓ_φ.

(v) This follows from (8.1) together with Corollary 6.6(iii). ■

We now show that one can even obtain all asymptotically exact upper estimates for the Hilbert numbers of the finite-dimensional inclusions id_{uv}^n by using (8.1). The lower ones can be found in [20]. Note that the case $1 \le v < u' \le \infty$ follows from the one below by the duality of Hilbert numbers.

PROPOSITION 8.5. *Let $1 \le u' \le v \le \infty$ and $1 \le k \le n$. Then*

$$h_k(\mathrm{id}_{uv}^n) \asymp \begin{cases} \min(n^{1/v-1/u}, n^{1/v}k^{-1/2}, nk^{-1}), & 1 \le u' \le v \le 2, \\ \min(n^{1/v-1/u}, n^{1/v}k^{-1/2}), & 2 \le v \le u \le \infty, \\ \min(k^{(u/2)(1/v-1/u)}, n^{1/v}k^{-1/2}), & 2 \le u < v \le \infty, \\ k^{1/v-1/u}, & 2 \le u' \le v \le \infty. \end{cases}$$

Proof. Let $1 \le u' \le v \le 2$. Then $\pi_{2,2,2}(\mathrm{id}_{uv}^n) \asymp n^{1/v}$, hence $h_k(\mathrm{id}_{uv}^n) \prec n^{1/v}k^{-1/2}$. Moreover, $\pi_{1,2,2}(\mathrm{id}_{uv}^n) \asymp n$, which gives $h_k(\mathrm{id}_{uv}^n) \prec nk^{-1}$. Finally, by the monotonicity of Hilbert numbers, $h_k(\mathrm{id}_{uv}^n) \le h_1(\mathrm{id}_{uv}^n) = n^{1/v-1/u}$.

Let $2 \le u, v \le \infty$. Then $\pi_{2,2,2}(\mathrm{id}_{uv}^n) \asymp n^{1/v}$, and therefore $h_k(\mathrm{id}_{uv}^n) \prec n^{1/v}k^{-1/2}$. If $v \le u$, then as before $h_k(\mathrm{id}_{uv}^n) \le h_1(\mathrm{id}_{uv}^n) = n^{1/v-1/u}$. If $u < v$, then $h_k(\mathrm{id}_{uv}^n) \le h_k(\mathrm{id}_{uv}) \prec k^{(u/2)(1/v-1/u)}$ as already seen in the above corollary. The lower estimate in this special case also follows similarly to the above: if $k \le n^{2/u}$, then choose $1 \le \ell \le m$ such that $\ell^{2/u}/2 \le k \le \ell^{2/u}$. Then again [20, Proposition (2)] gives

$$h_k(\mathrm{id}_{uv}^n) \ge h_k(\mathrm{id}_{uv}^\ell) \succ \ell^{1/v-1/u} \ge 2^{-1}k^{(u/2)(1/v-1/u)}.$$

The estimate $h_k(\mathrm{id}_{uv}^n) \succ n^{1/v}k^{-1/2}$ for all $n^{2/u} \le k \le m$ from [20, Proposition (2)] gives the remaining lower estimate in this case.

Let $2 \le u' \le v \le \infty$. Then $\mathrm{id}_{uv} \in \Pi_{r,2,2}$, where $1/r = 1/u - 1/v$. Hence, $h_k(\mathrm{id}_{uv}^n) \prec k^{1/v-1/u}$. ∎

We finish with inclusions between finite-dimensional Schatten classes. Since ℓ_2^n is contained in both spaces involved, $h_k(\mathrm{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) = 1$ whenever E is continuously embedded into F, and $1 \le k \le n$. Proposition 7.7 together with (8.1) gives the following upper estimate:

PROPOSITION 8.6. *Let E_0, E_1 be 2-concave symmetric Banach sequence spaces satisfying* (7.3). *Then for $n \le k \le n^2$,*

$$h_k(\mathrm{id} : \mathcal{S}_{E_0}^n \hookrightarrow \mathcal{S}_{E_1^\times}^n) \prec \frac{k}{n}\,\frac{\lambda_{E_0}(n)\lambda_{E_1}(n)}{\lambda_{E_0}(k)\lambda_{E_1}(k)}.$$

The situation for the lower estimate is more satisfactory. Here, for a symmetric Banach sequence space G let $\lambda_G : [1,\infty) \to [1,\infty)$ be a monotone function extending $\lambda_G : \mathbb{N} \to [1,\infty)$.

PROPOSITION 8.7. *Let E and F be symmetric Banach sequence spaces such that E is 2-concave and F is 2-convex. Then for $n \le k \le n^2$,*

$$h_k(\mathrm{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \succ \frac{\lambda_F(k/n)}{\lambda_E(k/n)}.$$

Proof. We proceed similarly to the proof of [21, 4.2]. Choose $1 \le h \le n$ such that $nh - 1 \le k \le nh$. Identify $\mathcal{L}(\ell_2^n, \ell_2^h)$ and $\mathcal{L}(\ell_2^n, \ell_2^n)$ with the sets of all $n \times h$-matrices and $n \times n$-matrices, respectively. Furthermore, denote the space $\mathcal{L}(\ell_2^n, \ell_2^h)$ equipped with the Hilbert–Schmidt norm by $\mathcal{S}_2(\ell_2^n, \ell_2^h)$, and define $\mathcal{S}_F(\ell_2^n, \ell_2^h)$ likewise. Clearly, the natural injection $i_{2E} : \mathcal{S}_2(\ell_2^n, \ell_2^h) \hookrightarrow \mathcal{S}_E^n$ has norm asymptotically equivalent to $\lambda_E(h)/\sqrt{h}$. Now let $P_{F2} : \mathcal{S}_F^n \to \mathcal{S}_2(\ell_2^n, \ell_2^h)$ be the natural projection which cuts off the last $n - h + 1$ rows. Observe that any matrix in $\mathcal{L}(\ell_2^n, \ell_2^h)$ has at most h nonzero singular values. Since $s_k(P_{F2}A) \le s_k(A)$ for all $A \in \mathcal{L}(\ell_2^n)$, we have $\|P_{F2} : \mathcal{S}_F^n \to \mathcal{S}_F(\ell_2^n, \ell_2^h)\| \le 1$. Hence, for $A \in \mathcal{L}(\ell_2^n)$ and $\sigma = (\sigma_1, \ldots, \sigma_\ell)$ the nonzero singular values of $P_{F2}A$, with $\ell \le h$,

$$\|P_{F2}A \,|\, \mathcal{S}_2(\ell_2^n, \ell_2^h)\| = \|\sigma\|_2 \le \|\mathrm{id}_{F2}^h\|\,\|\sigma\|_F \le \frac{\lambda_F(h)}{\sqrt{h}}\,\|A \,|\, \mathcal{S}_F^n\|.$$

Thus, since $\mathrm{id}_{\mathcal{S}_2(\ell_2^n,\ell_2^h)} = P_{F2} \circ (\mathrm{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \circ i_{2E}$, it follows by the definition of h_k that

$$h_k(\mathrm{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \geq \frac{a_k(\mathrm{id}_{\mathcal{S}_2(\ell_2^n, \ell_2^h)})}{\lambda_E(h)/\lambda_F(h)} = \frac{\lambda_F(h)}{\lambda_E(h)} \geq \frac{1}{2} \frac{\lambda_F(k/n)}{\lambda_E(k/n)},$$

which gives the desired estimate. ∎

All the above together now gives the following examples. As usual, we set $\mathcal{S}_{p,q}^n := \mathcal{S}_{\ell_{p,q}}^n$.

COROLLARY 8.8. *Let $n \leq k \leq n^2$.*

(i) *Let $1 \leq u \leq 2 \leq v \leq \infty$. Then*

$$h_k(\mathrm{id} : \mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp (n/k)^{1/u-1/v}.$$

(ii) *Let $1 < u < 2 < v < \infty$ and $1 \leq r \leq 2 \leq s \leq \infty$. Then*

$$h_k(\mathrm{id} : \mathcal{S}_{u,r}^n \hookrightarrow \mathcal{S}_{v,s}^n) \asymp (n/k)^{1/u-1/v}.$$

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